A reliable treatment to solve nonlinear Fredholm integral equations with non-separable kernel

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Abstract

This work is devoted to solve integral equations formulated in terms of the kernel functions and Nemystkii operators. This type of equations appear in different applied problems such as electrostatics and radiative heat transfer problems. We deal with both cases separable and non-separable kernels by setting the theoretical semilocal convergence results for an adequate iterative scheme that can be useful for approximating the solution of the infinite dimensional problem. We pay special attention to non-separable kernels avoiding the solution given in previous works where the original nonlinear integral equation has been approximated by means of an equation with separable kernel. However, in this case, we introduce an approximation of the derivative operator that it is needed for applying the iterative scheme considered. Moreover, we study the localization and separation of possible solutions of nonlinear integral equation by means a result of semilocal convergence for the iterative scheme considered. The theoretical results obtained have been tested with some applied problems showing competitive results.

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1 Introduction

In this paper, we consider the integral equation given by

$$x(s) = f(s) + \lambda \int_a^b \mathcal{K}(s,t)[\mathcal{N}(x)](t)dt, \tag{1}$$

where $\mathcal{N}$ is a Nemystkii operator [9], $\mathcal{N} : \Omega \subseteq \mathcal{C}[a,b] \longrightarrow \mathcal{C}[a,b]$, with $[\mathcal{N}(x)](t) = N(x(t))$, being $N : \mathbb{R} \longrightarrow \mathbb{R}$ a derivable scalar function, $f : [a,b] \longrightarrow \mathbb{R}$ a continuous function and $\mathcal{K} : [a,b] \times [a,b] \longrightarrow R$ a continuous function in both arguments, λ a real parameter and x is a solution to be determined. $\mathcal{C}[a,b]$ denotes the space of continuous real functions in $[a,b]$.

These equations are related to boundary value problems for differential equations, since they can be reformulated as two-point boundary value problems or elliptic partial differential equations with nonlinear boundary conditions [4, 10]. Moreover, these equations appear in several applications to real world: the theory of elasticity, engineering, mathematical physics, potential theory, electrostatics and radiative heat transfer problems [3].

It is known that if the kernel $\mathcal{K}(s,t)$ of the integral equation given in (1) is non-separable, the choice of the iterative scheme to approximate a solution of (1) is restricted. So, our first aim in this paper is to approximate a solution of equation (1), with non-separable kernel, by means of an iterative scheme and considering directly the infinite dimensional case (namely, without using a process of discretization of the problem). To achieve this aim, an iterative scheme ad hoc for this problem is obtained.

If we pay attention to the iterative schemes that can be applied for approximating a solution $x^* \in \mathcal{C}[a,b]$ of (1), the method of successive approximations plays an important role (see, [1, 2, 11]). This method consists of applying the Fixed Point Theorem to the equation

$$x(s) = F(x)(s), \tag{2}$$

with $F : \Omega \subseteq \mathcal{C}[a,b] \longrightarrow \mathcal{C}[a,b]$, where Ω is a nonempty convex domain in $\mathcal{C}[a,b]$, with

$$F(x)(s) = f(s) + \lambda \int_a^b \mathcal{K}(s,t)[\mathcal{N}(x)](t)dt \tag{3}$$

and obtaining a sequence $\{x_{n+1} = F(x_n)\}_{n\in\mathbb{N}}$ that converges to a solution $x^* \in \mathcal{C}[a,b]$ of (1), i. e., a fixed point of F.

Observe that looking for a fixed point of equation (2) is equivalent to solving $G(x) = 0$, where $G : \Omega \subseteq \mathcal{C}([a,b]) \longrightarrow \mathcal{C}([a,b])$ and

$$G(x)(s) = x(s) - F(x)(s) = [(I - F)(x)](s). \tag{4}$$

In relation to the above, we can obtain the sequence of approximations $\{x_n\}$ by different ways, depending on the iterative schemes applied. Between these, the best-known iterative scheme with quadratic convergence is Newton's method, whose algorithm is the following:

$$\begin{cases} x_0 \text{ given in } \Omega, \\ x_{n+1} = x_n - [G'(x_n)]^{-1}G(x_n), \quad n = 0, 1, 2\ldots \end{cases} \tag{5}$$

However, for non-separable kernels, this iterative scheme sets out an important difficulty: we cannot calculate explicitly the operator $[G'(x_n)]^{-1} = (I - F'(x_n))^{-1}$ in each step of the algorithm. This problem was first studied in [8] where for solving this problem a Newton-type method is constructed. Then, we consider a two-steps iterative scheme so that it is modified as in [8].

Therefore, in this paper, we consider the two-steps iterative scheme with frozen first derivative given by the following algorithm:

$$\begin{cases} x_0 \text{ given in } \Omega, \\ y_n = x_n - [G^{'}(x_n)]^{-1}G(x_n) \\ x_{n+1} = y_n - [G^{'}(x_n)]^{-1}G(y_n), n \geq 0. \end{cases}$$

It is well known that if we compose Newton's method with itself twice, but taking into account the derivative frozen, we obtain an iterative scheme of order three. This is a classical result obtained by Traub, [12]. Moreover, being an iterative scheme of third order, it does not increase the expensive computation of derivatives because this iterative scheme only uses the same first derivative in each step. For this, it is easy to check that this iterative scheme is more efficient than Newton's method [12]. So, in this paper, we consider an iterative scheme of fixed point type for approximating a fixed point of F. The algorithm of this iterative scheme is

$$\begin{cases} x_0 \text{ given in } \Omega, \\ y_n = x_n - [I - F^{'}(x_n)]^{-1}(x_n - F(x_n)) \\ x_{n+1} = y_n - [I - F^{'}(x_n)]^{-1}(y_n - F(y_n)), n \geq 0. \end{cases} \tag{6}$$

Notice that this iterative scheme is the frozen two steps Newton method [7] applied to the equation $G(x)(s) = x(s) - F(x)(s) = 0$. So, we introduce a variant of iterative scheme (6) and obtain an iterative scheme to approximate a solution of (1) when the kernel is non-separable.

The paper it is organized as follows. First of all, in section 2 we describe an algorithm for solving the equation (1) in the case of separable kernel. In next section, we modify this iterative scheme for case of non-separable kernel by introducing an operator that approximates the inverse of the derivative. Section 4 it is devoted to present some numerical experiments that confirm the construction of this new procedure. So, in section 5 we perform a qualitative study of equation (1) by obtaining a result of existence and uniqueness. Moreover, a solution of (1) is successively approximated by the two-steps iterative scheme obtained from a modification of (6). Finally, we draw some conclusions of the developed work.

2 Motivation

In what follows, we consider $F : \Omega \subseteq \mathcal{C}[a,b] \longrightarrow \mathcal{C}[a,b]$, where Ω is a nonempty convex domain in $\mathcal{C}[a,b]$, and the Nemytskii operator $\mathcal{N} : \Omega \subseteq \mathcal{C}[a,b] \longrightarrow \mathcal{C}[a,b]$ such that $\mathcal{N}(x)(s) = N(x(s))$ for $x \in \mathcal{C}[a,b]$, where N is a derivable scalar function. Obviously, the operator F given in (3) is a Frechet differentiable operator and verifies, for $x, y, w \in \mathcal{C}[a,b]$,

$$[F'(x)y](s) = \lambda \int_a^b \mathcal{K}(s,t)[\mathcal{N}'(x)y](t)dt = \lambda \int_a^b \mathcal{K}(s,t)N'(x(t))y(t)dt.$$

When we want to apply iterative scheme (6) in the infinite dimensional case, we consider the problem of the construction of the operator $[I - F'(x_n)]^{-1}$ at each step. Then, assuming that $\mathcal{K}(s,t)$ is a separable kernel:

$$\mathcal{K}(s,t) = \sum_{i=1}^{m} \alpha_i(s)\beta_i(t),$$

if we denote $I_j = \int_a^b \beta_j(t)N'(x(t))y(t)\,dt$, we have

$$[(I - F'(x))(y)](s) = w(s) = y(s) - \lambda \sum_{j=1}^{m} \alpha_j(s)I_j$$

and

$$[(I - F'(x))^{-1}(w)](s) = .y(s) = w(s) + \lambda \sum_{j=1}^{m} \alpha_j(s)I_j. \tag{7}$$

Besides, the integrals I_j can be calculated independently of y. To do this, we multiply equality (7) by $\beta_i(s)N'(x(s))$ and we integrate in the s variable the equality obtained. So, we have

$$I_i - \lambda \sum_{j=1}^{m} \left(\int_a^b \beta_i(s)N'(x(s))\alpha_j(s)\,ds \right) I_j = \int_a^b \beta_i(s)N'(x(s))w(s)\,ds.$$

Now, if we denote

$$a_{ij}(x) = \int_a^b \beta_i(s)N'(x(s))\alpha_j(s)\,ds \quad \text{and} \quad b_i(x) = \int_a^b \beta_i(s)N'(x(s))w(s)\,ds,$$

we obtain the following linear system of equations

$$I_i - \lambda \sum_{j=1}^{m} a_{ij}(x)I_j = b_i(x), \quad i = 1, \ldots, m. \tag{8}$$

This system has a unique solution if

$$(-\lambda)^m \begin{vmatrix} a_{11}(x) - \frac{1}{\lambda} & a_{12}(x) & a_{13}(x) & \dots & a_{1m}(x) \\ a_{21}(x) & a_{22}(x) - \frac{1}{\lambda} & a_{23}(x) & \dots & a_{2m}(x) \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1}(x) & a_{m2}(x) & a_{m3}(x) & \dots & a_{mm}(x) - \frac{1}{\lambda} \end{vmatrix} \neq 0.$$

Then, we assume $\dfrac{1}{\lambda}$ is not an eigenvalue of the matrix $(a_{ij}(x))$. Thus, if $I_1, I_2, \ldots, I_m$ is the solution of system (8), we can define

$$[(I - F^{'}(x))^{-1}(w)](s) = w(s) + \lambda \sum_{j=1}^{m} \alpha_j(s) I_j,$$

and the iteration (6) can be applied, whose convergence was established in [6].

Now we wonder, what happens when the kernel is not separable?. In the infinite dimensional case, it may be considered to approximate the original nonlinear integral equation with non-separable kernel by means of another nonlinear integral equation with separable kernel, see [5]. Therefore, we can apply the iterative scheme (6) to solve the approximate nonlinear integral equation. But, as we can see below, the errors produced may not suggest applying this procedure.

Now, we assume that $x^*(s)$ is a solution of equation (1) and the kernel $\mathcal{K}(s,t)$ is non-separable, then the application of iterative scheme (6) for solving (1) is difficult. Taking into account this fact, if we can approximate $\mathcal{K}(s,t)$:

$$\mathcal{K}(s,t) = \widetilde{\mathcal{K}}(s,t) + \mathcal{R}(s,t), \tag{9}$$

where $\widetilde{\mathcal{K}}(s,t) = \sum_{i=1}^{m} \alpha_i(s)\beta_i(t)$, we consider the nonlinear integral equation

$$x(s) = f(s) + \lambda \int_a^b \widetilde{\mathcal{K}}(s,t)[\mathcal{N}(x)](t)dt, \tag{10}$$

and therefore, we apply the iterative scheme (6) to solve the nonlinear integral equation $\widetilde{G}(x)(s) = x(s) - \widetilde{F}(x)(s) = [(I - \widetilde{F})(x)](s) = 0$, with

$$\widetilde{F}(x)(s) = f(s) + \lambda \int_a^b \widetilde{\mathcal{K}}(s,t)[\mathcal{N}(x)](t)dt, \tag{11}$$

to obtain an approximation to $x^*(s)$. So, a solution of (10), which has separable kernel, is then approximated by iterative scheme (6) and following the procedure previously developed.

If we denote a solution of (10) by $\widetilde{x}(s)$ and we now look for it by means of iterative scheme (6), applying the above procedure for separable kernels, we obtain a sequence $\{\widetilde{x}_n\}$ that, under some conditions (see [**?**]), converges to $\widetilde{x}(s)$. But, as

$$\|x^*(s) - \widetilde{x}_n(s)\| \leq \|x^*(s) - \widetilde{x}(s)\| + \|\widetilde{x}(s) - \widetilde{x}_n(s)\|,$$

for obtaining an suitable error it is necessary that the quantity $\|x^*(s) - \widetilde{x}(s)\|$ is sufficiently small. Obviously, this depends of the value $\|\mathcal{R}(s,t)\|$. If, for example, $\mathcal{K}(s,t)$ is sufficiently derivable in some argument , we can apply the Taylor series to calculate the approximation given in (10) and then the error made by the Taylor series will allow us to establish how much $\{x_n\}$ approaches to x^*. Improving this approach will depend, in general, on the number of Taylor's development terms. Therefore, we will have to increase the operational cost of the procedure used.

Therefore, having into account the previous reasoning, it is clear that if we want apply the iterative scheme (6) for non-separable kernels working in the infinite dimensional case, we must modify this iterative scheme, as we can see in the next section.

3 Construction of an iterative scheme for non-separable kernels

Taking into account the procedures developed in the previous section, to apply the iterative scheme (6) in the infinite dimensional case, it is clear that this iterative scheme needs to be modified to make its implementation more effective. Thus, if we want to approximate a solution of the equation (1), with $\mathcal{K}(s,t)$ a non-separable kernel, we consider the operator $A : \Omega \subseteq \mathcal{C}[a,b] \longrightarrow \mathcal{L}(\mathcal{C}[a,b], \mathcal{C}[a,b])$, given by

$$[A(x)(y)](s) = y(s) - \lambda \int_a^b \widetilde{\mathcal{K}}(s,t) N'(x(t)) y(t) dt, \tag{12}$$

where $\widetilde{\mathcal{K}}(s,t)$ is a separable kernel with $\widetilde{\mathcal{K}}(s,t) = \sum_{i=1}^m \alpha_i(s)\beta_i(t)$ and $\mathcal{K}(s,t) = \widetilde{\mathcal{K}}(s,t) + \mathcal{R}(s,t)$. $\mathcal{L}(\mathcal{C}[a,b], \mathcal{C}[a,b])$ is the set of linear operators in $\mathcal{C}[a,b]$. Notice that, the operator $A(x)$ is an approximation of the operator $I - F'(x)$. Besides, if $\mathcal{K}(s,t)$ is a separable kernel, we will consider $\mathcal{K}(s,t) = \widetilde{\mathcal{K}}(s,t)$ and then $A(x) = I - F'(x)$.

From the previous reasoning, we consider the frozen two-step Newton-type method, given by the following algorithm:

$$\begin{cases} x_0 \text{ given in } \Omega, \\ y_n = x_n - A(x_n)^{-1}(x_n - F(x_n)) \\ x_{n+1} = y_n - A(x_n)^{-1}(y_n - F(y_n)), n \geq 0. \end{cases} \tag{13}$$

Notice that in the iterative scheme (13) we work directly with the operator F. So, the nonlinear integral with non-separable kernel is not approximate (see equation (11)) as when we have applied the iterative scheme (6) in the previous section. In this case we approximate de inverse of the derivative, that is, $(I - F'(x))^{-1}$ by means of $A(x)^{-1}$.

4 Numerical Experiments

We consider the following nonlinear Fredholm integral equation,

$$x(s) = \mathrm{e}^s(1 - \mathrm{e}^2) + 1 + \int_0^1 (s+2)\mathrm{e}^{st} x(t)^2 dt. \tag{14}$$

It is easy to check that $x^*(s) = \mathrm{e}^s$ is a solution.

Obviously, in this case $\Omega = \mathcal{C}[0,1]$ and the kernel $\mathcal{K}(s,t) = (s+2)\mathrm{e}^{st}$ is non-separable. Then, for example, if we take $m = 3$:

$$\mathcal{K}(s,t) = (s+2)\mathrm{e}^{st} = (s+2)\big(\sum_{i=0}^{2} \frac{s^i\, t^i}{i!} + \mathcal{R}(\theta,s,t)\big), \quad \mathcal{R}(\theta,s,t) = \frac{\mathrm{e}^{s\theta}}{3!} s^3\, t^3, \tag{15}$$

thus, if we consider

$$\widetilde{\mathcal{K}}(s,t) = (s+2) + s(s+2)t + s^2(s+2)\frac{t^2}{2},$$

we have

$$\mathcal{K}(s,t) = \widetilde{\mathcal{K}}(s,t) + \mathcal{R}(\epsilon,s,t), \text{ with } \widetilde{\mathcal{K}}(s,t) = \sum_{i=1}^{3} \alpha_i(s)\beta_i(t) \text{ and } \mathcal{R}(\theta,s,t) = \frac{\mathrm{e}^{s\theta}}{3!} s^3\, t^3,$$

for the real functions:

$$\alpha_1(s) = s+2, \;\; \alpha_2(s) = s(s+2), \;\; \alpha_3(s) = s^2(s+2),$$

$$\beta_1(t) = 1, \;\; \beta_2(t) = t, \;\; \beta_3(t) = \frac{t^2}{2}.$$

Then, if we apply the iterative scheme (6) for approximate the solution $x^*(s) = \mathrm{e}^s$ of equation (14), with the previous procedure, really we apply (6) for approximate a solution of the nonlinear integral equation with separable kernel, given by $\widetilde{G}(x)(s) = [(I - \widetilde{F})(x)](s) = 0$, with

$$\widetilde{F}(x)(s) = \mathrm{e}^s(1-\mathrm{e}^2) + 1 + \int_0^1 \left(s+2+s(s+2)t+s^2(s+2)\frac{t^2}{2}\right) x(t)^2 dt. \tag{16}$$

In this situation, we apply the following algorithm: Fixed $x_0 \in \mathcal{C}[0,1]$, for $n \geqslant 0$,

1. **First step**. Calculate the following value

$$z(x_n)(s) = x_n(s) - \widetilde{F}(x_n)(s).$$

2. **Second step**. Calculate the following integrals

$$a_{11}(x_n) = \int_0^1 2x_n(s)(s+2)ds, \qquad a_{12}(x_n) = \int_0^1 2x_n(s)s(s+2)ds,$$

$$a_{13}(x_n) = \int_0^1 x_n(s)s^2(s+2)ds, \qquad a_{21}(x_n) = \int_0^1 2x_n(s)s(s+2)ds,$$

$$a_{22}(x_n) = \int_0^1 2x_n(s)s^2(s+2)ds, \qquad a_{23}(x_n) = \int_0^1 x_n(s)s^3(s+2)ds,$$

$$a_{31}(x_n) = \int_0^1 2x_n(s)s^2(s+2)ds, \qquad a_{32}(x_n) = \int_0^1 2x_n(s)s^3(s+2)ds,$$

$$a_{33}(x_n) = \int_0^1 x_n(s)s^4(s+2)ds, \qquad b_1(x_n) = \int_0^1 2x_n(s)z(x_n)(s)ds,$$

$$b_2(x_n) = \int_0^1 2sx_n(s)z(x_n)(s)ds, \qquad b_3(x_n) = \int_0^1 s^2x_n(s)z(x_n)(s)ds.$$

3. **Third step**. To obtain I_1, I_2 and I_3, solve the following linear system

$$I_i - \sum_{j=1}^{3} a_{ij}(x_n)I_j = b_i(x_n), \quad i = 1, 2, 3.$$

4. **Fourth step**. $y_n(s) = x_n(s) - z(x_n)(s) - \sum_{j=1}^{3} \alpha_j(s)I_j$.
5. **Fifth step**.

$$z(y_n)(s) = y_n(s) - \widetilde{F}(y_n)(s).$$

6. **Sixth**. Calculate the following integrals

$$\hat{b}_1(x_n) = \int_0^1 2x_n(s)z(y_n)(s)ds,$$

$$\hat{b}_2(x_n) = \int_0^1 2sx_n(s)z(y_n)(s)ds,$$

$$\hat{b}_3(x_n) = \int_0^1 s^2x_n(s)z(y_n)(s)ds.$$

7. **Seventh step**. To obtain $\hat{I}_1, \hat{I}_2$ and $\hat{I}_3$, solve the following linear system

$$\hat{I}_i - \sum_{j=1}^{3} a_{ij}(x_n)\hat{I}_j = \hat{b}_i(x_n), \quad i = 1, 2, 3.$$

8. **Eighth step**. Calculate

$$x_{n+1}(s) = y_n(s) - z(y_n)(s) - \sum_{j=1}^{3} \alpha_j(s)\hat{I}_j.$$

Moreover, in other case, we consider for example $m = 6$. So, we have

$$\mathcal{K}(s,t) = \widetilde{\mathcal{K}}(s,t) + \mathcal{R}(\theta,s,t), \text{ with } \widetilde{\mathcal{K}}(s,t) = \sum_{i=1}^{6} \alpha_i(s)\beta_i(t) \text{ and } \mathcal{R}(\theta,s,t) = \frac{\mathrm{e}^{s\theta}}{6!}s^6\,t^6,$$

for the real functions:

$$\alpha_1(s) = s+2,\ \alpha_2(s) = s(s+2),\ \alpha_3(s) = s^2(s+2), \alpha_4 = s^3(s+2), \alpha_5 = s^4(s+2), \alpha_6 = s^5(s+2),$$

$$\beta_1(t) = 1,\ \beta_2(t) = t,\ \beta_3(t) = \frac{t^2}{2}, \beta_4(t) = \frac{t^3}{3!}, \beta_5(t) = \frac{t^4}{4!}, \beta_6(t) = \frac{t^5}{5!}.$$

So, we have

$$\begin{aligned} \widetilde{F}(x)(s) &= \mathrm{e}^s(1-\mathrm{e}^2) + 1 + \\ &+ \int_0^1 (s+2)\left(1 + st + s^2\frac{t^2}{2} + s^3\frac{t^3}{3!} + s^4\frac{t^4}{4!} + s^5\frac{t^5}{5!}\right) x(t)^2 dt. \end{aligned} \tag{17}$$

In the analysis of the numerical results we use the Taylor approximation for the non-separable kernel, see (15), by considering different order approximations from $m = 3$ and $m = 6$. In first place, we consider the iterative scheme (6), the algorithm where the nonlinear integral equation has been approximated by means of separable kernels, expressed by (16) and (17). In second place, we consider the iterative scheme (13), the algorithms where we have solved the problem by using the new approximation $A(x)$ for the operator $I - F'(x)$ (12), that is, we work with the original non-separable kernel. We have calculate the integrals resulting in the second and sixth's steps of the algorithm by applying Gauss-Legendre's formula, see Tables 1 and 2.

We run the algorithms by allowing a maximum of 10, 20 and 30 iterations for reaching a tolerance of 10^{-50}. Then, in the results we can see, for each value of m, the iterations needed k, the distance between the last two iterates, $\|x_n(s) - x_{n-1}(s)\|$ and the distance to the exact solution of the problem, given by $x^*(s) = \mathrm{e}^s$.

Results, in Tables 1 and 2, show that iterative scheme (6) always reaches the stopping criterion quickly due to the fact that it is a third order iterative scheme. However, iterative scheme (13) needs more iterations to reach it, for $m = 3$. For $m = 6$, the iterative scheme (13) does not need more iterations than the iterative scheme (6). For the starting guesses considered, it is obvious the improvement obtained by iterative scheme (13) respect to (6) for approximating the solution $x^*(s)$. This fact justify the construction of this new procedure (13).

5 Main results on the convergence

Now, to obtain a semilocal convergence result for (13), we assume that the following conditions are satisfied:

(I) $A(x_0)^{-1}$ exists for some $x_0 \in \Omega \subseteq \mathcal{C}[a,b]$, with $\|A(x_0)^{-1}\| \leq \beta$ and $\|A(x_0)^{-1}(x_0 - F(x_0))\| \leq \eta$.

(II) $\mathcal{N}'$ is ω-Lipschitz continuous operator such that

$$\|\mathcal{N}'(u) - \mathcal{N}'(v)\| \leq \omega(\|u - v\|) \text{ for } u, v \in \Omega, \tag{18}$$

maxiter	$m = 3$	Iter. scheme (6)	Iter. scheme (13)
	k	10	10
10	$\|x_n(s) - x_{n-1}(s)\|$	5.3796e-37	1.3637e-04
	$\|x_n(s) - x^*(s)\|$	2.6792e+00	2.0990e-05
	k	10	20
20	$\|x_n(s) - x_{n-1}(s)\|$	5.3796e-37	2.4190e-13
	$\|x_n(s) - x^*(s)\|$	2.6792e+00	4.7278e-15
	k	10	30
30	$\|x_n(s) - x_{n-1}(s)\|$	5.3796e-37	4.2878e-22
	$\|x_n(s) - x^*(s)\|$	2.6792e+00	3.2814e-14

maxiter	$m = 6$	Iter. scheme (6)	Iter. scheme (13)
	k	6	9
10	$\|x_n(s) - x_{n-1}(s)\|$	6.5839e-57	4.8355e-33
	$\|x_n(s) - x^*(s)\|$	1.2987e-02	3.2814e-14
	k	6	9
20	$\|x_n(s) - x_{n-1}(s)\|$	6.5839e-57	4.8355e-33
	$\|x_n(s) - x^*(s)\|$	1.2987e-02	3.2814e-14
	k	6	9
30	$\|x_n(s) - x_{n-1}(s)\|$	6.5839e-57	4.8355e-33
	$\|x_n(s) - x^*(s)\|$	1.2987e-02	3.2814e-14

Table 1: Numerical results by taking starting function $x_0(s) = 3/2\exp(s)$.

where $\omega : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a continuous and non-decreasing function satisfying $\omega(\alpha z) \leq h(\alpha)\omega(z)$ for $\alpha \in [0,1]$ and $z \in [0,+\infty)$, with $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ a continuous and non-decreasing function.

Note that condition of existence for the function h does not involve any restriction, since h always exists, such that $h(t) = 1$, as a consequence of ω being a non-decreasing function. We use it to sharpen the bounds that we obtain for particular expressions, as we will see later.

As first step, from the previous conditions, we easily obtain the following result for the operator $A(x)$ given in (12).

Lemma 1. *Under assumptions* (**I**) *and* (**II**), *if there exists* $R > 0$, *such that* $|\lambda|\beta\omega(R) < 1$, *then the operator* $A(x_n)^{-1}$ *exists with* $\|A(x_n)^{-1}\| \leq \mu(R)$ *for all* $x_n \in B(x_0, R)$, *where* $\mu(t) = \dfrac{\beta}{1 - \beta|\lambda|L\omega(t)}$ *and* $L = \max_{s\in[a,b]} \int_a^b |\widetilde{\mathcal{K}}(s,t)|dt$.

***Proof*.** Consider

$$\|I - A(x_0)^{-1}A(x_n)\| \leq \|A(x_0)^{-1}\|\|A(x_0) - A(x_n)\| \leq |\lambda|\beta L\|\mathcal{N}'(x_n) - \mathcal{N}'(x_0)\| \leq |\lambda|\beta L\omega(R) < 1.$$

Then, by Banach lemma, the result is proved. ■

maxiter	$m=3$	Iter. scheme (6)	Iter. scheme(13)
	k	6	10
10	$\|x_n(s)-x_{n-1}(s)\|$	3.4611e-57	4.9429e-07
	$\|x_n(s)-x^*(s)\|$	2.6792e+00	7.6028e-08
	k	6	20
20	$\|x_n(s)-x_{n-1}(s)\|$	3.4611e-57	8.7613e-16
	$\|x_n(s)-x^*(s)\|$	2.6792e+00	3.2680e-14
	k	6	30
30	$\|x_n(s)-x_{n-1}(s)\|$	3.4611e-57	1.5530e-24
	$\|x_n(s)-x^*(s)\|$	2.6792e+00	3.2814e-14
maxiter	$m=6$	Iter. scheme (6)	Iter. scheme (13)
	k	10	10
10	$\|x_n(s)-x_{n-1}(s)\|$	1.2701e+01	5.0350e-03
	$\|x_n(s)-x^*(s)\|$	1.0131e+01	1.5643e-07
	k	20	16
20	$\|x_n(s)-x_{n-1}(s)\|$	6.6250e+00	3.7707e-32
	$\|x_n(s)-x^*(s)\|$	1.1192e+00	3.2814e-14
	k	27	16
30	$\|x_n(s)-x_{n-1}(s)\|$	7.4536e-39	3.7707e-32
	$\|x_n(s)-x^*(s)\|$	1.2987e-02	3.2814e-14

Table 2: Numerical results by taking starting function $x_0(s)=2$.

In what follows, we tested a technical lemma to subsequently obtain recurrence relations for the sequences $\{x_n\}$ and $\{y_n\}$. For this, we consider (9).

Lemma 2. *Under assumptions* (**I**) *and* (**II**), *if* $x_n, y_n \in B(x_0,R)\subseteq\Omega$, *then*

(a) $\|x_{n+1}-y_n\| \le \phi(\|y_n-x_n\|,R)\|y_n-x_n\||$,

(b) $\|x_{n+1}-x_n\| \le (1+\phi(\|y_n-x_n\|,R))\|y_n-x_n\|$,

(c) $\|y_{n+1}-x_{n+1}\| \le \xi\Big(\|y_n-x_n\|,\|x_{n+1}-y_n\|,R\Big)\|x_{n+1}-y_n\|$,

where $\phi(t,u)=\mu(u)|\lambda|\Big(LD\omega(t)+M(\epsilon+\omega(u))\Big)$, $\xi(t,u,v)=\mu(v)|\lambda|\Big[L\Big(\omega(t)+D\omega(u)\Big)+M(\epsilon+\omega(v))\Big]$, $M=\max_{s\in[a,b]}\int_a^b\|\mathcal{R}(s,t)\|\|dt$, $D=\int_0^1 h(t)dt$ *and* $\epsilon=\|\mathcal{N}'(x_0)\|$.

***Proof*.** Using (13), we have

$$\|x_{n+1}-y_n\| \le \|A(x_n)^{-1}\|\|y_n-F(y_n)\| \le \beta_R\|y_n-F(y_n)\|.$$

Consider

$$\begin{aligned}(y_n - F(y_n))(s) &= y_n(s) - F(x_n)(s) - \left(\int_0^1 [F^{'}(x_n + \tau(y_n - x_n))(y_n - x_n)]d\tau\right)(s).\\ &= (x_n - F(x_n))(s) + (y_n - x_n)(s) - \left(\int_0^1 [F^{'}(x_n + \tau(y_n - x_n))(y_n - x_n)]d\tau\right)(s)\\ &= [\big(I - A(x_n)\big)(y_n - x_n)](s) - \left(\int_0^1 [F^{'}(x_n + \tau(y_n - x_n))(y_n - x_n)]d\tau\right)(s).\end{aligned}$$

Therefore,

$$\|y_n - F(y_n)\| \leq \left\| \int_0^1 \left(\lambda \int_a^b \widetilde{\mathcal{K}}(s,t)\mathcal{N}^{'}(x_n)(t)dt - F^{'}(x_n + \tau(y_n - x_n))\right) d\tau \right\| \|y_n - x_n\|.$$

Since $[F^{'}(x)y](s) = \lambda \int_a^b \mathcal{K}(s,t)\mathcal{N}^{'}(x)(t)y(t)dt$ and $\mathcal{K}(s,t) = \widetilde{\mathcal{K}}(s,t) + \mathcal{R}(s,t)$, from (9), we obtain

$$\begin{aligned}\|y_n - F(y_n)\| &\leq \left(\left\| \int_0^1 \left(\lambda \int_a^b \widetilde{\mathcal{K}}(s,t)\mathcal{N}^{'}(x_n)dt - \lambda \int_a^b \widetilde{\mathcal{K}}(s,t)\mathcal{N}^{'}(x_n + \tau(y_n - x_n)(t)dt\right) d\tau \right\|\right.\\ &+ \left.\left\| \int_0^1 \lambda \int_a^b \mathcal{R}(s,t)\mathcal{N}^{'}(x_n + \tau(y_n - x_n))(t)dtd\tau \right\|\right) \|y_n - x_n\|\\ &\leq |\lambda|\Big(LD\omega(\|y_n - x_n\|) + M(\epsilon + \omega(R))\Big)\|y_n - x_n\| = \phi(\|y_n - x_n\|, R)\|y_n - x_n\|,\end{aligned}$$

since that $\|\mathcal{N}^{'}(x_n + \tau(y_n - x_n))\| \leq \|\mathcal{N}^{'}(x_0)\| + \omega(R)$.

Hence,

$$\|x_{n+1} - y_n\| \leq \|A(x_n)^{-1}\|\|y_n - F(y_n)\| \leq \mu(R)\phi(\|y_n - x_n\|, R)\|y_n - x_n\|.$$

So, clearly

$$\|x_{n+1} - x_n\| \leq \|x_{n+1} - y_n\| + \|y_n - x_n\| \leq (1 + \mu(R)\phi(\|y_n - x_n\|, R))\|y_n - x_n\|.$$

Hence items (**a**) and (**b**) are proved respectively.

Now, to prove (**c**), we consider

$$\begin{aligned}(x_{n+1} - F(x_{n+1}))(s) &= x_{n+1}(s) - F(y_n)(s) - (F(x_{n+1}) - F(y_n))(s)\\ &= x_{n+1}(s) - F(y_n)(s) - \left(\int_0^1 [F^{'}(y_n + \tau(x_{n+1} - y_n]))(x_{n+1} - y_n)\, d\tau\right)(s).\end{aligned}$$

Using (13), we have $A(x_n)\,(x_{n+1}-y_n) = -y_n + F(y_n)$, i.e. $F(y_n) = y_n + A(x_n)\,(x_{n+1}-y_n)\,.$ Therefore, we have

$$\begin{aligned}
(x_{n+1}-F(x_{n+1}))(s) &= x_{n+1}(s) - y_n(s) - [A(x_n)\,(x_{n+1}-y_n)](s) \\
&- \left(\int_0^1 [F^{'}\,(y_n+\tau(x_{n+1}-y_n))\,(x_{n+1}-y_n)]d\tau\right)(s) \\
&= [(I-A(x_n))\,(x_{n+1}-y_n)](s) \\
&- \left(\int_0^1 [F^{'}\,(y_n+\tau(x_{n+1}-y_n))\,(x_{n+1}-y_n)]d\tau\right)(s).
\end{aligned}$$

Thus,

$$\begin{aligned}
\|x_{n+1}-F(x_{n+1})\| &\le \left\| \int_0^1 \left(\lambda \int_a^b \widetilde{\mathcal{K}}(s,t)\mathcal{N}^{'}(x_n)(t)dt - F^{'}\,(y_n+\tau\,(x_{n+1}-y_n))\,d\tau\right) \right\| \|x_{n+1}-y_n\| \\
&\le \left\| \int_0^1 \left(\lambda \int_a^b \widetilde{\mathcal{K}}(s,t)\mathcal{N}^{'}(x_n)(t)dt \right.\right. \\
&- \left.\left. \lambda \int_a^b \mathcal{K}(s,t)\mathcal{N}^{'}\,(y_n+\tau\,(x_{n+1}-y_n))\,(t)dt\right) d\tau \right\| \|x_{n+1}-y_n\| \\
&\le \left\| \int_0^1 \left(\lambda \int_a^b \widetilde{\mathcal{K}}(s,t)\left(\mathcal{N}^{'}\,(y_n+\tau(x_{n+1}-y_n)) - \mathcal{N}^{'}(x_n)\right)(t)dt\right) d\tau \right\| \|x_{n+1}-y_n\| \\
&+ \left\| \int_0^1 \left(\lambda \int_a^b \mathcal{R}(s,t)\mathcal{N}^{'}\,(y_n+\tau(x_{n+1}-y_n))\,(t)dt\right) d\tau \right\| \|x_{n+1}-y_n\| \\
&\le \left(\left\| \int_0^1 \lambda \int_a^b \widetilde{\mathcal{K}}(s,t)\left(\mathcal{N}^{'}\,(y_n+\tau(x_{n+1}-y_n)) - \mathcal{N}^{'}(y_n) + \mathcal{N}^{'}(y_n) - \mathcal{N}^{'}(x_n)\right)(t)d \right.\right. \\
&+ \left.\left. \left\| \int_0^1 \lambda \int_a^b \mathcal{R}(s,t)\mathcal{N}^{'}\,(y_n+\tau(x_{n+1}-y_n))\,(t) \right\| \right) \|x_{n+1}-y_n\| \\
&\le |\lambda| \Big(L\Big(D\omega(\|x_{n+1}-y_n\|) + \omega(\|y_n-x_n\|)\Big) + M(\epsilon+\omega(R))\Big) \|x_{n+1}-y_n\|.
\end{aligned}$$

Hence,

$$\|y_{n+1}-x_{n+1}\| \le \|A(x_{n+1})^{-1}\|\|x_{n+1}-F(x_{n+1})\| \le \xi\big(\|y_n-x_n\|, \|x_{n+1}-y_n\|, R\big)\|x_{n+1}-y_n\|.$$

■

Now, we establish two real sequences to obtain the recurrence relations to prove the existence of a solution of (1).

In first place, notice that $\mu(t)$ is non-decreasing in $(0,R]$. Therefore $\phi(t,u)$ is non-decreasing in $\mathbb{R}\times(0,R]$ and $\xi(t,u,v)$ is non-decreasing in $\mathbb{R}\times\mathbb{R}\times(0,R]$.

In second place, to derive the two real sequences, we define the scalar parameters $p_0 = \eta$, $q_0 = \phi(p_0, R)p_0$, $P = 1 + \phi(p_0, R)$ and $Q = \xi(p_0, q_0, R)\phi(p_0, R)$.

In third place, for $n = 0$, we have

$$\|y_0 - x_0\| = \|A(x_0)^{-1}(x_0 - F(x_0))\| \leq \eta = p_0,$$

$$\|x_1 - y_0\| \leq \phi(\|y_0 - x_0\|, R)\|y_0 - x_0\| \leq \phi(p_0, R)p_0 = q_0,$$

$$\|x_1 - x_0\| \leq \|x_1 - y_0\| + \|y_0 - x_0\| \leq (1 + \phi(p_0, R))p_0 = Pp_0.$$

Then, if $Pp_0 < R$, we obtain that $x_1, y_0 \in B(x_0, R)$.

On the one hand, using Lemma 2, for $n = 0$, we get

$$\|y_1 - x_1\| = \|A(x_1)(x_1 - F(x_1))\| \leq \xi\,(p_0, q_0, R)\,\phi(p_0, R)p_0 = Qp_0,$$

so, we define $p_1 = Qp_0$. Moreover, if we assume that $Q < 1$, then $p_1 < p_0$.

On the other hand, as $\|x_2 - y_1\| \leq \phi(p_1, R)p_1$, we define $q_1 = \phi(p_1, R)p_1$ and, obviously, $q_1 < q_0$.

Since $P > 1$ and using Lemma 2, we have

$$\|y_1 - x_0\| \leq \|y_1 - x_1\| + \|x_1 - x_0\| \leq Qp_0 + Pp_0 < (1 + Q)Pp_0,$$

$$\begin{aligned} \|x_2 - x_1\| &\leq \|x_2 - y_1\| + \|y_1 - x_1\| \leq \big(1 + \phi\,(\|y_1 - x_1\|, R)\,\big)\|y_1 - x_1\| \\ &\leq \big(1 + \phi\,(p_0, R)\,\big)\xi\,(p_0, q_0, R)\,\phi\,(p_0, R)\,p_0 = PQp_0, \end{aligned}$$

$$\|x_2 - x_0\| \leq \|x_2 - x_1\| + \|x_1 - x_0\| \leq PQp_0 + Pp_0 = (1 + Q)Pp_0.$$

Then, if $(1 + Q)Pp_0 < R$, we obtain that $x_2, y_1 \in B(x_0, R)$.

Proceeding in this way, for $n \geqslant 1$, we can define the following scalar sequences

$$p_n = Qp_{n-1} \text{ and } q_n = \phi(p_n, R)p_n.$$

It is easy to check that both sequences, $\{p_n\}$ and $\{q_n\}$, are decreasing.

Lemma 3. *Under assumptions* (**I**) *and* (**II**), *if the equation*

$$t = \frac{1 + \phi(\eta, t)}{1 - \xi(\eta, \phi(\eta, t)\eta, t)\phi(\eta, t)}\eta \tag{19}$$

has at least one positive real root and the smallest positive real root, denoted by R, *satisfies* $\beta\ |\lambda|\ M\ \omega(R) < 1$, $Q = \xi(\eta, \phi(\eta, R)\eta, R)\phi(\eta, R) < 1$ *and* $B(x_0, R) \subseteq \Omega$, *then*

($\mathbf{i_n}$) $\|y_n - x_n\| \leq p_n$ *and* $y_n \in B(x_0, R)$,

$(\mathbf{ii_n})$ $\|x_{n+1} - y_n\| \leq q_n$,

$(\mathbf{iii_n})$ $\|x_{n+1} - x_n\| \leq PQ^n p_0$ *and* $x_n \in B(x_0, R)$.

Proof. We will prove this lemma by using mathematical induction. For $n = 1$, we have already established $(\mathbf{i_n}), (\mathbf{ii_n})$ and $(\mathbf{iii_n})$. We assume that $(\mathbf{i_n}) - (\mathbf{iii_n})$ holds for $n = 1, 2, \ldots, k$. Then, for $k+1$, we have

$$\|y_{k+1} - x_{k+1}\| \leq \xi(p_k, q_k, R)\phi(p_k, R)p_k = Qp_k = p_{k+1}$$

and

$$\begin{aligned} \|y_{k+1} - x_0\| &\leq \|y_{k+1} - x_{k+1}\| + \|x_{k+1} - x_0\| \leq Qp_k + (1 + Q + \ldots + Q^k)Pp_0 \\ &< (1 + Q + \ldots + Q^{k+1})Pp_0 < \frac{P}{1-Q}p_0 = R. \end{aligned}$$

Thus, $y_{k+1} \in B(x_0, R)$. Using Lemma 2, we get

$$\|x_{k+2} - y_{k+1}\| \leq \phi(p_{k+1}, R)p_{k+1} = q_{k+1}.$$

Therefore,

$$\begin{aligned} \|x_{k+2} - x_{k+1}\| &\leq \|x_{k+2} - y_{k+1}\| + \|y_{k+1} - x_{k+1}\| \leq (1 + \phi(p_{k+1}, R))\, p_{k+1} \\ &\leq (1 + \phi(p_0, R))\, p_{k+1} = Pp_{k+1}. \end{aligned}$$

Now,

$$\begin{aligned} \|x_{k+2} - x_0\| &\leq \|x_{k+2} - x_{k+1}\| + \|x_{k+1} - x_0\| \leq Pp_{k+1} + (1 + Q + \ldots + Q^k)Pp_0 \\ &\leq (1 + Q + \ldots + Q^{k+1})Pp_0 < \frac{P}{1-Q}p_0 = R. \end{aligned}$$

Hence $x_{k+2} \in B(x_0, R)$. ■

5.1 Localization of a solution for equation (1)

Theorem 4. *Under assumptions* (**I**), (**II**) *and Lemma 3, the sequence generated by* (13) *converges to a fixed point* x^* *of* (2), *for the starting point* x_0, *and* $x^* \in \overline{B(x_0, R)}$.

Proof. To prove this result, it is sufficient to prove that the sequence $\{x_n\}$ is a Cauchy sequence. Using Lemma 2, we get

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \sum_{j=n}^{n+m-1} \|x_{j+1} - x_j\| \leq \sum_{j=n}^{n+m-1} PQ^j p_0 \\ &\leq Pp_0 \sum_{j=n}^{n+m-1} Q^j \leq Pp_0 \frac{Q^n - Q^{n+m}}{1-Q}. \end{aligned} \tag{20}$$

As $Q<1$, hence $\{x_n\}$ is a Cauchy sequence which converges to x^*. Taking $n=0$ and $m\to\infty$ in (20), we get $\|x_0-x^*\|\leq R$, and $x^*\in\overline{B(x_0,R)}$.

Now, to prove x^* is a fixed point of (2), we consider $\|x_n-F(x_n)\|\leq\|A(x_n)\|\|A(x_n)^{-1}(x_n-F(x_n))\|=\|A(x_n)\|\|x_{n+1}-x_n\|$ and the operators $\{\|A(x_n)\|\}$ are bounded. Taking $n\to\infty$ and using the continuity of the operator F, we get that x^* is a fixed point of operator F.

■

5.2 Uniqueness of solution for equation (1)

Observe that a fixed point of (2) is a solution of equation (1) and reciprocally. For this, we establish the uniqueness of the fixed point which proves the uniqueness of the solution of (1).

Theorem 5. *Under conditions of Theorem 4, if the equation*

$$\beta\,|\lambda|\left(\int_0^1 L\omega((1-\tau)R+\tau t)+Mt\right)=1 \tag{21}$$

has at least one positive real root and the biggest positive real root is denoted by $\overline{R}$, then the equation (2) has a unique fixed point in $B(x_0,\overline{R})\cap\Omega$.

Proof. In order to prove the uniqueness part, let y^* be another fixed point in $B(x_0,\overline{R})$. Then

$$0=A(x_0)^{-1}\big(y^*-F(y^*)-x^*+F(x^*)\big)=\int_0^1 A(x_0)^{-1}\left(I-F'(x^*+\tau(y^*-x^*))\right)d\tau(y^*-x^*).$$

If we prove that the operator $S=\displaystyle\int_0^1 A(x_0)^{-1}\left(I-F'(x^*+\tau(y^*-x^*))\right)d\tau$ is invertible then $x^*=y^*$.

Now,

$$\begin{aligned}
(S-I)(x)(s) &= \int_0^1 A(x_0)^{-1}\left[I-F'(x^*+\tau(y^*-x^*))\right]d\tau x(s)-x(s)=\\
&= \int_0^1 A(x_0)^{-1}\left(x(s)-[A(x_0)x](s)-\lambda\int_a^b \mathcal{K}(s,t)[N'(x^*+\tau(y^*-x^*))x](t)dt\right)d\tau=\\
&= \int_0^1 A(x_0)^{-1}\left[\lambda\int_a^b \widetilde{\mathcal{K}}(s,t)[N'(x_0)-N'(x^*+\tau(y^*-x^*))x](t)dt\right.\\
&- \left.\lambda\int_a^b R(s,t)[N'(x^*+\tau(y^*-x^*))x](t)dt\right]d\tau.
\end{aligned}$$

$x_0(s)$	β	η	R	$\bar{R}$
1/10	1.0259	0.6171	0.9159	3.2979
cos(s)/10	1.0223	0.6145	0.8998	3.3127
s^2/10	1.0072	0.6046	0.8485	3.3760
sin(s)/10	1.0107	0.5908	0.8120	3.3611

Table 3: Radius of semilocal convergence balls.

Then,

$$\begin{aligned}\|S-I\| &\leq \beta|\lambda|\int_0^1\left[L\left\|N'(x_0)-N'(x^*+\tau(y^*-x^*))\right\|+M(\epsilon+\omega((1-\tau)R+\tau\overline{R}))\right]d\tau\\ &\leq \beta\,|\lambda|\left(\int_0^1(L+M)\omega((1-\tau)R+\tau\overline{R})\,d\tau+M\epsilon\right).\end{aligned}$$

So, if (21) holds, the operator S has an inverse and, consequently, $y^*=x^*$. ■

5.3 Numerical example

In order to apply the semilocal convergence study performed in previous section, we consider the following nonlinear integral equation:

$$x(s)=\frac{1}{2}\sin(s)+\frac{1}{20}\int_0^1(s+2)\cos(st)x(t)^2dt, \tag{22}$$

with $\Omega=\mathcal{C}[0,1]$ and non-separable kernel $\mathcal{K}(s,t)=(s+2)\text{cos(st)}$. Then, we have:

$$\mathcal{K}(s,t)=(s+2)\text{cos(st)}=(s+2)\big(\sum_{i=0}^{2}\frac{(-1)^i(st)^{2i}}{(2i)!}+\mathcal{R}(\theta,s,t)\big),\;\;\mathcal{R}(\theta,s,t)=\frac{\cos(s\theta)}{6!}s^6\,t^6.$$

The application of our study gives us function $w(t)=2t$ and $h(t)=t$. All parameters involved in the theoretical study have been obtained, $D=0.5$, $M=0.0014$, $L=4.6250$ and $\epsilon=0.2$. In Table 3 we take different functions $x_0(s)$ as starting guesses. For each one, we can see the value of η and β and finally by solving the corresponding equation (19) and (21) we obtain the semilocal convergence radii R and the uniqueness radii $\bar{R}$.

Moreover, we have solved the nonlinear integral equation (22) by using iterative scheme (13). Then, in Table 4 we can see the results in all iterations, with $x_0(s)=0.1\sin(s)$,

n	$\|\|x_n(s) - x_{n-1}(s)\|\|$	$\|\|x_n(s) - F(x_n(s))\|\|$
1	6.2738e-01	1.4016e-01
2	6.6077e-04	1.8386e-08
3	1.1420e-12	5.4433e-26
4	5.8216e-39	1.415e-78
5	7.7156e-118	2.4855e-236

Table 4: Iterations with iterative scheme (13).

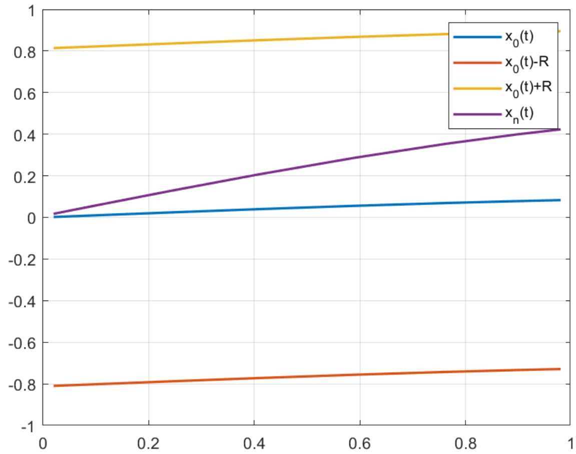

Figure 1: Graphic of different iterations.

until reaching a tolerance $\|x_n - x_{n-1}\| < 10^{-50}$. We can observe, by the values $\|x_n(s) - F(x_n(s))\|$, that we get a good approximation of the solution of (22) without needing a large number of iterations. In Figure 1, we can observe all the calculated iterations maintained in the domain of the existence obtained in the theoretical study previously carried out.

6 Conclusions

We have to point out the treatment for solving nonlinear integral equations with non-separable kernel introduced in this paper. For this purpose we have introduced a new approximation to the Frechet derivative that appears in the iterative method proposed. Different applied problems have been solved for corroborating the theoretical results obtained showing competitive results.

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