Eur. Phys. J. Plus (2022) 137:554
https://doi.org/10.1140/epjp/s13360-022-02691-x

THE EUROPEAN PHYSICAL JOURNAL PLUS

Regular Article

Probabilistic analysis of a foundational class of generalized second-order linear differential equations in classic mechanics

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Received: 30 December 2021 / Accepted: 5 April 2022

Abstract A number of relevant models in Classical Mechanics are formulated by means of the differential equation $y''(t) + At^{\beta}y(t) = 0$. In this paper, we improve the results recently established for a randomized reformulation of this model that includes a generalized derivative. The stochastic analysis permits solving that generalized model by computing reliable approximations of the probability density function of the solution, which is a stochastic process. The approach avoids constructing these approximations from limited statistical punctual information and the Principle of Maximum Entropy by directly constructing a sequence of approximations using the Probabilistic Transformation Method. We prove that these approximations converge to the exact density under mild conditions on the data. Finally, several numerical examples illustrate our theoretical findings.

1 Introduction

The homogeneous linear second-order differential equation

$$y''(t) + f(t)y(t) = 0, \tag{1.1}$$

plays a key role in a number of problems mainly appearing in Physics and Engineering. It describes simple harmonic motion or free undamped motion of a spring of mass m and constant $k > 0$. In this case, one obtains the simplest formulation of (1.1), where $f(t)$ is the *spring constant*, $f(t) = \varpi^2 := \frac{k}{m}$ (ϖ is called the circular frequency of the spring) [1]. On the other hand, there are real-world situations where it is more realistic to assume that the spring constant changes over time. For example, if the spring/mass system is in motion for a long period, the spring will weaken due to wear and tear; if the temperature of the environment is rapidly decreasing, this will affect the spring material. In these two scenarios, that fall within the so called aging spring theory, possible choices for $f(t)$ are $\frac{k}{m}\mathrm{e}^{-at}$, $\alpha > 0$ and $\frac{k}{m}t$, respectively [2, p. 197]. This latter choice, in a more general form, $f(t) = At$, leads to the so-called Airy's differential equation,

$$y''(t) + Aty(t) = 0, \tag{1.2}$$

that is encountered in the study of radiowave propagation (diffraction of radio waves around the surface of the Earth) and in physical optics (the diffraction of the light) [3, p. 89]. The differential equation (1.1) also appears in theory of elasticity to model the problem of determining when a uniform vertical column will buckle under its own weight. In this context, the independent variable t is better denoted by x representing the deflection with respect to the stable (vertical) position and $f(x) = \frac{g\rho}{EI}x$, where g denotes the gravitational acceleration, E is the Young's modulus of the material of the column, I is its cross-sectional moment of inertia, and ρ is the linear density of the column [4]. Many other applications of model (1.2) can be found in [5].

On the one hand, over the last decades classical ordinary differential equations, as (1.1), have been reformulated in terms of generalized or fractional derivatives aimed at better modelling complex phenomena such as acoustic attenuations in Acoustics [6], anomalous diffusion processes in complex media in Fluids [7] or memory effects in materials (viscoelasticity, aging, etc.) in Structural Mechanics [8]. On the other hand, when ordinary or fractional differential equations are applied to real-world problems, one requires setting their parameters (initial/boundary conditions, forcing term and/or coefficients) using the available information, which is usually collected from measurements and metadata that often contain epistemic and/or aleatoric uncertainties coming from

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measurement errors and/or lack of knowledge, respectively. This simple approach leads to consider fractional differential equations with uncertainties. One can delineate two main different forms of this type of equations depending on the way that uncertainty is considered in the equation, namely, Stochastic Fractional Differential Equations (SFDEs) and Random Fractional Differential Equations (RFDEs). In the former case, uncertainties are driven by one (o more) prefixed types of stochastic processes such as the fractional Brownian motion [9, Ch. 6] and/or the fractional Lèvy process [10]. In the framework of RFDEs, uncertainties are directly assigned to every parameter of the equation via specific probability distributions that must be previously chosen so that they properly represent the physical meaning of the corresponding model parameter and the solution captures the uncertainties in the model response [11].

In this paper, we are concerned in studying a generalization of (1.2), that takes into account the previous considerations, namely, a fractional derivative and a full randomization of model parameters. Additionally, a power-law generalized coefficient is also included. Specifically, in this contribution we deal with the following Random Fractional Initial Value Problem (RFIVP)

$$\begin{cases} ({}^C D_0^{\alpha} Y)(t) - B\, t^{\beta} Y(t) = 0, \qquad t > 0, \quad n = -\lfloor -\alpha \rfloor, \quad n-1 \le \alpha \le n, \\ \qquad\qquad Y^{(j)}(0) = A_j, \qquad j = 0, \dots, n-1, \end{cases} \tag{1.3}$$

where $\lfloor \cdot \rfloor$ stands for the floor function, $({}^C D_0^{\alpha} Y)(t)$ denotes the Caputo derivative of order $\alpha > 0$, interpreted in the random mean square sense [12], of the stochastic process $Y(t)$. The initial conditions $A_0, A_1, \dots, A_{n-1}$ and the coefficient B are assumed random variables belonging to the space $\mathrm{L}_2(\Omega)$, where $(\Omega, \mathcal{F}_{\Omega}, \mathbb{P})$ is a complete probability space, and whose elements are real random variables with finite absolute second-order moment, i.e., $\mathrm{L}_2(\Omega) = \{X : \Omega \longrightarrow \mathbb{R} \text{ such that } \mathbb{E}[|X|^2]) < \infty\}$ (here $\mathbb{E}[\cdot]$ denotes the expectation operator). It can be shown that $(\mathrm{L}_2(\Omega)|, < \cdot, \cdot >_{\mathrm{L}_2(\Omega)})$ is a Hilbert space being $< X, Y >_{\mathrm{L}_2(\Omega)} = \mathbb{E}[|XY|]$, $X, Y \in \mathrm{L}_2(\Omega)$ (so $||X||_2 = (\mathbb{E}[|X|^2])^{1/2} < \infty$) [13, Ch. 4].

To study the RFIVP (1.3), from a probabilistic standpoint, entails computing its solution but also determining the main statistical information of the solution, $Y(t)$, which is a stochastic process. It includes the computation of the first moments of $Y(t)$, such as the mean and the variance functions, but also its *fidis* (finite dimensional distributions), particularly the so-called first probability density function (1-PDF) [13].

Under the following hypotheses:

H1: $A_0, A_1, \dots, A_{n-1}$ and B are (mutually) independent random variables,
H2: There exist positive constants η, $\mathcal{H}$ and p, such that

$$||B^m||_2 \le \eta \mathcal{H}^{m-1}((m-1)!)^p, \qquad \forall m \ge m_0 \ge 1, \text{ integers},$$

some of the authors have recently studied the RFIVP (1.3) [14]. Specifically, in that article one constructs the solution stochastic process by means of the following finite sum of n random generalized power series

$$Y(t) = \sum_{j=0}^{n-1} Y_j(t), \tag{1.4}$$

where

$$Y_j(t) = \sum_{m=0}^{\infty} X_{m,j} t^{\gamma m + j}, \qquad X_{m,j} = B^m A_j g_{m,j}, \qquad g_{m,j} = \prod_{k=1}^{m} \frac{\Gamma(k\gamma + j + 1 - \alpha)}{\Gamma(k\gamma + j + 1)}, \qquad \gamma = \alpha + \beta. \tag{1.5}$$

In [14], besides constructing the foregoing solution and studying its convergence, in the mean square sense, approximations for the mean and for the variance functions of the solution were also obtained by truncating $Y_j(t)$ at certain order, say M. Then, taking advantage of this punctual statistical information, the Principle of Maximum Entropy (PME) [15] was applied to construct reliable approximations of the 1-PDF of the solution. The accuracy in the approximations of the 1-PDF, via this approach, heavily depends on controlling two main sources of errors, first the ones associated with the approximations of the mean and the variance, and secondly, the associated with the application of the PME with *only* these two (approximate) statistical moments. The aforementioned errors involved in the calculation of the mean and the variance could, theoretically, be reduced at expense of increasing the order of truncation, M, of the random generalized power series solution (1.4)–(1.5) and/or by constructing approximations for higher statistical moments. However, both strategies might be unaffordable in many cases. Motivated by these drawbacks, in this paper we propose a direct method to approximate the 1-PDF of the solution stochastic process of the RFIVP (1.3), that avoids approximating the first statistical moments of the solution, and the subsequent application of the PME.

Our subsequent study will be based on the representation of the solution given in (1.4)–(1.5), hence the foregoing assumptions **H1** and **H2** must be fulfilled. Specifically, hereinafter, we will assume the following assumptions:

A1: $A_0, A_1, \dots, A_{n-1}$ and B are (mutually) independent random variables whose respective domains will be denoted by $\mathcal{D}(A_j)$, $j = 0, 1, \dots, n-1$ and $\mathcal{D}(B)$.

A2: B is a bounded random variable that takes values away from the origin, i.e., there exist positive lower and upper bounds, b_l and b_u, such that

$$0 < b_l \le |B(\omega)| \le b_u < \infty, \quad \forall \omega \in \Omega.$$

Now, we give two key remarks. The first one justifies that assumption **A2** guarantees that hypothesis **H2** fulfils, while in the second one shows that the random variables A_j, $j = 0, 1, \ldots, n-1$, defining the initial conditions, have finite expectation. This latter fact will be used later.

Remark 1 In [16, Subsect. 3.3] one proves that the boundedness of random variable B is equivalent to the following condition about the exponential growth of its absolute moments

$$\exists\, \rho, \mathcal{H} > 0 : \quad ||B^m||_2 \le \rho \mathcal{H}^m, \qquad \forall m \ge m_0 \ge 1, \quad \text{integers},$$

Notice that, this condition can be obtained from the inequality stated in assumption **A2** just taking $p = 0$ and relabeling the constant η as $\rho = \eta\mathcal{H}$. Furthermore, as it is proved in [14], the random generalized power series solution (1.4)–(1.5) is mean square convergent on the whole real line. Almost surely convergence on the whole real line is also guaranteed since, by assumption **A2**, B is a bounded random variable.

Remark 2 As $A_0, A_1, \ldots, A_{n-1} \in \mathrm{L}_2(\Omega)$, then $\mathbb{E}[A_j^2] < \infty$, and by the Schwarz's inequality, $\mathbb{E}[|A_j|]) < (\mathbb{E}[A_j^2])^{1/2} < \infty$. This fact will used later.

The key tool to conduct our subsequent analysis is the Probabilistic Transformation Method (PTM) (also called the Random Variable Transformation technique). This result permits determining the PDF, say $f_{\mathbf{Y}}(\mathbf{y})$, of a random vector, $\mathbf{Y}$, that results from mapping another random vector, $\mathbf{Z}$, whose PDF, $f_{\mathbf{Z}}(\mathbf{z})$, is known.

Theorem 1 *(PTM) [13, p. 25]. Let us consider $\mathbf{Z} = (Z_1, \ldots, Z_k)$ and $\mathbf{Y} = (Y_1, \ldots, Y_k)$ two k-dimensional absolutely continuous random vectors defined on a common complete probability space $(\Omega, \mathcal{F}_\Omega, \mathbb{P})$. Let $\mathbf{r} : \mathbb{R}^k \to \mathbb{R}^k$ be a one-to-one deterministic transformation of $\mathbf{Z}$ into $\mathbf{Y}$, i.e., $\mathbf{Y} = \mathbf{r}(\mathbf{Z})$. Assume that $\mathbf{r}$ is continuous in $\mathbf{Z}$ and has continuous partial derivatives with respect to each Z_i, $1 \le i \le k$. Then, if $f_{\mathbf{Z}}(\mathbf{z})$ denotes the joint probability density function of random vector $\mathbf{Z}$, and $\mathbf{s} = \mathbf{r}^{-1} = (s_1(y_1, \ldots, y_k), \ldots, s_k(y_1, \ldots, y_k))$ represents the inverse mapping of $\mathbf{r} = (r_1(z_1, \ldots, z_k), \ldots, r_k(z_1, \ldots, z_k))$, the joint probability density function of random vector $\mathbf{Y}$ is given by*

$$f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{Z}}\left(\mathbf{s}(\mathbf{y})\right) |J|, \tag{1.6}$$

where $|J|$, which is assumed to be different from zero, is the absolute value of the Jacobian defined by the following determinant

$$J = \det\left(\frac{\partial \mathbf{s}}{\partial \mathbf{y}}\right) = \det \begin{pmatrix} \frac{\partial s_1(y_1,\ldots,y_k)}{\partial y_1} & \cdots & \frac{\partial s_k(y_1,\ldots,y_k)}{\partial y_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial s_1(y_1,\ldots,y_k)}{\partial y_k} & \cdots & \frac{\partial s_k(y_1,\ldots,y_k)}{\partial y_k} \end{pmatrix}. \tag{1.7}$$

In the setting of differential equations with uncertainties, the PTM method has been successfully applied to determine, exact or approximately, the 1-PDF of the solution to some classes of ordinary and partial differential equations [17–19] and [20], respectively, and differential equations with delay [21]. To the best of our knowledge, contributions with regard to fractional differential equations, are still scarce. In [22], one determines the 1-PDF of the solution for the simplest autonomous linear Caputo fractional differential equation with $0 < \alpha \le 1$ by taking advantage of the PTM technique.

2 Computing the 1-PDF of the solution stochastic process by applying PTM

To compute the 1-PDF of the solution of the RFIVP (1.4), by means of the application of Theorem 1, we will first consider the truncation of (1.4),

$$Y^M(t) = \sum_{j=0}^{n-1} Y_j^M(t), \tag{2.1}$$

where

$$Y_j^M(t) = \sum_{m=0}^{M} X_{m,j} t^{\gamma m + j} = A_j \sum_{m=0}^{M} B^m g_{m,j} t^{\gamma m + j}. \tag{2.2}$$

For t fixed, expression (2.1)–(2.2) can be interpreted as a continuous mapping of the random variables, $A_0, A_1, \ldots A_{n-1}$ and B, whose joint PDF, can be expressed as $f_{A_0,A_1,\ldots,A_{n-1},B} = f_{A_0} f_{A_1} \cdots f_{A_{n-1}} f_B$, by assumption **A1**. Then, in order to apply Theorem 1 with $k = n + 1$, we identify $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_n, Z_{n+1}) = (A_0, A_1, \ldots, A_{n-1}, B)$, and define $\mathbf{Y}$ via the following transformation $\mathbf{r} : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ of $\mathbf{Z}$ whose components are

$$
\begin{array}{rcl}
y_1 &=& r_1(z_1, z_2, \ldots, z_n, z_{n+1}) = z_1 \displaystyle\sum_{m=0}^{M} z_{n+1}^m g_{m,0} t^{\gamma m} + \sum_{j=1}^{n-1} \sum_{m=0}^{M} z_{j+1} z_{n+1}^m g_{m,j} t^{\gamma m + j}, \\
y_2 &=& r_2(z_1, z_2, \ldots, z_n, z_{n+1}) = z_2, \\
\vdots & \vdots & \quad \vdots \quad \vdots \, \vdots \\
y_n &=& r_n(z_1, z_2, \ldots, z_n, z_{n+1}) = z_n, \\
y_{n+1} &=& r_{n+1}(z_1, z_2, \ldots, z_n, z_{n+1}) = z_{n+1}.
\end{array}
\tag{2.3}
$$

The inverse mapping of $\mathbf{r}$, $\mathbf{s} = \mathbf{r}^{-1}$, is given by

$$
\begin{array}{rcl}
z_1 &=& s_1(y_1, y_2, y_3, \ldots, y_n, y_{n+1}) = \dfrac{y_1 - \sum_{j=1}^{n-1} \sum_{m=0}^{M} y_{j+1} y_{n+1}^m g_{m,j} t^{\gamma m + j}}{\sum_{m=0}^{M} y_{n+1}^m g_{m,0} t^{\gamma m}}, \\
z_2 &=& s_2(y_1, y_2, y_3, \ldots, y_n, y_{n+1}) = y_2, \\
\vdots & \vdots & \quad \vdots \quad \vdots \, \vdots \\
z_n &=& s_n(y_1, y_2, y_3, \ldots, y_n, y_{n+1}) = y_n, \\
z_{n+1} &=& s_{n+1}(y_1, y_2, y_3, \ldots, y_n, y_{n+1}) = y_{n+1},
\end{array}
\tag{2.4}
$$

being its Jacobian

$$
J = \left| \frac{\partial s_1}{\partial y_1}(y_1, y_2, y_3, \ldots y_n, y_{n+1}) \right| = \frac{1}{\left| \sum_{m=0}^{M} y_{n+1}^m g_{m,0} t^{\gamma m} \right|} \neq 0.
$$

The expression in the denominator of the Jacobian depends on the random variable B, which is absolutely continuous, so the denominator is distinct from zero with probability one (w.p. 1), hence it is well-defined.

Applying Theorem 1, one gets

$$
\begin{aligned}
& f_{Y_1, Y_2, \ldots, Y_n, Y_{n+1}}(y_1, y_2, \ldots, y_n, y_{n+1}) \\
&= f_{Z_1}\left(\frac{y_1 - \sum_{j=1}^{n-1} \sum_{m=0}^{M} y_{j+1} y_{n+1}^m g_{m,j} t^{\gamma m + j}}{\sum_{m=0}^{M} y_{n+1}^m g_{m,0} t^{\gamma m}} \right) f_{Z_2}(y_2) \cdots f_{Z_n}(y_n) f_{Z_{n+1}}(y_{n+1}) \frac{1}{\left| \sum_{m=0}^{M} y_{n+1}^m g_{m,0} t^{\gamma m} \right|}.
\end{aligned}
\tag{2.5}
$$

For t fixed, $Y_1 = Y^M(t)$, and marginalizing with respect to $Y_2 = A_1, \ldots, Y_n = A_{n-1}, Y_{n+1} = B$, the 1-PDF of the truncated solution (2.1)–(2.2) is given by

$$
\begin{aligned}
f_{Y^M(t)}(y) = \int_{\mathcal{D}(A_1)} \cdots \int_{\mathcal{D}(A_{n-1})} \int_{\mathcal{D}(B)} & f_{A_0}\left(\frac{y - \sum_{j=1}^{n-1} \sum_{m=0}^{M} a_j b^m g_{m,j} t^{\gamma m + j}}{\sum_{m=0}^{M} b^m g_{m,0} t^{\gamma m}} \right) \\
& \cdot f_{A_1}(a_1) \cdots f_{A_{n-1}}(a_{n-1}) f_B(b) \frac{1}{\left| \sum_{m=0}^{M} b^m g_{m,0} t^{\gamma m} \right|} \, \mathrm{d}a_1 \cdots \mathrm{d}a_{n-1} \mathrm{d}b.
\end{aligned}
\tag{2.6}
$$

So far, the 1-PDF, $f_{Y^M(t)}(y)$, of the truncated solution, $Y^M(t)$, has been obtained via a semi-explicit expression (in terms of a multidimensional integral). Now, we will impose mild conditions on the data so that $f_{Y^M(t)}(y)$ converges to $f_{Y(t)}(y)$, where

$$
\begin{aligned}
f_{Y(t)}(y) = \int_{\mathcal{D}(A_1)} \cdots \int_{\mathcal{D}(A_{n-1})} \int_{\mathcal{D}(B)} & f_{A_0}\left(\frac{y - \sum_{j=1}^{n-1} \sum_{m=0}^{\infty} a_j b^m g_{m,j} t^{\gamma m + j}}{\sum_{m=0}^{\infty} b^m g_{m,0} t^{\gamma m}} \right) \\
& \cdot f_{A_1}(a_1) \cdots f_{A_{n-1}}(a_{n-1}) f_B(b) \frac{1}{\left| \sum_{m=0}^{\infty} b^m g_{m,0} t^{\gamma m} \right|} \, \mathrm{d}a_1 \cdots \mathrm{d}a_{n-1} \mathrm{d}b.
\end{aligned}
\tag{2.7}
$$

For the sake of clarity in the presentation of the forthcoming development, we introduce the following notation

$$
s_j^M(t) = \sum_{m=0}^{M} b^m g_{m,j} t^{\gamma m + j}, \quad s_j(t) = \sum_{m=0}^{\infty} b^m g_{m,j} t^{\gamma m + j}, \qquad j = 0, 1, \ldots n - 1.
\tag{2.8}
$$

Then, expressions (2.6) and (2.7), can be written, respectively, as

$$
\begin{aligned}
f_{Y^M(t)}(y) = \int_{\mathcal{D}(A_1)} \cdots \int_{\mathcal{D}(A_{n-1})} \int_{\mathcal{D}(B)} f_{A_0}\left(\frac{y - \sum_{j=1}^{n-1} a_j s_j^M(t)}{s_0^M(t)}\right) \\
\cdot f_{A_1}(a_1) \cdots f_{A_{n-1}}(a_{n-1}) f_B(b) \frac{1}{\left|s_0^M(t)\right|} \mathrm{d}a_1 \cdots \mathrm{d}a_{n-1} \mathrm{d}b
\end{aligned}
\tag{2.9}
$$

and

$$
\begin{aligned}
f_{Y(t)}(y) = \int_{\mathcal{D}(A_1)} \cdots \int_{\mathcal{D}(A_{n-1})} \int_{\mathcal{D}(B)} f_{A_0}\left(\frac{y - \sum_{j=1}^{n-1} a_j s_j(t)}{s_0(t)}\right) \\
\cdot f_{A_1}(a_1) \cdots f_{A_{n-1}}(a_{n-1}) f_B(b) \frac{1}{|s_0(t)|} \mathrm{d}a_1 \cdots \mathrm{d}a_{n-1} \mathrm{d}b.
\end{aligned}
\tag{2.10}
$$

Now, let us derive a lower bound for $S_0(t)$ and $S_0^M(t)$, and an upper bound for $S_j(t)$ and $S_j^M(t)$, $j = 1, \ldots n-1$, in a certain neighborhood, that will be required later when studying the convergence $f_{Y^M(t)}(y) \longrightarrow f_{Y(t)}(y)$ as $M \to \infty$. Let us first observe that, from the initial condition and the form of the solution, one gets

$$
A_0 = Y(0) = Y_0(0) = \sum_{m=0}^{\infty} X_{m,0} t^{\gamma m}\Big|_{t=0} = \sum_{m=0}^{\infty} B^m A_0 g_{m,0} t^{\gamma m}\Big|_{t=0} = A_0 \sum_{m=0}^{\infty} B^m g_{m,0} t^{\gamma m}\Big|_{t=0} = A_0 S_0(0).
$$

Observe that in the above expression, we have used capital letter to denote the series $S_0(t)$ evaluated at $t = 0$ because of in this setting it represents a random power series. Later, when this series as well as $S_j(t)$, $j = 1, \ldots, n-1$ are considered via its samples or trajectories, for consistency with the notation, hereinafter they will be denoted using lower-case letters, i.e., $s_j(t)$, $j = 0, 1, \ldots, n-1$. Since the random variable A_0 is different from zero w.p. 1 (because is absolutely continuous), then by the last expression one deduces $S_0(0) = 1$ w.p. 1. From this fact and taking into account that $S_0(t)$ is a random power series evaluated at $t := t^{\gamma}$, so continuous w.p. 1, it is guaranteed that

$$
\exists\, \delta_0 > 0 : \; 0 < m_{s,0} \le \min\{|s_0^M(t)|, |s_0(t)|\}, \qquad \forall t : |t| \le \delta_0, \quad \forall M \ge 0 \text{ integer.} \tag{2.11}
$$

On the other hand, by Remark 1 and the definition of $S_j^M(t)$, $j = 1, \ldots, n-1$, which are random power series evaluated at $t := t^{\gamma m + j}$ and convergent on the whole real line, it is known that each $S_j^M(t)$ is almost surely uniformly convergent in every compact set of $\mathbb{R}$. This guarantees that

$$
\exists\, M_{s,j} > 0 : \; \max\{|s_j^M(t)|, |s_j(t)|\} \le M_{s,j}, \qquad \forall t : |t| \le T, T > 0, \; \forall M > 0, \text{integer } j = 1, \ldots, n-1. \tag{2.12}
$$

Finally, as $S_j^M(t)$ converges uniformly to $S_j(t)$, for each $j = 0, 1, \ldots, n-1$ and given $\epsilon_j > 0$, then

$$
\exists\, M_0 > 0 \text{ integer} : \; |s_j^M(t) - s_j(t)| < \epsilon_j, \quad \forall M \ge M_0 \text{ integer and } \forall t : |t| \le T, \; T > 0. \tag{2.13}
$$

Hereinafter, and according to (2.11), we will work in a neighborhood of $t = 0$ (possibly small), where the RFIVP (1.3) is formulated, and where the bounds (2.11)–(2.13) are guaranteed. We will prove the convergence $f_{Y^M(t)}(y) \longrightarrow f_{Y(t)}(y)$ as $M \to \infty$ for each t in that neighborhood. To this end, besides assumptions **A1** and **A2**, we will suppose that

– **A3**: The PDF of the first random initial condition A_0, f_{A_0}, is Lipschitz in $\mathbb{R}$, i.e., exists $L_0 > 0$ such that

$$
|f_{A_0}(a_2) - f_{A_0}(a_1)| \le L_0 |a_2 - a_1|, \qquad \forall a_2, a_1 \in \mathbb{R}.
$$

Let us observe that

$$
\begin{aligned}
\left|f_{Y(t)}(y) - f_{Y^M(t)}(y)\right| \le \int_{\mathcal{D}(A_1)} \cdots \int_{\mathcal{D}(A_{n-1})} int_{\mathcal{D}(B)} \Bigg| f_{A_0}\left(\frac{y - \sum_{j=1}^{n-1} a_j s_j(t)}{s_0(t)}\right) \frac{1}{|s_0(t)|} \\
- f_{A_0}\left(\frac{y - \sum_{j=1}^{n-1} a_j s_j^M(t)}{s_0^M(t)}\right) \frac{1}{\left|s_0^M(t)\right|} \Bigg| f_{A_1}(a_1) \cdots f_{A_{n-1}}(a_{n-1}) f_B(b) \, \mathrm{d}a_1 \cdots \mathrm{d}a_{n-1} \mathrm{d}b.
\end{aligned}
\tag{2.14}
$$

Now, we first add and subtract $f_{A_0}\left(\frac{y - \sum_{j=1}^{n-1} a_j S_j^M(t)}{S_0^M(t)}\right) \frac{1}{|S_0(t)|}$ inside the absolute value. We then arrange the terms and we apply the triangular inequality. Finally, we take into account that any PDF is a non-negative function to remove the unnecessary absolute values. The last expression then writes

$$\left|f_{Y(t)}(y) - f_{Y^M(t)}(y)\right| \le \int_{\mathcal{D}(A_1)} \cdots \int_{\mathcal{D}(A_{n-1})} \int_{\mathcal{D}(B)} \left\{ \underbrace{f_{A_0}\left(\frac{y - \sum_{j=1}^{n-1} a_j s_j^M(t)}{s_0^M(t)}\right)}_{\text{(I)}} \underbrace{\left|\frac{1}{|s_0(t)|} - \frac{1}{|s_0^M(t)|}\right|}_{\text{(II)}} \right.$$
$$\left. + \underbrace{\left|f_{A_0}\left(\frac{y - \sum_{j=1}^{n-1} a_j s_j(t)}{s_0(t)}\right) - f_{A_0}\left(\frac{y - \sum_{j=1}^{n-1} a_j s_j^M(t)}{s_0^M(t)}\right)\right|}_{\text{(III)}} \underbrace{\frac{1}{|s_0(t)|}}_{\text{(IV)}} \right\}$$
$$\cdot f_{A_1}(a_1)) \cdots f_{A_{n-1}}(a_{n-1}) f_B(b)\, \mathrm{d}a_1 \cdots \mathrm{d}a_{n-1} \mathrm{d}b. \tag{2.15}$$

We are going to bound the terms (I)–(IV). For the term (I), let us denote by $F_0 = f_{A_0}(0)$ (remember that by assumption **A3** this value exists), then first applying **A3**, secondly the triangular inequality together with bounds (2.11) and (2.12), one gets

$$\begin{aligned} f_{A_0}\left(\frac{y - \sum_{j=1}^{n-1} a_j s_j^M(t)}{s_0^M(t)}\right) &\le \left|f_{A_0}\left(\tfrac{y - \sum_{j=1}^{n-1} a_j s_j^M(t)}{s_0^M(t)}\right) - f_{A_0}(0)\right| + F_0 \\ &\le L_0 \left|\tfrac{y - \sum_{j=1}^{n-1} a_j s_j^M(t)}{s_0^M(t)}\right| + F_0 \\ &\le \tfrac{L_0}{m_{s,0}} \left(|y| + \textstyle\sum_{j=1}^{n-1} |a_j| M_{s,j}\right) + F_0. \end{aligned}$$

Let us now bound the term (II) by applying (2.11) and (2.13) for $j = 0$,

$$\left|\frac{1}{|s_0(t)|} - \frac{1}{|s_0^M(t)|}\right| = \frac{\left||s_0^M(t)| - |s_0(t)|\right|}{|s_0(t)||s_0^M(t)|} \le \frac{|s_0^M(t) - s_0(t)|}{|s_0(t)||s_0^M(t)|} \le \frac{\epsilon_0}{m_{s,0}^2}.$$

The term (IV) can be similarly bounded applying (2.11)

$$\frac{1}{|s_0(t)|} \le \tfrac{1}{m_{s,0}}.$$

Finally, let us bound the term (III). To this end, we first apply assumption **A3**, and secondly, the triangular inequality together with bounds (2.11)–(2.13),

$$\begin{aligned} &\left|f_{A_0}\left(\frac{y - \sum_{j=1}^{n-1} a_j s_j(t)}{s_0(t)}\right) - f_{A_0}\left(\frac{y - \sum_{j=1}^{n-1} a_j s_j^M(t)}{s_0^M(t)}\right)\right| \\ &\le L_0 \left|\frac{y - \sum_{j=1}^{n-1} a_j s_j(t)}{s_0(t)} - \frac{y - \sum_{j=1}^{n-1} a_j s_j^M(t)}{s_0^M(t)}\right| \\ &= L_0 \left|\frac{y s_0^M(t) - s_0^M(t) \sum_{j=1}^{n-1} a_j s_j(t) - y s_0(t) + s_0(t) \sum_{j=1}^{n-1} a_j s_j^M(t)}{s_0(t) s_0^M(t)}\right| \\ &= L_0 \left|\frac{y\left(s_0^M(t) - s_0(t)\right) + \sum_{j=1}^{n-1} a_j \left(s_0(t) s_j^M(t) - s_0^M(t) s_j(t)\right)}{s_0(t) s_0^M(t)}\right| \\ &\le L_0 \frac{|y||s_0^M(t) - s_0^M(t)| + \sum_{j=1}^{n-1} |a_j| \left|s_0(t) s_j^M(t) - s_0^M(t) s_j(t)\right|}{|s_0(t)||s_0^M(t)|} \\ &= L_0 \frac{|y||s_0^M(t) - s_0^M(t)| + \sum_{j=1}^{n-1} |a_j| \left|s_0(t) s_j^M(t) - s_j(t) s_0(t) + s_j(t) s_0(t) - s_0^M(t) s_j(t)\right|}{|s_0(t)||s_0^M(t)|} \\ &\le L_0 \frac{|y||s_0^M(t) - s_0^M(t)| + \sum_{j=1}^{n-1} |a_j| \left(|s_0(t)||s_j^M(t) - s_j(t)| + |s_j(t)||s_0(t) - s_0^M(t)|\right)}{|s_0(t)||s_0^M(t)|} \\ &\le L_0 \frac{|y|\epsilon_0 + \sum_{j=1}^{n-1} |a_j| \left(M_{s,0}\epsilon_j + M_{s,j}\epsilon_0\right)}{m_{s,0}^2}. \end{aligned}$$

Substituting the previous bounds in (2.15), one gets

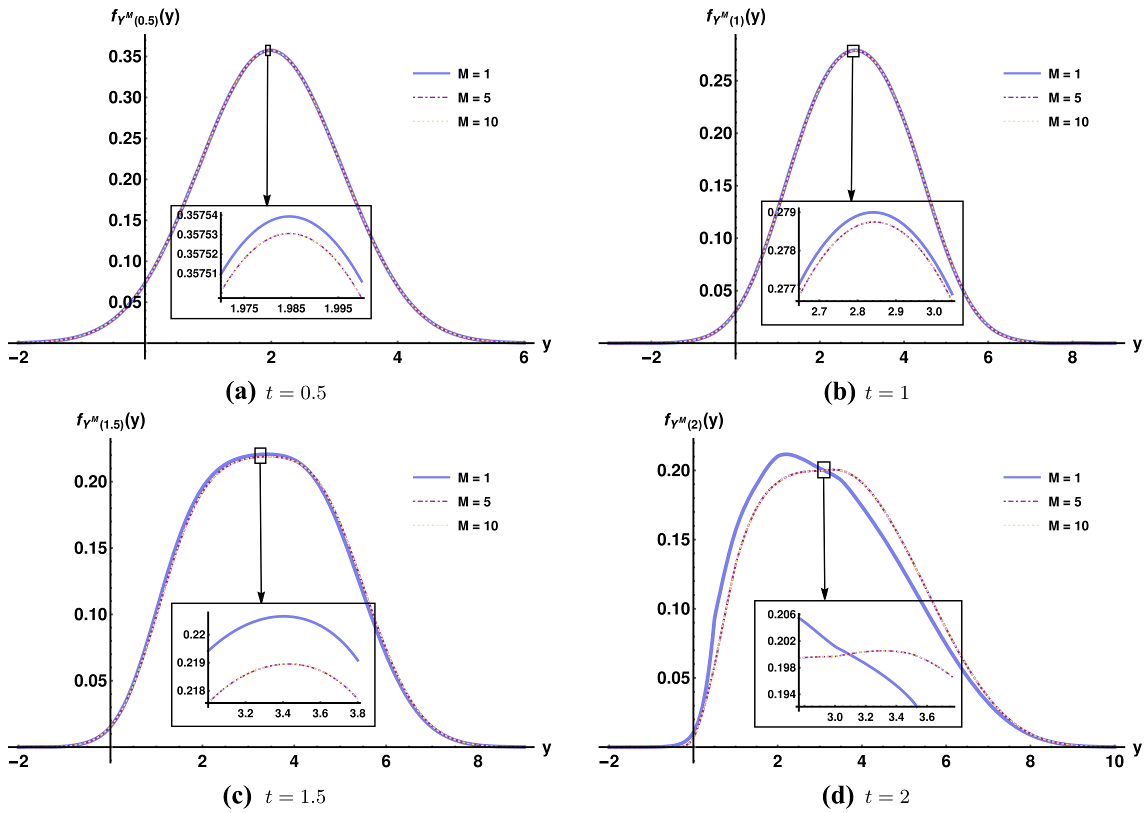
$$\left|f_{Y(t)}(y)-f_{Y^M(t)}(y)\right| \le \int_{\mathcal{D}(A_1)}\cdots\int_{\mathcal{D}(A_{n-1})}\int_{\mathcal{D}(B)}\left\{\left(\frac{L_0}{m_{s,0}}\left(|y|+\sum_{j=1}^{n-1}|a_j|M_{s,j}\right)+F_0\right)\frac{\epsilon_0}{m_{s,0}^2}\right.$$
$$\left.+\left(L_0\frac{|y|\varepsilon_0+\sum_{j=1}^{n-1}|a_j|\left(M_{s,0}\epsilon_j+M_{s,j}\epsilon_0\right)}{m_{s,0}^2}\right)\frac{1}{m_{s,0}}\right\}$$
$$\cdot f_{A_1}(a_1)\cdots f_{A_{n-1}}(a_{n-1})f_B(b)\,\mathrm{d}a_1\cdots\mathrm{d}a_{n-1}\mathrm{d}b.$$

If we denote $\epsilon := \max \epsilon_j$, and $\mathcal{M} := \max M_{s,j}$, $j = 0, 1, \ldots, n-1$, one gets

$$\left|f_{Y(t)}(y)-f_{Y^M(t)}(y)\right| \le \int_{\mathcal{D}(A_1)}\cdots\int_{\mathcal{D}(A_{n-1})}\int_{\mathcal{D}(B)}\left\{\left(\frac{L_0}{m_{s,0}}\left(|y|+\mathcal{M}\sum_{j=1}^{n-1}|a_j|\right)+F_0\right)\frac{\epsilon}{m_{s,0}^2}\right.$$
$$\left.+L_0\frac{|y|\epsilon+2\mathcal{M}\epsilon\sum_{j=1}^{n-1}|a_j|}{m_{s,0}^3}\right\}$$
$$\cdot f_{A_1}(a_1)\cdots f_{A_{n-1}}(a_{n-1})f_B(b)\,\mathrm{d}a_1\cdots\mathrm{d}a_{n-1}\mathrm{d}b$$
$$=\int_{\mathcal{D}(A_1)}\cdots\int_{\mathcal{D}(A_{n-1})}\int_{\mathcal{D}(B)}\left(\frac{L_0|y|\epsilon}{m_{s,0}^3}+\frac{L_0\mathcal{M}\epsilon}{m_{s,0}^3}\sum_{j=1}^{n-1}|a_j|+\frac{F_0\epsilon}{m_{s,0}^2}\right.$$
$$\left.+\frac{L_0|y|\epsilon}{m_{s,0}^3}+\frac{2L_0\mathcal{M}\epsilon}{m_{s,0}^3}\sum_{j=1}^{n-1}|a_j|\right)$$
$$\cdot f_{A_1}(a_1)\cdots f_{A_{n-1}}(a_{n-1})f_B(b)\,\mathrm{d}a_1\cdots\mathrm{d}a_{n-1}\mathrm{d}b$$
$$=\epsilon\left(\frac{2L_0|y|}{m_{s,0}^3}+\frac{F_0}{m_{s,0}^2}\right)\int_{\mathcal{D}(A_1)}\cdots\int_{\mathcal{D}(A_{n-1})}\int_{\mathcal{D}(B)} f_{A_1}(a_1)\cdots f_{A_{n-1}}(a_{n-1})f_B(b)\,\mathrm{d}a_1\cdots\mathrm{d}a_{n-1}\mathrm{d}b$$
$$+\frac{3L_0\mathcal{M}\epsilon}{m_{s,0}^3}\sum_{j=1}^{n-1}\int_{\mathcal{D}(A_1)}\cdots\int_{\mathcal{D}(A_{n-1})}\int_{\mathcal{D}(B)}|a_j|f_{A_1}(a_1)\cdots f_{A_{n-1}}(a_{n-1})f_B(b)\,\mathrm{d}a_1\cdots\mathrm{d}a_{n-1}\mathrm{d}b$$
$$=\epsilon\left(\frac{2L_0|y|}{m_{s,0}^3}+\frac{F_0}{m_{s,0}^2}\right)\underbrace{\left(\int_{\mathcal{D}(A_1)}f_{A_1}(a_1)\,\mathrm{d}a_1\right)}_{=1}\cdots\underbrace{\left(\int_{\mathcal{D}(A_{n-1})}f_{A_{n-1}}(a_{n-1})\,\mathrm{d}a_{n-1}\right)}_{=1}\underbrace{\left(\int_{\mathcal{D}(B)}f_B(b)\,\mathrm{d}b\right)}_{=1}$$
$$+\frac{3L_0\mathcal{M}\epsilon}{m_{s,0}^3}\sum_{j=1}^{n-1}\underbrace{\left(\int_{\mathcal{D}(A_1)}f_{A_1}(a_1)\,\mathrm{d}a_1\right)}_{=1}\cdots\underbrace{\left(\int_{\mathcal{D}(A_{j-1})}f_{A_{j-1}}(a_{j-1})\,\mathrm{d}a_{j-1}\right)}_{=1}\cdots\underbrace{\left(\int_{\mathcal{D}(A_j)}|a_j|f_{A_j}(a_j)\,\mathrm{d}a_j\right)}_{=\mathbb{E}[|A_j|]}\cdots$$
$$\cdot\underbrace{\left(\int_{\mathcal{D}(A_{n-1})}f_{A_{n-1}}(a_{n-1})\,\mathrm{d}a_{n-1}\right)}_{=1}\underbrace{\left(\int_{\mathcal{D}(B)}f_B(b)\,\mathrm{d}b\right)}_{=1}f_{A_1}(a_1)\cdots f_{A_{n-1}}(a_{n-1})f_B(b)\,\mathrm{d}a_1\cdots\mathrm{d}a_{n-1}\mathrm{d}b$$
$$=\epsilon\left(\frac{2L_0|y|}{m_{s,0}^3}+\frac{F_0}{m_{s,0}^2}\right)+\epsilon\frac{3L_0\mathcal{M}}{m_{s,0}^3}\sum_{j=1}^{n-1}\mathrm{E}[|A_j|]$$
$$=\epsilon\left(\frac{2L_0|y|}{m_{s,0}^3}+\frac{F_0}{m_{s,0}^2}+\frac{3L_0\mathcal{M}}{m_{s,0}^3}\sum_{j=1}^{n-1}\mathrm{E}[|A_j|]\right).$$

Taking into account Remark 2, the above sum of the expectations is finite. So, we the following result has been established:

Theorem 2 *Consider the random fractional initial value problem* (1.3)*, where its data satisfies the assumptions* ***A1*–*A3****. Then, the probability density function,* $f_{Y^M(t)}(y)$*, defined by* (2.6)*, corresponding to the approximate solution of order* M *given in* (2.1)–(2.2)*, converges to the probability density function,* $f_{Y(t)}(y)$*, given by* (2.7) *of the exact solution* (1.4)–(1.5) *as* $M \to \infty$ *for each* (t, y).



(a) $t = 0.5$ **(b)** $t = 1$ **(c)** $t = 1.5$ **(d)** $t = 2$

Fig. 1 Approximations of $f_{Y^M(t)}$, given in (2.6), of 1-PDF of the solution stochastic process at the time instants $t \in \{0.5, 1, 1.5, 2\}$ for different orders of truncation $M \in \{1, 5, 10\}$. Example 1 with $\alpha = 1.9$

Table 1 Approximations of the mean and variance at time instants $t \in \{0.5, 1, 1.5, 2\}$ using different orders of truncation $M \in \{1, 5, 10\}$. These values have been calculated using (3.1) and (3.2), respectively. Example 1 with $\alpha = 1.9$

$\mathbb{E}[Y^M(t)] \mid \mathbb{V}[Y^M(t)]$	$t = 0.5$	$t = 1$	$t = 1.5$	$t = 2$
$M = 1$	1.9841 | 1.22372	2.84533 | 1.7836	3.37137 | 2.39698	3.26288 | 3.04415
$M = 5$	1.9842 | 1.22378	2.84807 | 1.7872	3.40555 | 2.42417	3.46492 | 2.99021
$M = 10$	1.9842 | 1.22378	2.84807 | 1.7872	3.40555 | 2.42417	3.46492 | 2.99021

3 Examples

This section is devoted to illustrate the previous theoretical findings with two numerical examples, where we graphically show the convergence of the sequence of PDFs, $f_{Y^M(t)}(y)$, to $f_{Y(t)}(y)$, as M increases. Additionally, as assumptions **A1** and **A2** guarantee hypotheses **H1** and **H2** fulfill, then $Y^M(t)$, given by (2.1)–(2.2), is mean square convergent to $Y(t)$, given by (1.4)–(1.5). As a consequence, the mean and the variance of the solution exist, and then they can be approximated by

$$\mathbb{E}[Y^M(t)] = \int_{-\infty}^{\infty} y f_{Y^M(t)}(y)\,\mathrm{d}y \tag{3.1}$$

and

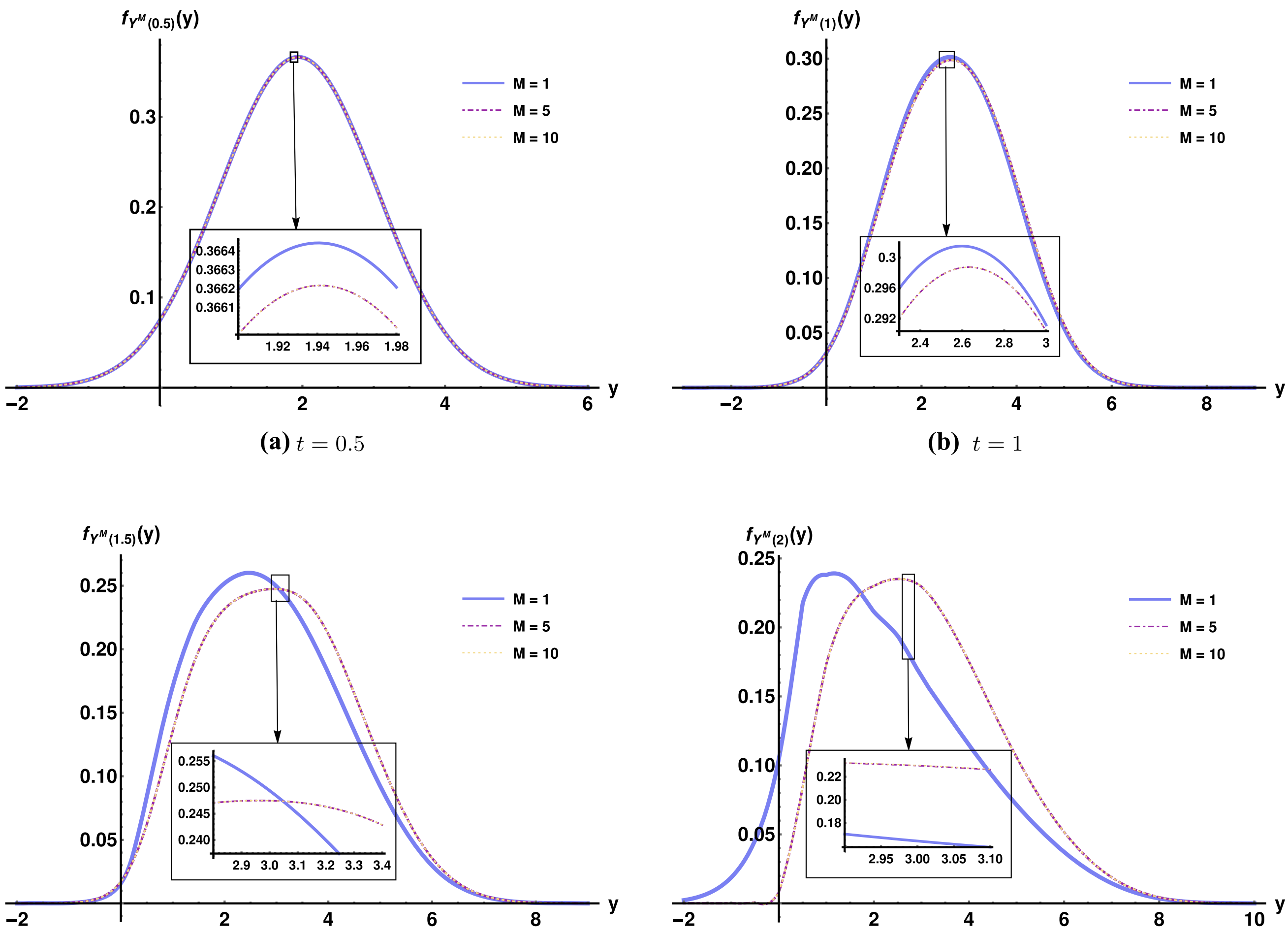
$$\mathbb{V}[Y^M(t)] = \mathbb{E}[(Y^M(t))^2] - (\mathbb{E}[Y^M(t)])^2 = \int_{-\infty}^{\infty} y^2 f_{Y^M(t)}(y)\,\mathrm{d}y - \left(\int_{-\infty}^{\infty} y f_{Y^M(t)}(y)\,\mathrm{d}y\right)^2, \tag{3.2}$$

respectively, (see [13, Th.4.2.1] and [13, Th. 4.3.1], respectively).

Example 1 The first example aims at showing that the results obtained with the approach presented in this paper are in full agreement with the ones obtained in [14]. To show the results given in [14, Example 1] agree with the ones in the present example, let us assume that the values of A_0 and A_1 and B are close to that example, so ensuring assumptions **A1**–**A3** fulfill. Firstly, let us consider $\beta = 1$ and $\alpha = 1.9$. In [14] the random variable B is considered as a negative beta distribution of parameters 2 and 3. According to

Table 2 Values of the L_1-norm defined in (3.3) for different orders of truncation, $M \in \{1, 5, 10\}$, at the time instants $t \in \{0.5, 1, 1.5, 2\}$. The order of the fractional derivative is $\alpha = 1.9$. Example 1

$e_M(t)$; $\alpha = 1.9$	$t = 0.5$	$t = 1$	$t = 1.5$	$\mathbf{t = 2}$
$M = 1$	0.00003521008	0.00184215438	0.01988455842	0.11839990407
$M = 5$	$2.5804011 \cdot 10^{-17}$	$1.4576956 \cdot 10^{-12}$	$1.6507047 \cdot 10^{-9}$	$4.0953768 \cdot 10^{-7}$
$M = 10$	0.	0.	0.	$3.4694469 \cdot 10^{-17}$



(a) $t = 0.5$ **(b)** $t = 1$ **(c)** $t = 1.5$ **(d)** $t = 2$

Fig. 2 Approximations of $f_{Y^M(t)}$, given in (2.6), of 1-PDF of the solution stochastic process at the time instants $t \in \{0.5, 1, 1.5, 2\}$ for different orders of truncation $M \in \{1, 5, 10\}$. Example 1 with $\alpha = 1.2$

Assumption **A2** the random variable B must be bounded and away from 0. To guarantee this assumption is satisfied while the spirit of [14, Example 1] holds true, we here take B as a negative Beta distribution of parameters 2 and 3 truncated at the interval [0.05, 1], i.e., $B \sim -\mathrm{Be}_{[0.05,1]}(2, 3)$. The initial conditions A_0 and A_1 are taken as in [14, Example 1]. So A_0 is a Normal distribution with mean 1 and standard deviation 1, i.e., $A_0 \sim \mathrm{N}(1; 1)$, hence $\mathbb{E}[A_0] = 1$ $\mathbb{E}[A_0^2] = 2$. It is important to remark that the PDF of a Gaussian distribution has bounded derivative on the whole real line, so applying the Mean Value theorem it is straightforward to check that is globally Lipschitz continuous [23]. As a consequence, assumption **A3** fulfills. The random variable A_1 is assumed to have a uniform distribution of mean 2 and standard deviation 1, i.e., $A_1 \sim \mathrm{U}(2 - \sqrt{3}, 2 + \sqrt{3})$. Additionally, we will assume that A_0, A_1 and B are independent, so assumption **A1** holds.

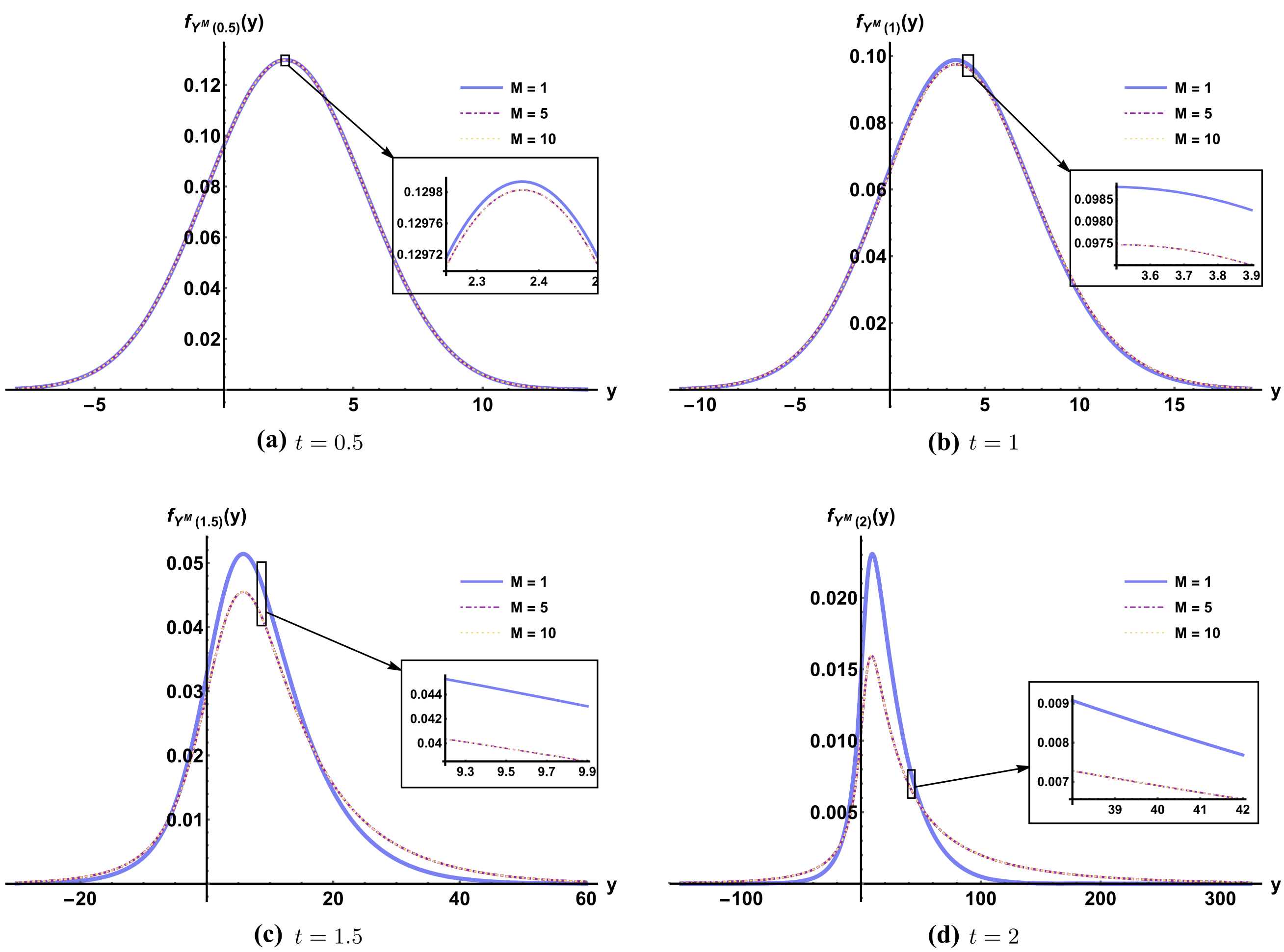
In Fig. 1, we graphically show the approximations, $f_{Y^M(t)}$, of the 1-PDF (2.6) at the time instants $t \in \{0.5, 1, 1.5, 2\}$ for different orders of truncation, $M \in \{1, 5, 10\}$. In each panel, we have performed a zoom to better illustrate the convergence as M increases.

Table 3 Approximations of the mean and variance at time instants $t \in \{0.5, 1, 1.5, 2\}$ using different orders of truncation $M \in \{1, 5, 10\}$. These values have been calculated using (3.1) and (3.2), respectively. Example 1 with $\alpha = 1.2$

$\mathbb{E}[Y^M(t)] \mid \mathbb{V}[Y^M(t)]$	$t = 0.5$	$t = 1$	$t = 1.5$	$\mathbf{t = 2}$
$M = 1$	1.94084 | 1.16605	2.62433 | 1.53783	2.82825 | 1.99607	2.3131 | 3.18559
$M = 5$	1.94185 | 1.16743	2.65148 | 1.56298	3.01537 | 2.03536	3.03015 | 2.53089
$M = 10$	1.94185 | 1.16743	2.65148 | 1.56298	3.01539 | 2.03536	3.03099 | 2.52984

Table 4 Values of the L_1-norm defined in (3.3) for different orders of truncation, $M \in \{1, 5, 10\}$, at the time instants $t \in \{0.5, 1, 1.5, 2\}$. The order of the fractional derivative is $\alpha = 1.2$. Example 1

$e_M(t); \alpha = 1.2$	$t = 0.5$	$t = 1$	$\mathbf{t = 1.5}$	$\mathbf{t = 2}$
$M = 1$	0.00093398498	0.02108848639	0.13985203151	0.48984846327
$M = 5$	$6.5739242 \cdot 10^{-12}$	$6.1373116 \cdot 10^{-8}$	0.00001457034	0.00091111341
$M = 10$	0.	$3.1576341 \cdot 10^{-16}$	$5.6654917 \cdot 10^{-12}$	$8.4097494 \cdot 10^{-9}$



(a) $t = 0.5$ **(b)** $t = 1$ **(c)** $t = 1.5$ **(d)** $t = 2$

Fig. 3 Approximations of $f_{Y^M(t)}$, given in (2.6), of 1-PDF of the solution stochastic process at the time instants $t \in \{0.5, 1, 1.5, 2\}$ for different orders of truncation $M \in \{1, 5, 10\}$. Example 2 with $\alpha = 2.5$

In Table 1, we collect the approximations of the mean and the variance of the solution using (3.1) and (3.2), respectively. We point out these figures agree with the ones obtained in [14, Example 1]. To numerically verify the convergence of the 1-PDFs as M increases, Table 2 collects the L_1-norm of the difference between consecutive approximations of the 1-PDF with respect the order of truncation, i.e.

$$e_M(t) := \| f_{Y_{M+1}(t)}(y) - f_{Y_M(t)}(y) \|_1 = \int_{\mathcal{D}_t} | f_{Y_{M+1}(t)}(y) - f_{Y_M(t)}(y) | \, \mathrm{d}y. \tag{3.3}$$

Here, we have taken the same domain of integration, $\mathcal{D}_t$, as the one used for plotting the PDF for for t fixed. As we can observe, the error decreases with M, so correctly illustrating convergence as expected.

We complete this example showing the corresponding approximations when $\alpha = 1.2$ at the same foregoing time instants and using the same orders of truncation. The approximations of the 1-PDF of the solution are shown in Fig. 2, while Table 3 collects the figures for the mean and the variance. In Table 4 we collect the $L1$-norm for the difference for consecutive approximations. We can graphically and numerically observe convergence as the order of truncation increases, respectively.

Example 2 In this second example we deal with the case that $\alpha = 2.5$, so $n = -\lfloor -2.5 \rfloor = 3$, and three initial conditions, A_0, A_1 and A_2 are then required. We take $A_0 \sim \mathrm{N}(2; 3^2)$, hence **A3** is fulfilled. The random variables A_1 and A_2 are assumed $A_1 \sim \mathrm{Be}(1, 1)$ and $A_2 \sim \mathrm{U}(2, 4)$. In order to guarantee assumption **A2** holds true, we will assume that B has a Gamma distribution of parameters $(2, 2)$ truncated at the interval $[0.05, 15]$, i.e., $A_2 \sim \mathrm{Ga}_{[0.05,15]}(2, 2)$. In Fig. 3 we show the approximations $f_{Y^M(t)}$ for the PDF of the solution at the time instants $t \in \{0.5, 1, 1.5, 2\}$ for different orders of truncation. In each panel of Fig. 3 is graphically observed convergence as M increases. From the plots, and particularly looking at the magnified plots, we can observe that, for t fixed, the approximations improve when M increases, while for M fixed, the approximations deteriorate as t increases, as expected.

4 Conclusions

In this paper we have provided an alternative approach to approximate the first probability density function of the solution stochastic process of a generalized model formulated via a differential equation that appears in a number of relevant problems of Classical Mechanics. The proposed approach avoids double approximations, and hence their associated errors, based on first approximating the moments of the solution and secondly applying the Principle of Maximum Entropy. The mild conditions assumed to conduct our study permit applying our theoretical results to a wide variety of real-world scenarios where probability distributions must be inferred from all the model data. Our study can be extended to other problems in Physics formulated by means of generalizable differential equations with uncertainties as well as to study the randomized versions of recent techniques that have demonstrated to be very useful to deal with linear and nonlinear problems in Physics [24] and [25].

Acknowledgements This paper has been supported by the grant PID2020–115270GB–I00 funded by MCIN/AEI/10.13039/501100011033 and by the grant AICO/2021/302 (Generalitat Valenciana). The author CP was partially supported by CMUP (UID/-MAT/00144/2013), which is funded by Fundação para a Ciência e Tecnologia (FCT) (Portugal) with national (MEC) and European structural funds European Regional Development Fund (FEDER), under the partnership agreement PT2020.

Declarations

Conflict of interest The authors declare that they do not have any conflict of interest.

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