Solving random fractional second-order linear equations via the mean square Laplace transform: Theory and statistical computing

C. Burgos[a], J.-C. Cortés[a,], L. Villafuerte[b], R.J. Villanueva[a]

[a]*Instituto Universitario de Matemática Multidisciplinar, Universitat Politècnica de València, Camino de Vera s/n, 46022, Valencia, Spain*
[b]*Department of Mathematics, University of Texas at Austin, USA*

Abstract

This paper deals with random fractional differential equations of the form, ${}^C D^{\alpha}_{0+} X(t) + A\dot{X}(t) + BX(t) = 0$, $t > 0$, with initial conditions, $X(0) = C_0$ and $\dot{X}(0) = C_1$, where ${}^C D^{\alpha}_{0+} X(t)$ stands for the Caputo fractional derivative of $X(t)$. We consider the case that the fractional differentiation order is $1 < \alpha < 2$. For the sake of generality, we further assume that C_0, C_1, A and B are random variables satisfying certain mild hypotheses. Then, we first construct a solution stochastic process, via a generalized power series, which is mean square convergent for all $t > 0$. Secondly, we provide explicit approximations of the expectation and variance functions of the solution. To complete the random analysis and from this latter key information, we take advantage of the Principle of Maximum Entropy to calculate approximations of the first probability density function of the solution. All the theoretical findings are illustrated via numerical experiments.

Keywords. Random fractional differential equations, random mean square calculus, Principle of Maximum Entropy, mean square Laplace transform.

1. Introduction and preliminaries

Fractional differential equations are widely applied for describing many physical phenomena such as seepage flow in porous media [1], fluid dynamic traffic [2], viscoelastic damping in certain types of materials like polymers [3], spreads of diseases [4], etc., just to list a few ones. Although many types of fractional

Email addresses: `clabursi@posgrado.upv.es` (C. Burgos), `jccortes@imm.upv.es` (J.-C. Cortés), `altuzar@math.utexas.edu` (L. Villafuerte), `rjvillan@imm.upv.es` (R.J. Villanueva)

derivatives have been introduced in the extant literature, a common characteristic is that they are defined via nonlocal operators, which results particularly effective for modelling phenomena with memory or after-effects.

In [5] one analyzes the main advantages and disadvantages of different types of fractional derivatives. In the present contribution, we study a class of initial value problem formulated via the Caputo fractional derivative. As explicitly indicated in [5], *the Caputo derivative is very useful in dealing with real-world problems formulated through initial and boundary conditions.* The contributions listed in the previous paragraph illustrate this assertion (see [6, 7] for further examples). In [8, 9, 10], one can find interesting theoretical and practical applications of fractional derivatives.

On the other hand, the presence of uncertainties is ubiquitous when mathematical modelling real-world problems. Randomness appears from different sources such as, for example, the lack of knowledge of complex phenomena or the measurement errors associated to experiments or samples required to calibrate the corresponding mathematical model. As a result, the formulation of fractional differential equations with uncertainties leads to two main classes of approaches, namely, stochastic fractional differential equations (SFDEs) and random fractional differential equations (RFDEs).

SFDEs are differential equations driven by the fractional Brownian motion (fBm), which is Gaussian, so with unbounded trajectories. The analysis of this class of equations is usually done via the Wick-Itô-Skorohod calculus [11, 12]. fBm is neither a semimartingale (except when the Hurst exponent $H = 1/2$) nor a Markov process, so the classical mathematical machineries for stochastic calculus are not applicable in the fBm case [13]. SFDEs have found interesting applications, particularly in Finance where abrupt changes may occur [14]. Nevertheless, it must be said that applying SFDEs implicitly entails assuming the uncertainties are Gaussian. Obviously, this limits the application of SFDEs in real-world problems where other stochastic patterns are more suitable.

A complementary approach to SFDEs are RFDEs. These class of equations are more flexible when applying in practice, since it permits separately randomizing all the information (initial/boundary conditions, source term and/or coefficients) defining the fractional differential equation. Furthermore, one can assign a wide range of probability distributions to each term of the differential equation instead of prefixing a single stochastic pattern to the whole model, as it is done when using SFDEs via the fBm. An appealing option to choice the distributions for model inputs is utilizing parametric distributions (uniform, beta, exponential, Gaussian, etc.), since it provides great flexibility to better capture uncertainties. As accurately indicated in recent contributions, parametric RDEs are becoming at the forefront useful tools to model real-world problems [15, 16]. For example, in [16] explicitly one states: "Random Ordinary Differential Equations are being used in the biological sciences, where non-Gaussian and bounded noise are often more realistic than the conventionally used Itô calculus".

While there are many contributions dealing with SFDEs, the study of RFDEs is still scarce and further analysis is required. Some recent contributions on RFDEs include the study both of theoretical aspects [17, 18, 19, 20] and applications [21, 22, 23].

The aim of this paper is contributing to advance in the study of parametric RFDEs by studying a class of second order linear equations that, in the classical scenario (deterministic setting and entire order derivative) are vital to any serious investigation of the classical areas of mathematical physics. Indeed, one cannot go very far in the development of fluid mechanics, heat conduction, wave motion, or electromagnetic phenomena without finding it necessary to solve second order linear differential equations. In particular, this class of equations is ubiquitous in the study of oscillations of mechanical and electrical systems.

For the sake of completeness, down below we briefly collect the main definitions and results that will be required throughout this paper. Given a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, throughout this paper we will work in the Hilbert space, $(\mathrm{L}^2(\Omega), \langle\cdot,\cdot\rangle)$, whose elements are real random variables, $X : \Omega \longrightarrow \mathbb{R}$, having finite second-order moment, i.e. $\mathbb{E}[|X|^2] < \infty$. Here, $\mathbb{E}[\cdot]$ denotes the expectation operator. The norm, $\|\cdot\|_2$, usually termed 2-norm, is defined as $\|X\|_2 = (\mathbb{E}[|X|^2])^{1/2} < \infty$, for each $X \in \mathrm{L}^2(\Omega)$ and inferred by the inner product $\langle X, Y\rangle = \mathbb{E}[|XY|]$, $X, Y \in \mathrm{L}^2(\Omega)$. Such random variables are called 2-random variables. A 2-stochastic process, $X(t) \equiv \{X(t) : t \in \mathcal{T}\}$, is a set of 2-random variables indexed by t. This index usually represents the time and then it lies within a positive interval, say $\mathcal{T} = [0, T]$, $T > 0$. Notice that the ω-notation (i.e., sample dependence) for both random variables $X \equiv X(\omega)$ and stochastic processes, $X(t) \equiv X(t;\omega)$, $\omega \in \Omega$ has been hidden, as usual. The concepts of 2-continuity, 2-differentiability and 2-integrability are defined in terms of the stochastic convergence inferred from the 2-norm. The corresponding convergence is called mean square convergence [24], [25, Ch.4]. A key property of mean square convergence, that will be applied later, is stated in the following result.

Lemma 1.1. *[25] If $\{X_n : n \geq 0\}$ is a sequence of 2-random variables such that X_n is mean square convergent to $X \in \mathrm{L}^2(\Omega)$, then*

$$\mathbb{E}[X_n] \xrightarrow[n\to\infty]{} \mathbb{E}[X] \quad \text{and} \quad \mathbb{V}[X_n] \xrightarrow[n\to\infty]{} \mathbb{V}[X], \tag{1.1}$$

where $\mathbb{V}[X_n] = \mathbb{E}\left[(X_n)^2\right] - (\mathbb{E}[X_n])^2$ denotes the variance of X_n.

Observe that any sequence of 2-random variables, that is mean square convergent, will have both convergent mean and convergent variance. Later on, we will show that the solution of the random model under study will be a mean square convergent series, and therefore by Lemma 1.1, it will have convergent mean and convergent variance.

In this paper we deal with the following random initial value problem (RIVP)

$$\left\{ \begin{array}{lcl} {}^{C}D^{\alpha}_{0^+}X(t) + A\dot{X}(t) + BX(t) & = & 0, \qquad t > 0, \quad 1 < \alpha < 2, \\ & & \\ X(0) = C_0, \quad \dot{X}(0) = C_1, & & \end{array} \right. \tag{1.2}$$

where A, B, C_0 and C_1 are random variables satisfying certain properties that will be specified later. Here, ${}^{C}D^{\alpha}_{0^+}X(t)$ denotes the mean square Caputo derivative of the stochastic process $X(t)$ (see [26, p. 290], [19], and references therein), defined by

$$
{}^{C}D_{0+}^{\alpha}X(t) = \frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)}\int_0^t (t-\tau)^{\lceil\alpha\rceil-\alpha-1}(X(\tau))^{(\lceil\alpha\rceil)}\mathrm{d}\tau, \quad t>0, \tag{1.3}
$$

where $\lceil\alpha\rceil$ is the ceiling function at α (so $\lceil\alpha\rceil=2$ and $X(\tau)^{(\lceil\alpha\rceil)}=\ddot{X}(\tau)$ stands for the mean square derivative of order 2) and $\Gamma(\cdot)$ denotes the classical Gamma function. The equation (1.2) is a generalization of the classical damped simple harmonic oscillator. This is because when α converges to 2, the deterministic Caputo derivative of a function, converges to its second derivative, see [27, p. 79].

Assuming that $\{X(t): t\geq 0\}$ is twice mean square differentiable, and denoting by $\Gamma_X(t_1,t_2) := \mathbb{E}\left[X(t_1)X(t_2)\right]$ its correlation function (which, according to Schwarz inequality always exists for a 2-stochastic process), it is straightforward to prove, by applying [25, Th. 4.5.1, p.100], that its mean square Caputo derivative, ${}^{C}D_{0+}^{\alpha}X(t)$, exists, if and only if, the following double integral exists and is finite

$$
\int_0^t\int_0^t (t-\tau_1)^{1-\alpha}(t-\tau_2)^{1-\alpha}\Gamma_{\ddot{X}}(\tau_1,\tau_2)\mathrm{d}\tau_1\mathrm{d}\tau_2, \quad t>0. \tag{1.4}
$$

Remark 1.2. *In the particular case that $\Gamma_{\ddot{X}}(\tau_1,\tau_2)$ is continuous at every point (τ,τ), $\tau\in[0,t]$, (which, according to [25, Th.4.3., p. 90], is equivalent to say that the 2-stochastic process $X(t)$ is such that its $\ddot{X}(t)$ is mean square continuous), the integral* (1.4) *exists and is finite, and, as a consequence, the mean square Caputo derivative, ${}^{C}D_{0+}^{\alpha}X(t)$, exists. Of course, this result is also true in the case that mean square continuity is replaced by mean square piecewise continuity.*

Furthermore, taking into account that (see [25, formula (4.133), p.98])

$$
\Gamma_{\ddot{X}}(\tau_1,\tau_2) = \frac{\partial^4\Gamma_X(\tau_1,\tau_2)}{\partial\tau_1^2\partial\tau_2^2},
$$

the existence of ${}^{C}D_{0+}^{\alpha}X(t)$ can be given in terms of the existence and finiteness of the following double integral of the correlation function

$$
\int_0^t\int_0^t (t-\tau_1)^{1-\alpha}(t-\tau_2)^{1-\alpha}\frac{\partial^4\Gamma_X(\tau_1,\tau_2)}{\partial\tau_1^2\partial\tau_2^2}\mathrm{d}\tau_1\mathrm{d}\tau_2, \quad t>0. \tag{1.5}
$$

Example 1.3. *Let U be a Beta random variable of parameters $(2,1)$, i.e. $U\sim Be(2,1)$ (so $U\in \mathrm{L}^2(\Omega)$) and define $X(t):=Ut^3$, $t\geq 0$. Observe that $\ddot{X}(t)=6tU$, $\mathbb{E}[U^2]=\frac{1}{2}$, $\Gamma_{\ddot{X}}(\tau_1,\tau_2)=\mathbb{E}[\ddot{X}(\tau_1)\ddot{X}(\tau_2)]=36\tau_1\tau_2\mathbb{E}[U^2]=18\tau_1\tau_2$ and*

$$
\int_0^t\int_0^t (t-\tau_1)^{1-\alpha}(t-\tau_2)^{1-\alpha}18\tau_1\tau_2\mathrm{d}\tau_1\mathrm{d}\tau_2 = \frac{18t^{6-2\alpha}}{(\alpha^2-5\alpha+6)^2}<\infty.
$$

Then, applying condition (1.4)*, it is justified that the mean square Caputo derivative of $X(t)$ exists. Moreover, it can be shown that ${}^{C}D_{0+}^{\alpha}X(t)=\frac{6U}{\Gamma(4-\alpha)}t^{3-\alpha}$.*

2. Basic properties of the mean square Laplace transform of second-order stochastic processes

In this section, we introduce the basic definitions, results and operational rules, corresponding to mean square calculus, that will be required to solve the RIVP (1.2). Some preliminary results about the mean square Laplace transform, that complement the ones given hereafter, were established by some of the authors in [28].

Definition 2.1. *The Laplace transform of a 2-stochastic process, $\{X(t) : t \geq 0\}$, is defined by*

$$\mathcal{L}\{X(t); s\} := \int_0^\infty \mathrm{e}^{-st} X(t)\mathrm{d}t, \qquad s \in \mathcal{S} \subset \mathbb{R}, \tag{2.1}$$

provided the improper integral exists in $\mathrm{L}^2(\Omega)$. The variable s can be allowed to be a complex number, but for our purposes of solving random fractional differential equations, we will only require that s be a real parameter.
Similarly as it is done in the deterministic context, to examine conditions under which a 2-stochastic process has a mean square Laplace transform, we define the concept of exponential order in the mean square sense that extends its deterministic counterpart.

Definition 2.2. *A 2-stochastic process, $\{X(t) : t \geq 0\}$, is said to be of exponential order $s_0 \geq 0$, if and only if, there exist positive constants M and T_0 such that $\|X(t)\|_2 \leq M\mathrm{e}^{s_0 t}$, for all $t \geq T_0 > 0$.*

Lemma 2.3. *If the 2-stochastic process, $X(t)$, is of exponential order $s_0 \geq 0$, then*

$$\lim_{T\to\infty} \mathrm{e}^{-sT} \|X(T)\|_2 = 0, \quad s > s_0. \tag{2.2}$$

Proof. *Since $X(t)$ is of exponential order $s_0 \geq 0$, applying Definition 2.2 for $t = T$ large enough, one gets, $0 \leq \mathrm{e}^{-sT} \|X(T)\|_2 \leq M\mathrm{e}^{T(s_0-s)}$. Since $\lim_{T\to\infty} M\mathrm{e}^{T(s_0-s)} = 0, \quad s > s_0$, one straightforwardly follows the result.* †

Then, we are ready to establish sufficient conditions in order for the existence of the mean square Laplace transform of a 2-stochastic process is guaranteed, as well as to determine the domain of the transform.

Proposition 2.4. *Let $\{X(t) : t \geq 0\}$ be a mean square piecewise continuous stochastic process of exponential order $s_0 \geq 0$. Then, its mean square Laplace transform, $\mathcal{L}\{X(t); s\}$, exists for $s > s_0$.*

Proof. To establish the result we show that the integral (2.1) is mean square convergent for every $s > s_0$. Now, taking into account Definition 2.2, we split the improper integral in two parts,

$$\mathcal{L}\{X(t); s\} = \int_0^{T_0} \mathrm{e}^{-st} X(t)\mathrm{d}t + \int_{T_0}^\infty \mathrm{e}^{-st} X(t)\mathrm{d}t.$$

The first integral exists because it can be written as a sum of mean square integrals over intervals on which the stochastic process $\mathrm{e}^{-st}X(t)$ is mean square continuous. Since $s > s_0$, for the second integral one gets

$$\left\| \int_{T_0}^{\infty} \mathrm{e}^{-st} X(t)\, \mathrm{d}t \right\|_2 \leq \int_{T_0}^{\infty} \mathrm{e}^{-st} \left\| X(t) \right\|_2 \, \mathrm{d}t \leq M \int_{T_0}^{\infty} \mathrm{e}^{-(s-s_0)t} \, \mathrm{d}t = M \frac{\mathrm{e}^{-(s-s_0)T_0}}{s-s_0} < \infty.$$

Notice that the first inequality is legitimized by [25, expression (4.149), p.102] for mean square Riemann integrals. Finally, notice that we have also shown that $\left\| \mathcal{L}\{X(t); s\} \right\|_2 < \infty$, therefore the Laplace transform of a 2-stochastic process satisfying the hypotheses of the theorem is also a 2-stochastic process, i.e. $\mathcal{L}\{X(t); s\} \in \mathrm{L}^2(\Omega)$ for each $s > s_0$. †

Remark 2.5. *As it also happens in the deterministic context, the two conditions of Prop. 2.4, namely, (1) mean square piecewise continuity and (2) exponential order for a 2-stochastic process $X(t)$, are sufficient but not necessary for the existence of a mean square Laplace transform. Indeed, let U be a Uniform random variable on the unit interval, $U \sim U([0,1])$ (so, $U \in \mathrm{L}^2(\Omega)$), then the 2-stochastic process $X(t) = U/\sqrt{t}$ is obviously not mean square piecewise continuous on $[0, \infty)$, while $X(t) = 2Ut\mathrm{e}^{t^2} \cos\left(\mathrm{e}^{t^2}\right)$ is not of exponential order. However, it is easy to show that in both cases the mean square Laplace transform exists. We here omit the proof since it is similar to the deterministic case [29, p.15].*

Remark 2.6. *Similarly as we have previously shown for the existence of the mean square Caputo derivative of a 2-stochastic process, $X(t)$, in terms of its correlation function, $\Gamma_X(t_1, t_2)$ (see condition (1.5)), by applying [25, Th. 4.5.1, p.100] one also deduces the mean square Laplace transform exists, if and only, if the following double integral is convergent*

$$\int_0^{\infty} \int_0^{\infty} \mathrm{e}^{-s(t_1+t_2)} \Gamma_X(t_1, t_2) \mathrm{d}t_1 \mathrm{d}t_2.$$

Notice that by Jensen and Schwarz inequalities one gets

$$|\Gamma_X(t_1, t_2)| = |\mathbb{E}[X(t_1)X(t_2)]| \leq \mathbb{E}[|X(t_1)X(t_2)|] \leq \left\| X(t_1) \right\|_2 \left\| X(t_2) \right\|_2,$$

and by Fubini's theorem one gets,

$$\begin{aligned} & \int_0^{\infty} \int_0^{\infty} \mathrm{e}^{-s(t_1+t_2)} |\Gamma_X(t_1, t_2)| \mathrm{d}t_1 \mathrm{d}t_2 \\ \leq & \int_0^{\infty} \int_0^{\infty} \mathrm{e}^{-s(t_1+t_2)} \left\| X(t_1) \right\|_2 \left\| X(t_2) \right\|_2 \mathrm{d}t_1 \mathrm{d}t_2 \\ = & \left(\int_0^{\infty} \mathrm{e}^{-st_1} \left\| X(t_1) \right\|_2 \mathrm{d}t_1 \right) \left(\int_0^{\infty} \mathrm{e}^{-st_2} \left\| X(t_2) \right\|_2 \mathrm{d}t_2 \right). \end{aligned}$$

So, it is sufficient that the following integral exists and is finite to guarantee the existence of the mean square Laplace transform

$$\int_0^\infty \mathrm{e}^{-st}\|X(t)\|_2 \mathrm{d}t < \infty. \tag{2.3}$$

In the following example, we compute the Laplace transform of a particular 2-stochastic process $X(t)$ that will be required later on. We will apply condition (2.3) to check that its mean square Laplace transform exists.

Example 2.7. *Let U be a 2-random variable and define $\{X(t) := Ut^k, t \geq 0\}$ with $k > -1$. Hence, for $s > 0$,*

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-st}\|X(t)\|_2 \mathrm{d}t &= \int_0^\infty \mathrm{e}^{-st} t^k \|U\|_2 \mathrm{d}t,\\ &= \int_0^\infty \mathrm{e}^{-v}\frac{1}{s}\left(\frac{v}{s}\right)^k \|U\|_2 \mathrm{d}v\\ &= \|U\|_2 \frac{1}{s^{k+1}} \int_0^\infty \mathrm{e}^{-v} v^{(k+1)-1}\mathrm{d}v\\ &= \|U\|_2 \frac{1}{s^{k+1}}\Gamma(k+1) < \infty,\end{aligned} \tag{2.4}$$

where in the second step we have made the change of variables $v = st$ and in last step we have used the integral representation of the classical Gamma function, $\Gamma(x) = \int_0^\infty \mathrm{e}^{-v} v^{x-1}\mathrm{d}v$, $x > 0$ since $k > -1$. According to (2.3), the Laplace transform of the 2-stochastic process, $X(t) = Ut^k$, $k > -1$, is well defined in $\mathrm{L}_2(\Omega)$. If $k \geq 0$, this fact can also be checked by applying Proposition 2.4. For every $k \geq 0$, the 2-stochastic process, $X(t) = Ut^k$, $U \in \mathrm{L}^2(\Omega)$ is, obviously, mean square continuous and of exponential order with $s_0 = k \geq 0$. Indeed, let us denote $\|U\|_2 = M < \infty$, then taking into account that $\ln(t) \leq t$, $t > 0$, one gets

$$\|X(t)\|_2 = \|U\|_2 \, t^k = M\mathrm{e}^{k\ln(t)} \leq M\mathrm{e}^{kt}, \quad t > 0.$$

Further, in similar manner as we have proved, using condition (2.3)*, the existence of the Laplace transform of $X(t)$ in* (2.4)*, it can be shown that*

$$\mathcal{L}\{X(t); s\} = U\frac{1}{s^{k+1}}\Gamma(k+1), \quad s > 0.$$

The linearity of the mean square Laplace transform, defined in (2.1), follows from the linearity of the mean square Riemann integral [25, Ch.4]. However, when the constants of the linear combination are random variables, according to Lemma 1 of [30], some conditions must be imposed on the data in order to guarantee the resulting stochastic process lies in $\mathrm{L}^2(\Omega)$. The proof is a consequence of the following key inequality that involves the random Lebesgue spaces $\mathrm{L}^2(\Omega)$ and $\mathrm{L}^4(\Omega)$ [31],

$$\|XY\|_2 \leq \|X\|_4 \, \|Y\|_4, \quad X, Y \in \mathrm{L}^4(\Omega). \tag{2.5}$$

Lemma 2.8. *Let U_1 and U_2 be 4-random variables and let $\{X_1(t) : t \geq 0\}$ and $\{X_2(t) : t \geq 0\}$ be 4-stochastic processes. If $\mathcal{L}\{X_1(t); s\}$ and $\mathcal{L}\{X_2(t); s\}$ belong to $\mathrm{L}^4(\Omega)$, then*

$$\mathcal{L}\{U_1X_1(t) + U_2X_2(t); s\} = U_1\mathcal{L}\{X_1(t); s\} + U_2\mathcal{L}\{X_2(t); s\}, \tag{2.6}$$

and $\mathcal{L}\{U_1X_1(t) + U_2X_2(t); s\} \in \mathrm{L}^2(\Omega)$.

Proof. Since $U_1, U_2 \in \mathrm{L}^4(\Omega)$ and $\mathcal{L}\{X_1(t); s\}$, $\mathcal{L}\{X_2(t); s\} \in \mathrm{L}^4(\Omega)$ for each $t \geq 0$, then, from (2.5), one obtains $U_1\mathcal{L}\{X_1(t); s\}$, $U_2\mathcal{L}\{X_2(t); s\} \in \mathrm{L}^2(\Omega)$. Consequently, $U_1\mathcal{L}\{X_1(t); s\}+U_2\mathcal{L}\{X_2(t); s\} \in \mathrm{L}^2(\Omega)$. Similarly, for each $t \geq 0$, $U_1X_1(t) + U_2X_2(t) \in \mathrm{L}^2(\Omega)$. Further, the linearity of the mean square integral and Lemma 1 of [30] imply the identity.†

Remark 2.9. *It can be proved that if U_1 and U_2 in Lemma 2.8 are bounded random variables, then U_1, U_2, $\mathcal{L}\{X_1(t); s\}, \mathcal{L}\{X_2(t); s\} \in \mathrm{L}^2(\Omega)$, and the conclusion of Lemma 2.8 holds. Indeed, let $M_1, M_2 > 0$ such that $|U_1(\omega)| \leq M_1$ and $|U_2(\omega)| \leq M_2$ for every $\omega \in \Omega$. Setting $M = \max\{M_1, M_2\}$, we have*

$$\mathbb{E}\left[|U_1\mathcal{L}\{X_1(t); s\}|^2\right] = \int_\Omega |U_1\mathcal{L}\{X_1(t); s\}|^2 \,\mathrm{d}\mathbb{P} \leq M^2 \int_\Omega |\mathcal{L}\{X_1(t); s\}|^2 \,\mathrm{d}\mathbb{P} < +\infty,$$

which proves that $U_1\mathcal{L}\{X_1(t); s\} \in \mathrm{L}^2(\Omega)$. Analogously, it can be shown that $U_2\mathcal{L}\{X_2(t); s\} \in \mathrm{L}^2(\Omega)$. Hence $U_1\mathcal{L}\{X_1(t); s\} + U_2\mathcal{L}\{X_2(t); s\} \in \mathrm{L}^2(\Omega)$. The equality in Lemma 2.8 is inferred from the linearity of the mean square integral and Proposition 1 of [19]. Throughout this paper, we will apply the linearity of mean square Laplace transform in the case that coefficients U_1 and U_2 of the linear combination (2.6) *are bounded random variables.*

A key result in solving random differential equations is knowing the Laplace transform of the mean square derivative, $\dot{X}(t)$, of a 2-stochastic process, $X(t)$, whose mean square Laplace transform exists.

Proposition 2.10. *Let $\{X(t) : t \geq 0\}$ be a 2-stochastic process satisfying the following conditions:*

i) $X(t)$ is mean square differentiable (so, continuous),

ii) $\dot{X}(t)$ is mean square piecewise continuous,

iii) $X(t)$ is of exponential order $s_0 \geq 0$.

Then,

$$\mathcal{L}\{\dot{X}(t); s\} = s\mathcal{L}\{X(t); s\} - X(0), \quad s > s_0 \geq 0.$$

Proof. Using the definition of the Laplace transform and the formula of integration by parts for improper mean square integrals (see [25, p. 104]), we have

$$\mathcal{L}\{\dot{X}(t); s\} = \int_0^\infty \mathrm{e}^{-st}\dot{X}(t)\mathrm{d}t = \lim_{T\to\infty} \mathrm{e}^{-sT}X(T) - X(0) + s\int_0^\infty X(t)\mathrm{e}^{-st}\mathrm{d}t.$$

As $X(t)$ is of exponential order s_0, Remark 2.3 implies that $\lim_{T\to\infty} \mathrm{e}^{-sT} X(T) = 0$ in the mean square sense. Hence,

$$\mathcal{L}\{\dot{X}(t); s\} = s\mathcal{L}\{X(t); s\} - X(0).\dagger$$

Remark 2.11. *If the 2-stochastic process $X(t)$ is twice mean square differentiable such that $X(t)$ and $\dot{X}(t)$ are both mean square continuous and of exponential order and $\ddot{X}(t)$ is mean square piecewise continuous, then applying twice Proposition 2.10, one obtains*

$$\begin{aligned}\mathcal{L}\{\ddot{X}(t); s\} &= s\mathcal{L}\{\dot{X}(t); s\} - \dot{X}(0) = s(s\mathcal{L}\{X(t); s\} - X(0)) - \dot{X}(0) \\ &= s^2\mathcal{L}\{X(t); s\} - sX(0) - \dot{X}(0).\end{aligned} \tag{2.7}$$

One of the most attractive applications of the Laplace transform is solving differential equations, in particular, fractional differential equations (see [32] and references therein). The next result gives the mean square Laplace transform of the mean square Caputo fractional derivative of a 2-stochastic process.

Proposition 2.12. *Let $X(t)$ be a 2-stochastic process satisfying the hypotheses of Remark 2.11. Let ${}^{C}D^{\alpha}_{0^+}X(t)$, $1 < \alpha < 2$, denote its mean square Caputo fractional derivative. If the pathwise integral $\int_0^\infty \mathrm{e}^{-s\tau}|\ddot{X}(\tau)|\mathrm{d}\tau$ exists and is finite, then*

$$\mathcal{L}\{{}^{C}D^{\alpha}_{0^+}X(t); s\} = s^{\alpha}\mathcal{L}\{X(t); s\} - s^{\alpha-1}X(0) - s^{\alpha-2}\dot{X}(0),$$

and $\mathcal{L}\{{}^{C}D^{\alpha}_{0^+}X(t); s\}$ belongs to $\mathrm{L}^2(\Omega)$.

Proof. By definition of the mean square Laplace transform and the definition of the Caputo derivative (see (1.3)) one gets,

$$\begin{aligned}\mathcal{L}\{{}^{C}D^{\alpha}_{0^+}X(t); s\} &= \int_0^\infty \mathrm{e}^{-st}\left({}^{C}D^{\alpha}_{0^+}X(t)\right)\mathrm{d}t \\ &= \int_0^\infty \mathrm{e}^{-st}\frac{1}{\Gamma(2-\alpha)}\int_0^t (t-\tau)^{1-\alpha}\ddot{X}(\tau)\mathrm{d}\tau\mathrm{d}t \\ &= \int_0^\infty \int_\tau^\infty \mathrm{e}^{-st}\frac{1}{\Gamma(2-\alpha)}(t-\tau)^{1-\alpha}\ddot{X}(\tau)\mathrm{d}t\mathrm{d}\tau \\ &\overset{u=t-\tau}{=} \int_0^\infty \int_0^\infty \mathrm{e}^{-s(u+\tau)}\frac{1}{\Gamma(2-\alpha)}u^{1-\alpha}\ddot{X}(\tau)\mathrm{d}u\mathrm{d}\tau \\ &= \int_0^\infty \mathrm{e}^{-s\tau}\frac{1}{\Gamma(2-\alpha)}\ddot{X}(\tau)\left[\int_0^\infty \mathrm{e}^{-su}u^{1-\alpha}\mathrm{d}u\right]\mathrm{d}\tau.\end{aligned} \tag{2.8}$$

Since it is known, from the deterministic scenario, that $\mathcal{L}\{u^k; s\} = \frac{\Gamma(k+1)}{s^{k+1}}$, $k > -1$, taking $k = 1-\alpha > -1$ the $\mathcal{L}\{{}^{C}D^{\alpha}_{0^+}X(t); s\}$ yields

$$\mathcal{L}\{{}^{C}D^{\alpha}_{0^+}X(t); s\} = \int_0^\infty \mathrm{e}^{-s\tau}\frac{1}{\Gamma(2-\alpha)}\ddot{X}(\tau)\mathcal{L}\{u^{1-\alpha}; s\}\mathrm{d}\tau$$

$$
\begin{aligned}
&= \int_0^\infty \mathrm{e}^{-s\tau} \frac{1}{\Gamma(2-\alpha)} \ddot{X}(\tau) \frac{\Gamma(2-\alpha)}{s^{2-\alpha}} \mathrm{d}\tau \\
&= s^{\alpha-2} \mathcal{L}\{\ddot{X}(\tau); s\} \\
&= s^{\alpha-2} (s^2 \mathcal{L}\{X(t); s\} - sX(0) - \dot{X}(0)) \\
&= s^{\alpha} \mathcal{L}\{X(t); s\} - s^{\alpha-1} X(0) - s^{\alpha-2} \dot{X}(0),
\end{aligned} \tag{2.9}
$$

where we have applied identity (2.7) derived in Remark 2.11. Observe that by hypothesis, for each $\omega \in \Omega$ the integral $\int_0^\infty \mathrm{e}^{-s\tau} |\ddot{X}(\tau)| \mathrm{d}\tau$ exists and is finite. Further, $\int_0^\infty \mathrm{e}^{-st} t^{1-\alpha} \mathrm{d}t < \infty$. Hence, Fubini's Theorem justifies the commutation of the integrals made in the third step of (2.8), see [33, p. 26], [34, p. 46] and [35, p. 11]. Finally, $\mathcal{L}\{{}^C D_{0+}^{\alpha} X(t); s\}$ belongs to $\mathrm{L}^2(\Omega)$ because for each $t \geq 0$, $X(t)$, $\dot{X}(t)$, $\mathcal{L}\{X(t); s\}$ belong to $\mathrm{L}^2(\Omega)$. †

Remark 2.13. *Notice that according to Remark 1.2, in Proposition 2.12 it is guaranteed the existence of the mean square Caputo derivative,* ${}^C D_{0+}^{\alpha} X(t)$*, since we are assuming that* $\ddot{X}(t)$ *is mean square piecewise continuous.*

3. Some applications of the mean square Laplace transform of second-order stochastic processes

The Binomial and Geometric series will be used to find a mean square analytic solution of the RIVP (1.2). Hereinafter, we will assume that the coefficients, A and B, and the initial conditions, C_0 and C_1, of (1.2) satisfy the following assumptions:

A1. A and B are bounded random variables, that is, there are positive numbers M_A and M_B such that $|A(\omega)| \leq M_A$ and $|B(\omega)| \leq M_B$, for all $\omega \in \Omega$. So, $A, B \in \mathrm{L}^2(\Omega)$.

A2. C_0 and C_1 are second order random variables, i.e. $C_0, C_1 \in \mathrm{L}^2(\Omega)$.

A3. C_0, C_1, A and B are independent random variables.

Applying the mean square Laplace transform to the fractional differential equation of RIVP (1.2) yields

$$
\mathcal{L}\{{}^C D_{0+}^{\alpha} X(t) + A\dot{X}(t) + BX(t); s\} = 0.
$$

By linearity of the mean square Laplace transform and Remark 2.9 (with $U_1 = A$ and $U_2 = B$ that are bounded random variables by assumption A1), we have

$$
\mathcal{L}\{{}^C D_{0+}^{\alpha} X(t); s\} + A\mathcal{L}\{\dot{X}(t); s\} + B\mathcal{L}\{X(t); s\} = 0.
$$

Now, taking into account Propositions 2.10–2.12, we obtain

$$
s^{\alpha}\mathcal{L}\{X(t); s\} - s^{\alpha-1}X(0) - s^{\alpha-2}\dot{X}(0) + As\mathcal{L}\{X(t); s\} - AX(0) + B\mathcal{L}\{X(t); s\} = 0.
$$

Solving for $\mathcal{L}\{X(t); s\}$ and taking into account that $X(0) = C_0$ and $\dot{X}(0) = C_1$,

$$\mathcal{L}\{X(t); s\} = \frac{s^{\alpha-1}C_0 + s^{\alpha-2}C_1 + AC_0}{s^\alpha + As + B}. \tag{3.1}$$

Now, we would like to find a domain, $\mathcal{S}$, for which the Laplace transform of the 2-stochastic process, $X(t)$, can be written as a mean square convergent series. To this end, we will first prove the following technical result which is established for each $\omega \in \Omega$.

Theorem 3.1. *Let A and B be random variables satisfying assumption A1. If $1 < \alpha < 2$ and $s > K_1 := \max\left\{M_A^{\frac{1}{\alpha-1}}, (M_A + M_B)^{\frac{1}{\alpha-1}}, 1\right\}$, then for all $\omega \in \Omega$,*

i) $|A(\omega)|s^{1-\alpha} < 1$.

ii) $\left|\frac{B(\omega)s^{-1}}{s^{\alpha-1}+A(\omega)}\right| < 1$.

iii) $\frac{1}{s^\alpha+A(\omega)s+B(\omega)} = \sum_{n,m\geq 0} \frac{\Gamma(m+n+1)}{\Gamma(n+1)\Gamma(m+1)}(-A(\omega))^m(-B(\omega))^n s^{-(m(\alpha-1)+\alpha(n+1))}$.

Proof. By hypothesis $s > M_A^{\frac{1}{\alpha-1}}$ and so $s^{\alpha-1} > M_A \geq |A(\omega)|$, for all $\omega \in \Omega$. Thus, it follows that $1 > \frac{|A(\omega)|}{s^{\alpha-1}} = s^{1-\alpha}|A(\omega)|$. This proves *i*).
On the one hand, $s > (M_A + M_B)^{\frac{1}{\alpha-1}}$, so $s^{\alpha-1} > M_A + M_B$. Consequently, $s^{\alpha-1} - M_A > M_B$. On the other hand, $-|A(\omega)| \geq -M_A$ and so $s^{\alpha-1} - |A(\omega)| \geq s^{\alpha-1} - M_A > M_B > 0$. Hence,

$$\begin{aligned}\left|\frac{B(\omega)s^{-1}}{s^{\alpha-1}+A(\omega)}\right| &\leq \frac{M_B s^{-1}}{|s^{\alpha-1}+A(\omega)|} < \frac{(s^{\alpha-1}-|A(\omega)|)s^{-1}}{|s^{\alpha-1}+A(\omega)|} \\ &< \frac{|s^{\alpha-1}+A(\omega)|s^{-1}}{|s^{\alpha-1}+A(\omega)|} = \frac{1}{s} < 1,\end{aligned} \tag{3.2}$$

since $s > K_1 \geq 1$. Hence, ii) holds.
In the following the Geometric series, $\sum_{n\geq 0} x^n = \frac{1}{1-x}$, $|x| < 1$, and an implication of the Binomial series (see [36, p. 201]),

$$(1-x)^{-\lambda} = \sum_{m\geq 0} \frac{\Gamma(m+\lambda)}{m!\Gamma(\lambda)} x^m, \qquad |x| < 1,\ \lambda > 0, \tag{3.3}$$

will be used. First, observe that

$$\frac{1}{s^\alpha + As + B} = \frac{s^{-1}}{s^{\alpha-1} + A + Bs^{-1}} = \frac{s^{-1}}{(s^{\alpha-1}+A)\left(1 + \frac{Bs^{-1}}{s^{\alpha-1}+A}\right)}. \tag{3.4}$$

For each ω in Ω, part ii) and the Geometric series with ratio $x = \frac{Bs^{-1}}{s^{\alpha-1}+A}$ ($|x| < 1$), imply

$$\frac{1}{s^\alpha + A(\omega)s + B(\omega)} = \frac{s^{-1}}{s^{\alpha-1}+A(\omega)} \sum_{n\geq 0} \frac{(-B(\omega))^n s^{-n}}{(s^{\alpha-1}+A(\omega))^n}$$

$$= \sum_{n\geq 0} \frac{(-B(\omega))^n s^{-n-1}}{(s^{\alpha-1}(1+A(\omega)s^{1-\alpha}))^{n+1}}$$
$$= \sum_{n\geq 0} (-B(\omega))^n s^{-\alpha n-\alpha}(1+A(\omega)s^{1-\alpha})^{-(n+1)}. \tag{3.5}$$

Further, as a consequence of (3.3) with $\lambda = n+1 > 0$, $x = -A(\omega)s^{1-\alpha}$ (that satisfies $|x| < 1$ by part i)), it follows

$$\frac{1}{s^\alpha + A(\omega)s + B(\omega)} = \sum_{n,m\geq 0} (-B(\omega))^n(-A(\omega))^m \frac{\Gamma(m+n+1)}{\Gamma(m+1)\Gamma(n+1)} s^{-((\alpha-1)m+\alpha(n+1))}.\dagger \tag{3.6}$$

For every $\omega \in \Omega$, let us denote $A := A(\omega)$ and $B := B(\omega)$ to alleviate the forthcoming notation. Setting

$$\phi_{n,m}(A,B) = (-B)^n(-A)^m \frac{\Gamma(m+n+1)}{\Gamma(m+1)\Gamma(n+1)} = (-1)^{n+m} B^n A^m \frac{\Gamma(m+n+1)}{m!n!}, \tag{3.7}$$

then according to (3.1) and (3.6), we have proved that for each $\omega \in \Omega$,

$$\begin{aligned} \mathcal{L}\{X(t);s\} = &\sum_{n,m\geq 0} \phi_{n,m}(A,B) C_0 s^{-((\alpha-1)m+n\alpha+1)} \\ &+ \sum_{n,m\geq 0} \phi_{n,m}(A,B) C_1 s^{-(m(\alpha-1)+n\alpha+2)} \\ &+ \sum_{n,m\geq 0} \phi_{n,m}(A,B) A C_0 s^{-(m(\alpha-1)+(n+1)\alpha)} \end{aligned} \tag{3.8}$$

converges for all $s > K_1$, being K_1 given in Theorem 3.1. Expression (3.8) will be used to show that $\mathcal{L}\{X(t);s\}$ is a mean square convergent series. To conduct this analysis, the following inequalities for the Gamma function will be applied [37]

$$\Gamma(p+q) \geq pq\Gamma(p)\Gamma(q) = \Gamma(p+1)\Gamma(q+1), \qquad p,q \geq 1, \tag{3.9}$$

and

$$\Gamma(p+q) \leq (\Gamma(2p)\Gamma(2q))^{\frac{1}{2}}, \qquad p,q > 0. \tag{3.10}$$

We will start by showing that the first term of the expansion given by (3.8) is mean square convergent, and for the other ones, the reasoning is analogous. By assumption A1, is clear that $\|(-A)^m\|_2 = \|A^m\|_2 = (\mathbb{E}[|A^{2m}|])^{\frac{1}{2}} \leq (\mathbb{E}[(|A|^m)^2])^{\frac{1}{2}} \leq (\mathbb{E}[(M_A)^m]^2)^{\frac{1}{2}} = (M_A)^m$. Similarly, $\|(-B)^n\|_2 \leq (M_B)^n$. Applying (3.10) to $p = n > 0$ and $q = m+1 > 0$ integers, we have $\Gamma(n+m+1) \leq (\Gamma(2n)\Gamma(2(m+1)))^{\frac{1}{2}} = ((2n-1)!(2m+1)!)^{\frac{1}{2}}$, since n and m are positive integers. Next, setting

$$\delta_1(n) = \frac{\|C_0\|_2 (M_B)^n ((2n-1)!)^{\frac{1}{2}}}{n!} s^{-(\alpha n+1)}, \quad \delta_2(m) = \frac{(M_A)^m\,((2m+1)!)^{\frac{1}{2}}}{m!} s^{-m(\alpha-1)},$$

and using assumption A3, yields

$$\|\phi_{n,m}(A,B)C_0\|_2 s^{-(m(\alpha-1)+\alpha n+1)} \leq \delta_1(n)\delta_2(m). \tag{3.11}$$

Hence, we only need to show that both deterministic series $\sum_{n\geq 0}\delta_1(n)$ and $\sum_{m\geq 0}\delta_2(m)$ are convergent in a common domain to be determined. To this end, we will use the ratio test. Since

$$\lim_{n\to\infty}\frac{\delta_1(n+1)}{\delta_1(n)} = \lim_{n\to\infty}\frac{M_B}{n+1}\left((2n+1)(2n)\right)^{\frac{1}{2}} s^{-\alpha} = \frac{2M_B}{s^{\alpha}}, \tag{3.12}$$

and

$$\lim_{m\to\infty}\frac{\delta_1(m+1)}{\delta_2(m)} = \lim_{m\to\infty}\frac{M_A}{m+1}\left((2m+3)(2m+2)\right)^{\frac{1}{2}} s^{-(\alpha-1)} = \frac{2M_A}{s^{\alpha-1}}. \tag{3.13}$$

This proves that first series in expression (3.8) is mean square convergent for

$$s > K_2 := \max\left\{(2M_B)^{\frac{1}{\alpha}}, (2M_A)^{\frac{1}{\alpha-1}}, K_1\right\},$$

where K_1 is defined in Theorem 3.1.
Analogously, it can be shown that the remaining two infinite series in (3.8) are mean square convergent for $s > K_2$. We summarize our findings in the next theorem.

Theorem 3.2. *Consider the RIVP given by (1.2). Under assumptions A1 − −A3, the Laplace transform of the mean square solution $X(t)$, of (1.2) can be written as the mean square convergent series given by (3.8) for $s > K_2 := \max\left\{(2M_B)^{\frac{1}{\alpha}}, (2M_A)^{\frac{1}{\alpha-1}}, K_1\right\}$, where K_1 is defined in Theorem 3.1.*

Setting $\nu := \alpha - 1 \in (0,1)$, with the aid of Theorem 3.2, we will show that

$$\begin{aligned} X(t) = &\sum_{n,m\geq 0}\phi_{n,m}(A,B)C_0\frac{t^{m\nu+n\alpha}}{\Gamma(m\nu+n\alpha+1)} \\ &+ \sum_{n,m\geq 0}\phi_{n,m}(A,B)C_1\frac{t^{m\nu+n\alpha+1}}{\Gamma(m\nu+n\alpha+2)} \\ &+ \sum_{n,m\geq 0}\phi_{n,m}(A,B)AC_0\frac{t^{m\nu+n\alpha+\alpha-1}}{\Gamma(m\nu+n\alpha+\alpha)} \end{aligned} \tag{3.14}$$

is a mean square solution of (1.2) for all $t > 0$. To achieve this goal, two facts will be proven for all $t > 0$: the m.s. convergence and the term-wise application of the Laplace transform of the double series (3.14) to justify its Laplace transform is given by (3.8).

Theorem 3.3. *If the random variables A, B, C_0 and C_1 satisfy conditions A1–A3, then the stochastic process $X(t)$, given by (3.14), is a mean square solution of RIVP (1.2) for all $t > 0$.*

Proof. Recall that $1 < \alpha < 2$, so $0 < \nu = \alpha - 1 < 1$. First, we will show that $X(t)$ belongs to $\mathrm{L}^2(\Omega)$ for each $t > 0$. Inequality (3.10) with $p = n+1$ and $q = m$ implies $\Gamma(n+m+1) \leq ((2n+1)!)^{1/2}((2m-1)!)^{1/2}$, and inequality (3.9) with $p = m\nu$ and $q = n\alpha+1$, gives $\Gamma(m\nu + n\alpha + 1) \geq \Gamma(m\nu+1)\Gamma(n\alpha+2)$. Hence

$$\left\| \phi_{n,m}(A,B) C_0 \frac{t^{m\nu+n\alpha}}{\Gamma(m\nu+n\alpha+1)} \right\|_2 \leq \psi_1(n;t)\psi_2(m;t) \tag{3.15}$$

with

$$\psi_1(n;t) = \frac{\|C_0\|_2 (M_B)^n \left[(2n-1)!\right]^{\frac{1}{2}}}{n!\Gamma(n\alpha+2)} t^{n\alpha}, \qquad \psi_2(m;t) = \frac{(M_A)^m \left[(2m+1)!\right]^{\frac{1}{2}}}{m!\Gamma(m\nu+1)} t^{m\nu}.$$

To study the mean square convergence of first infinite series in (3.14), we will apply the ratio test. So, we calculate

$$\lim_{n\to\infty} \frac{\psi_1(n+1;t)}{\psi_1(n;t)} = \lim_{n\to\infty} \frac{M_B \left[(2n+1)(2n)\right]^{\frac{1}{2}}}{n+1} \frac{\Gamma(n\alpha+2)}{\Gamma(n\alpha+2+\alpha)} t^{\alpha}.$$

The Stirling's formula, $\Gamma(x+1) \approx x^x e^{-x}\sqrt{2\pi x}$ as $x \to \infty$, implies

$$\begin{aligned}\lim_{n\to\infty} \frac{\Gamma(n\alpha+2)}{\Gamma(n\alpha+2+\alpha)} &= \lim_{n\to\infty} \frac{(n\alpha+1)^{n\alpha+1}\mathrm{e}^{-(n\alpha+1)}\sqrt{2\pi(n\alpha+1)}}{(n\alpha+1+\alpha)^{n\alpha+1+\alpha}\mathrm{e}^{-(n\alpha+1+\alpha)}\sqrt{2\pi(n\alpha+1+\alpha)}} \\ &= \lim_{n\to\infty} \left(\frac{n\alpha+1}{n\alpha+1+\alpha}\right)^{n\alpha+1} \left(\frac{1}{n\alpha+1+\alpha}\right)^{\alpha} \mathrm{e}^{\alpha} \left(\frac{n\alpha+1}{n\alpha+1+\alpha}\right)^{\frac{1}{2}}.\end{aligned} \tag{3.16}$$

Since

$$\lim_{n\to\infty} \left(\frac{n\alpha+1}{n\alpha+1+\alpha}\right)^{n\alpha+1} = \mathrm{e}^{-\alpha}, \; \lim_{n\to\infty} \left(\frac{n\alpha+1}{n\alpha+1+\alpha}\right)^{\frac{1}{2}} = 1, \; \lim_{n\to\infty} \left(\frac{1}{n\alpha+1+\alpha}\right)^{\alpha} = 0,$$

from (3.16) yields $\lim_{n\to\infty} \frac{\Gamma(n\alpha+2)}{\Gamma(n\alpha+2+\alpha)} = 0$. Moreover, $\lim_{n\to\infty} \frac{M_B\left[(2n+1)(2n)\right]^{\frac{1}{2}}}{n+1} t^{\alpha} = 2M_B t^{\alpha}$, then for every $t > 0$ finite, one gets

$$\lim_{n\to\infty} \frac{\psi_1(n+1;t)}{\psi_1(n;t)} = 0. \tag{3.17}$$

Similarly,

$$\lim_{m\to\infty} \frac{\psi_2(m+1;t)}{\psi_2(m;t)} = \lim_{m\to\infty} \frac{M_A \left[(2m+3)(2m+2)\right]^{\frac{1}{2}}}{m+1} \frac{\Gamma(m\nu+1)}{\Gamma(m\nu+1+\nu)} t^{\nu} = 0, \tag{3.18}$$

since

$$\lim_{n\to\infty} \frac{\Gamma(m\nu+1)}{\Gamma(m\nu+1+\nu)} = \lim_{m\to\infty} \left(\frac{m\nu}{m\nu+\nu}\right)^{m\nu} \left(\frac{1}{m\nu+\nu}\right)^{\nu} \mathrm{e}^{\nu} \left(\frac{m\nu}{m\nu+\nu}\right)^{\frac{1}{2}} = 0$$

and

$$\lim_{m\to\infty} \frac{M_A\left[(2m+3)(2m+2)\right]^{\frac{1}{2}}}{(m+1)} t^{\nu} = 2M_A t^{\nu}.$$

As a consequence of (3.17) and (3.18), the ratio test implies that both majorant series $\sum_{n\geq 0}\psi_1(n;t)$ and $\sum_{m\geq 0}\psi_2(m;t)$ are convergent for all $t>0$. It follows, by (3.15), that the first series of the stochastic process $X(t)$, given by (3.14), is mean square convergent for all $t>0$. Following a similar analysis for the remaining two series defining $X(t)$, it is shown that (3.14) is mean square convergent for all $t>0$. This proves that $X(t)$ belongs to $\mathrm{L}^2(\Omega)$. The mean square convergence of $X(t)$ for all $t>0$ and the mean square continuity of the each term in the expansion (3.14) imply that $X(t)$ is mean square continuous for all $t>0$. Consequently, we only need to show that the Laplace transform of (3.14) is (3.8) to validate that $X(t)$ given (3.14) is a mean square solution of (1.2). To this end, we prove that the term-wise application for the Laplace transform of the expansion (3.14) is valid for all $t>0$. Since the first series in (3.8) is mean square convergent for $s>K_2$ (see Theorem 3.2), its general term tends to zero, so it must be bounded, i.e., there exists $K_3>0$ such that

$$\frac{\|\phi_{n,m}(A,B)\|_2}{s^{m(\alpha-1)}s^{n\alpha}} \leq K_3, \tag{3.19}$$

for all $n,m\in\mathbb{N}\cup\{0\}$ and for all $s>K_2$. Let r be such that $r>K_2>0$, then

$$\|\phi_{n,m}(A,B)\|_2 \leq K_3 r^{m\nu} r^{n\alpha}, \quad r>K_2>0, \quad \nu=\alpha-1>0. \tag{3.20}$$

Taking into account that $\Gamma(m\nu+\alpha n+1)\geq\Gamma(m\nu+2)\Gamma(n\alpha+1)>0$ (derived from (3.9)), one gets

$$\frac{\|\phi_{n,m}(A,B)\|_2}{\Gamma(m\nu+\alpha n+1)} \leq \frac{K_3 r^{m\nu} r^{n\alpha}}{\Gamma(m\nu+2)\Gamma(\alpha n+1)}, \quad r>K_2>0. \tag{3.21}$$

Let us define the tail for the first series in (3.14),

$$\begin{aligned} X^T_{N,M}(t) := & \sum_{n,m\geq 0} \phi_{n,m}(A,B)C_0 \frac{t^{m\nu+n\alpha}}{\Gamma(m\nu+n\alpha+1)} \\ & - \sum_{n=0}^{N}\sum_{m=0}^{M} \phi_{n,m}(A,B)C_0 \frac{t^{m\nu+n\alpha}}{\Gamma(m\nu+n\alpha+1)}. \end{aligned} \tag{3.22}$$

Since the first series defining $X(t)$ in (3.14) is mean square convergent, for M and N large enough, applying (3.20) it follows

$$\begin{aligned} \|X^T_{N,M}(t)\|_2 \leq & \sum_{n\geq N+1, m\geq M+1} \left\|\phi_{n,m}(A,B)\right\|_2 \left\|C_0\right\|_2 \frac{t^{m\nu+n\alpha}}{\Gamma(m\nu+\alpha n+1)} \\ \leq & K_4 \left(\sum_{m\geq M+1} \frac{(rt)^{m\nu}}{\Gamma(m\nu+2)}\right)\left(\sum_{n\geq N+1} \frac{(rt)^{n\alpha}}{\Gamma(n\alpha+1)}\right), \end{aligned} \tag{3.23}$$

with $K_4 := K_3 \|C_0\|_2 > 0$. For $m > \frac{1}{\nu}$ (so $m\nu + 1 > 2$) and using that $\Gamma(x)$ is increasing for $x > 2$, one gets $\frac{1}{\Gamma(m\nu+2)} \le \frac{1}{\Gamma(m\nu+1)}$. Then for $M > \frac{1}{\nu} - 1$,

$$\begin{aligned}\|X^T_{N,M}(t)\|_2 &\le K_4 \left(\sum_{m\ge M+1} \frac{(rt)^{m\nu}}{\Gamma(m\nu+1)} \right) \left(\sum_{n\ge N+1} \frac{(rt)^{n\alpha}}{\Gamma(n\alpha+1)} \right) \\ &= K_4 \left(\mathrm{E}_\nu((rt)^\nu) - \sum_{m=0}^{M} \frac{(rt)^{m\nu}}{\Gamma(m\nu+1)} \right) \left(\mathrm{E}_\alpha((rt)^\alpha) - \sum_{n=0}^{N} \frac{(rt)^{n\alpha}}{\Gamma(n\alpha+1)} \right),\end{aligned} \tag{3.24}$$

where $\mathrm{E}_\gamma(z) := \sum_{k=0}^\infty \frac{z^k}{\Gamma(\gamma k+1)}$, $z > 0$, denotes the Mittag-Leffler function. Setting

$$\mathrm{I}_1(t) := \mathrm{E}_\nu((rt)^\nu)\mathrm{E}_\alpha((rt)^\alpha), \qquad \mathrm{I}_2(t) := \mathrm{E}_\nu((rt)^\nu) \sum_{n=0}^{N} \frac{(rt)^{n\alpha}}{\Gamma(n\alpha+1)},$$

$$\mathrm{I}_3(t) := \sum_{m=0}^{M} \frac{(rt)^{m\nu}}{\Gamma(m\nu+1)} \mathrm{E}_\alpha((rt)^\alpha), \qquad \mathrm{I}_4(t) := \sum_{m=0}^{M} \frac{(rt)^{m\nu}}{\Gamma(m\nu+1)} \sum_{n=0}^{N} \frac{(rt)^{n\alpha}}{\Gamma(n\alpha+1)},$$

we have $\|X^T_{N,M}(t)\|_2 \le K_4 \left(\mathrm{I}_1(t) - \mathrm{I}_2(t) - \mathrm{I}_3(t) + \mathrm{I}_4(t)\right)$. Hence

$$\begin{aligned}\|\mathcal{L}\{X^T_{N,M}(t); s\}\|_2 &\le \mathcal{L}\{\|X^T_{N,M}(t)\|_2; s\} \\ &\leqslant K_4 \left(\mathcal{L}\{\mathrm{I}_1(t); s\} - \mathcal{L}\{\mathrm{I}_2(t); s\} - \mathcal{L}\{\mathrm{I}_3(t); s\} + \mathcal{L}\{\mathrm{I}_4(t); s\}\right).\end{aligned} \tag{3.25}$$

It is known that the Mittag-Leffler function satisfies the following inequality ([38, p. 21]),

$$\text{exists } C > 0: \quad \mathrm{E}_\gamma(\rho t^\gamma) \le C \mathrm{e}^{\rho^{\frac{1}{\gamma}} t}, \qquad t,\, \rho \ge 0, \quad 0 < \gamma < 2. \tag{3.26}$$

By a weaker form of Lebesgue's dominated convergence theorem ([36, p. 167]) and inequality (3.26), the following commutation of the Laplace transform and the double series on the third step is justified (see Appendix)

$$\begin{aligned}\mathcal{L}\{\mathrm{I}_1(t); s\} &= \mathcal{L}\{\mathrm{E}_\nu((rt)^\nu)\mathrm{E}_\alpha((rt)^\alpha); s\} \\ &= \mathcal{L}\left\{ \sum_{n,m\ge 0} \frac{r^{m\nu+n\alpha}}{\Gamma(m\nu+1)} \frac{t^{m\nu+n\alpha}}{\Gamma(n\alpha+1)}; s \right\} \\ &= \sum_{n,m\ge 0} \mathcal{L}\left\{ \frac{r^{m\nu+n\alpha}}{\Gamma(m\nu+1)} \frac{t^{m\nu+n\alpha}}{\Gamma(n\alpha+1)}; s \right\}.\end{aligned} \tag{3.27}$$

Next, by Example 2.7 it is inferred

$$\mathcal{L}\{\mathrm{I}_1(t); s\} = \sum_{n,m\ge 0} \frac{r^{m\nu+n\alpha}}{\Gamma(m\nu+1)\Gamma(n\alpha+1)} \frac{\Gamma(m\nu+n\alpha+1)}{s^{m\nu+n\alpha+1}}. \tag{3.28}$$

Further, inequality (3.26) also implies that the double series in (3.28) converges for $s > 2r > K_2 > 0$ (see Appendix). Similarly, it can be deduced that

$$\mathcal{L}\{\mathrm{I}_2(t); s\} = \sum_{n=0}^{N} \sum_{m\geq 0} \frac{r^{m\nu+n\alpha}}{\Gamma(m\nu+1)\Gamma(n\alpha+1)} \frac{\Gamma(m\nu+n\alpha+1)}{s^{m\nu+n\alpha+1}},$$
$$\mathcal{L}\{\mathrm{I}_3(t); s\} = \sum_{m=0}^{M} \sum_{n\geq 0} \frac{r^{m\nu+n\alpha}}{\Gamma(m\nu+1)\Gamma(n\alpha+1)} \frac{\Gamma(m\nu+n\alpha+1)}{s^{m\nu+n\alpha+1}},$$
$$\mathcal{L}\{\mathrm{I}_4(t); s\} = \sum_{n=0}^{N} \sum_{m=0}^{M} \frac{r^{m\nu+n\alpha}}{\Gamma(m\nu+1)\Gamma(n\alpha+1)} \frac{\Gamma(m\nu+n\alpha+1)}{s^{m\nu+n\alpha+1}}. \tag{3.29}$$

Consequently from (3.25)–(3.29), we have $\lim_{N,M\to\infty} \|\mathcal{L}\{X^T_{N,M}(t); s\}\|_2 = 0$. Hence from (3.22) we have that

$$\begin{aligned}
&\mathcal{L}\left\{\sum_{n,m\geq 0} \phi_{n,m}(A,B)C_0 \frac{t^{m(\alpha-1)+n\alpha}}{\Gamma(m(\alpha-1)+n\alpha+1)}; s\right\} \\
&= \lim_{N,M\to\infty} \mathcal{L}\left\{\sum_{n=0}^{N}\sum_{m=0}^{M} \phi_{n,m}(A,B)C_0 \frac{t^{m(\alpha-1)+n\alpha}}{\Gamma(m(\alpha-1)+n\alpha+1)}; s\right\} \\
&= \sum_{n,m\geq 0} \phi_{n,m}(A,B)C_0 \frac{\mathcal{L}\{t^{m(\alpha-1)+n\alpha}; s\}}{\Gamma(m(\alpha-1)+n\alpha+1)} \\
&= \sum_{n,m\geq 0} \phi_{n,m}(A,B)C_0 s^{-(m(\alpha-1)+n\alpha+1)}.
\end{aligned} \tag{3.30}$$

That is, the application of the Laplace transform can be done term by term for the first term of the expansion of $X(t)$ given by (3.14). The analysis for the rest of the terms is analogous. †

Remark 3.4. *Since $r, t, \Gamma(m\nu+1) > 0$, from (3.24) and (3.26) we have*

$$\begin{aligned}
\|X^T_{N,M}(t)\|_2 &\leq K_4 \left(\sum_{m\geq M+1} \frac{(rt)^{m\nu}}{\Gamma(m\nu+1)}\right)\left(\sum_{n\geq N+1} \frac{(rt)^{n\alpha}}{\Gamma(n\alpha+1)}\right) \\
&\leq K_4 \mathrm{E}_\nu((rt)^\nu)\mathrm{E}_\alpha((rt)^\alpha) \leq K_4 C_1 \mathrm{e}^{\left((r^\nu)^{\frac{1}{\nu}}\right)t} C_2 \mathrm{e}^{\left((r^\alpha)^{\frac{1}{\alpha}}\right)t} \\
&= K_4 C_1 C_2 \mathrm{e}^{2rt},
\end{aligned} \tag{3.31}$$

for all $N \geq 0$ and for all $M > \frac{1}{\nu} - 1$. This shows that $X^T_{N,M}(t)$ is of exponential order $2r > 0$. Since $\sum_{n=0}^{N}\sum_{m=0}^{M} \phi_{n,m}(A,B)C_0 \frac{t^{m\nu+n\alpha}}{\Gamma(m\nu+n\alpha+1)}$ is of exponential order, it follows that the first term of the mean square solution, $X(t)$, given by (3.14), is of exponential order. A similar analysis with the rest of the terms of $X(t)$ proves that $X(t)$ is of exponential order parameter s_0 for some $s_0 > 0$, so it admits a mean square Laplace transform.

4. Approximating the mean and the variance of the solution stochastic process

So far, we have established sufficient conditions to show the mean square convergence of the double sequence of approximations

$$
\begin{aligned}
X_{N,M}(t) \quad = \quad & \sum_{n=0}^{N}\sum_{m=0}^{M} \phi_{n,m}(A,B)C_0 \frac{t^{m\nu+n\alpha}}{\Gamma(m\nu+n\alpha+1)} \\
& + \sum_{n=0}^{N}\sum_{m=0}^{M} \phi_{n,m}(A,B)C_1 \frac{t^{m\nu+n\alpha+1}}{\Gamma(m\nu+n\alpha+2)} \\
& + \sum_{n=0}^{N}\sum_{m=0}^{M} \phi_{n,m}(A,B)AC_0 \frac{t^{m\nu+n\alpha+\alpha-1}}{\Gamma(m\nu+n\alpha+\alpha)}
\end{aligned} \tag{4.1}
$$

to the solution stochastic process, $X(t)$, of the RIVP (1.2). Now, we will take advantage of the key property stated in Lemma 1.1 to compute reliable approximations of the mean and of the variance of the solution. To this end, we need to compute the mean and the variance of $X_{N,M}(t)$. To obtain the approximation of the mean, we take the expectation operator in (4.1) (substituing the explicit expression for $\phi_{n,m}(A,B)$ given in (3.7)), apply the linearity of the expectation and utilize that C_0, C_1, A and B are independent random variables (assumption A3). This yields

$$
\begin{aligned}
\mathbb{E}[X_{N,M}(t)] \quad = \quad & \sum_{n=0}^{N}\sum_{m=0}^{M} \frac{(-1)^{n+m}\mathbb{E}[B^n]\mathbb{E}[A^m]\Gamma(m+n+1)}{\Gamma(m+1)\Gamma(n+1)} \\
& \cdot \left\{ \mathbb{E}[C_0]\frac{t^{m\nu+n\alpha}}{\Gamma(m\nu+\alpha n+1)} + \mathbb{E}[C_1]\frac{t^{m\nu+n\alpha+1}}{\Gamma(m\nu+\alpha n+2)} \right\} \\
& + \sum_{n=0}^{N}\sum_{m=0}^{M} (-1)^{n+m}\mathbb{E}[B^n]\mathbb{E}[A^{m+1}]\mathbb{E}[C_0] \\
& \cdot \frac{\Gamma(m+n+1)}{\Gamma(m+1)\Gamma(n+1)} \frac{t^{m\nu+\alpha n+\alpha-1}}{\Gamma(m\nu+\alpha n+\alpha)}.
\end{aligned} \tag{4.2}
$$

To calculate the approximation of the variance, $\mathbb{V}[X_{N,M}(t)] = \mathbb{E}[(X_{N,M}(t))^2] - \mathbb{E}^2[X_{N,M}(t)]$, we first need to calculate the second order moment, $\mathbb{E}[(X_{N,M}(t))^2]$. Using again assumption A3 and the properties of the expectaion operator, after some technical calculations one obtains

$$
\begin{aligned}
\mathbb{E}[(X_{N,M}(t))^2] = & \sum_{n=0}^{N} \left(\sum_{m=0}^{M} \left(\frac{(n+m)!}{n!m!} \right) \mathbb{E}[B^{2n}] \left[\mathbb{E}[A^{2m}] \left\{ \mathbb{E}[C_0^2] \left(\frac{t^{m\nu+\alpha n}}{\Gamma(m\nu+\alpha n+1)} \right)^2 \right.\right.\right. \\
& + \mathbb{E}[C_1^2] \left(\frac{t^{m\nu+\alpha n+1}}{\Gamma(m\nu+\alpha n+2)} \right)^2 \\
& \left. + 2\mathbb{E}[C_0]\mathbb{E}[C_1] \left(\frac{t^{m\nu+\alpha n}}{\Gamma(m\nu+\alpha n+1)} \right) \left(\frac{t^{m\nu+\alpha n+1}}{\Gamma(m\nu+\alpha n+2)} \right) \right\}
\end{aligned}
$$

$$+ \mathbb{E}[A^{2m+2}]\mathbb{E}[C_0^2]\left(\frac{t^{m\nu+\alpha n+\alpha-1}}{\Gamma(m\nu+\alpha n+\alpha)}\right)^2$$

$$+ 2\mathbb{E}[C_0^2]\mathbb{E}[A^{2m+1}]\left(\frac{t^{m\nu+\alpha n}}{\Gamma(m\nu+\alpha n+1)}\right)\left(\frac{t^{m\nu+\alpha n+\alpha-1}}{\Gamma(m\nu+\alpha n+\alpha)}\right)$$

$$+ 2\mathbb{E}[C_0]\mathbb{E}[C_1]\mathbb{E}[A^{2m+1}]\left(\frac{t^{m\nu+\alpha n+1}}{\Gamma(m\nu+\alpha n+2)}\right)\left(\frac{t^{m\nu+\alpha n+\alpha-1}}{\Gamma(m\nu+\alpha n+\alpha)}\right)\Bigg]$$

$$+ 2\sum_{m_1=1}^{M}\sum_{m_2=0}^{m_1-1}(-1)^{2n+m_1+m_2}\mathbb{E}[B^{2n}]\frac{\Gamma(n+m_1+1)}{\Gamma(n+1)\Gamma(m_1+1)}\frac{\Gamma(n+m_2+1)}{\Gamma(n+1)\Gamma(m_2+1)}$$

$$\cdot\Bigg[\mathbb{E}[A^{m_1+m_2}]\Bigg\{\mathbb{E}[C_0^2]\left(\frac{t^{m_1\nu+\alpha n}}{\Gamma(m_1\nu+\alpha n+1)}\right)\left(\frac{t^{m_2\nu+\alpha n}}{\Gamma(m_2\nu+\alpha n+1)}\right)$$

$$+ \mathbb{E}[C_0]\mathbb{E}[C_1]\left(\frac{t^{m_1\nu+\alpha n}}{\Gamma(m_1\nu+\alpha n+1)}\right)\left(\frac{t^{m_2\nu+\alpha n+1}}{\Gamma(m_2\nu+\alpha n+2)}\right)$$

$$+ \mathbb{E}[C_0]\mathbb{E}[C_1]\left(\frac{t^{m_1\nu+\alpha n+1}}{\Gamma(m_1\nu+\alpha n+2)}\right)\left(\frac{t^{m_2\nu+\alpha n}}{\Gamma(m_2\nu+\alpha n+1)}\right)$$

$$+ \mathbb{E}[C_1^2]\left(\frac{t^{m_1\nu+\alpha n+1}}{\Gamma(m_1\nu+\alpha n+2)}\right)\left(\frac{t^{m_2\nu+\alpha n+1}}{\Gamma(m_2\nu+\alpha n+2)}\right)\Bigg\}$$

$$+ \mathbb{E}[A^{m_1+m_2+1}]\mathbb{E}[C_0^2]\left(\frac{t^{m_1\nu+\alpha n}}{\Gamma(m_1\nu+\alpha n+1)}\right)\left(\frac{t^{m_2\nu+\alpha n+\alpha+1}}{\Gamma(m_2\nu+\alpha n+\alpha)}\right)$$

$$+ \mathbb{E}[C_0]\mathbb{E}[C_1]\mathbb{E}[A^{m_1+m_2+1}]\left(\frac{t^{m_1\nu+\alpha n+1}}{\Gamma(m_1\nu+\alpha n+2)}\right)\left(\frac{t^{m_2\nu+\alpha n+\alpha-1}}{\Gamma(m_2\nu+\alpha n+\alpha)}\right)$$

$$+ \mathbb{E}[C_0^2]\mathbb{E}[A^{m_1+m_2+1}]\left(\frac{t^{m_1\nu+\alpha n+\alpha-1}}{\Gamma(m_1\nu+\alpha n+\alpha)}\right)\left(\frac{t^{m_2\nu+\alpha n}}{\Gamma(m_2\nu+\alpha n+1)}\right)$$

$$+ \mathbb{E}[C_0]\mathbb{E}[C_1]\mathbb{E}[A^{m_1+m_2+1}]\left(\frac{t^{m_1\nu+\alpha n+\alpha-1}}{\Gamma(m_1\nu+\alpha n+\alpha)}\right)\left(\frac{t^{m_2\nu+\alpha n+1}}{\Gamma(m_2\nu+\alpha n+2)}\right)$$

$$+ \mathbb{E}[C_0^2]\mathbb{E}[A^2]\left(\frac{t^{m_1\nu+\alpha n+\alpha-1}}{\Gamma(m_1\nu+\alpha n+\alpha)}\right)\left(\frac{t^{m_2\nu+\alpha n+\alpha-1}}{\Gamma(m_2\nu+\alpha n+\alpha)}\right)\Bigg]\Bigg)$$

$$+ 2\sum_{n_1=1}^{N}\sum_{n_2=0}^{n_1-1}\sum_{m_1=0}^{M}\sum_{m_2=0}^{M}(-1)^{n_1+m_1+n_2+m_2}\mathbb{E}[B^{n_1+n_2}]\frac{\Gamma(m_1+n_2+1)}{\Gamma(m_1+1)\Gamma(n_1+1)}$$

$$\cdot\frac{\Gamma(m_2+n_2+1)}{\Gamma(m_2+1)\Gamma(n_2+1)}\Bigg[\mathbb{E}[A^{m_1+m_2}]\Bigg\{\mathbb{E}[C_0^2]\left(\frac{t^{m_1\nu+\alpha n_1}}{\Gamma(m_1\nu+\alpha n_1+1)}\right)\left(\frac{t^{m_2\nu+\alpha n_2}}{\Gamma(m_2\nu+\alpha n_2+1)}\right)$$

$$+ \mathbb{E}[C_0]\mathbb{E}[C_1]\left(\frac{t^{m_1\nu+\alpha n_1}}{\Gamma(m_1\nu+\alpha n_1+1)}\right)\left(\frac{t^{m_2\nu+\alpha n_2+1}}{\Gamma(m_2\nu+\alpha n_2+2)}\right)$$

$$+ \mathbb{E}[C_0]\mathbb{E}[C_1]\left(\frac{t^{m_1\nu+\alpha n_1+1}}{\Gamma(m_1\nu+\alpha n_1+2)}\right)\left(\frac{t^{m_2\nu+\alpha n_2}}{\Gamma(m_2\nu+\alpha n_2+1)}\right)$$

$$+ \mathbb{E}[C_1^2]\left(\frac{t^{m_1\nu+\alpha n_1+1}}{\Gamma(m_1\nu+\alpha n_1+2)}\right)\left(\frac{t^{m_2\nu+\alpha n_2+1}}{\Gamma(m_2\nu+\alpha n_2+2)}\right)\Bigg\}$$

$$
\begin{aligned}
&+ \mathbb{E}[A^{m_1+m_2+1}] \Bigg\{ \mathbb{E}[C_0^2] \left(\frac{t^{m_1\nu+\alpha n_1}}{\Gamma(m_1\nu+\alpha n_1+1)} \right) \left(\frac{t^{m_2\nu+\alpha n_2+\alpha-1}}{\Gamma(m_2\nu+\alpha n_2+\alpha)} \right) \\
&+ \mathbb{E}[C_0]\mathbb{E}[C_1] \left(\frac{t^{m_1\nu+\alpha n_1+1}}{\Gamma(m_1\nu+\alpha n_1+2)} \right) \left(\frac{t^{m_2\nu+\alpha n_2+\alpha-1}}{\Gamma(m_2\nu+\alpha n_2+\alpha)} \right) \\
&+ \mathbb{E}[C_0^2] \left(\frac{t^{m_1\nu+\alpha n_1+\alpha-1}}{\Gamma(m_1\nu+\alpha n_1+\alpha)} \right) \left(\frac{t^{m_2\nu+\alpha n_2}}{\Gamma(m_2\nu+\alpha n_2+1)} \right) \\
&+ \mathbb{E}[C_0]\mathbb{E}[C_1] \left(\frac{t^{m_1\nu+\alpha n_1+\alpha-1}}{\Gamma(m_1\nu+\alpha n_1+\alpha)} \right) \left(\frac{t^{m_2\nu+\alpha n_2+1}}{\Gamma(m_2\nu+\alpha n_2+2)} \right) \Bigg\} \\
&+ \mathbb{E}[A^{m_1+m_2+2}]\mathbb{E}[C_0^2] \left(\frac{t^{m_1\nu+\alpha n_1+\alpha-1}}{\Gamma(m_1\nu+\alpha n_1+\alpha)} \right) \left(\frac{t^{m_2\nu+\alpha n_2+\alpha-1}}{\Gamma(m_2\nu+\alpha n_2+\alpha)} \right) \Bigg] .
\end{aligned} \tag{4.3}
$$

Remark 4.1. *In the previous development we have approximated the mean and the variance, however it is worthwhile to point out that higher moments,* $(\mathbb{E}[X(t)])^k$*,* $k = 3, 4, \ldots$*, can also be calculated using the same reasoning, but the resulting expressions become somewhat cumbersome.*

Remark 4.2. *It is interesting to point out that hypothesis A3 can be relaxed assuming weaker conditions, for example, that* $\{C_0, C_1\}$ *and* $\{A, B\}$ *are independent random variables. However, assumption A3 is not too much restrictive from a practical standpoint. We have assumed hypothesis A3 just to facilitate the process of boundedness to rigurously prove the mean square convergence stated in Th. 3.3. If the aforementioned weaker hypothesis is assumed, then the approximations of the mean and of the second-order moment given in* (4.2) *and* (4.3)*, respectively, would be expressed in terms of the joint moments of* $\{C_0, C_1\}$ *and* $\{A, B\}$*. For example, in the expression* (4.2) *would appear the joint moment* $\mathbb{E}[B^n A^m]$ *instead of* $\mathbb{E}[B^n]\mathbb{E}[A^m]$*, etc.*

5. Numerical examples

This section is addressed to numerically illustrate our previous theoretical findings. First, we will compute approximations for the mean and the variance of the solution stochastic process of RIVP (1.2) via the expressions obtained in the foregoing section. Secondly, we will take advantage these approximations together with the Principle of Maximum Entropy (PME) to construct approximations for the first probability density function (1-PDF) of the solution.

Example 5.1. *Let us consider the RIVP* (1.2) *assuming the following hypotheses with regard to its parameters and initial conditions:*

- $\alpha = 1.5 \in (1, 2)$,

- C_0 *and* C_1 *such that* $\mathbb{E}[C_0] = \mathbb{E}[C_1] = 0.5$ *and* $\mathbb{E}[C_0^2] = \mathbb{E}[C_1^2] = 0.5$ *(so they are second order random variables),*
- A *and* B *are random variables with Beta distributions,* $A \sim Be(10, 20)$ *and* $B \sim Be(20, 30)$ *(so they are bounded second order random variables),*
- C_0*,* C_1*,* A *and* B *are independent.*

To calculate the approximations of the mean and the variance of $X(t)$*, we will apply expressions* (4.2) *and* (4.3) *taking into account the following expressions for the moments of* A *and* B *[39]*

$$\mathbb{E}[A^k] = \prod_{r=0}^{k-1} \frac{10+r}{10+20+r}, \quad \mathbb{E}[B^k] = \prod_{r=0}^{k-1} \frac{20+r}{20+30+r}, \quad k = 1, 2, \ldots. \tag{5.1}$$

Figure 1 shows the approximations for the mean and for the variance of the truncated solution (4.1)*,* $X_{N,M}(t)$*, considering different values of* M *and* N *in the time interval* $[0, 3]$*. To easily visualize the convergence, we zoom-up the results on the right-piece of the interval where it is supposed the accuracy of approximations becomes worse. However, the plots show very good results even taking small order of truncation* $M = N = 12$*.*

Considering the foregoing approximations of the mean and the variance with $M = N = 12$*, we now provide approximations of the 1-PDF of the solution* $X(t)$ *at this time instants using the PME technique [40]. According to this method, for each* t*, the PDF is sought in the form*

$$f_{X(t)}(x) = \mathrm{e}^{-1-\lambda_{0,t}-\lambda_{1,t}x-\lambda_{2,t}x^2}, \tag{5.2}$$

where $\lambda_{0,t}$*,* $\lambda_{1,t}$ *and* $\lambda_{2,t}$ *are determined solving the variational optimization problem*

$$\textit{Max.} \quad \mathcal{S}(f) = -\int_{a_1}^{a_2} f_{X(t)}(x) \ln(f_{X(t)}(x))\,\mathrm{d}x, \tag{5.3}$$

subject to the following constraints

$$\int_{a_1}^{a_2} f_{X(t)}(x)\,\mathrm{d}x = 1, \quad \int_{a_1}^{a_2} x f_{X(t)}(x)\,\mathrm{d}x = \mathbb{E}[X_{N,M}(t)], \quad \int_{a_1}^{a_2} x^2 f_{X(t)}(x)\,\mathrm{d}x = \mathbb{E}[X_{N,M}^2(t)]. \tag{5.4}$$

The integration interval $[a_1, a_2]$ *has been taken as* $a_1 = \mathbb{E}[X_{N,M}(t)] - 10\sqrt{\mathbb{V}[X_{N,M}(t)]}$ *and* $a_2 = \mathbb{E}[X_{N,M}(t)] + 10\sqrt{\mathbb{V}[X_{N,M}(t)]}$*. According to the general Bienaymé-Chebyshev inequality [39], more than* 99.9% *of the probability is contained within this interval regardless the probability distribution of* $X(t)$*. Table 1 collects the values of* $\lambda_{0,t}, \lambda_{1,t}, \lambda_{2,t}$ *at* $t \in \{0.5, 1, 1.5, 2, 2.5, 3\}$*. Figure 2 shows the 1-PDF surface of the approximate solution* $X_{N,M}(t)$ *on the time interval* $t \in [0, 3]$*. We can see that the variance increases as* t *does which is in full agreement with the results shown in Figure 1 (right panel).*

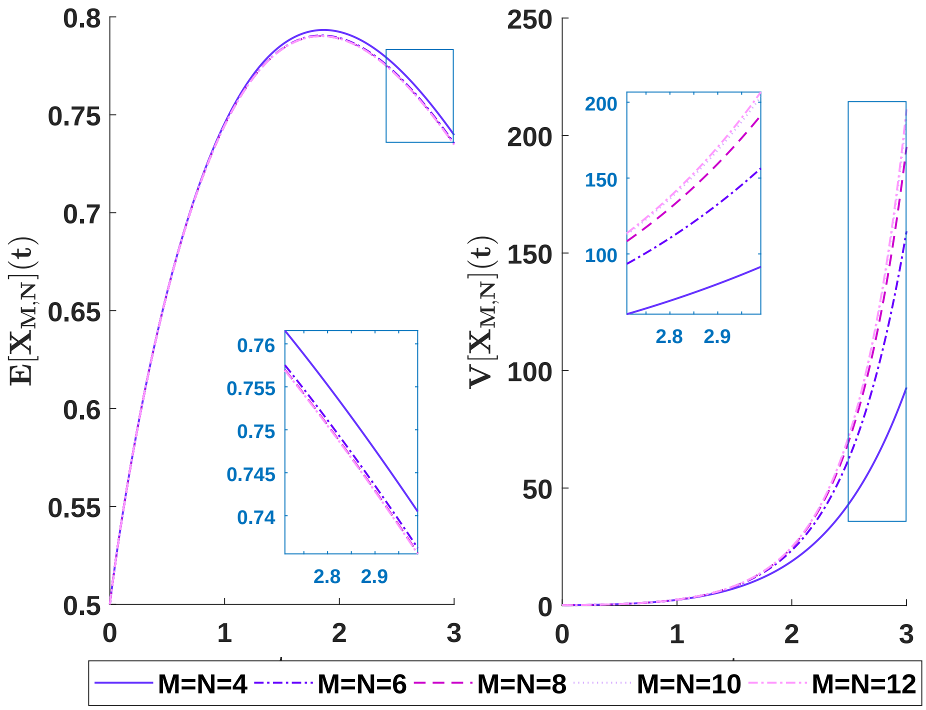

Figure 1: Approximations of the mean (left panel) and for the variance (right panel) of the solution stochastic process of the RIVP (1.2) considering different order of truncation (M, N), in the context of Example 5.1.

	$\lambda_{0,t}$	$\lambda_{1,t}$	$\lambda_{2,t}$
$t = 0.5$	2.86574e-02	-5.58862e-01	4.22942e-01
$t = 1.0$	6.00364e-01	-1.56599e-01	1.04757e-01
$t = 1.5$	1.15845e+00	-5.00819e-02	3.17505e-02
$t = 2.0$	1.69615e+00	-1.68397e-02	1.05534e-02
$t = 2.5$	2.22623e+00	-5.69139e-03	3.62312e-03
$t = 3.0$	2.75796e+00	-1.89774e-03	1.24712e-03

Table 1: Values of parameters $\lambda_{0,t}$, $\lambda_{1,t}$, $\lambda_{2,t}$ of the 1-PDF (5.2) resulting after solving the optimization problem (5.3) and (5.4) at the time instants $t \in \{0.5, 1, 1.5, 2, 2.5, 3\}$. Example 5.1.

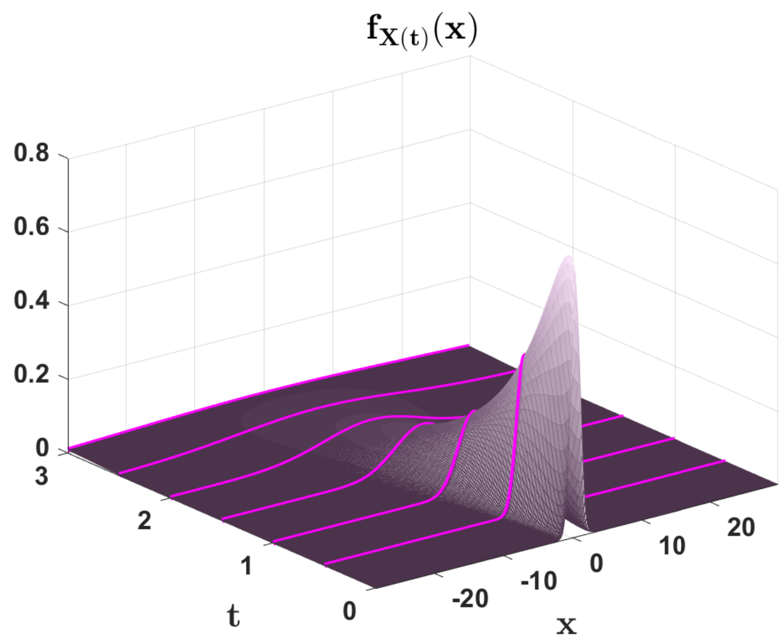

Figure 2: 1-PDF surface of the approximate solution $X(t)$ to random fractional IVP (1.2) using PME method on the time interval $t \in [0, 3]$ in the context of Example 5.1. The lines highlighted in magenta represent the PDF at $t \in \{0.5, 1, 1.5, 2, 2.5, 3\}$. They have been calculated using (5.2) with the λ_i-values collected in Table 1.

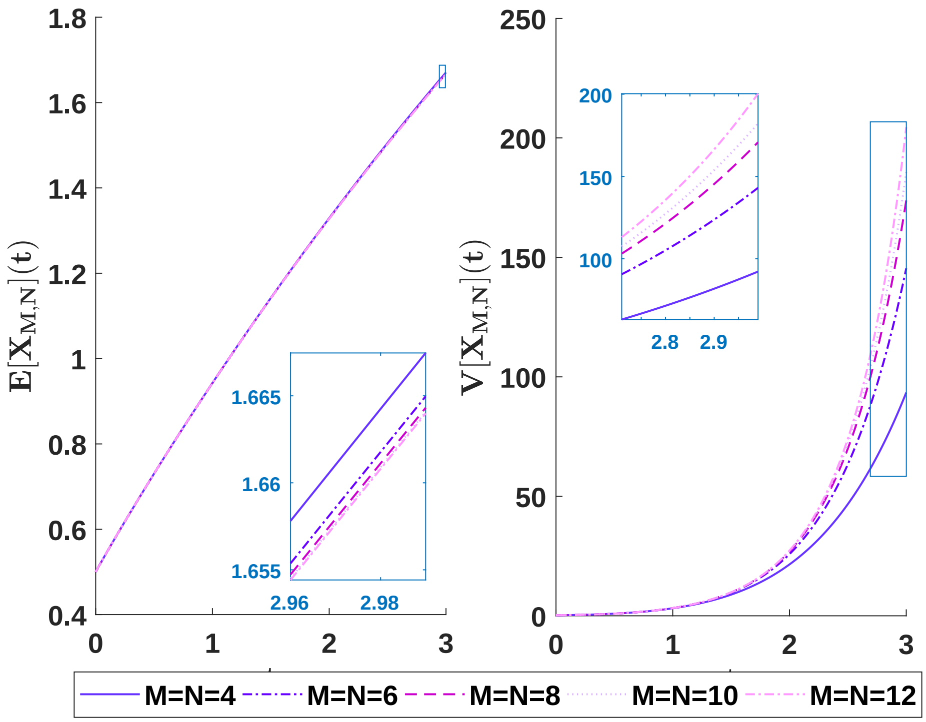

Figure 3: Approximations for the mean (left panel) and for the variance (right panel) of the solution stochastic process to the random (1.2) considering different order of truncation (M, N). Example 5.2.

Example 5.2. *In this example, we keep the same data as in Example 5.1, except that we change the probability distributions of random variables A and B, which clearly are the most influential parameters in the equation. Specifically, we assume that A and B have truncated exponential distributions on the interval $[0, 20]$ with parameters 10 and 20, respectively, i.e. $A \sim Exp_{[0,20]}(10)$ and $B \sim Exp_{[0,20]}(20)$. In this way, all the assumptions A1-A3 are fulfilled. To compute the first moments of $X_{N,M}(t)$ via (4.2) and (4.3) will utilize that*

$$\mathbb{E}[A^k] = 10^{-k}\left(k! - \Gamma(1+k, 200)\right), \quad \mathbb{E}[B^k] = 20^{-k}\left(k! - \Gamma(1+k, 400)\right), \quad k = 1, 2, \ldots,$$

where $\Gamma(p, q)$ is the incomplete Gamma function defined by $\Gamma(p, q) = \int_q^\infty t^{p-1}e^{-t}\mathrm{d}t$ [41].

Figure 3 shows the approximations for the mean and for the variance of the truncated solution (4.1) considering different values of M and N. From this graphical representation we can observe that approximations are better as N and M increase as expected despite uncertainty propagates rapidly as observed in the plot of the variance.

Similarly as in Example 5.1, we have approximated the 1-PDF of the solution via the PME. In Table 2 we collect the values for $\lambda_{0,t}$, $\lambda_{1,t}$ and $\lambda_{2,t}$ of the 1-PDF,

	$\lambda_{0,t}$	$\lambda_{1,t}$	$\lambda_{2,t}$
$t = 0.5$	1.83532e-01	-4.30221e-01	2.93877e-01
$t = 1.0$	7.39317e-01	-1.54518e-01	8.16821e-02
$t = 1.5$	1.25723e+00	-6.16776e-02	2.68857e-02
$t = 2.0$	1.75189e+00	-2.57862e-02	9.64095e-03
$t = 2.5$	2.24213e+00	-1.07863e-02	3.55176e-03
$t = 3.0$	2.74480e+00	-4.36089e-03	1.28802e-03

Table 2: Values of parameters $\lambda_{0,t}$, $\lambda_{1,t}$, $\lambda_{2,t}$ of the 1-PDF (5.2) resulting after solving the optimization problem (5.3) and (5.4) at the time instants $t \in \{0.5, 1, 1.5, 2, 2.5, 3\}$. Example 5.2.

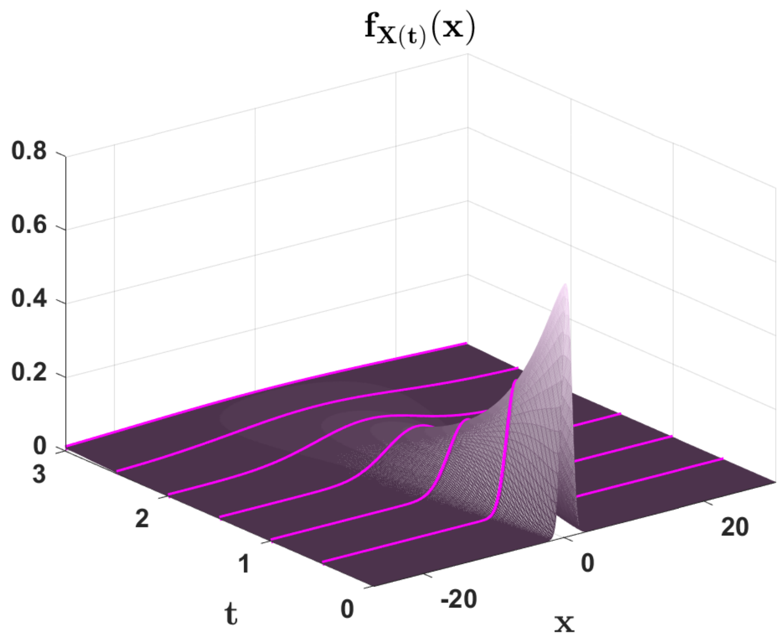

Figure 4: 1-PDF surface of the approximate solution $X(t)$ to random fractional IVP (1.2) using PME method on the time interval $t \in [0, 3]$ in the context of Example 5.2. The lines highlighted in magenta represent the PDF at $t \in \{0.5, 1, 1.5, 2, 2.5, 3\}$. They have been calculated using (5.2) with the λ_i-values collected in Table 2.

according to the general expression (5.2)*, at* $t \in \{0.5, 1, 1.5, 2, 2.5, 3\}$*. Figure 4 shows the approximate 1-PDF surface of the solution. We can observe that this graphical representation is in full agreement with the results shown in Figure 3 for the mean and the variance of the solution. In particular, we notice that the variance increases as time does.*

Remark 5.3. *In the two previous examples, we have applied the PME to ap-*

proximate the 1-PDF using the two first moment as restrictions. As indicated, in Remark 4.1 we could also obtain higher moments and this information can be added when applying the PME to improve the approximation of the 1-PDF.

6. Conclusions

The paper has focussed on the extension of an important class of fractional linear differential equations (in the Caputo sense) to the random framework. We have seen that random mean square calculus provides a suitable setting to construct an approximate solution stochastic process to model (1.2). Even more important, this approach has the distinctive advantage that the approximations for the mean and the variance of the solutions will converge to their respective exact values. This is a key information when studying differential equations with uncertainties. Additionally, the approximations of these two probability moments have also allowed us to approximate the first probability density function of the solution via the Principle of Maximum Entropy, a more complete probabilisitic information of the solution which is barely given in practice. We think that our approach can contribute to continue studying other classes of fractional differential equations with randomness, such as the random fractional 3-term equation, that is, when the derivatives of order 1 and 2 are considered fractional derivatives.

Appendix

In this appendix, we collect some useful results from Analysis that will clarify the steps in the proof of Theorem 3.3.

Theorem 6.1. *[36, p. 167]. If g and f_n, $n = 1, 2, 3, \dots$ are defined on $(0, \infty)$, are Riemann-integrable on $[t, T]$ whenever $0 < t < T < \infty$, $|f_n| \le g$, $f_n \to f$ uniformly on every compact subset of $(0, \infty)$, and*

$$\int_0^\infty g(x)\, \mathrm{d}x < \infty,$$

then

$$\lim_{n\to\infty} \int_0^\infty f_n(x)\, \mathrm{d}x = \int_0^\infty f(x)\, \mathrm{d}x.$$

Further, the next classical result gives sufficient conditions on which a series product converges.

Theorem 6.2. *Suppose that*

a) $\sum_{n=0}^{\infty} a_n$ *converges absolutely,*

b) $\sum_{n=0}^{\infty} a_n = S_1$,

c) $\sum_{n=0}^{\infty} b_n = S_2$,

d) $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, $n = 0, 1, 2, 3, \ldots$.

Then

$$\sum_{n=0}^{\infty} c_n = S_1 S_2.$$

Finally, it is well known that the Mittag-Leffler function

$$E_\gamma(z) := \sum_{k\geq 0} \frac{z^k}{\Gamma(\gamma k + 1)} \tag{6.1}$$

converges for all $\mathcal{R}e\,\gamma > 0$, where $\mathcal{R}e$ stands for real part of γ, see [42, p. 17]. In the following we will show the assertion given by (3.27). Let $a_m = \frac{(rt)^{\nu m}}{\Gamma(m\nu+1)}$ and $b_n = \frac{(rt)^{\alpha n}}{\Gamma(\alpha n+1)}$ and $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ in Theorem 6.2. Since $\sum_{m\geq 0} a_m = E_\nu((rt)^\nu)$ and $\sum_{n\geq 0} b_n = E_\alpha((rt)^\alpha)$ are absolutely convergent for all $t > 0$ and for $\nu, \alpha, r > 0$, Theorem 6.2 implies that

$$\sum_{n\geq 0} c_n = E_\nu((rt)^\nu) E_\alpha((rt)^\alpha).$$

On the other hand, in Theorem 6.1 set $f_N = e^{-st}\sum_{n=0}^{N} c_n$ and $g = e^{-st}E_\nu((rt)^\nu)E_\alpha((rt)^\alpha)$. Further, observe that f_N and g are Riemann integrable on $[t, T]$ for $0 < t < T < \infty$. Since $c_n > 0$ for all n because $\nu, \alpha, r, t > 0$ and $\Gamma(x) > 0$ for $x > 0$, we have

$$|f_N| = e^{-st}\sum_{n=0}^{N} c_n \leq e^{-st}E_\nu((rt)^\nu)E_\alpha((rt)^\alpha := g(t).$$

The absolutely convergence of E_ν and E_α on $(0, \infty)$ implies that $f_N \to e^{-st}E_\nu((rt)^\nu)E_\alpha((rt)^\alpha)$ uniformly for every compact set in $(0, \infty)$. Now, by (3.26) there exist two constants C_1 and C_2 such that

$$\int_0^\infty g(t)dt \leq \int_0^\infty e^{-st}C_1 e^{(r^\nu)^{\frac{1}{\nu}}t} C_2 e^{(r^\alpha)^{\frac{1}{\alpha}}t} dt = C_1 C_2 \int_0^\infty e^{(2r-s)t}dt. \tag{6.2}$$

The above inequality shows that the improper integral on the left hand side converges for $2r < s$. Further,

$$\begin{aligned}
\lim_{N\to\infty}\left(\int_0^\infty f_N(t)\mathrm{d}t\right) &= \lim_{N\to\infty}\left(\int_0^\infty e^{-st}\sum_{n=0}^{N} c_n\,\mathrm{d}t\right) \\
&= \lim_{N\to\infty}\left(\sum_{n=0}^{N}\int_0^\infty e^{-st}c_n\,\mathrm{d}t\right) \\
&= \lim_{N\to\infty}\left(\sum_{n=0}^{N}\int_0^\infty e^{-st}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)\mathrm{d}t\right) \\
&= \lim_{N\to\infty}\sum_{n=0}^{N}\sum_{k=0}^{n}\mathcal{L}\{a_k b_{n-k}; s\}.
\end{aligned}$$

$$(6.3)$$

Now,

$$\begin{aligned}\int_0^\infty \left(\lim_{N\to\infty} f_N(t)\right) \mathrm{d}t &= \int_0^\infty \left(\lim_{N\to\infty} e^{-st} \sum_{n=0}^{N} c_n\right) \mathrm{d}t \\ &= \int_0^\infty e^{-st} E_\nu((rt)^\nu) E_\alpha((rt)^\alpha)\, \mathrm{d}t \\ &= \mathcal{L}\{E_\nu((rt)^\nu) E_\alpha((rt)^\alpha)\, \mathrm{d}t; s\}.\end{aligned} \tag{6.4}$$

Therefore, Theorem 6.1 implies that

$$\lim_{N\to\infty} \left(\int_0^\infty f_N(t)\mathrm{d}t\right) = \int_0^\infty \left(\lim_{N\to\infty} f_N(t)\right) \mathrm{d}t$$

which shows

$$\sum_{n,m\geq 0} \mathcal{L}\left\{\frac{(rt)^{\nu m}}{\Gamma(m\nu+1)} \frac{(rt)^{\alpha n}}{\Gamma(\alpha n+1)}; s\right\} = \mathcal{L}\left\{\sum_{n,m\geq 0} \frac{(rt)^{\nu m}}{\Gamma(m\nu+1)} \frac{(rt)^{\alpha n}}{\Gamma(\alpha n+1)}; s\right\}$$

for $s > 2r$.

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