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A front-fixing method for American option pricing on zero-coupon bond under the Hull and White model

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A new efficient numerical method is proposed for valuation of American option on zero-coupon bond using Hull and White model. By applying the front-fixing transformation suggested by Holmes and Yang, the original free boundary problem is transformed into a new fixed boundary partial differential equation (PDE) problem, where the optimal stopping boundary is one of the unknowns of the problem. The numerical finite difference scheme for the transformed problem is constructed. Stability and convergence rate is studied empirically. Numerical simulation of the computation of both the option price and the optimal stopping boundary are illustrated with examples and the comparison with the Hull and White tree method.

KEYWORDS
American option pricing, finite difference method, front-fixing method, numerical simulations, zero-coupon bond

MSC CLASSIFICATION
60G40; 65N06; 65N12

1 | INTRODUCTION

In recent decades, new interest rate securities, such as bond futures, options on bonds and bonds with option features, become more and more popular. The payoff of such products and, consequently, the value of these derivatives strongly depend on the interest rates. Thus, it is necessary to design liable models and effective numerical algorithms to provide fast and fair price of the interest rate security.[1]

There are several empirical models for the pricing interest rate derivatives; however, their implementations are quite slow due to the large number of discrete tenor rates evaluations. Moreover, these techniques are not adequate for some securities with an early optimal exercise possibility, such as American option on bonds. Thus, theoretical models become more appealing, for instance, the Hull and White (HW) model[2] that has an analytical expression for the bond value and for the European bond option price.[3] The most widely used methods for evaluating interest rate securities are binomial and trinomial trees.[4,5]

American option has an advantage of the early exercise opportunity, which leads to a free-boundary problem. Thus, the valuation model can be written as a linear complementarity problem (LCP), which can be solved numerically, as proposed in Falcó et al,[6] by applying LU decomposition and a modified backward substitution with a projection. Another very common method for free-boundary problems is the penalty method, which transforms the LCP into a nonlinear partial differential equation (PDE) problem by adding a compensating penalty term. This approach together with the

finite volume method for American options on bonds has been used in previous studies.[7,8] In ShuJin and ShengHong,[9] the integral representation of the early exercise rate is derived. A finite element method (FEM) based on a new formulation of the original free-boundary problem has been proposed in Allegretto et al.[10]

An alternative technique for treating free boundaries is the front-fixing transformation. Instead of the free-boundary domain, a transformed nonlinear PDE is posed in a fixed domain. This method has the advantage that the early exercise boundary and the option price can be computed simultaneously after immobilizing the boundary using a transformation; see Holmes et al[11] and references therein. In Holmes and Yang,[12] a linear transformation instead of the typical Landau type transformation[13] is used; the solution is computed by using a FEM combined with a Newton method for the computation of the early exercise boundary.

In the present paper, we propose a combination of linear front-fixing transformation[12] with an explicit finite difference method (FF-FDM). The FEM is advisable when the domain is geometrically complex. In other cases, the FDM is easier to implement and computationally less expensive when the method is explicit because it avoids the use of iterative methods and collateral drawbacks such as how to stop in the iterative process and others.

The remaining part of the paper proceeds as follows: In Section 2, the HW zero-coupon model is described and American option on zero-coupon bond is posed paying special attention to the boundary conditions. Numerical algorithm for the problem is proposed in Section 3. Section 4 provides numerical results and comparison with the HW tree method. Stability and numerical convergence rate are studied in Section 4 as well. The concluding remarks are given is Section 5.

2 | AMERICAN PUT OPTION ON ZERO-COUPON BOND MODEL

In this paper, we consider a zero-coupon bond with the maturity period T, such that at any time moment $t < T$, the value of the bond denoted by $P(t, T)$ is defined as follows:

$$P(t, T) = \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(u)du\right)\right], \tag{1}$$

where $\mathbb{Q}$ is a martingale measure and $r(t)$ is a short rate at moment t. In the literature, there exist various models concerning different underlying stochastic processes $r(t)$. One of the most popular models is the Vasicek model,[14] where the short rate follows the elastic random walk (Ornstein–Uhlenbeck process). In present paper, the restricted HW model[2] is considered, which is an extension of the Vasicek model in the sense that the drift is some deterministic function on time, that is,

$$dr_t = (\theta_t - ar_t)dt + \sigma dW_t, \tag{2}$$

where $a > 0, \sigma > 0$ are some constant parameters, θ_t is a time-dependent function and W_t is a geometric Brownian motion. An explicit expression of θ_t is proposed in Brigo and Mercurio[15]:

$$\theta(t) = \frac{\partial f(0, t)}{\partial t} + af(0, t) + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right), \tag{3}$$

where $f(0, t)$ is the forward rate observed on the market defined by

$$f(0, t) = -\frac{\partial \ln P(0, t)}{\partial t}. \tag{4}$$

For the HW model, the analytical expression for the bond value is known[2]:

$$P(r; t, T) = A(t, T)\exp(-B(t, T)r), \tag{5}$$

where

$$B(t, T) = \frac{1}{a}\left(1 - e^{-a(T-t)}\right), \tag{6}$$

$$A(t, T) = \frac{P(0, T)}{P(0, t)}\exp\left(B(t, T)f(0, t) - \frac{\sigma^2}{4a}B^2(t, T)(1 - e^{-2at})\right). \tag{7}$$

Now, let us consider a European put option written on a zero-coupon bond $P(r; t, T)$ (5), with the maturity S and the strike price K. Under assumptions of the HW short rate model and no arbitrage market, the option value $V(r, t)$ is described by the following PDE:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial r^2} + (\theta(t) - ar(t)) \frac{\partial V}{\partial r} - r(t)V = 0, \ r \in \mathbb{R}, \ 0 < t < S, \tag{8}$$

subject to the terminal and boundary conditions

$$V(r, S) = g(r, S), \ \lim_{r \to -\infty} V(r, t) = 0, \ \lim_{r \to -\infty} V(r, t) = g(r, t), \tag{9}$$

where $g(r, t)$ is a payoff function $g(r, t) = [K - P(r; t, T)]^+$.

American style options give the opportunity to exercise the option at any time before the maturity, which leads to a free boundary problem. Denoting by r^* the optimal exercise boundary, that is, the value of $r(t)$ such that for $0 \le t < S \le T$, one gets $V(r, t) > g(r, t)$ for $r < r^*(t)$. Hence, the price of the American put option written in a zero-coupon bond is the solution of the following free-boundary PDE problem:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial r^2} + (\theta(t) - ar(t)) \frac{\partial V}{\partial r} - r(t)V = 0, \ -\infty < r < r^*(t), \ 0 < t < S, \tag{10}$$

$$V(r, S) = g(r, S), \ \lim_{r \to -\infty} V(r, t) = 0, \tag{11}$$

$$V(r^*, t) = g(r^*(t), t), \ \frac{\partial V}{\partial r}(r^*(t), t) = \frac{\partial g}{\partial r}(r^*(t), t). \tag{12}$$

Equation (10) is written for $r < r^*(t)$, which is not known a priori, and thus, additional boundary conditions (12) are imposed to determine the optimal exercise boundary $r^*(t)$.

3 | FRONT-FIXING NUMERICAL METHOD

A linear front-fixing transformation is applied to the free boundary problem (10–12). First of all, we truncate the computational domain: Following Holmes and Yang,[12] let L be a positive number large enough to guarantee that at any fixed time moment t, the bond price P is greater than the strike K, that is, that $g(r, t) = 0$ for $r < r^*(t) - L$. Thus, the infinite domain for r can be reduced to the finite interval $[-L + r^*(t), r^*(t)]$. Hence, PDE problem (10) is to be solved in a bounded time-dependent domain with the unknown optimal exercise boundary $r^*(t)$. Further, the moving interval $[-L + r^*, r^*]$ is transformed to the fixed $[0, L]$ by introducing a new variable[12]:

$$x = r + L - r^*. \tag{13}$$

At the terminal moment $t = S$, the option price is defined by the payoff function (11) and has to be calculated at the initial moment $t = 0$, which corresponds to the moment of signing the option. Using the time inverse $\tau = S - t$, the problem (10–12) can be written as an initial value problem.

Finally, let us denote the inverse optimal exercise boundary by $r_f(\tau) = r^*(t)$ and the transformed option value by $U(x, \tau) = V(r, t)$, then the original PDE (10) takes the form

$$\frac{\partial U}{\partial \tau} = \frac{\sigma^2}{2} \frac{\partial^2 U}{\partial x^2} + \left(\varphi(\tau) - a(x - L + r_f) + r_f' \right) \frac{\partial U}{\partial x} - (x - L + r_f)U, \tag{14}$$

for $(x, \tau) \in [0, L] \times (0, S)$, where $\varphi(\tau) = \theta(t)$.

Transformed payoff function is defined as follows:

$$G(x, \tau) = g(r, t) = [K - P(r; S - \tau, T)]^+. \tag{15}$$

Boundary and terminal conditions (11) are transformed as follows:

$$U(x, 0) = G(x, 0), \ U(0, \tau) = 0, \ U(L, \tau) = G(L, \tau), \ \frac{\partial U}{\partial x}(L, \tau) = \frac{\partial G}{\partial x}(L, \tau). \tag{16}$$

Nonlinear PDE problem (14–16) in the fixed domain $[0, L] \times (0, S)$ has no analytical solution; thus, a numerical method should be employed. In this paper, an explicit FDM is proposed. This method is easy to implement and does not require many computational resources. Although the method is conditionally stable and small enough time step should be chosen to preserve the stability, in Section 4, we show that the method is fast and efficient.

3.1 | Finite difference method

In this subsection, we describe the explicit centred in space FDM for the problem (14–16). Let us introduce the uniform grid with the step sizes:

$$h = \frac{L}{M}, \ k = \frac{S}{N}. \tag{17}$$

Thus, the computational nodes are

$$x_j = jh, \ j = 0, \ \dots, M; \ \tau^n = kn, \ n = 0, \ \dots, N. \tag{18}$$

Let us introduce the following notation for the approximated values at the nodes of the grid: $U_j^n \approx U(x_j, \tau^n)$ and $r_f^n \approx r_f(\tau^n)$. We use the centred finite difference of the second order for the approximation of the spatial derivatives and the forward difference of the first order for the temporal derivatives, which results in the following system of nonlinear algebraic equations:

$$
\begin{aligned}
\frac{U_j^{n+1} - U_j^n}{k} &= \frac{\sigma^2}{2} \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2} - \left(x_j - L + r_f^n \right) U_j \\
&+ \left(\varphi(\tau^n) - a(x_j - L + r_f^n) + \frac{r_f^{n+1} - r_f^n}{k} \right) \frac{U_{j+1}^n - U_{j-1}^n}{2h}, \ j = 1, \ \dots, M-1; \ n = 0, \ \dots, N-1.
\end{aligned}
\tag{19}
$$

Equations (19) can be written in the following form, which defines explicitly U_j^{n+1}:

$$U_j^{n+1} = a_j^n U_{j+1}^n + b_j^n U_j^n + c_j^n U_{j-1}^n, \ j = 1, \ \dots, M-1; \ n = 0, \ \dots, N-1, \tag{20}$$

where

$$a_j^n = \frac{k}{2} \left[\frac{\sigma^2}{h^2} + \frac{\zeta_j^n}{h} \right], \tag{21}$$

$$b_j^n = 1 - k \left[\frac{\sigma^2}{h^2} + \left(x_j - L + r_f^n \right) \right], \tag{22}$$

$$c_j^n = \frac{k}{2} \left[\frac{\sigma^2}{h^2} - \frac{\zeta_j^n}{h} \right], \tag{23}$$

$$\zeta_j^n = \varphi(\tau^n) - a(x_j - L + r_f^n) + \frac{r_f^{n+1} - r_f^n}{k}. \tag{24}$$

Note that the coefficients a_j^n, b_j^n and c_j^n defined by (21–24) are time dependent, because they contain the transformed optimal exercise boundary r_f^n and r_f^{n+1}. In order to calculate these coefficients at each time moment τ^n, r_f^{n+1} should be computed beforehand by using the initial and boundary conditions.

Initial condition (16) is discretised as follows for $j = 0, \ \dots, M$,

$$U_j^0 = G(x_j - L + r_f^0, S). \tag{25}$$

Boundary conditions (16) for $n = 1, \ldots, N$, are written as

$$U_0^n = 0, \quad U_M^n = G(r_f^n, \tau^n), \tag{26}$$

$$\frac{U_{M+1}^n - U_{M-1}^n}{2h} = \frac{G(r_f^n + h, \tau^n) - G(r_f^n - h, \tau^n)}{2h}. \tag{27}$$

Equation (27) is used to approximate the value of option at the point $x_{m+1} = L + h$ situated outside the computational domain:

$$U_{M+1}^n = U_{M-1}^n + \left(G(r_f^n + h, \tau^n) - G(r_f^n - h, \tau^n) \right). \tag{28}$$

Denoting

$$\Delta G^n = G(r_f^n + h, \tau^n) - G(r_f^n - h, \tau^n), \tag{29}$$

one gets the approximation of U_{M+1}^n in the following form:

$$U_{M+1}^n = U_{M-1}^n + \Delta G^n. \tag{30}$$

This expression allows us to write Equation (20) at the point x_M by using the values only in the computational domain:

$$U_M^{n+1} = a_M^n (U_{M-1} + \Delta G^n) + b_M^n U_M^n + c_M^n U_{M-1}^n. \tag{31}$$

From (21) and (24), $a_M^n + c_M^n = \frac{\sigma^2 k}{h^2}$. Thus, reordering the terms in (31), one gets

$$U_M^{n+1} = \frac{\sigma^2 k}{h^2} U_{M-1}^n + b_M^n U_M^n + a_M^n \Delta G^n. \tag{32}$$

Note that a_M^n contains unknown value r_f^{n+1}; thus, an additional condition is necessary to define both the optimal stopping boundary r_f^{n+1} and U_M^{n+1}.

From (16), $U(L, \tau) = G(r_f(\tau), \tau) = K - P(r^*; t, T)$. Deriving this expression with respect to τ, the following equation is obtained:

$$\frac{\partial U}{\partial \tau}(L, \tau) = \frac{\partial P}{\partial t}(r^*; t, T) + \frac{\partial P}{\partial r^*}(r^*; t, T) r_f'(\tau). \tag{33}$$

Forward finite difference approximation of (33) at the node (x_M, τ^n) takes the form

$$\frac{U_M^{n+1} - U_M^n}{k} = \frac{P(r^*; S - \tau^n, T) - P(r^*; S - \tau^{n+1}, T)}{k}$$
$$+ \frac{P(r^* + h; S - \tau^n, T) - P(r^* - h; S - \tau^n, T)}{2h} \frac{r_f^{n+1} - r_f^n}{k}, \tag{34}$$

that leads to the following

$$U_M^{n+1} = U_M^n + P(r^*; S - \tau^n, T) - P(r^*; S - \tau^{n+1}, T)$$
$$+ \frac{P(r^* + h; S - \tau^n, T) - P(r^* - h; S - \tau^n, T)}{2h} \left(r_f^{n+1} - r_f^n \right). \tag{35}$$

Finally, equalling the right-hand sides of (32) and (35), the explicit expression for the optimal stopping boundary r_f^{n+1} is found:

$$r_f^{n+1} = r_f^n + \frac{h}{\Delta G^n} \left(P(r^*; S - \tau^n, T) - P(r^*; S - \tau^{n+1}, T) \right)$$
$$+ \frac{k}{\Delta G^n} \left(\frac{\sigma^2}{h} (U_M^n - U_{M-1}^n) + h U_M^n r_f^n \right) - \frac{k}{2} \left(\frac{\sigma^2}{h} + \varphi(\tau^n) - a r_f^n \right). \tag{36}$$

From boundary conditions (26), the optimal stopping boundary r_f^{n+1} might as well be found as a solution of a nonlinear equation:

$$F(r_f^{n+1}) = U_M^{n+1}(r_f^{n+1}) - G(r_f^{n+1}, \tau^n) = 0. \tag{37}$$

This nonlinear equation can be solved by an iterative method,[16] which may increase significantly the computational time.

4 | RESULTS AND DISCUSSION

The proposed FF-FDM method is implemented by using MatLAB2018a. In this section, the numerical results are reported and discussed. We provide empirical study of stability and convergence rate of the method. The results are compared with the well-known Hull and White tree (HWT).[17,18] All the simulations are performed with MatLAB2018a on MacBook Pro with processor 2.5-Ghz Intel i5.

In our simulations, we use estimated yield curve for OIS EUR (EUONIA) data that were downloaded on 24 May 2019, from the Thomson Reuters Eikon for maturities from 3 months to 50 years. In Table 1, we give the data, and in Figure 1, the yield curve is plotted. The continuous function for the real data is calculated by the cubic spline.

Example 1. Let us consider an American option on zero-coupon bond $P(r; t, T)$, where $T = 8$, with the bond face value $B = 1$. The maturity of the option is $S = 5$ years and the strike price $K = 0.97$. The rest of constant parameters for the model are $a = 1\%$ and $\sigma = 0.5\%$.

The numerical solution is calculated by the proposed front-fixing finite difference method (FF-FDM) with spatial step size $h = 10^{-2}$ and $k = 10^{-4}$. The front-fixing transformation allows us to calculate the optimal exercise boundary $r_f(\tau)$ simultaneously with the option price by the explicit formula (36). The optimal exercise boundary $r_f(\tau)$ for $\tau \in [0, S]$ is plotted in Figure 2. An important point of interest in American option pricing is the optimal exercise boundary close to maturity, which corresponds to τ close to 0 due to the time inverse. The calculated values of $r_f(\tau)$ for $\tau \in [0, 0.05]$ are presented in Figure 3. Observing both plots given in Figures 2 and 3, one can conclude that the optimal exercise boundary calculated by the explicit formula is monotone increasing concave function with respect to time to maturity τ, which agrees with the results by Holmes and Yang.[12]

TABLE 1 EUR OIS on 24 May 2019 with maturities from 3 months to 50 years

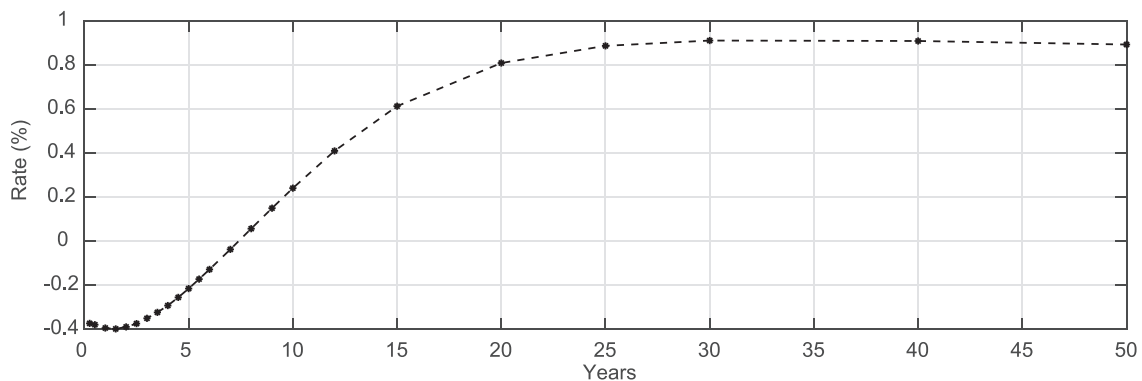| Maturity (year)	Rate (%)	Maturity (year)	Rate (%)	Maturity (year)	Rate (%)
0.25	−0.374	4	−0.293	10	0.240
0.5	−0.380	4.5	−0.256	12	0.409
1	−0.395	5	−0.216	15	0.612
1.5	−0.399	5.5	−0.173	20	0.808
2	−0.390	6	−0.129	25	0.886
2.5	−0.375	7	−0.038	30	0.910
3	−0.351	8	0.056	40	0.908
3.5	−0.324	9	0.149	50	0.892

FIGURE 1 Interpolated by the cubic spline yield curve for data in Table 1

Numerical results for the American option value for Example 1 is presented in Figure 4. The solution is plotted in original variables r and t; thus, the terminal conditions correspond to $t = S$ (broken line). The option value $V(r, t) \geq 0$; thus, the proposed FF-FDM preserves the non-negativity of the numerical solution. Moreover, the option value is non-decreasing function with respect to time to maturity and non-decreasing in space, as expected.

We compare the proposed FF-FDM with the well-known HWT.[17,18] The option price at the moment $\tau = S$ that corresponds to the moment of writing the option is reported in Table 2, as well as CPU time for both methods. The relative error of both methods is plotted in Figure 5, which shows similar speed of convergence. Corresponding CPU time is presented
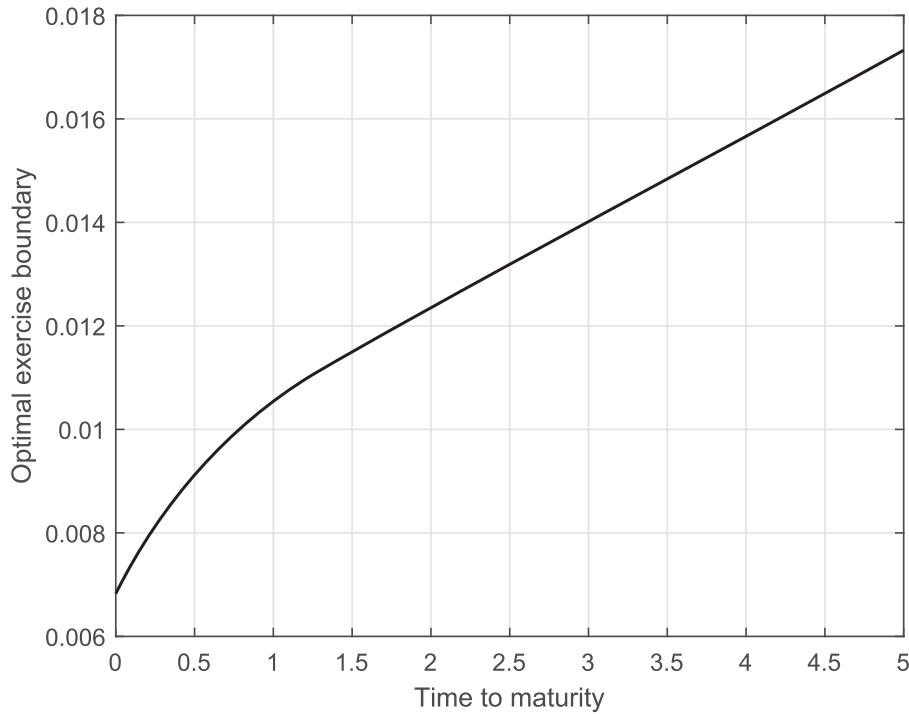

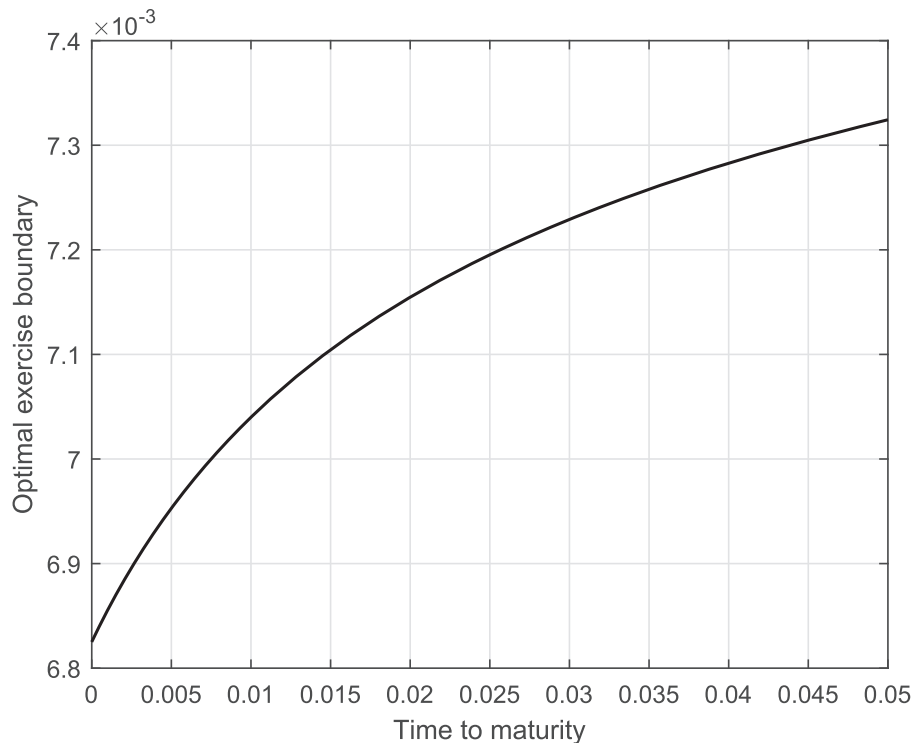**FIGURE 2** Optimal exercise boundary $r_f(\tau)$ for Example 1

FIGURE 3 Computed values $r_f(\tau)$ for $\tau \in [0, 0.05]$ of the optimal exercise boundary $r_f(\tau)$ for Example 1

FIGURE 4 American option value $V(r, t)$ calculated by the proposed front-fixing transformation-finite difference method (FF-FDM) method for Example 1
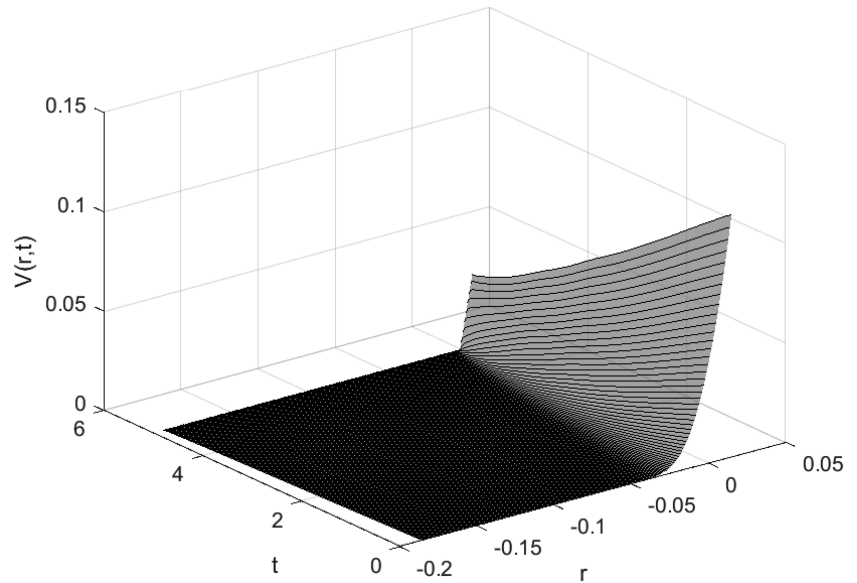

TABLE 2 Numerical solution by the proposed FF-FDM method and HWT method[18] for Example 1 and corresponding CPU time obtained with fixed spatial step size $h = 0.01$ and various temporal step size k

	FF- FDM		HWT	
k	**Price**	**CPU time (s)**	**Price**	**CPU time (s)**
0.5000	0.9012	0.0951	0.8814	0.0433
0.1000	0.8990	0.2286	0.8767	0.0935
0.0500	0.8590	0.7532	0.8459	0.1865
0.0250	0.8489	1.2825	0.8467	0.2001
0.0100	0.8379	2.5183	0.8464	1.9470
0.0050	0.8339	3.2447	0.8387	5.9920
0.0025	0.8371	8.6183	0.8373	39.7846
0.0020	0.8371	12.9246	0.8371	65.8110
0.0010	0.8371	16.1918	0.8369	93.7910

Abbreviations: FF-FDM, front-fixing transformation-finite difference method; HWT, Hull and White tree.

in Figure 6. For the simulations, the fixed spatial step size $h = 0.01$ is chosen, and the temporal step size k is varying. Observing the results, one can found that both methods present an empirical convergence when the time step size k is decreasing. However, the computational time of the proposed method is lower than the computational time of the HWT for small step sizes. Thus, the efficiency of the proposed method comparing with HWT is shown.

It is known from the literature that explicit FDM is conditionally stable. Thus, the empirical study of stability is necessary. The idea of the empirical study of stability is to consider the parabolic mesh ratio $C = k/h^2$, and varying this value establishes a maximum value to guarantee the stability of proposed method.

Iterative process of the search of maximum parabolic mesh ratio for the stable numerical solution starts with $C = 1$, that is for fixed $h = 10^{-2}$ results in $k = 10^{-4}$. Increasing C up to 495, we still have a stable solution. However, for $C = 500$, the stability is broken. Thus, it is recommended to choose the temporal step size $k < Ch^2$, where $C \leq 495$. It is important to note that although k satisfies the stability condition increasing k lead to less accurate solution (see Table 2). Thus, it is still recommended to fix k of the same order as h^2 to guarantee the stable and accurate numerical solution.

In order to complete the study of proposed FF-FDM method, the numerical convergence rate is analysed. First, let us define the infinite-norm error as

$$\epsilon(k, h) = \left\| V_k^h - V_{ref} \right\|_{h, \infty}, \tag{38}$$

where V_h^k is the approximated numerical solution and V_{ref} is the reference value. Because analytical solution for the American option pricing problem is unknown, we consider the numerical solution on the refined grid ($h = 0.001$, $k = 0.0001$) as the reference value of the solution.
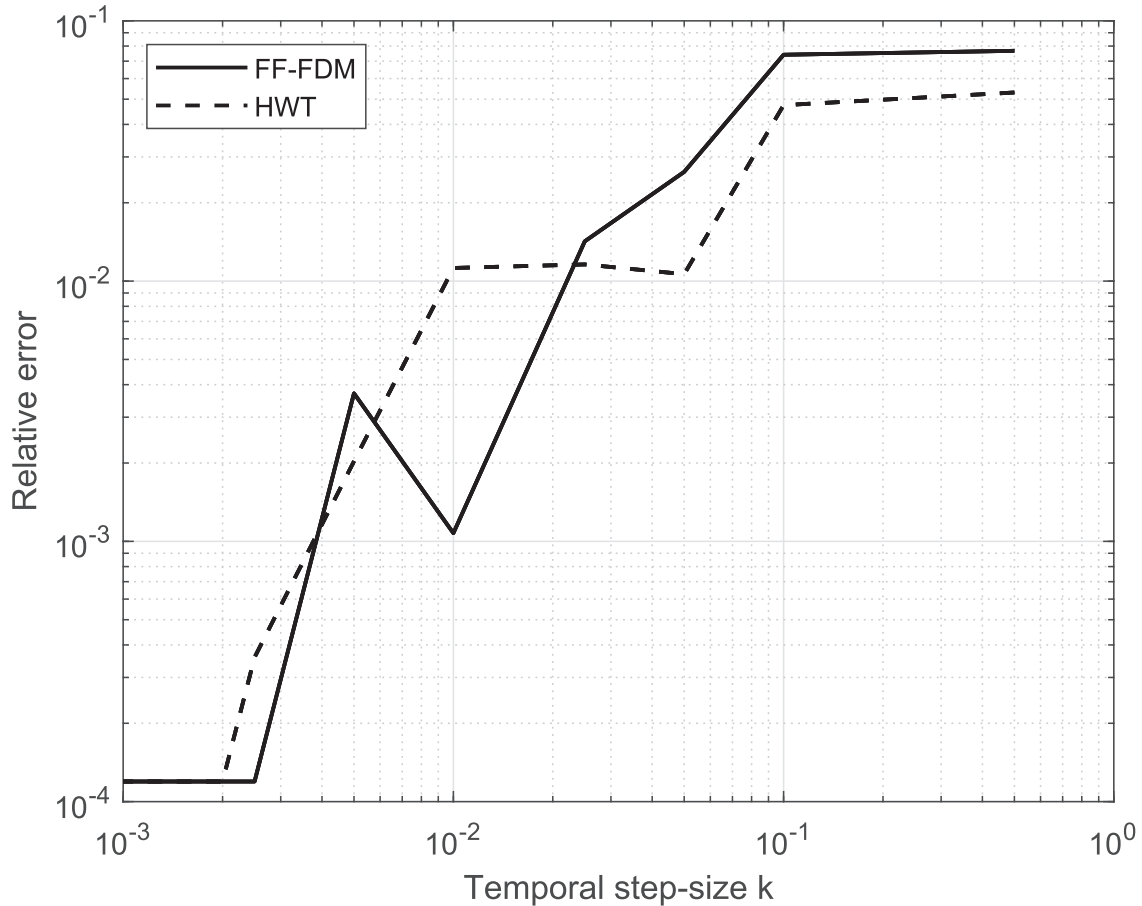
FIGURE 5 Comparison of the proposed front-fixing transformation-finite difference method (FF-FDM) and Hull and White tree (HWT)[18] methods in terms of the relative error

Let us define the ratio of errors as follows:

$$R(h,k) = \frac{\epsilon(2k,h)}{\epsilon(k,h)} = \frac{\left\| V_{2k}^h - V_{ref} \right\|}{\left\| V_k^h - V_{ref} \right\|}. \tag{39}$$

Then the convergence rate is $\alpha = \log_2 R(h,k)$. First, let us set $h = 0.01$ and vary k. The results are reported in Table 3. As it was expected, the method shows the first order of numerical convergence in time in accordance with the chosen forward FD approximation of the temporal derivatives.

We repeat the numerical convergence analysis for fixed $k = 0.00125$ and various h. The results collected in Table 4 show that the FF-FDM is of the second order in space. It also agrees with the theoretical convergence, because for the spatial derivatives, the centred approximation of the second order is used.

5 | CONCLUSIONS

In this article, we have proposed a new efficient numerical method for pricing American option on zero-coupon bond under the HW model. Due to the early exercise opportunity, the option price is written as a nonlinear free-boundary PDE problem of Black–Scholes type. The free boundary is treated by the front-fixing transformation, which casts the PDE system on a fixed domain. Numerical solution for the resulting nonlinear problem is constructed by an explicit finite difference method. Numerical results are in agreement with previous results of the FEM by Holmes and Yang[12] with the additional advantage that here the FDM results are computationally less expensive and easier to implement. However, it is important to point out that the FEM captures better the singularity of the optimal stopping boundary close to maturity.
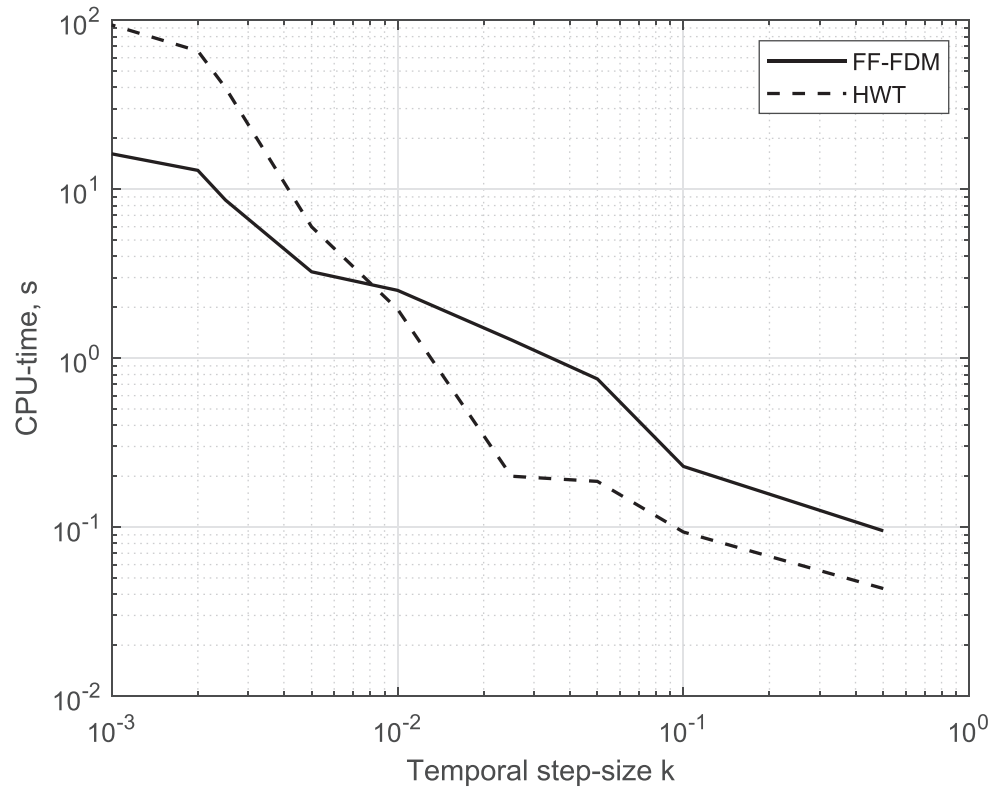
FIGURE 6 Comparison of the proposed front-fixing transformation-finite difference method (FF-FDM) and Hull and White tree (HWT)[18] methods in terms of the CPU time

TABLE 3 Numerical convergence analysis of FF-FDM method for Example 1 with fixed $h = 0.01$

k	$\epsilon(h, k)$	$R(h, k)$	α	CPU time (s)
0.005000	0.036880	-	-	5.4219
0.002500	0.019928	1.850702	0.888073	7.1563
0.001250	0.010344	1.926505	0.945986	13.7344
0.000625	0.005332	1.939868	0.955959	29.1094

Abbreviation: FF-FDM, front-fixing transformation-finite difference method.

TABLE 4 Numerical convergence analysis of FF-FDM method for Example 1 with fixed $k = 0.00125$

h	$\epsilon(h, k)$	$R(h, k)$	α	CPU time (s)
0.010000	0.010344	-	-	14.4219
0.005000	0.002935	3.524715	1.817506	14.8420
0.002500	0.000742	3.955965	1.984030	15.7500
0.001250	0.000189	3.935333	1.976486	19.0156

Abbreviation: FF-FDM, front-fixing transformation-finite difference method.

Numerical results are provided to show the potential advantages of the proposed method. Stability and convergence rate have been studied numerically. The proposed FF-FDM method is found to be of the second order in space and the first order in time corresponding to the finite difference stencil used in the scheme. Numerical results have been compared with the HWT method noting that our approach is less time consuming.

Thus, the proposed combination of the front-fixing transformation and explicit FDM is a new promising method for solving nonlinear free-boundary PDE problem of American options pricing on zero-coupon bond. This approach could be applied in future to more complicated models, such as jump-extended Cox–Ingersoll–Ross mode (CIR) models becoming partial integro-differential problems.

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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