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On self-normalising subgroups of finite groups

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Abstract

The aim of this paper is to characterise the classes of groups in which every subnormal subgroup is normal, permutable, or S-permutable by the embedding of the subgroups (respectively, subgroups of prime power order) in their normal, permutable, or S-permutable closure, respectively.

Keywords: finite group, permutability, Sylow permutability, permutable closure, subnormal closure, *PST*-group, *PT*-group, *T*-group.

Mathematics Subject Classification (2000): 20D10, 20D20, 20D35

Dedicated to Derek J. S. Robinson on the occasion of his seventieth birthday.

1 Introduction and statement of results

In this paper, we consider only finite groups.

A subgroup H of a group G is said to be *permutable* in G if H permutes with every subgroup of G . A group G is said to be a *PT-group* (respectively, *T-group*) if permutability (respectively, normality) is a transitive relation in G . By a result of Ore [13], *PT*-groups are exactly those groups where all subnormal subgroups are permutable. *PST*-groups are also defined via a transitivity property, namely with respect to S-permutability ([11]): a subgroup of a group G is called *S-permutable* if it permutes with all the Sylow

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subgroups of G . By a result of Kegel ([11, Satz 1]), every S-permutable subgroup is subnormal and hence PST -groups are exactly those groups in which all subnormal subgroups are S-permutable.

Note that the class of T -groups is a proper subclass of the class of PT -groups, which in turns forms a proper subclass of the class of PST -groups. These classes have been studied in detail, with many characterisations available (see [1, 2, 3, 4, 5, 6, 7, 14, 15]).

The basic structure of soluble T -, PT -, and PST -groups were established by Gaschütz, Zacher, and Agrawal, respectively, and are presented in the following theorem.

Theorem 1. *Let L be the nilpotent residual of a group G . Then:*

1. *(Agrawal, [1]) G is a soluble PST -group if and only if L is an abelian Hall subgroup of odd order of G on which G acts by conjugation as a group of power automorphisms.*
2. *(Zacher, [15]) G is a soluble PT -group if and only if G is a soluble PST -group with modular Sylow subgroups.*
3. *(Gaschütz, [10]) G is a soluble T -group if and only if G is a soluble PST -group with Dedekind Sylow subgroups.*

The results of the present paper spring from a recent characterisation of soluble T -groups due to the third author. There he proves that a group is a soluble T -group if and only if every subgroup is self-normalising in its normal closure ([12, Theorem 3.1]). It is natural to wonder whether or not it is possible to get similar characterisations of soluble PT - and PST -groups by using permutable and S-permutable versions of the normal closure. This is the goal of the present paper.

Since the intersection of S-permutable subgroups of a group G is again a S-permutable subgroup of G , it seems reasonable to replace the normal closure of a subgroup H by the intersection of all S-permutable subgroups of G containing H in the PST -case. Unfortunately, the intersection of permutable subgroups of G is not permutable in general. Therefore the intersection of all permutable subgroups of G containing a given subgroup H is not the smallest permutable subgroup of G containing H in general. This is the main reason why the proofs concerning PT -groups are much more involved than the corresponding ones for T - and PST -groups. Despite this fact, the above subgroup will play a central role in our approach.

Definition 2. Let H be a subgroup of a group G .

1. The permutable closure $A_G(H)$ of H in G is the intersection of all permutable subgroups of G containing H .
2. The S-permutable closure $B_G(H)$ of H in G is the intersection of all S-permutable subgroups of G containing H .

Applying [11], $B_G(H)$ is S-permutable in G . However, $A_G(H)$ is not permutable in general, as the following example shows:

Example 3. Suppose that p is a prime and $m > 1$ is a natural number. Let $G = \langle a, b, c \mid a^{p^m} = b^p = c^p = 1, b^a = bc, c^a = c^b = c \rangle$. The subgroups $H_1 = \langle b, c \rangle$ and $H_2 = \langle b, ca^{p^{m-1}} \rangle$ are permutable in G , but the intersection $H = H_1 \cap H_2$ is not permutable in G . For the subgroup H , the permutable closure $A_G(H) = H$ is not permutable in G .

However we have:

Theorem 4. *Assume that G is a group such that every subgroup is self-normalising in its permutable closure. Then $A_G(H)$ is a permutable subgroup of G for each subgroup H of G .*

Theorem 4 is a consequence of a stronger result:

Theorem 5. *Let p be a prime. If every p -subgroup of a group G is self-normalising in its permutable closure, then $A_G(H)$ is a permutable subgroup of G for every p -subgroup H of G .*

These results justify the study of the class of groups in which every subgroup is self-normalising in its permutable closure. It turns out that this class is the class of all soluble PT -groups.

If we fix a prime p , the class of groups for which every p -subgroup is self-normalising in its permutable closure is a subclass of a class which can be considered as a local version of the class of all soluble PT -groups.

Definition 6 ([5]). A group G satisfies \mathcal{X}_p if and only if each subgroup of a Sylow p -subgroup P of G is permutable in the normaliser $N_G(P)$.

Theorem 7. *If p is a prime and G is a group in which every p -subgroup is self-normalising in its permutable closure, then G satisfies \mathcal{X}_p .*

Theorem 8. *For a group G , the following statements are equivalent:*

1. G is a soluble PT -group.
2. H is abnormal in $A_G(H)$ for every subgroup H of G .

3. $N_G(H) \cap A_G(H) = H$ for every subgroup H of G .
4. For every prime p and every p -subgroup H of G , H is abnormal in $A_G(H)$.
5. For every prime p and every p -subgroup H of G , we have $N_G(H) \cap A_G(H) = H$.

Theorem 7 follows from the local strategy we use in the *PST*-case. In fact, this local point of view leads to the local defining property of the class of soluble *PST*-groups.

Recall that if p is a prime, a group G satisfies property \mathcal{Y}_p if for each pair of p -subgroups H and K of G such that H is contained in K , then H is S -permutable in $N_G(K)$ ([4]). A group G satisfies \mathcal{C}_p if every subgroup of a Sylow p -subgroup P of G is normal in $N_G(P)$ ([14]).

A group G is a soluble *PST*-group if and only if G satisfies \mathcal{Y}_p for all primes ([4, Theorem 4]). Similar results hold for soluble *PT*-groups and property \mathcal{X}_p ([5]), and soluble *T*-groups and property \mathcal{C}_p ([14]). These results are consequences of the following:

Theorem 9 ([4, Theorem 3]). *Let p be a prime. A group G satisfies \mathcal{X}_p (respectively, \mathcal{C}_p) if and only if G satisfies \mathcal{Y}_p and the Sylow p -subgroups of G are modular (respectively, Dedekind).*

If p is a prime and every p -subgroup is self-normalising in its permutable closure, then Sylow subgroups are modular. Hence Theorem 7, by virtue of Theorem 9, is the permutable local version of the following:

Theorem 10. *If p is a prime and G is a group such that every p -subgroup of G is self-normalising in its S -permutable closure, then G satisfies \mathcal{Y}_p .*

The converse of the above result does not hold.

Example 11. Suppose that p and q are two primes such that q divides $p - 1$. Let E be an extraspecial group of order p^3 and exponent p . Let i and j be two numbers such that $ij \equiv 1 \pmod{p - 1}$ and i and j have order q modulo p . Let $\{x, y\}$ be a generating system for E and let z be an automorphism of order q of E given by $x^z = x^i$, $y^z = y^j$. Let $G = [E]\langle z \rangle$ be the corresponding semidirect product, then the S -permutable closure of $Z = \langle z \rangle$ is G , but $N_G(Z) = \langle z, [x, y] \rangle$. However, the group G satisfies \mathcal{Y}_q because it is q -nilpotent.

Note that if H is a p -subgroup of G and $N_G(H) \cap \langle H^G \rangle = H$, where $\langle H^G \rangle$ is the normal closure of H in G , we have that H is a Sylow p -subgroup

of $\langle H^G \rangle$. Hence if the above condition holds for every p -subgroup of G , it follows that the Sylow p -subgroups of G are Dedekind groups. Therefore applying Theorems 9 and 10 we have:

Corollary 12. *Let p be a prime and let G be a group. If every p -subgroup of G is self-normalising in its normal closure, then G satisfies property \mathcal{C}_p .*

We are now in a position to give characterisations of groups in which every subgroup is self-normalising in its S-permutable closure: they turn out to be the soluble *PST*-groups.

Theorem 13. *Let G be a group. Any two of the following five statements are equivalent:*

1. G is a soluble *PST*-group.
2. For every subgroup H of G , H is abnormal in $B_G(H)$.
3. For every subgroup H of G , the equality $N_G(H) \cap B_G(H) = H$ holds.
4. If p is any prime and H is a p -subgroup of G , then H is abnormal in $B_G(H)$.
5. If p is any prime and H is a p -subgroup of G , it follows that $N_G(H) \cap B_G(H) = H$.

Combining Theorem 13 and Corollary 12, we have:

Corollary 14 ([12, Theorem 3.1]). *Any two of the following assertions about a group G are equivalent:*

1. G is a soluble *T*-group.
2. H is abnormal in $\langle H^G \rangle$ for all subgroups H of G .
3. $N_G(H) \cap \langle H^G \rangle = H$ for all subgroups H of G .
4. H is abnormal in $\langle H^G \rangle$ for every p -subgroup H of G and every prime p .
5. $N_G(H) \cap \langle H^G \rangle = H$ for every p -subgroup H of G and every prime p .

2 Proofs

We begin the section with a pair of lemmas, which will be used several times in subsequent proofs.

Lemma 15. *Let G be a group and let N be a normal subgroup of G . If H is a subgroup of G , then $A_G(H)N/N$ and $B_G(H)N/N$ are subgroups of $A_{G/N}(HN/N)$ and $B_{G/N}(HN/N)$, respectively. If N is contained in $A_G(H)$, then $A_G(H)/N = A_{G/N}(HN/N)$. The same is true for $B_G(H)$.*

Proof. The assertions made in the lemma follow from the fact that a subgroup X of G containing N is permutable (respectively, S-permutable) in G if and only if X/N is permutable (respectively, S-permutable) in G/N . \square

Lemma 16. *Let H be a subgroup of a group G and let S be a subgroup of G containing H . Then $A_S(H) \leq A_G(H)$ and $B_S(H) \leq B_G(H)$.*

Proof. Note that if X is a permutable (respectively, S-permutable) subgroup of G containing H , then $X \cap S$ is a permutable (respectively, S-permutable) subgroup of S containing H . \square

Corollary 17. *The classes of all groups in which every subgroup is self-normalising in its permutable (respectively, S-permutable) closure are closed under taking subgroups and factor groups.*

Proof of Theorem 5. Assume that every p -subgroup of G is self-normalising in its permutable closure. We prove that $A_G(H)$ is permutable in G by induction on the order of G . If N is a minimal normal subgroup of G contained in $A_G(H)$, then $A_G(H)/N = A_{G/N}(HN/N)$ by Lemma 15. Since the hypotheses of the theorem hold in G/N , we have that $A_G(H)/N$ is permutable in G/N by induction. Hence $A_G(H)$ is permutable in G , as required. Therefore we can suppose that $\text{Core}_G(A_G(H)) = 1$. On the other hand, applying [11, Satz 2], $A_G(H)$ is S-permutable in G and so $A_G(H)$ is nilpotent by a result of Deskins [8, Theorem 1]. Since H is self-normalising in $A_G(H)$, it follows that $A_G(H)$ is a p -group and so $H = A_G(H)$ is S-permutable in G . By [11, Satz 1], H is subnormal in G . Let P be a Sylow p -subgroup of G . Then H is contained in P . If P were a proper subgroup of G , then H would be permutable in P because P inherits the hypotheses of the theorem. It would imply the permutability of H in G , as required. Hence we may assume that $P = G$. Then a minimal normal subgroup N of G is central. Since H is self-normalising in G , it follows that N must be contained in H , contradicting the fact that $\text{Core}_G(H) = 1$. Therefore H is permutable in P and so in G . \square

Proof of Theorem 10. Let H and K be p -subgroups of G such that H is contained in K and let P be a Sylow p -subgroup of G containing K . Write $T = N_G(K)$. We must prove that H is S-permutable in T . Applying Lemma 16, H is self-normalising in $B_T(K)$. If H is not S-permutable in T , then H is a proper subgroup of $B_T(H)$. It implies that H is a proper subgroup of its normaliser in $B_T(H)$ as H is a subnormal subgroup of T . This contradiction shows that H is S-permutable in $N_G(K)$ and G has property \mathcal{Y}_p . \square

Proof of Theorem 13. 1 implies 2. We suppose that the implication is not true and derive a contradiction. Let (G, H) be a counterexample with $|G|$ minimal. Assume that $A = B_G(H)$ is a proper subgroup of G . Then, by Theorem 1, A is a soluble PST -group and so H is abnormal in $B_A(H)$. Since $B_A(H)$ is S-permutable in A and A is S-permutable in G , it follows that $B_A(H)$ is S-permutable in G because G is a PST -group. Consequently $A = B_A(H)$ and then H is abnormal in A . This contradiction shows that $A = G$. If $N := \text{Core}_G(H) \neq 1$, we have that H/N is abnormal in G/N by the minimal choice of G . Hence H is abnormal in G . Thus we can assume that $N := \text{Core}_G(H) = 1$. Applying Agrawal's theorem (Theorem 1), $G = LM$, where L is the nilpotent residual of G , $L \cap M = 1$, and L is an abelian normal Hall subgroup of G of odd order acted upon by conjugation as a group of power automorphisms by M . It implies that every subgroup of L is normal in G and so $L \cap H = 1$. Hence we can assume, without loss of generality, that H is contained in M . Since M is nilpotent, we have that LH is S-permutable in G . Thus $B_G(H) = G = LH$ and $H = M$. Applying [9, IV, 5.18 and III, 4.6], H is a Carter subgroup of G . Applying [9, IV, Section 3 and 4.6, and I, 6.21], H is abnormal in G . This is the desired contradiction.

On the other hand, by virtue of [9, I, 6.21], every abnormal subgroup is self-normalising. Therefore 2 implies 3 and 4 implies 5. It is clear that 2 implies 4 and 3 implies 5.

To complete the proof we now show that 5 implies 1. Assume that every p -subgroup is self-normalising in its S-permutable closure for each prime p . By Theorem 10, G satisfies \mathcal{Y}_p for each prime p . Applying [4, Theorem 4], we conclude that G is a soluble PST -group. \square

Proof of Theorem 8. The same arguments to those used in the proof of Theorem 13 replacing Agrawal's result by Zacher's result (see Theorem 1) show that 1 implies 2. It is clear that 2 implies 3 and 4 implies 5. Obviously 2 implies 4 and 3 implies 5.

Now if every p -subgroup is self-normalising in its permutable closure for each prime p , then G satisfies \mathcal{X}_p for each prime p by virtue of Theorem 7. Applying [5, Theorem A], G is a soluble PT -group. \square

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References

- [1] R. K. Agrawal. Finite groups whose subnormal subgroups permute with all Sylow subgroups. *Proc. Amer. Math. Soc.*, 47(1):77–83, 1975.
- [2] M. J. Alejandro, A. Ballester-Bolinches, and M. C. Pedraza-Aguilera. Finite soluble groups with permutable subnormal subgroups. *J. Algebra*, 240(2):705–722, 2001.
- [3] A. Ballester-Bolinches and R. Esteban-Romero. Sylow permutable subnormal subgroups of finite groups II. *Bull. Austral. Math. Soc.*, 64(3):479–486, 2001.
- [4] A. Ballester-Bolinches and R. Esteban-Romero. Sylow permutable subnormal subgroups of finite groups. *J. Algebra*, 251(2):727–738, 2002.
- [5] J. C. Beidleman, B. Brewster, and D. J. S. Robinson. Criteria for permutability to be transitive in finite groups. *J. Algebra*, 222(2):400–412, 1999.
- [6] J. C. Beidleman, H. Heineken, and M. F. Ragland. Strong Sylow bases and mutually permutable products. Preprint, 2008.
- [7] R. A. Bryce and J. Cossey. The Wielandt subgroup of a finite soluble group. *J. London Math. Soc.*, 40(2):244–256, 1989.
- [8] W. E. Deskins. On quasinormal subgroups of finite groups. *Math. Z.*, 82:125–132, 1963.
- [9] K. Doerk and T. Hawkes. *Finite Soluble Groups*, volume 4 of *De Gruyter Expositions in Mathematics*. Walter de Gruyter, Berlin, New York, 1992.
- [10] W. Gaschütz. Gruppen, in denen das Normalteilersein transitiv ist. *J. reine angew. Math.*, 198:87–92, 1957.
- [11] O. H. Kegel. Sylow-Gruppen und Subnormalteiler endlicher Gruppen. *Math. Z.*, 78:205–221, 1962.

- [12] Y. Li. Finite groups with NE-subgroups. *J. Group Theory*, 9:49–58, 2006. 2006.
- [13] O. Ore. Contributions to the theory of groups of finite order. *Duke Math. J.*, 5:431–460, 1939.
- [14] D. J. S. Robinson. A note on finite groups in which normality is transitive. *Proc. Amer. Math. Soc.*, 19:933–937, 1968.
- [15] G. Zacher. I gruppi risolubli finiti in cui i sottogruppi di composizione coincidono con i sottogruppi quasi-normali. *Atti Accad. Naz. Lincei Rend. cl. Sci. Fis. Mat. Natur. (8)*, 37:150–154, 1964.