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On some classes of supersoluble groups

A. Ballester-Bolinches* J.C. Beidleman† R. Esteban-Romero‡

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Dedicated to Professor Hermann Heineken on the occasion of his seventieth birthday

Abstract

Finite groups G for which for every subgroup H and for all primes q dividing the index $|G : H|$ there exists a subgroup K of G such that H is contained in K and $|K : H| = q$ are called $\mathcal{Y}$-groups. Groups in which subnormal subgroups permute with all Sylow subgroups are called PST-groups. In this paper a local version of the $\mathcal{Y}$-property leading to a local characterisation of $\mathcal{Y}$-groups, from which the classical characterisation emerges, is introduced. The relationship between PST-groups and $\mathcal{Y}$-groups is also analysed.

1 Introduction and statement of results

In this paper, only finite groups will be taken into account.

A well-known theorem of Lagrange (see [11, I, 2.7]) states that given a subgroup H of a group G, the order of G is the product of the order $|H|$ of H and the index $|G : H|$ of H in G. In particular, the order of any subgroup divides the order of the group. The converse, namely, if d divides the order of a group G, then G has a subgroup of order d, is not true in general. Groups satisfying this condition are often called CLT-*groups*. The

*Departament d'Àlgebra, Universitat de València, Dr. Moliner, 50, E-46100 Burjassot, València, Spain, e-mail: Adolfo.Ballester@uv.es

†Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506-0027, USA, e-mail: clark@ms.uky.edu

‡Departament de Matemàtica Aplicada-IMPA, Universitat Politècnica de València, Camí de Vera, s/n, E-46022 València, Spain, e-mail: resteban@mat.upv.es

alternating group of order 12, having no subgroups of order 6, is an example of a non-CLT-group.

On the other hand, abelian groups contain subgroups of every possible order, and it is not difficult to prove that a group is nilpotent if and only if it contains a normal subgroup of each possible order [10]. Ore [13] and Zappa [16] obtained a similar characterisation for supersoluble groups:

Theorem 1. *A group G is supersoluble if and only if each subgroup $H \leq G$ contains a subgroup of order d for each divisor d of $|H|$.*

Of course, we can state Theorem 1 in the following equivalent way, more easily treated:

Theorem 2. *A group G is supersoluble if and only if each subgroup $H \leq G$ contains a subgroup of index p for each prime divisor p of $|H|$.*

A proof of this theorem can be found in [14, Chapter 1, 4.3]. It must be noted that CLT-groups are not necessarily supersoluble, as the symmetric group of order 4 shows.

The condition on a group G given in Theorem 2, namely

> for all $H \leq G$ and for all primes q dividing $|H|$, there exists a subgroup K of G such that $K \leq H$ and $|H : K| = q$,

has a dual formulation:

> for all $H \leq G$ and for all primes q dividing $|G : H|$, there exists a subgroup K of G such that $H \leq K$ and $|K : H| = q$.

Groups satisfying the latter condition have been studied by some authors. Following [14, Chapter 1, 4], we will call them $\mathcal{Y}$-groups.

Definition 3. A group G is said to be a *$\mathcal{Y}$-group* if for all subgroups H of G and all primes q dividing the index $|G : H|$ of H in G, there exists a subgroup K of G with $H \leq K$ and $|K : H| = q$.

Note that a group G is a $\mathcal{Y}$-group if and only if for every subgroup H of G and for every natural number d dividing $|G : H|$ there exists a subgroup K of G such that $H \leq K$ and $|K : H| = d$. The following characterisation of $\mathcal{Y}$-groups appears in [14, Chapter 1, 4.3].

Theorem 4. *Let $L = G^{\mathfrak{N}}$ be the nilpotent residual of the group G. Then G is a $\mathcal{Y}$-group if and only if L is a nilpotent Hall subgroup of G such that for all subgroups H of L, $G = L\,\mathrm{N}_G(H)$.*

From Theorem 4, we see that if $G \in \mathcal{Y}$ and X is a normal subgroup of L, then X is normal in G. In particular, $\mathcal{Y}$-groups are supersoluble. Moreover, if $G \in \mathcal{Y}$, then L must have odd order.

Further results on $\mathcal{Y}$-groups can be found in [14, Chapter 6, 6.1]. For example, a soluble group G is a $\mathcal{Y}$-group if and only if every subgroup of G can be written as an intersection of subgroups of G of coprime prime-power indices.

On the other hand, we say that a subgroup H of a group G is S-*permutable* in G when it permutes with every Sylow subgroup of G. According to [12], S-permutable subgroups are subnormal and the set of all S-permutable subgroups of a group G is a sublattice of the lattice of all subnormal subgroups of G.

A group G is said to be a PST-*group* when every subnormal subgroup of G is S-permutable, that is, when S-permutability is a transitive relation. Some interesting subclasses of the class of all PST-groups are the class of PT-*groups* or groups in which permutability is a transitive relation and the class of T-groups or groups in which normality is a transitive relation.

Soluble PST-groups were studied by Agrawal [1], and, more recently, by Alejandre, the first author, and Pedraza-Aguilera in [2], by the first and the last author [3, 4], and by the second author and Heineken [6], among others. The approach followed in these papers began with a paper of Bryce and Cossey [7] in which a local version of some of the results on T-groups was presented.

Let us recall the classical theorem of Agrawal:

Theorem 5. *A group G is a soluble* PST-*group if and only if G has an abelian normal Hall subgroup N of odd order such that G/N is nilpotent and the elements of G induce power automorphisms in N.*

If we add in this result "G/N nilpotent modular group," we obtain the characterisation of soluble PT-groups given by Zacher [15], and if we put "G/N Dedekind," we get Gaschütz's characterisation of soluble T-groups [9].

Agrawal's theorem has the virtue of showing that the class of soluble PST-groups is closed under taking subgroups. A consequence of Theorems 4 and 5, Gaschütz's characterisation, and Dedekind theorem [11, III, 7.12] is:

Corollary 6. *Let G be a group.*

1. *If G is a soluble* PST-*group, then G is a $\mathcal{Y}$-group.*

2. *Assume that $G \in \mathcal{Y}$. Then G is a soluble* PST-*group if and only if the nilpotent residual of G is abelian.*

3. *Assume that $G \in \mathcal{Y}$. Then G is a soluble* T*-group if and only if all Sylow subgroups of G are Dedekind.*

For a prime p, Bryce and Cossey [7] defined the class T_p of all soluble groups G for which every subnormal p'-perfect subgroup of G is normal. They proved:

Theorem 7. *A soluble group is a* T*-group if and only if it is a* T_p*-group for all primes p.*

In [2], Alejandre, the first author, and Pedraza-Aguilera introduced in the soluble universe the class PST_p of all soluble groups G in which every p'-perfect subnormal subgroup in G permutes with every Hall p'-subgroup of G. This condition is equivalent to G being p-supersoluble and having all its p-chief factors isomorphic when regarded as modules over G (see [2]). This result not only holds in the soluble universe, but also in the p-soluble one.

Theorem 8. *A soluble group G is a* PST*-group if and only if G is a* PST_p*-group for all primes p.*

The second author and Heineken defined in [6] the class T''_p, for a prime p, of all soluble groups G in which every p'-perfect subnormal subgroup of G is S-permutable in G and proved:

Theorem 9. *A soluble group G is a* PST*-group if and only if it is a* T''_p*-group for all primes p.*

A similar result holds for PT-groups replacing S-permutability by permutability.

In [3, Theorem A], the following local version of Agrawal's result was obtained. For each group X and every prime p, $X(p)$ denotes the p-nilpotent residual of X, that is, the smallest normal subgroup N of X such that X/N is p-nilpotent, while $\mathrm{O}_{p'}(X)$ denotes the largest normal p'-subgroup of X.

Theorem 10. *A p-soluble group G is a* PST_p*-group if, and only if, one of the following two conditions holds:*

1. *G is p-nilpotent, or*

2. *the subgroup $G(p)/\mathrm{O}_{p'}\big(G(p)\big)$ is an abelian normal Sylow p-subgroup of $G/\mathrm{O}_{p'}\big(G(p)\big)$ in which the elements of $G/\mathrm{O}_{p'}\big(G(p)\big)$ induce power automorphisms.*

Theorem 5 follows from Theorems 8 and 10, as shown in [3].

These local results, together with Corollary 6, encourages us for the search of local versions of the $\mathcal{Y}$-property, leading to a local characterisation of $\mathcal{Y}$-groups, running parallel to the characterisations for PST_p-groups, p a prime. This is the aim of the present paper.

In the sequel, p will denote a fixed prime number.

Definition 11. We say that G *satisfies* $\mathcal{Z}_p$ when for every p-subgroup X of G and for every power of a prime q, q^m, dividing $|G : X \mathrm{O}_{p'}(G)|$, there exists a subgroup K of G containing $X \mathrm{O}_{p'}(G)$ such that $|K : X \mathrm{O}_{p'}(G)| = q^m$.

Note that if $q = p$, the condition is obviously satisfied in every group.

Definition 12. Let G be a group. We say that G *satisfies* $\mathcal{Z}'_p$ if G satisfies either of the following conditions:

1. G is p-nilpotent, or
2. $G(p)/\mathrm{O}_{p'}\bigl(G(p)\bigr)$ is a Sylow p-subgroup of $G/\mathrm{O}_{p'}\bigl(G(p)\bigr)$ and for every p-subgroup H of $G(p)$, we have that $G = G(p)\,\mathrm{N}_G(H)$.

Our first main result can be regarded as the analogue of Theorem 10:

Theorem 13. *Let G be a p-soluble group. Then G satisfies $\mathcal{Z}_p$ if and only if G satisfies $\mathcal{Z}'_p$.*

Combining Theorems 10 and 13, we have:

Theorem 14. *A p-soluble group G satisfies* PST_p *if and only if G satisfies $\mathcal{Z}_p$ and G has an abelian Sylow p-subgroup.*

Our second main result is the analogue of Theorem 8.

Theorem 15. *Let G be a soluble group. G satisfies $\mathcal{Y}$ if and only if G satisfies $\mathcal{Z}_p$ for every prime p.*

Recall that class of groups which is closed under taking epimorphic images and subdirect products is called a *formation*. In [5] it has been proved that the largest formation contained in the class PST_p is the class $\mathfrak{E}_{p'}\mathfrak{S}_p$ of all p-nilpotent groups. As a consequence, the class of all nilpotent groups is the largest formation contained in the class of all PST-groups. A similar result can be obtained for the class of all groups satisfying $\mathcal{Z}_p$:

Theorem 16. *The class of all p-nilpotent groups is the largest formation contained in the class $\mathcal{Z}_p$.*

As a consequence:

Corollary 17. *The class of all nilpotent groups is the largest formation contained in the class $\mathcal{Y}$.*

We bring the paper to a close by giving an alternative proof of Theorem 4 which is based on our local approach. We also show that the class $\mathcal{Y}$ is a proper subclass of the class of all supersoluble groups and that the classes $\mathcal{Z}_p$ are not subgroup-closed in general.

The notation used in this paper is standard. For notation not explained, we address the reader to the book [8].

2 Proofs

The following lemmas turn out to be crucial in the proofs of our main results. The first and second ones are very useful in induction arguments. The proof of the first lemma is a routine check.

Lemma 18. *Let G be a group and let N be a normal subgroup of G. Then:*

1. *If N is a p'-subgroup, then G satisfies $\mathcal{Z}_p$ if and only if G/N satisfies $\mathcal{Z}_p$.*
2. *If N is a p-group and G satisfies $\mathcal{Z}_p$, then G/N satisfies $\mathcal{Z}_p$.*
3. *If G is a group, G is not p-nilpotent, and N is a normal subgroup of G contained in $\mathrm{O}_{p'}\big(G(p)\big)$, then G is satisfies $\mathcal{Z}'_p$ if and only if G/N satisfies $\mathcal{Z}'_p$.*
4. *If G is a group satisfying $\mathcal{Z}'_p$ and N is a normal p-subgroup of G, then G/N satisfies $\mathcal{Z}'_p$.*

Lemma 19. *If G is a group and N is a normal p'-subgroup of G, then G satisfies $\mathcal{Z}'_p$ if and only if G/N satisfies $\mathcal{Z}'_p$.*

Proof. Assume that G satisfies $\mathcal{Z}'_p$. If G is p-nilpotent, then G/N is also p-nilpotent and so G/N satisfies $\mathcal{Z}'_p$. Suppose that G is not p-nilpotent. According to Lemma 18, we can suppose that $\mathrm{O}_{p'}\big(G(p)\big) = 1$ by changing N by $N\,\mathrm{O}_{p'}\big(G(p)\big)/\,\mathrm{O}_{p'}\big(G(p)\big)$ if needed. Then $(G/N)(p) = G(p)N/N$ is a Sylow p-subgroup of G/N. It is clear that $G/N = \mathrm{N}_G(H)G(p)/N = \mathrm{N}_{G/N}(HN/N)(G/N)(p)$ for every p-subgroup HN/N of G/N.

Conversely, assume that there exists a group G having a normal p'-subgroup N such that G/N satisfies $\mathcal{Z}'_p$, but G does not satisfy $\mathcal{Z}'_p$. We choose G of minimal order. We can suppose that N is a minimal normal subgroup of G and N is not contained in $G(p)$. In this case, $(G/N)(p) = \big(G(p)N\big)/N$. Moreover, $\mathrm{O}_{p'}\big((G/N)(p)\big) = \Big(\mathrm{O}_{p'}\big(G(p)\big)N\Big)/N$. It follows that $G(p)/\mathrm{O}_{p'}\big(G(p)\big)$ is a Sylow p-subgroup of $G/\mathrm{O}_{p'}\big(G(p)\big)$. Let H be a p-subgroup of G. Then

$$G/N = \mathrm{N}_{G/N}(HN/N)\big((G/N)(p)\big) = \big(\mathrm{N}_G(H)N/N\big)\big(G(p)N/N\big).$$

This implies that $G = \mathrm{N}_G(H)G(p)N$. But N centralises $G(p)$ and so H. It follows that $G = \mathrm{N}_G(H)G(p)$, and G satisfies condition $\mathcal{Z}'_p$. □

Lemma 20. *Let G be a p-soluble group satisfying $\mathcal{Z}_p$. Then G is p-supersoluble.*

Proof. Assume that the result is false. Consider a p-soluble group G of minimal order such that G satisfies $\mathcal{Z}_p$, but G is not p-supersoluble. From the p-solubility of G, we can assume that G has a unique minimal normal subgroup, N say, and that N is a non-cyclic p-group. Moreover $\mathrm{O}_{p'}(G) = 1$. Let Z be a subgroup of N of order p contained in the centre of a Sylow p-subgroup P of G. Let q be a prime different from p dividing $n = |G : Z|$ and let q^m be the largest power of q dividing n. Since G has property $\mathcal{Z}_p$, there exists a subgroup K of G such that $|K : Z| = q^m$. Moreover Z is a Sylow p-subgroup of K and is subnormal in G. This implies that Z is normal in K. In particular, Z is normalised by a Sylow q-subgroup of G. Since this happens for all primes $q \neq p$ and Z is normal in P, we have that Z is a normal subgroup of G. Hence $N = Z$ because N is a minimal normal subgroup of G and so N has order p. This contradiction shows that no such counterexample exists and the result is proved. □

Lemma 21. *If G is p-soluble and satisfies $\mathcal{Z}'_p$, then G is p-supersoluble.*

Proof. Assume that there exist p-soluble groups satisfying $\mathcal{Z}'_p$ which is not p-supersoluble, and among them we choose a group G of minimal order. Clearly G is not p-nilpotent. By Lemma 19, we have that $\mathrm{O}_{p'}(G) = 1$. Thus $G(p)$ is a normal Sylow p-subgroup of G. Let N be a minimal normal subgroup of G, then N is a p-group and N cannot be cyclic, by Lemma 18 and the minimality of G. Let M be a minimal normal subgroup of $G(p)$ contained in N. We have that $G = \mathrm{N}_G(M)G(p) = \mathrm{N}_G(M)$. Therefore M is normal in G. This contradiction shows that no such counterexample can exist and the lemma is proved. □

The following lemma is fundamental to understand the class $\mathcal{Y}$ and its local versions.

Lemma 22. *Let G be a p-soluble group satisfying $\mathcal{Z}_p$ and let H be a subgroup of $\mathrm{O}_p(G)$. Then H is normalised by a Hall p'-subgroup of G.*

Proof. Consider a prime q different from p. There exists a subgroup K of G such that $H\,\mathrm{O}_{p'}(G) \leq K$ and $|K : H\,\mathrm{O}_{p'}(G)| = q^m$, where q^m is the largest power of q dividing $|K : H\,\mathrm{O}_{p'}(G)|$. A Sylow q-subgroup K_q of K is a Sylow q-subgroup of G and H is a subnormal Sylow p-subgroup of K. Therefore H is normal in K and, in particular, K_q normalises H. Since this happens for every $q \neq p$, we have that $\langle K_q \mid q \neq p\rangle$ normalises H. In particular, there exists a Hall p'-subgroup of G normalising H. $\square$

Proof of Theorem 13. Assume that G satisfies $\mathcal{Z}_p$. We show that G also satisfies $\mathcal{Z}_p'$. Since p-nilpotent groups satisfy $\mathcal{Z}_p'$, there is no loss of generality in supposing that G is not p-nilpotent. By Lemma 20, we have that G is p-supersoluble. In particular, the derived subgroup G' of G is p-nilpotent by [11, VI, 9.1(a)]. Since $G(p) \leq G'$, we have that $G(p)$ is p-nilpotent, too. Moreover, by Lemma 19, we can assume that $\mathrm{O}_{p'}(G) = 1$. It is clear then that $G(p)$ is a p-group.

We prove that $G(p)$ is a Sylow p-subgroup of G. Assume that this is false and derive a contradiction. In this case, G has both central and non-central p-chief factors. Bearing in mind that G' is p-nilpotent and $\mathrm{O}_{p'}(G') = 1$, we obtain that G' is a p-group. Hence G has a normal Sylow p-subgroup, L say. Consider a chief series of G passing through L and $G(p)$, and consider two chief factors of that series, $K/G(p)$ and $G(p)/M$, say. Then $K/G(p)$ is central in G and $G(p)/M$ is not central in G. Applying [8, IV, 6.7], $K/G(p)$ is not cyclic. Therefore K/M is a p-elementary abelian group of order p^2, because G is p-supersoluble.

Let H be a Hall p'-subgroup of G. It is clear that K/M is an H-module over $\mathrm{GF}(p)$. By Maschke's theorem [8, A, 11.4], $K/M = G(p)/M \times C/M$, where C/M is normalised by H. Let $G(p)/M = \langle aM\rangle$ and $C/M = \langle bM\rangle$. Note that C/M is H-isomorphic to $K/G(p)$, and so C/M is in fact centralised by H. Consider $D = M\langle ab\rangle$. By Lemma 18, we have that G/M satisfies $\mathcal{Z}_p$. By Lemma 22, there exists a Hall p'-subgroup H_1 of G normalising D.

Since all Hall p'-subgroups are conjugate by [11, VI, 1.7], there exists an element $g \in G$ such that $H_1 = H^g$. Moreover, $G = HL$ and so $g = hx$ with $h \in H$ and $x \in L$. Hence $H_1 = H^x$ with $x \in L$.

Let y be an element of H such that $a^yM \neq aM$. Then $a^yM = a^iM$, for some natural number $i \geq 2$. Note that $K = \langle a, b\rangle M$ and $K^{x^{-1}} = K$. Consequently $\langle a, b\rangle M = \langle a^{x^{-1}}, b^{x^{-1}}\rangle M$ and, bearing in mind that $K/M \cong C_p \times C_p$, there exist two natural numbers m and n with $1 \leq m \leq p-1$ and $1 \leq n \leq p-1$ and $(ab)^{x^{-1}}M = a^m b^n M$. Since $\langle (ab)^{x^{-1}}\rangle M$ is normalised by

H, we have that $\big((ab)^{x^{-1}}\big)^y M = \big((ab)^{x^{-1}}\big)^j M$ for a natural number j with $1 \le j \le p-1$. Then $(a^m b^n)^y M = (a^m b^n)^j M$, and so $(a^{im} b^n)M = (a^{jm} b^{jn})M$. It follows that $i = j = 1$. This contradiction proves that $G(p)$ is a Sylow p-subgroup of G.

Now we prove that $G = G(p)\,\mathrm{N}_G(H)$ for all p-subgroups H of G. Note that if H is a p-subgroup of G, then H is subnormal in G and then H is normalised by a Hall p'-subgroup of G by Lemma 22. Hence $G = G(p)\,\mathrm{N}_G(H)$.

Conversely, suppose that G satisfies $\mathcal{Z}'_p$. Let us show that G satisfies $\mathcal{Z}_p$. By Lemmas 18 and 19, we can assume that $\mathrm{O}_{p'}(G) = 1$. If G is p-nilpotent, then G is a p-group and G clearly satisfies $\mathcal{Z}_p$. Now assume that G is not p-nilpotent. In this case, $G(p)$ is a Sylow p-subgroup of G and for every p-subgroup H of G, $G = G(p)\,\mathrm{N}_G(H)$. Let H be a p-subgroup of G. Then H is normalised by a Hall p'-subgroup T of G. Let q^r be a power of the prime q dividing $|G : H|$. If $q = p$, then H is clearly contained in a subgroup K such that $|K : H| = q^r$. If $q \ne p$, then the Sylow q-subgroup T_q of T has a subgroup X of order q^r. Hence $|HX : H| = q^r$. Therefore, G satisfies $\mathcal{Z}_p$. □

Proof of Theorem 15. Only the sufficiency of the condition is in doubt. We argue by induction on $|G|$. Let G be a soluble group satisfying $\mathcal{Z}_p$ for all primes p, and assume that for all soluble groups T satisfying $\mathcal{Z}_p$ for all primes p with $|T| < |G|$, T satisfies $\mathcal{Y}$. By Lemma 20, G is p-supersoluble for all primes p. Therefore G is supersoluble.

Let q be the largest prime dividing $|G|$. Then G has a normal Sylow q-subgroup Q. Since $Q \le \mathrm{O}_{p'}(G)$ for every prime $p \ne q$, we have that G/Q satisfies $\mathcal{Z}_p$ for every $p \ne q$ by Lemma 18. Clearly, G/Q satisfies $\mathcal{Z}_q$. The induction hypothesis implies that G/Q is a $\mathcal{Y}$-group. Consequently, if $G_{q'}$ is a Hall q'-subgroup of G, then $G_{q'}$, being isomorphic to G/Q, satisfies $\mathcal{Y}$.

Let H be a subgroup of G and let r be a prime number dividing the $|G : H|$. We prove that there exists a subgroup K of G such that $H \le K$ and $|K : H| = r$. Consider a Hall q'-subgroup $H_{q'}$ of G. It follows that $H = H_{q'}(H \cap Q)$ and $H \cap Q$ is a normal Sylow q-subgroup of H. In particular, $H_{q'} \le \mathrm{N}_G(H \cap Q)$. On the other hand, since $H \cap Q \le \mathrm{O}_q(G)$ and G satisfies $\mathcal{Z}_q$, by Lemma 22, there exists a Hall q'-subgroup $G_{q'}$ of G such that $G_{q'} \le \mathrm{N}_G(H \cap Q)$. Since $G_{q'}$ is a Hall q'-subgroup of $\mathrm{N}_G(H \cap Q)$ and $H_{q'}$ is a q'-subgroup of $\mathrm{N}_G(H \cap Q)$, there exists an element $x \in \mathrm{N}_G(H \cap Q)$ such that $H_{q'} \le G_{q'}^x$. Since $G_{q'}^x$ normalises $H \cap Q$, there is no loss of generality in assuming that $H_{q'} \le G_{q'}$.

Suppose that $r \ne q$. Since $G_{q'}$ satisfies $\mathcal{Y}$, there exists a subgroup L of $G_{q'}$ such that $H_{q'} \le L$ and $|L : H_{q'}| = r$. In this case, $|(H \cap Q)L : H| = r$. Assume that $r = q$. Consider a piece of chief series $1 = N_0 \trianglelefteq N_1 \trianglelefteq N_2 \trianglelefteq \cdots \trianglelefteq N_m = Q$ of G and let i be the first index such that N_i is not contained in $H \cap Q$. It follows that $|N_i H : H| = q$. Consequently, G is a $\mathcal{Y}$-group. This completes the proof. $\square$

Proof of Theorem 16. Let $\mathfrak{F}$ be a formation contained in the class $\mathcal{Z}_p$. Assume that G is a group such that $G \in \mathfrak{F}$, but G is not p-nilpotent. Since $\mathfrak{F}$ is a formation, we have that $H = G/O_{p'}\big(G(p)\big)$ belongs to $\mathfrak{F}$. By Theorem 13, $H(p)$ is a normal Sylow p-subgroup of H such that for every subgroup X of $H(p)$, $H = H(p)\,\mathrm{N}_H(X)$. On the other hand, since $\mathfrak{F}$ is a formation, we have that $H \times H \in \mathfrak{F}$. Let D be a diagonal subgroup of $H(p) \times H(p)$. Since $H \times H$ satisfies $\mathcal{Z}_p$, we have that D is normalised by a Hall p'-subgroup of $H \times H$ by Lemma 22. This Hall p'-subgroup can be assumed to be equal to $Q \times Q$, where Q is a Hall p'-subgroup of H. But Q does not centralise $H(p)$ because H is not p-nilpotent. Let $y \in H$ and $h_1 \in H(p)$ such that $h_2 = h_1^y \neq h_1$. We have that $(h_1, h_1)^{(1,y)} = (h_1, h_2) \in D$ and so $(h_1, h_1)^{-1}(h_1, h_2) = (1, h_1^{-1}h_2)$ is a non-trivial element of D in the second factor of $H(p) \times H(p)$. This contradiction shows that $\mathfrak{F}$ must consist only of p-nilpotent groups, as desired. $\square$

Proof of Theorem 4. Assume that G satisfies $\mathcal{Y}$. It is clear that G is supersoluble, because all maximal subgroups of G have prime index (see [11, VI, 9.5]). Moreover G satisfies $\mathcal{Z}_p$ for all primes p by Theorem 15. By Theorem 13, G satisfies $\mathcal{Z}'_p$ for all primes p. Then G is p-supersoluble for all primes p. In particular, G is supersoluble. Hence the derived subgroup G' of G is nilpotent. In particular, $G^{\mathfrak{N}}$ is nilpotent.

Now we prove that $G^{\mathfrak{N}}$ is a Hall subgroup of G. Since $G^{\mathfrak{N}} = \langle G(p) \mid p \in \mathbb{P}\rangle$, we have that $G^{\mathfrak{N}}$ contains a Sylow p-subgroup of G for every prime p such that G is not p-nilpotent. If $G^{\mathfrak{N}}$ is not a Hall subgroup of G, there exists a prime q such that the Sylow q-subgroup of $G^{\mathfrak{N}}$ is non-trivial and is not a Sylow q-subgroup of G. In particular, G is not q-nilpotent. This implies that a Sylow q-subgroup of G is contained in $G^{\mathfrak{N}}$, which is impossible. Therefore $G^{\mathfrak{N}}$ is a Hall subgroup of G. Let π be the set of primes dividing $G^{\mathfrak{N}}$ and consider a subgroup H of $G^{\mathfrak{N}}$. Since $G^{\mathfrak{N}}$ is nilpotent, we have that H is nilpotent and H can be expressed as a direct product $H = H_{p_1} \times \cdots \times H_{p_r}$ of its Sylow subgroups. We prove by induction on k that $H_{p_1} \times \cdots \times H_{p_k}$, $1 \leq k \leq r$, is normalised by a Hall π'-subgroup of G. The result is clear if $k = 1$, because $G = G(p_1)\,\mathrm{N}_G(H_{p_1}) = G^{\mathfrak{N}}\,\mathrm{N}_G(H_{p_1})$ inasmuch as G satisfies $\mathcal{Z}'_{p_1}$. Assume now that $H_{p_1} \times \cdots \times H_{p_{k-1}}$ is normalised by a Hall π'-subgroup T of G. The subgroup H_{p_k} is normalised by a Hall π'-subgroup T_1 of G. Since all Hall π'-subgroups of G are conjugate and $G = G^{\mathfrak{N}}T$, there exists an element $g \in G^{\mathfrak{N}}$ such that $T_1 = T^g$. Since $G^{\mathfrak{N}}$ is nilpotent, $g = g_{p_1} \cdots g_{p_t}$, where $t = |\pi|$. Moreover, $(H_{p_1} \times \cdots \times H_{p_{k-1}})^{g_{p_k}^{-1}} = H_{p_1} \times \cdots \times H_{p_{k-1}}$ and $H_{p_k}^{g^{-1}} = H_{p_k}^{g_{p_k}^{-1}}$. Consequently both $H_{p_1} \times \cdots \times H_{p_{k-1}}$ and H_{p_k} are normalised by the Hall π'-subgroup $T^{g_{p_k}}$. It follows that $H_{p_1} \times \cdots \times H_{p_k}$ is normalised by $T^{g_{p_k}}$. By induction, it follows that H is normalised by a Hall π'-subgroup of G and so $G = G^{\mathfrak{N}} \mathrm{N}_G(H)$.

Conversely, assume that the nilpotent residual L of G is a nilpotent Hall subgroup of G and that for all subgroups H of L, $G = L\,\mathrm{N}_G(H)$. In this case, it is clear that G is soluble. We prove that G satisfies $\mathcal{Z}'_p$ for all primes p. Let p be a prime and suppose that G is not p-nilpotent. Then a Sylow p-subgroup P of G is contained in L and so in the p-nilpotent residual $G(p)$ of G, because otherwise G would have central p-chief factors. Let H be a p-subgroup of $G(p)$, then H is centralised by the Hall p'-subgroup X of L. Clearly, $G(p) = P$ because L is nilpotent. Therefore $G = L\,\mathrm{N}_G(H) = G(p)X\,\mathrm{N}_G(H) = G(p)\,\mathrm{N}_G(H)$. This shows that G satisfies $\mathcal{Z}'_p$ for all primes p. Applying Theorems 13 and 15, it follows that G is a $\mathcal{Y}$-group. The proof of the theorem is now complete. □

3 Examples

The details of the following two examples can be found in [14, pages 201 and 202].

Example 23. Let $G = \langle x, y, z \mid x^3 = z^3 = y^2 = (xy)^2 = 1, xz = zx, yz = zy\rangle$. G is isomorphic to $S_3 \times C_3$, where S_3 is the symmetric group of degree 3 and C_3 is the cyclic group of order 3. Put $H = \langle xz\rangle$ and note that $|G : H| = 6$ but G does not contain a subgroup K such that $|K : H| = 2$. Thus G does not satisfy $\mathcal{Z}_2$. In particular, G is not a $\mathcal{Y}$-group.

Example 24. Let $X = \langle a, b, c \mid a^3 = b^3 = c^3 = 1, [a, b] = c, ca = ac, bc = cb\rangle$ be the nonabelian group of order 27 of exponent 3. Then X has an automorphism t of order 2 such that $a^t = a^{-1}$, $b^t = b^{-1}$, and $c^t = c$. Let $G = [X]\langle t\rangle$ be the semidirect product of X by $\langle t\rangle$. Then G has order 54, $\Phi(G) = \mathrm{Z}(G) = \langle c\rangle$, and G is a $\mathcal{Y}$-group. Also note that $\langle a, t\rangle \cong S_3$, so that $\langle a, t\rangle \times \langle c\rangle$ is a subgroup of G which is isomorphic to $S_3 \times C_3$ and hence is not a $\mathcal{Y}$-group. Note that G satisfies $\mathcal{Z}_2$, but $\langle a, t\rangle \times \langle c\rangle$ does not satisfy $\mathcal{Z}_2$. Then property $\mathcal{Z}_2$ is not inherited by subgroups. Moreover, subgroups of $\mathcal{Y}$-groups are not necessarily $\mathcal{Y}$-groups.

In particular, X is a $\mathcal{Y}$-group which is not a PST-group.

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