An optimal and low computational cost fractional Newton-type method for solving nonlinear equations*

Giro Candelario[a], Alicia Cordero[b,*], Juan R. Torregrosa[b], María P. Vassileva[a]

[a] *Área de Ciencias Básicas y Ambientales, Instituto Tecnológico de Santo Domingo, Santo Domingo, Dominican Republic*
[b] *Instituto de Matemáticas Multidisciplinar, Universitat Politècnica de València, València, Spain*

Abstract

In recent papers, some fractional Newton-type methods have been proposed by using the Riemann-Liouville and Caputo fractional derivatives in their iterative schemes, with order 2α or $1+\alpha$. In this manuscript, we introduce the Conformable fractional Newton-type method by using the so-called fractional derivative. The convergence analysis is made, proving its quadratic convergence, and the numerical results confirm the theory and improve the results obtained by classical Newton's method. Unlike previous fractional Newton-type methods, this one involves a low computational cost, and the order of convergence is at least quadratic.

Keywords: Nonlinear equations, Conformable fractional derivatives, Newton's method, Quadratic convergence, Computational cost, Stability

1. Introduction

Fractional calculus is not a recent area of research, as it dates from XVII century. However, in last years it has been shown as a fruitful source of tools for solving real problems with certain hereditary properties that are successfully modelled by means of this kind of derivatives [1]. Recently, two fractional Newton-type methods for solving nonlinear equations $f(x)=0$ were designed in [4] with iterative schemes:

$$x_{k+1} = x_k - \Gamma(\alpha+1)\frac{f(x_k)}{cD^{\alpha}_{a^+}f(x_k)},\ k=0,1,2,\ldots \tag{1}$$

and

$$x_{k+1} = x_k - \Gamma(\alpha+1)\frac{f(x_k)}{D^{\alpha}_{a^+}f(x_k)},\ k=0,1,2,\ldots \tag{2}$$

with Caputo and Riemann-Liouville derivatives respectively, being $\Gamma(\alpha+1)$ a damping parameter. The order of convergence of these methods was 2α, being α the order of derivative. Another two fractional Newton-type methods were proposed the last year in [5] as shown in the following iterative schemes:

$$x_{k+1} = x_k - \left(\Gamma(\alpha+1)\frac{f(x_k)}{cD^{\alpha}_{a^+}f(x_k)}\right)^{1/\alpha}, \quad k=0,1,2,\ldots \tag{3}$$

and

$$x_{k+1} = x_k - \left(\Gamma(\alpha+1)\frac{f(x_k)}{D^{\alpha}_{a^+}f(x_k)}\right)^{1/\alpha}, \quad k=0,1,2,\ldots \tag{4}$$

by using also Caputo and Riemann-Liouville derivatives respectively. The order of convergence of these methods was $1+\alpha$. When $\alpha=1$, we obtain the classical Newton-Raphson method for each case above.

The use of Caputo and Riemann-Liouville fractional derivatives require the evaluation of special functions as Gamma and Mittag-Leffler functions, which both involve a high computational cost to compute them. Theoretically, the order of

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*Corresponding author
Email address: acordero@mat.upv.es (Alicia Cordero)

convergence of these methods tends to be quadratic when $\alpha \approx 1$, but in practice, the approximated computational order of convergence (ACOC, see [6]) is linear if α is different from 1.

In order to design a new iterative scheme improving these aspects, we introduce the conformable fractional derivative, which is used in the iterative scheme of the fractional Newton-type method proposed in this paper.

The left conformable fractional derivative (see [2, 3]) starting from a of a function $f : [a, \infty) \longrightarrow \mathbb{R}$ of order $\alpha \in (0, 1]$, $\alpha, a, x \in \mathbb{R}$, is defined as

$$(T^a_\alpha f)(x) = \lim_{\varepsilon \longrightarrow 0} \frac{f(x + \varepsilon(x-a)^{1-\alpha}) - f(x)}{\varepsilon}. \tag{5}$$

If this limit exists, f is said to be α-differentiable. If, moreover, f is differentiable, then $(T^a_\alpha f)(x) = (x-a)^{1-\alpha} f'(x)$. If f is α-differentiable in (a, b), for some $b \in \mathbb{R}$, $(T^a_\alpha f)(a) = \lim_{x \to a^+} (T^a_\alpha f)(x)$.

The left conformable fractional derivative holds the property of non fractional derivative, $T^a_\alpha C = 0$, being C a constant. Conformable derivative is the most natural definition of fractional derivative, also, it does not require the evaluation of special functions, which involves a low computational cost compared with existing fractional Newton-type methods. The following result provides a Taylor power series of $f(x)$ with conformable fractional derivative.

Theorem 1 (Theorem 4.1, [3])**.** Let $f(x)$ be an infinitely α-differentiable function for $\alpha \in (0, 1]$, at the neighborhood of x_0 with conformable derivative starting from x_0. The fractional power series for $f(x)$ is:

$$f(x) = \sum_{k=0}^{\infty} \frac{(T^{x_0}_\alpha f)^{(k)}(x_0)(x-x_0)^{k\alpha}}{\alpha^k k!}, \quad x_0 < x < x_0 + R^{1/\alpha}, R > 0, \tag{6}$$

where $(T^{x_0}_\alpha f)^{(k)}(x_0)$ is the conformable fractional derivative applied k times.

From (6), another approach is obtained to provide a Taylor power series of $f(x)$, where the conformal derivative starts at a different point from where it is evaluated, what is more convenient for our purposes.

Theorem 2 (Theorem 4.1, [7])**.** Let $f(x)$ be an infinitely α-differentiable function for $\alpha \in (0, 1]$, at the neighborhood of a_1 with conformable derivative starting from a. The fractional power series for $f(x)$ is:

$$f(x) = f(a_1) + \frac{(T^a_\alpha f)(a_1)\delta_1}{\alpha} + \frac{(T^a_\alpha f)^{(2)}(a_1)\delta_2}{2\alpha^2} + R_2(x, a_1, a), \tag{7}$$

being $\delta_1 = H^\alpha - L^\alpha$, $\delta_2 = H^{2\alpha} - L^{2\alpha} - 2L^\alpha \delta_1, \ldots$, and $H = x - a$, $L = a_1 - a$.

It is easy to prove that $\delta_2 = \delta_1^2$, $\delta_3 = \delta_1^3$, etc. So, the Taylor power series (7) can be rewritten as

$$f(x) = f(a_1) + \frac{(T^a_\alpha f)(a_1)\delta_1}{\alpha} + \frac{(T^a_\alpha f)^{(2)}(a_1)\delta_1^2}{2\alpha^2} + R_2(x, a_1, a), \tag{8}$$

In next section, the conformable fractional Newton-type method is obtained from Taylor power expansion (8). In Section 3, the convergence analysis of the proposed scheme is made, proving its quadratical convergence. Sections 4 and 5 are respectively devoted to the numerical and stability tests. Finally, some conclusions are stated in Section 6.

2. Deduction of the method

To obtain a fractional Newton-type method from (8), let us regard the approximation of this Taylor power series to order one evaluated at the solution $\bar{x}$, as shown in the following expression:

$$f(x) \approx f(\bar{x}) + \frac{(T^a_\alpha f)(\bar{x})\delta_1}{\alpha}. \tag{9}$$

Knowing that $f(\bar{x}) = 0$, and $\delta_1 = H^\alpha - L^\alpha$, being $H = x - a$ and $L = a_1 - a$, $a_1 = \bar{x}$, expression (9) can be rewritten as

$$f(x) \approx \frac{(T^a_\alpha f)(\bar{x})}{\alpha} \left[(x-a)^\alpha - (\bar{x}-a)^\alpha\right]. \tag{10}$$

So, from $(\bar{x} - a)^\alpha$, $\bar{x}$ can be isolated as

$$\bar{x} \approx a + \left((x-a)^\alpha - \alpha \frac{f(x)}{(T^a_\alpha f)(\bar{x})}\right)^{1/\alpha}. \tag{11}$$

Considering the iterates x_k and x_{k+1} as approximations of the solution $\bar{x}$, we obtain the Conformable fractional Newton-type method as

$$x_{k+1} = a + \left((x_k - a)^\alpha - \alpha\frac{f(x_k)}{(T_\alpha^a f)(x_k)}\right)^{1/\alpha}, \quad k = 0, 1, 2, \ldots \tag{12}$$

Let us call this method TFN. In next Section the order of convergence of this method is proven. This is the first optimal fractional method according to Kung-Traub's conjecture (see [8]).

3. Convergence analysis

Theorem 3. Let $f : D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function in the interval D containing the zero $\bar{x}$ of $f(x)$. Let $(T_\alpha^a f)(x)$ be the conformable fractional derivative of $f(x)$ starting from a, with order α, for any $\alpha \in (0, 1]$. Let us suppose $(T_\alpha^a f)(x)$ is continuous and not null at $\bar{x}$. If an initial approximation x_0 is sufficiently close to $\bar{x}$, then the local order of convergence of the conformable fractional Newton-type method

$$x_{k+1} = a + \left((x_k - a)^\alpha - \alpha\frac{f(x_k)}{(T_\alpha^a f)(x_k)}\right)^{1/\alpha}, \quad k = 0, 1, 2, \ldots$$

is at least 2, being $0 < \alpha \leq 1$, and the error equation is

$$e_{k+1} = \alpha(\bar{x} - a)^{\alpha-1} C_2 e_k^2 + O\left(e_k^2\right),$$

being $C_j = \dfrac{1}{j!\alpha^j}\dfrac{(T_\alpha^a f)^{(j)}(\bar{x})}{(T_\alpha^a f)(\bar{x})}$ for $j = 2, 3, 4, \ldots$

Proof. By using the Taylor power expansion (8) of $f(x_k)$ around $\bar{x}$, and regarding $x_k = e_k + \bar{x}$,

$$\begin{aligned} f(x_k) &= (T_\alpha^a f)(\bar{x})\left[\delta_1 + C_2\delta_1^2 + C_3\delta_1^3\right] + O\left(e_k^4\right) \\ &= (T_\alpha^a f)(\bar{x})\Big[\left((e_k + \bar{x} - a)^\alpha - (\bar{x} - a)^\alpha\right) + C_2\left((e_k + \bar{x} - a)^\alpha - (\bar{x} - a)^\alpha\right)^2 \\ &\quad + C_3\left((e_k + \bar{x} - a)^\alpha - (\bar{x} - a)^\alpha\right)^3\Big] + O\left(e_k^4\right), \end{aligned}$$

being $C_j = \dfrac{1}{j!\alpha^j}\dfrac{(T_\alpha^a f)^{(j)}(\bar{x})}{(T_\alpha^a f)(\bar{x})}$ for $j = 2, 3, 4, \ldots$

Let us notice that the expansion of Newton's binomial theorem for fractional powers is given by

$$(x + y)^r = \sum_{k=0}^{\infty}\binom{r}{k}x^{r-k}y^k, \quad k \in \{0\} \cup \mathbb{N},$$

where the generalized binomial coefficient is (see [9])

$$\binom{r}{k} = \frac{\Gamma(r+1)}{k!\Gamma(r-k+1)}, \quad k \in \{0\} \cup \mathbb{N}.$$

Thus,

$$\begin{aligned} f(x_k) &= (T_\alpha^a f)(\bar{x})\Big[\left(\alpha(\bar{x} - a)^{\alpha-1}\right)e_k + \left(\frac{\alpha}{2}(\alpha - 1)(\bar{x} - a)^{\alpha-2} + \alpha^2(\bar{x} - a)^{2\alpha-2}C_2\right)e_k^2 \\ &\quad + \Big(\frac{\alpha}{6}(\alpha - 1)(\alpha - 2)(\bar{x} - a)^{\alpha-3} + \alpha^2(\alpha - 1)(\bar{x} - a)^{2\alpha-3}C_2 \\ &\quad + \alpha^3(\bar{x} - a)^{3\alpha-3}C_3\Big)e_k^3\Big] + O\left(e_k^4\right). \end{aligned}$$

Knowing that $(T_\alpha^a f)(x) = (x - a)^{1-\alpha}f'(x)$, and using again the generalized binomial theorem, the conformal derivative of $f(x_k)$ is developed as

$$\begin{aligned} (T_\alpha^a f)(x_k) &= (T_\alpha^a f)(\bar{x})\Big[\left(\alpha + \left(2\alpha^2(\bar{x} - a)^{\alpha-1}C_2\right)e_k\right. \\ &\quad + \Big(2\alpha(\alpha - 1)(\alpha - 2)(\bar{x} - a)^{-2} + \alpha^2(\alpha - 1)(\bar{x} - a)^{\alpha-2}C_2 + 3\alpha^3(\bar{x} - a)^{2\alpha-2}C_3 \\ &\quad - \alpha(1 - \alpha)^2(\bar{x} - a)^{-2} + \frac{\alpha^2}{2}(\alpha - 1)(\bar{x} - a)^{-2}\Big)e_k^2\Big] + O\left(e_k^3\right). \end{aligned}$$

	CFN method					TFN method				
α	$\bar{x}$	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1}-x_k\rvert$	iter	ACOC	$\bar{x}$	$\lvert f(x_{k+1})\rvert$	$\lvert x_{k+1}-x_k\rvert$	iter	ACOC
1	$\bar{x}_3$	$4.16\cdot 10^{-12}$	$3.47\cdot 10^{-8}$	11	2.00	$\bar{x}_3$	$4.16\cdot 10^{-12}$	$3.47\cdot 10^{-8}$	11	2.00
0.9	$\bar{x}_3$	$7.96\cdot 10^{-5}$	$8.11\cdot 10^{-9}$	68	0.98	$\bar{x}_3$	$6.18\cdot 10^{-13}$	$7.17\cdot 10^{-9}$	11	2.00
0.8	$\bar{x}_1$	$1.94\cdot 10^{-5}$	$9.99\cdot 10^{-9}$	123	0.99	$\bar{x}_3$	$4.18\cdot 10^{-12}$	$1.41\cdot 10^{-9}$	11	2.00
0.7	$\bar{x}_2$	$1.1\cdot 10^{-14}$	$9.94\cdot 10^{-9}$	389	1.00	$\bar{x}_3$	$1.6\cdot 10^{-12}$	$2.67\cdot 10^{-10}$	11	2.00
0.6	-	-	-	500	-	$\bar{x}_3$	$1.6\cdot 10^{-12}$	$4.81\cdot 10^{-11}$	11	2.00
0.5	-	-	-	500	-	$\bar{x}_3$	$2.26\cdot 10^{-12}$	$8.3\cdot 10^{-12}$	11	2.00
0.4	-	-	-	500	-	$\bar{x}_3$	$2.91\cdot 10^{-9}$	$8.89\cdot 10^{-7}$	10	2.01
0.3	-	-	-	500	-	$\bar{x}_3$	$4.62\cdot 10^{-10}$	$3.54\cdot 10^{-7}$	10	2.01
0.2	-	-	-	500	-	$\bar{x}_3$	$7.36\cdot 10^{-11}$	$1.38\cdot 10^{-7}$	10	2.00
0.1	-	-	-	500	-	$\bar{x}_3$	$2.26\cdot 10^{-12}$	$5.27\cdot 10^{-8}$	10	2.00

Table 1: CFN2 and TFN results for $f_1(x)$ with initial estimation $x_0=-2.2$

Then,

$$\frac{f(x_k)}{(T^a_\alpha f)(x_k)} = (\bar{x}-a)^{\alpha-1}e_k + \left(\frac{1}{2}(\alpha-1)(\bar{x}-a)^{\alpha-2} - \alpha(\bar{x}-a)^{2\alpha-2}C_2\right)e_k^2 + O\left(e_k^3\right),$$

and we have

$$(x_k-a)^\alpha - \alpha\frac{f(x_k)}{(T^a_\alpha f)(x_k)} = (\bar{x}-a)^\alpha + \alpha^2(\bar{x}-a)^{2\alpha-2}C_2e_k^2 + O\left(e_k^3\right).$$

Using again the generalized binomial theorem:

$$\left((x_k-a)^\alpha - \alpha\frac{f(x_k)}{(T^a_\alpha f)(x_k)}\right)^{1/\alpha} = \bar{x}-a+\alpha(\bar{x}-a)^{\alpha-1}C_2e_k^2 + O\left(e_k^3\right),$$

and using $x_{k+1}=e_{k+1}+\bar{x}$,

$$e_{k+1}+\bar{x} = a+\bar{x}-a+\alpha(\bar{x}-a)^{\alpha-1}C_2e_k^2+O\left(e_k^3\right).$$

So, the error equation is

$$e_{k+1} = \alpha(\bar{x}-a)^{\alpha-1}C_2e_k^2+O\left(e_k^3\right)$$

and the order of convergence is proven to be at least 2. □

In next section we make some numerical tests with several nonlinear equations.

4. Numerical tests

We compare the methods CFN described in (3) and TFN. It is important to point out that, in all tests, a comparison with the classical Newton-Raphson method is being made when $\alpha=1$. We use Matlab R2019b with double precision arithmetics, $|f(x_{k+1})|<10^{-8}$ or $|x_{k+1}-x_k|<10^{-8}$ as stopping criterium, and a maximum of 500 iterations. Nevertheless, only $|f(x_{k+1})|$ is shown in the tables, as well as the number of iterations, the reached root and the estimated computational order of convergence (ACOC, see [6]). For CFN method we use $a=0$ (as well it was in [5], the program made in [10] for computing of Gamma function, and the program provided by Igor Podlubny in Mathworks for the calculation of Mittag-Leffler function is used. For TFN method we consider $a=-10$ for each test.

In order to compare the performance of the methods, we use four test functions. The first one is $f_1(x)=-12.84x^6-25.6x^5+16.55x^4-2.21x^3+26.71x^2-4.29x-15.21$ with roots $\bar{x}_1=0.82366+0.24769i$, $\bar{x}_2=0.82366-0.24769i$, $\bar{x}_3=-2.62297$, $\bar{x}_4=-0.584$, $\bar{x}_5=-0.21705+0.99911i$ and $\bar{x}_6=-0.21705-0.99911i$. In Table 1, we can see that TFN method requires less iterations than CFN method for the same values of α, even less than classical Newton-Raphson method when $\alpha\leq 0.4$, whereas CFN method is not able to converge. It can also be observed that ACOC is 1 if $\alpha\neq 1$ in CFN2 method, whereas ACOC keeps being 2 or even greater if $\alpha\neq 1$ in TFN method.

The second test function is $f_2(x)=ix^{1.8}-x^{0.9}-16$, with roots $\bar{x}_1=2.90807-4.24908i$ and $\bar{x}_2=-3.85126+1.74602i$. In Table 2 we can observe a similar behavior as in Table 1: TFN method requires less iterations than CFN method for the same values of α, even less than classical Newton-Raphson method when $\alpha\leq 0.4$. We can also see that ACOC is 1 if $\alpha\neq 1$ in CFN2 method, whereas ACOC keeps being 2 or even greater than 2 if $\alpha\neq 1$ in TFN method. Our third test function is $f_3(x)=e^x-1$ with only real root $\bar{x}_1=0$. In Table 3 we can see that TFN scheme requires less iterations than CFN method for the same values of α with best error estimations. Again, ACOC keeps showing quadratic convergence

	CFN method					TFN method				
α	$\bar{x}$	$\|f(x_{k+1})\|$	$\|x_{k+1}-x_k\|$	iter	ACOC	$\bar{x}$	$\|f(x_{k+1})\|$	$\|x_{k+1}-x_k\|$	iter	ACOC
1	$\bar{x}_1$	$3.3\cdot10^{-14}$	$2.63\cdot10^{-7}$	8	2.00	$\bar{x}_1$	$3.3\cdot10^{-14}$	$2.63\cdot10^{-7}$	8	2.00
0.9	$\bar{x}_1$	$3.41\cdot10^{-7}$	$8.54\cdot10^{-9}$	53	0.98	$\bar{x}_1$	$8.73\cdot10^{-9}$	$1.27\cdot10^{-4}$	8	2.00
0.8	$\bar{x}_1$	$3.21\cdot10^{-6}$	$9.84\cdot10^{-9}$	209	1.00	$\bar{x}_1$	$8.44\cdot10^{-9}$	$1.22\cdot10^{-4}$	9	2.00
0.7	-	-	-	500	-	$\bar{x}_1$	$3.3\cdot10^{-14}$	$4.55\cdot10^{-9}$	12	2.00
0.6	-	-	-	500	-	$\bar{x}_2$	$3.66\cdot10^{-15}$	$3.42\cdot10^{-9}$	11	2.00
0.5	-	-	-	500	-	$\bar{x}_2$	$1.64\cdot10^{-11}$	$6.34\cdot10^{-6}$	8	2.00
0.4	-	-	-	500	-	$\bar{x}_2$	$9.57\cdot10^{-15}$	$2.12\cdot10^{-7}$	6	2.01
0.3	-	-	-	500	-	$\bar{x}_2$	$1.85\cdot10^{-14}$	$4.65\cdot10^{-9}$	6	2.00
0.2	-	-	-	500	-	$\bar{x}_2$	$2.1\cdot10^{-10}$	$2.33\cdot10^{-5}$	5	1.99
0.1	-	-	-	500	-	$\bar{x}_2$	$1.07\cdot10^{-11}$	$5.22\cdot10^{-6}$	5	2.02

Table 2: CFN2 and TFN results for $f_2(x)$ with initial estimation $x_0 = 0.5$

	CFN method					TFN method				
α	$\bar{x}$	$\|f(x_{k+1})\|$	$\|x_{k+1}-x_k\|$	iter	ACOC	$\bar{x}$	$\|f(x_{k+1})\|$	$\|x_{k+1}-x_k\|$	iter	ACOC
1	$\bar{x}_1$	0	$1.52\cdot10^{-8}$	4	2.00	$\bar{x}_1$	0	$1.52\cdot10^{-8}$	4	2.00
0.9	$\bar{x}_1$	$2.61\cdot10^{-9}$	$2.52\cdot10^{-8}$	8	1.00	$\bar{x}_1$	$1.78\cdot10^{-15}$	$1.62\cdot10^{-8}$	4	2.00
0.8	$\bar{x}_1$	$9.56\cdot10^{-9}$	$4.43\cdot10^{-8}$	10	1.00	$\bar{x}_1$	$1.78\cdot10^{-15}$	$1.73\cdot10^{-8}$	4	2.00
0.7	$\bar{x}_1$	$4.6\cdot10^{-9}$	$1.36\cdot10^{-8}$	13	1.00	$\bar{x}_1$	0	$1.84\cdot10^{-8}$	4	2.00
0.6	$\bar{x}_1$	$9.8\cdot10^{-9}$	$2.07\cdot10^{-8}$	15	1.00	$\bar{x}_1$	$1.78\cdot10^{-15}$	$1.95\cdot10^{-8}$	4	2.00
0.5	$\bar{x}_1$	$7.52\cdot10^{-9}$	$1.21\cdot10^{-8}$	18	1.00	$\bar{x}_1$	$1.78\cdot10^{-15}$	$2.08\cdot10^{-8}$	4	2.00
0.4	$\bar{x}_1$	$7.64\cdot10^{-9}$	$9.71\cdot10^{-9}$	21	1.00	$\bar{x}_1$	0	$2.21\cdot10^{-8}$	4	2.00
0.3	$\bar{x}_1$	$9.75\cdot10^{-9}$	$1\cdot10^{-8}$	24	1.00	$\bar{x}_1$	0	$2.34\cdot10^{-8}$	4	2.00
0.2	$\bar{x}_1$	$8.17\cdot10^{-9}$	$6.89\cdot10^{-9}$	28	1.00	$\bar{x}_1$	$3.55\cdot10^{-15}$	$2.49\cdot10^{-8}$	4	2.00
0.1	$\bar{x}_1$	$9.51\cdot10^{-9}$	$6.64\cdot10^{-9}$	32	1.00	$\bar{x}_1$	$5.33\cdot10^{-15}$	$2.64\cdot10^{-8}$	4	2.00

Table 3: CFN2 and TFN results for $f_3(x)$ with initial estimation $x_0 = 0.2$

if $\alpha \neq 1$ in TFN method. The last function is $f_4(x) = \sin 10x - 0.5x + 0.2$ with real roots $\bar{x}_1 = -1.4523$, $\bar{x}_2 = -1.3647$, $\bar{x}_3 = -0.87345$, $\bar{x}_4 = -0.6857$, $\bar{x}_5 = -0.27949$, $\bar{x}_6 = -0.021219$, $\bar{x}_7 = 0.31824$, $\bar{x}_8 = 0.64036$, $\bar{x}_9 = 0.91636$, $\bar{x}_{10} = 1.3035$, $\bar{x}_{11} = 1.5118$, $\bar{x}_{12} = 1.9756$ and $\bar{x}_{13} = 2.0977$. In Table 4 can also be observed that TFN method requires less iterations than CFN method for the same values of α. We can see that ACOC keeps being 2 if $\alpha \neq 1$ in TFN method, in contrast with the performance of CFN scheme. In next section we analyze the dependence on initial estimates of CFN and TFN methods on the same test functions.

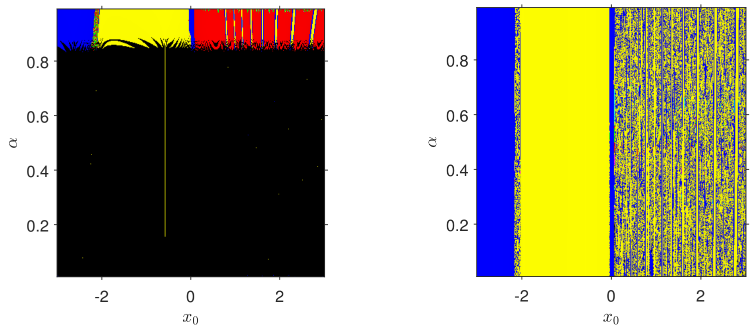
5. Numerical stability

In order to study the stability of fractional Newton-type methods tested in Section 4 we analyze the dependence on initial estimates by using convergence planes as defined in [11].

	CFN method					TFN method				
α	$\bar{x}$	$\|f(x_{k+1})\|$	$\|x_{k+1}-x_k\|$	iter	ACOC	$\bar{x}$	$\|f(x_{k+1})\|$	$\|x_{k+1}-x_k\|$	iter	ACOC
1	$\bar{x}_{12}$	$1.94\cdot10^{-11}$	$7.01\cdot10^{-7}$	4	1.99	$\bar{x}_{12}$	$1.94\cdot10^{-11}$	$7.01\cdot10^{-7}$	4	1.99
0.9	$\bar{x}_{12}$	$2.51\cdot10^{-7}$	$8.57\cdot10^{-9}$	35	0.98	$\bar{x}_{12}$	$1.92\cdot10^{-11}$	$6.98\cdot10^{-7}$	4	1.99
0.8	$\bar{x}_{12}$	$1.86\cdot10^{-6}$	$9.65\cdot10^{-9}$	125	0.99	$\bar{x}_{12}$	$1.9\cdot10^{-11}$	$6.96\cdot10^{-7}$	4	1.99
0.7	$\bar{x}_{12}$	$1.06\cdot10^{-5}$	$9.99\cdot10^{-9}$	426	1.00	$\bar{x}_{12}$	$1.89\cdot10^{-11}$	$6.93\cdot10^{-7}$	4	1.99
0.6	-	-	-	500	-	$\bar{x}_{12}$	$1.87\cdot10^{-11}$	$6.9\cdot10^{-7}$	4	1.99
0.5	-	-	-	500	-	$\bar{x}_{12}$	$1.85\cdot10^{-11}$	$6.87\cdot10^{-7}$	4	1.99
0.4	-	-	-	500	-	$\bar{x}_{12}$	$1.84\cdot10^{-11}$	$6.85\cdot10^{-7}$	4	1.99
0.3	-	-	-	500	-	$\bar{x}_{12}$	$1.82\cdot10^{-11}$	$6.82\cdot10^{-7}$	4	1.99
0.2	-	-	-	500	-	$\bar{x}_{12}$	$1.81\cdot10^{-11}$	$6.79\cdot10^{-7}$	4	1.99
0.1	-	-	-	500	-	$\bar{x}_{12}$	$1.79\cdot10^{-11}$	$6.76\cdot10^{-7}$	4	1.99

Table 4: CFN2 and TFN results for $f_4(x)$ with initial estimation $x_0 = 2$

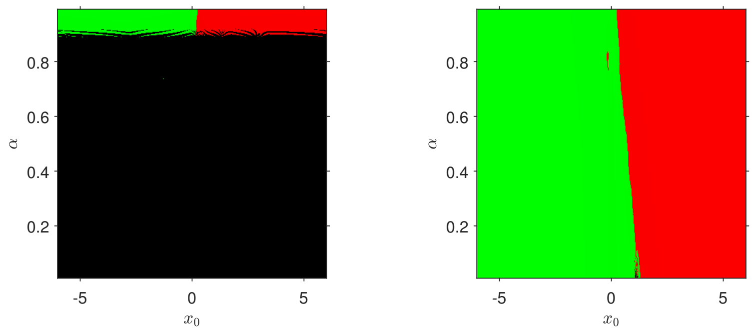
To construct the convergence planes, we regard the initial estimates in horizontal axis and values of $\alpha \in (0, 1]$ in vertical axis. Each color represents a different solution found, and it is painted in black when no solution was found. Each plane is made with a 400×400 grid, a maximum of 500 iteration, and tolerance of 0.001.
In Figures 1, 2 and 4 we can see that TFN method has a much higher percentage of convergence than CFN2 method. In case of Figure 3, CFN method has a higher percentage of convergence than TFN method due to the values of initial estimates used are very close to $a = -10$; the behavior of TFN can be improved by regarding lower values of a.



(a) CFN, $-3 \leq x_0 \leq 3$, 13.73% convergence

(b) TFN, $-3 \leq x_0 \leq 3$, 99.92% convergence

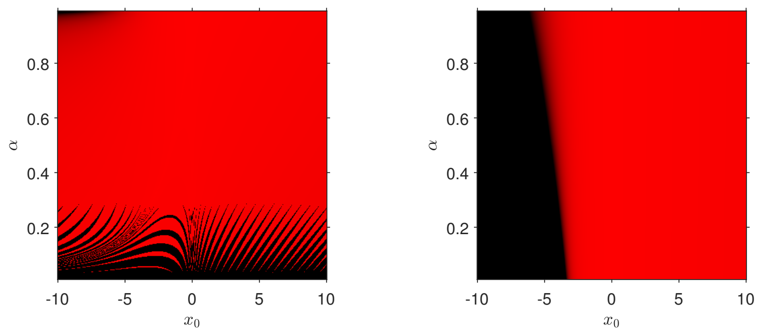
Figure 1: Convergence planes of CFN and TFN on $f_1(x)$



(a) CFN, $-6 \leq x_0 \leq 6$, 8.97% convergence

(b) TFN, $-6 \leq x_0 \leq 6$, 99.74% convergence

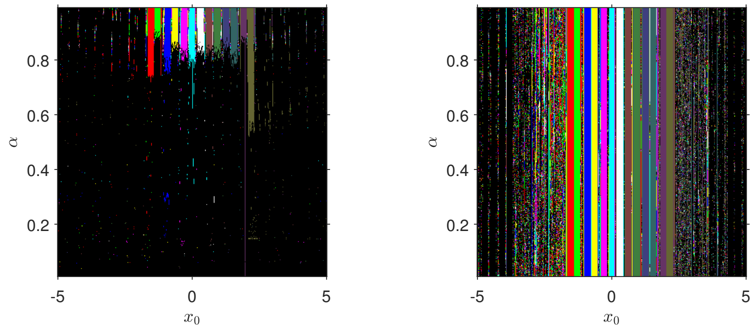
Figure 2: Convergence planes of CFN and TFN on $f_2(x)$



(a) CFN, $-10 \leq x_0 \leq 10$, 85.54% convergence

(b) TFN, $-10 \leq x_0 \leq 10$, 73.12% convergence

Figure 3: Convergence planes of CFN and TFN on $f_3(x)$



(a) CFN, $-5 \leq x_0 \leq 5$, 8.69% convergence (b) TFN, $-5 \leq x_0 \leq 5$, 58.21% convergence

Figure 4: Convergence planes of CFN and TFN on $f_4(x)$

6. Concluding Remarks

The first optimal fractional Newton-type method was designed by using Conformable derivative. The fractional derivative used has the most natural definition, so, in this method the evaluation of special functions is not required. This involves a low computational cost compared with the existing fractional Newton-type methods. Also, the order of convergence of this method is quadratic, unlike the existing ones. Numerical tests were made, and the dependence on initial estimates was analyzed, confirming the theoretical results. It can be concluded that this method shows a better numerical behavior than kind of fractional Newton-type methods previously proposed, even than classical Newton-Raphson method in some cases. It was also observed that is possible to obtain both, real and complex roots, with real initial estimates, and that is possible to obtain different roots not only by choosing a different initial estimate, but also by choosing a different value of α.

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