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Symmetry in the Multidimensional Dynamical Analysis of Iterative Methods with Memory

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Abstract: In this paper, new tools for the dynamical analysis of iterative schemes with memory for solving nonlinear systems of equations are proposed. These tools are in concordance with those of the scalar case and provide interesting results about the symmetry and wideness of the basins of attraction on different iterative procedures with memory existing in the literature.

1. Introduction

Iterative methods for solving nonlinear systems of equations are a fundamental tool for solving problems in applied mathematics since in most of the problems it is sometimes complicated or impossible to solve these systems. For this reason, iterative methods are used since, by giving an initial estimate close enough to the solution, they obtain an approximation of the solution.

As mentioned above, it is necessary for the initial estimate to be close to the solution to ensure convergence, but this does not always happen, which is why the dynamical study is becoming increasingly important, since this is how we can see the behaviour of the initial estimations.

The study of the stability of iterative fixed-point methods can be carried out by means of real or complex dynamics tools applied to a rational operator that result from applying the iterative scheme to low-degree polynomials. These dynamical techniques can be used to compare or to deepen known iterative methods, as can be seen in [1–3], to analyze the qualitative properties of new iterative methods without memory (see, for example, [4–7]) or with memory (see, for instant, [8,9]). They also change if the method is multidimensional, as we can see in [8,10–15].

In this paper, we are going to lay the foundations for our future work in the study of the dynamics of iterative methods with memory for approximating the solutions of nonlinear systems.

In the first section, we present the theoretical concepts and the obtained results. Then, in the second section, we apply these results to some multidimensional iterative schemes with memory. We choose two different systems to see the behaviour of these iterative schemes.

2. Theoretical Concepts

Let $F(x) = 0$ be a system of nonlinear equations where $F : \mathbb{R}^n \to \mathbb{R}^n$.

The standard form of an iterative method with memory that uses only two previous iterations to calculate the next one is:

$$x^{(k+1)} = \phi\left(x^{(k-1)}, x^{(k)}\right), \ k \geq 1,$$

where $x^{(0)}$ and $x^{(1)}$ are the initial estimations.

From here on, we assumed that when introducing memory to the iterative method, the operator ϕ depends not only on $x^{(k)}$ but also depends on the variable of the previous step, $x^{(k-1)}$, since otherwise a study of the dynamics of a iterative method without memory for systems would be carried out.

A function defined from $\mathbb{R}^n$ to $\mathbb{R}^n$ cannot have fixed points since to be a fixed point of a function, the point and its image by the function must coincide. Therefore, an auxiliary function O is defined as follows:

$$O\left(x^{(k-1)}, x^{(k)}\right) = \left(x^{(k)}, x^{(k+1)}\right) = \left(x^{(k)}, \phi\left(x^{(k-1)}, x^{(k)}\right)\right), \ k = 1, 2, \ldots$$

If $\left(x^{(k-1)}, x^{(k)}\right)$ is a fixed point of O, then

$$O\left(x^{(k-1)}, x^{(k)}\right) = \left(x^{(k-1)}, x^{(k)}\right),$$

and by the definition of O, one has

$$\left(x^{(k-1)}, x^{(k)}\right) = \left(x^{(k)}, x^{(k+1)}\right).$$

Thus, the discrete dynamical system $O: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ is defined as

$$O(z, x) = (x, \phi(z, x)),$$

where ϕ is the operator associated with the vectorial iterative method with memory.

Then, a point (z, x) is a fixed point of O if $z = x$ and $x = \phi(z, x)$. If (z, x) is a fixed point of operator O that does not satisfy $F(x) = 0$, it is called a strange fixed point.

The basin of attraction of a fixed point (z^*, x^*) is defined as the set of pre-images of any order such that

$$\mathcal{A}(z^*, x^*) = \{(w, y) \in \mathbb{R}^{n \times n} : O^m(w, y) \to (z^*, x^*), \ m \to \infty\}.$$

To study the caracter of the fixed points, we use the following result from [16].

Theorem 1. *Let $G: \mathbb{R}^m \to \mathbb{R}^m$ be of class C^2 and x a fixed point. Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be the eigenvalues of $G'(x)$, where G' is the Jacobian matrix of G.*

- *If $|\lambda_j| < 1$, for $j = 1, 2, \ldots, m$, then x is attracing.*
- *If $|\lambda_j| = 0$, for $j = 1, 2, \ldots, m$, then x is super attracing.*
- *If one eigenvalue λ_{j_0} has $|\lambda_{j_0}| > 1$, then x is repelling or saddle.*
- *If $|\lambda_j| > 1$, for $j = 1, 2, \ldots, m$, then x is repelling.*

If one eigenvalue λ of $G'(x)$ satisfies $|\lambda| = 1$, then x is not hyperbolic and we cannot conclude anything about the character of this fixed point.

We want to obtain a more specific result for determining the character of the fixed points (z, x) of operator O. To do this, we calculate the Jacobian matrix of O, denoted by O' which has size $2n \times 2n$. The result is matrix

$$
O'(z, x) = \begin{pmatrix} 0_{n \times n} & & & I_{n \times n} & & \\ \dfrac{\partial \phi_1}{\partial z_1} & \cdots & \dfrac{\partial \phi_1}{\partial z_n} & \dfrac{\partial \phi_1}{\partial x_1} & \cdots & \dfrac{\partial \phi_1}{\partial x_n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \dfrac{\partial \phi_n}{\partial z_1} & \cdots & \dfrac{\partial \phi_n}{\partial z_n} & \dfrac{\partial \phi_n}{\partial x_1} & \cdots & \dfrac{\partial \phi_n}{\partial x_n} \end{pmatrix}.
$$

We denote the matrices

$$
\frac{\partial \phi}{\partial z} = \begin{pmatrix} \dfrac{\partial \phi_1}{\partial z_1} & \cdots & \dfrac{\partial \phi_1}{\partial z_n} \\ \vdots & & \vdots \\ \dfrac{\partial \phi_n}{\partial z_1} & \cdots & \dfrac{\partial \phi_n}{\partial z_n} \end{pmatrix}
$$

and

$$
\frac{\partial \phi}{\partial x} = \begin{pmatrix} \dfrac{\partial \phi_1}{\partial x_1} & \cdots & \dfrac{\partial \phi_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial \phi_n}{\partial x_1} & \cdots & \dfrac{\partial \phi_n}{\partial x_n} \end{pmatrix}.
$$

So, matrix $O'(z, x)$ is defined as a block matrix

$$
O'(z, x) = \begin{pmatrix} 0_{n \times n} & I_{n \times n} \\ \dfrac{\partial \phi}{\partial z} & \dfrac{\partial \phi}{\partial x} \end{pmatrix}.
$$

We need to obtain the eigenvalues of the Jacobian matrix $O'(z, x)$ evaluated at the fixed points, for determining the character of them (Theorem 1). It is easy to see that

$$
\lambda I_{2n \times 2n} - O'(z, x) = \begin{pmatrix} \lambda I_{n \times n} & -I_{n \times n} \\ -\dfrac{\partial \phi}{\partial z} & \lambda I_{n \times n} - \dfrac{\partial \phi}{\partial x} \end{pmatrix}.
$$

By applying a result of [17] for calculating the determinant of a block matrix, we obtain

$$
\det(\lambda I_{2n \times 2n} - O'(z, x)) = \det\left(\lambda \left(\lambda I_{n \times n} - \frac{\partial \phi}{\partial x} \right) - \frac{\partial \phi}{\partial z} \right).
$$

Then, λ is an eigenvalue of $O'(z, x)$ if

$$
\det\left(\lambda \left(\lambda I_{n \times n} - \frac{\partial \phi}{\partial x} \right) - \frac{\partial \phi}{\partial z} \right) = 0,
$$

which is the same as

$$
\det\left(\lambda^2 I_{n \times n} - \lambda \frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial z} \right) = 0.
$$

These calculations are summarized in the following result.

Theorem 2. *The eigenvalues of $O'(z, x)$ are those satisfying:*

$$
\det\left(\lambda^2 I_{n \times n} - \lambda \frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial z} \right) = 0.
$$

In particular, $\lambda = 0$ is an eigenvalue of $O'(z, x)$ if 0 is an eigenvalue of $\dfrac{\partial \phi}{\partial z}$ since

$$0 = \det\left(-O'(z, x)\right) = \det\left(-\frac{\partial \phi}{\partial z}\right).$$

Another concept relevant in a dynamical study is the critical points. In this case, we use the following definition of these type of points.

Definition 1. *Vector (z, x) is a critical point of $O(z, x)$ if all the eigenvalues of $O'(z, x)$ are 0.*

This is a restrictive definition of a critical point since it is usually sufficient that the determinant of the Jacobian matrix cancels out, but in this case, if we do not use the above definition, we obtain critical point surfaces because of the form of the operator.

To study the eigenvalues of $O'(z, x)$, we use Theorem 2. In particular, for at least one of the eigenvalues to be 0, it must be satisfied that $\det(O'(z, x)) = 0$; this is equivalent to

$$\det(O'(z, x)) = \det\begin{pmatrix} 0_{n \times n} & I_{n \times n} \\ \dfrac{\partial \phi}{\partial z} & \dfrac{\partial \phi}{\partial x} \end{pmatrix} = \det\left(-\frac{\partial \phi}{\partial z}\right) = (-1)^n \det\left(\frac{\partial \phi}{\partial z}\right).$$

From this, we obtain that:

Theorem 3. *The determinant of $O'(z, x)$ is zero if and only if it satisfies*

$$\det\left(\frac{\partial \phi}{\partial z}(z, x)\right) = 0.$$

3. Experimental Results

In this section, we present the dynamical study of two simple vectorial methods with memory: Kurchatov's scheme, ref. [18], whose expression is as follows

$$x^{(k+1)} = x^{(k)} - [2x^{(k)} - x^{(k-1)}, x^{(k-1)}; F]^{-1} F\left(x^{(k)}\right), \quad k = 1, 2, \ldots \tag{1}$$

and Steffensen's method with memory, ref. [19], whose expression is

$$x^{(k+1)} = x^{(k)} - [x^{(k)} + \gamma^{(k)} F\left(x^{(k)}\right), x^{(k)}; F]^{-1} F\left(x^{(k)}\right),$$
$$\text{where } \gamma^{(k)} = -[x^{(k)}, x^{(k-1)}; F]^{-1}, \quad k = 1, 2, \ldots \tag{2}$$

This study is performed on two polynomial systems of different degrees.

3.1. Uncoupled Third Order System

We perform this dynamical study on a system of cubical polynomials in order to use graphical tools. However, these results can be easily extended to higher-dimension systems. The system, denoted by $p(x) = 0$, is as follows:

$$\begin{cases} x_1^3 - 1 = 0, & (3) \\ x_2^3 - 1 = 0, & (4) \end{cases}$$

where $(x_1, x_2) \in \mathbb{R}^2$. The only root with real components of this systems is $(1, 1)$.

We choose a polynomial of degree 3 instead of one of degree 2 as it is usually done in dynamical studies because the polynomial system of second degree gives us an operator that does not depend on the previous iteration $x^{(k-1)}$, so the study is the same as that of a iterative method without memory for solving nonlinear systems.

To make the study simpler, we denote $x^{(k-1)}$ by z and $x^{(k)}$ by x as was done in the theoretical study.

3.1.1. Kurchatov's Method

Operator ϕ_K of Kurchatov's method on the cubical system is

$$\phi_K(z, x) = \begin{pmatrix} x_1 + \frac{1-x_1^3}{4x_1^2 - 2x_1 z_1 + z_1^2} \\ x_2 + \frac{1-x_2^3}{4x_2^2 - 2x_2 z_2 + z_2^2} \end{pmatrix}. \tag{5}$$

Theorem 4. *The only fixed point of operator $O_K(z, x) = (x, \phi_K(z, x))$ has equal components $z = x = (1, 1)$ and has superattractor character.*

Proof. We calculate matrices $\frac{\partial \phi_K}{\partial z}$ and $\frac{\partial \phi_K}{\partial x}$ appearing in the dynamical study,

$$\frac{\partial \phi_K}{\partial z}(z, x) = \begin{pmatrix} -\frac{2(x_1^3 - 1)(x_1 - z_1)}{(4x_1^2 - 2x_1 z_1 + z_1^2)^2} & 0 \\ 0 & -\frac{2(x_2^3 - 1)(x_2 - z_2)}{(4x_2^2 - 2x_2 z_2 + z_2^2)^2} \end{pmatrix}$$

and

$$\frac{\partial \phi_K}{\partial x}(z, x) = \begin{pmatrix} \frac{9x_1^2 z_1^2 - 12x_1^3 z_1 + 12x_1^4 - 4x_1(z_1^3 + 2) + z_1(z_1^3 + 2)}{(4x_1^2 - 2x_1 z_1 + z_1^2)^2} & 0 \\ 0 & \frac{9x_2^2 z_2^2 - 12x_2^3 z_2 + 12x_2^4 - 4x_2(z_2^3 + 2) + z_2(z_2^3 + 2)}{(4x_2^2 - 2x_2 z_2 + z_2^2)^2} \end{pmatrix}.$$

If we evaluate these matrices at the fixed point, we get

$$\frac{\partial \phi_K}{\partial z}((1, 1), (1, 1)) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\frac{\partial \phi_K}{\partial x}((1, 1), (1, 1)) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

By Theorem 2, it follows that $\det(\lambda I - O_K'((1, 1), (1, 1))) = \det(\lambda^2 I)$. From the above relationship it follows that the only eigenvalue associated with the fixed point is $\lambda = 0$. So, fixed point $x = z = (1, 1)$ is a superattracting point. □

Regarding the critical points of operator $O_K(z, x)$, we have

Theorem 5. *Operator $O_K(z, x)$ has four types of critical points, denoted by $C_i(z, x)$, which have the following form.*

The notation of Table 1 is understood in such a way that, for example, the points C2(z, x) are those that verify that $x_1 = 1$ and $z_2 = x_2$, and the other components are arbitrary.

Table 1. Types of points of Kurchatovs's scheme that are critical points for $p(x) = 0$.

	$x_1 = 1$	$z_1 = x_1$
$x_2 = 1$	$C_1(z, x)$	$C_3(z, x)$
$z_2 = x_2$	$C_2(z, x)$	$C_4(z, x)$

Proof. To do this, we calculate the eigenvalues of $O_K'(z, x)$ for any point (z, x) and obtain those satisfying the condition that all their eigenvalues are 0.

It is obtained that the critical points are those $z = (z_1, z_2)$ and $x = (x_1, x_2)$ that satisfy these two expressions

$$\left(x_1^3 - 1\right)(x_1 - z_1) = 0 \quad \text{and} \quad \left(x_2^3 - 1\right)(x_2 - z_2) = 0. \tag{6}$$

It follows that the 4 types of points defined in Table 1 are critical points of operator $O_K(z, x)$. $\square$

We have that the points of type $C_1(z, x)$ are a preimage of the only fixed point since $\phi_K(z, (1, 1)) = (1, 1)$. We are only going to study the orbit of the points of type $C_2(z, x)$ and $C_4(z, x)$ since the points $C_1(z, x)$ converge to the fixed point for any value of z, and the fixed points of type $C_3(z, x)$ have a symmetrical study to that of the fixed points of type $C_2(z, x)$.

- Operator ϕ_K evaluated at the critical points of type $C_2(z, x)$ has the following form

$$\phi_K(C_2(z, x)) = \begin{pmatrix} 1 \\ x_2 + \frac{1 - x_2^3}{3x_2^2} \end{pmatrix}.$$

The convergence of these points only depends on x_2 and z_1, as we can see in the expression of ϕ_K. For this reason, we draw planes of convergence of these points with these two variables.

Now, we are going to describe how we generate the planes of convergence [4]. To draw these planes of the points of the type $C_2(z, x)$, what we are going to do is to see which of these points belong to the basins of attraction of the attractor fixed points, that is, which of these points converge to the attractor.

We make a mesh of 400×400 points of the set $[-2, 2] \times [-2.2]$. On one of the axes, we have x_2, and on the other, z_1, and with them we construct our points of type $C_2(z, x)$. We take each of these points $C_2(z, x)$ and apply the operator ϕ_K on it. If it converges to the only attractor fixed point, which is $(1, 1)$, then we paint it in orange. As convergence criterion, we have used that the distance from the iteration to the fixed point is less than 10^{-3} in less than 40 iterations. If this is not verified, the mesh point is painted black.

As can be seen in Figure 1 that we have a slower convergence when x_2 approaches the value 0 because of the shape of the operator, but we still have convergence. In the rest of the cases, the convergence to the point $(1, 1)$ is clear.
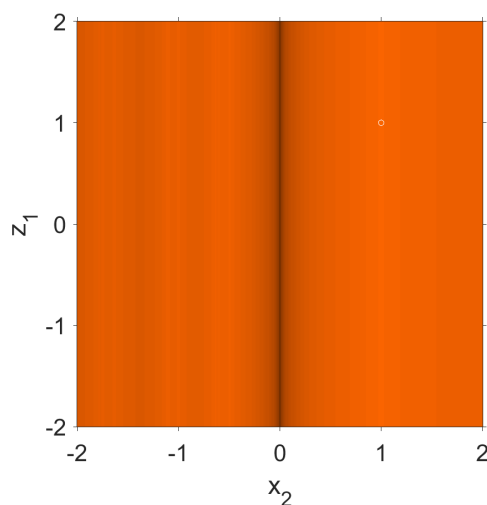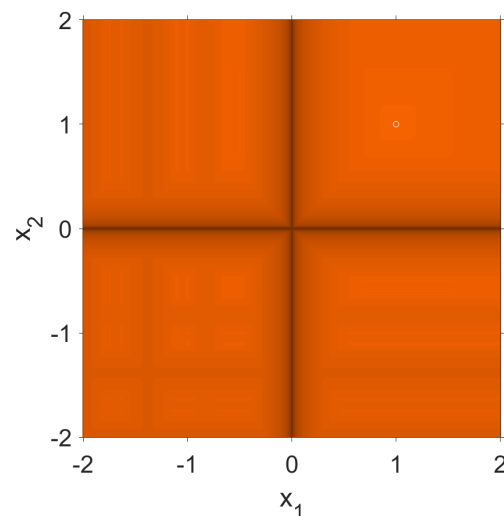

Figure 1. Behaviour of critical points $C_2(z, x)$.

- Next, we evaluate the operator ϕ_K at the critical points of type $C_4(z,x)$. In this case, the operator is

$$\phi_K(C_4(z,x)) = \begin{pmatrix} x_1 + \frac{1-x_1^3}{3x_1^2} \\ x_2 + \frac{1-x_2^3}{3x_2^2} \end{pmatrix}.$$

In this case, the critical points of type $C_4(z,x)$ depend on variables x_1 and x_2. For this reason, we draw the convergence plane of the critical points depending on these variables.

As in the previous case, it is shown in Figure 2 that if any of the variables approach the value 0 we have slow convergence but that in the rest of the points the convergence to the point $(1,1)$ is clear.

Figure 2. Behaviour of critical points $C_4(z,x)$.

To conclude the dynamical study of Kurchatov's method for this system, let us draw some dynamical planes in order to see the behaviour of the points in general.

To draw these planes, given that we have an operator with 4 variables, what we have done is to select a parameter a, so that $z = x - (a,a)$. We try different values of a to see which one gives the best results. Usually, testing with small values of a gives good results. Thus, our variables would be x_1 and x_2, and the variables z are a variation of these.

To make the dynamical planes, we have chosen a mesh of 400×400 points, where the chosen point of the mesh is the starting point. We study the orbit of the initial point. If the seed converges to $(1,1)$, it is painted orange, and if it does not converge, it is painted black. We define convergence to the point $(1,1)$ because the distance of the iteration is less than 10^{-3}, and this convergence is realised in, at most, 40 iterations.

We have tested with different values of a over a wide range and obtained that there are the same dynamical plane for different values of a, (Figure 3). As we can see in Figure 3, all initial points converge to the root $(1,1)$ showing the good stability properties of this iterative scheme with memory, even in this multidimensional case.

3.1.2. Steffensen's Scheme

In this part, we perform the dynamical study of Steffensen's scheme with memory for system $p(x) = 0$. Operator ϕ_S obtained by Steffensen's method is

$$\phi_S(z,x) = \begin{pmatrix} x_1 + \dfrac{\left(x_1^3-1\right)^2}{\left(x_1^2+x_1z_1+z_1^2\right)\left(\left(\frac{1-x_1^3}{x_1^2+x_1z_1+z_1^2}+x_1\right)^3-x_1^3\right)} \\[30pt] x_2 + \dfrac{\left(x_2^3-1\right)^2}{\left(x_2^2+x_2z_2+z_2^2\right)\left(\left(\frac{1-x_2^3}{x_2^2+x_2z_2+z_2^2}+x_2\right)^3-x_2^3\right)} \end{pmatrix}. \tag{7}$$
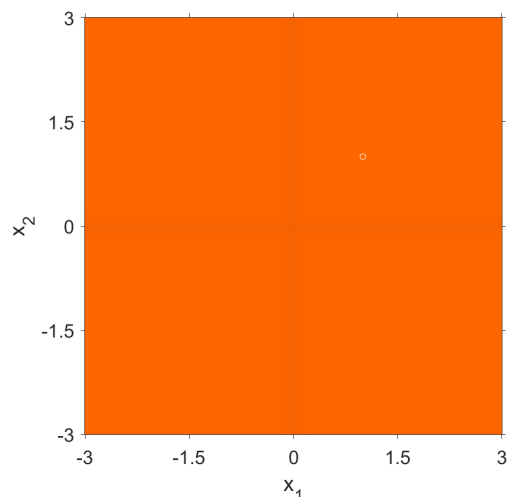

Figure 3. Dynamical plane of Kurchatov's scheme for $a = -0.1$.

Theorem 6. *Operator* $O_S(z,x) = (x, \phi_S(z,x))$ *has four fixed points, which are*

- *fixed point* $(z,x) = (S_1, S_1)$, *with* $S_1 = (1,1)$,
- *strange fixed point* $(z,x) = (S_2, S_2)$, *being* $S_2 = (0,0)$,
- *strange fixed point* $(z,x) = (S_3, S_3)$, *being* $S_3 = (1,0)$,
- *strange fixed point* $(z,x) = (S_4, S_4)$, *with* $S_4 = (0,1)$.

The strange fixed points are not hyperbolic, and the fixed point (S_1, S_1) *is a superattractor point.*

Proof. In order to study the character of these fixed points, we need to obtain matrices $\dfrac{\partial \phi_S}{\partial z}$ and $\dfrac{\partial \phi_S}{\partial x}$.

We denote by $GZ_i(z,x)$ for $i = 1, 2$ the following expression:

$$GZ_i(z,x) = \frac{\left(x_i^3 - 1\right)^2 (x_i + 2z_i)\left(x_i^2 + x_iz_i + z_i^2\right)\left(3x_i^2z_i + x_i^3 + 3x_iz_i^2 + 2\right)}{\left(6x_i^4z_i^2 + x_i^3\left(6z_i^3 + 1\right) + 3x_i^2\left(z_i^4 + z_i\right) + 3x_i^5z_i + x_i^6 + 3x_iz_i^2 + 1\right)^2}. \tag{8}$$

Thus,

$$\frac{\partial \phi_S}{\partial z}(z,x) = \begin{pmatrix} GZ_1(z,x) & 0 \\ 0 & GZ_2(z,x) \end{pmatrix}.$$

We denote by $GX_i(z,x)$ for $i = 1, 2$ expression:

$$GX_i(z,x) = \frac{36x_i^4z_i^2 + x_i^3\left(39z_i^3 + 7\right) + 12x_i^2\left(2z_i^4 + z_i\right) + 18x_i^5z_i}{\left(6x_i^4z_i^2 + x_i^3\left(6z_i^3 + 1\right) + 3x_i^2\left(z_i^4 + z_i\right) + 3x_i^5z_i + x_i^6 + 3x_iz_i^2 + 1\right)^2} \\ + \frac{4x_i^6 + 6x_iz_i^2\left(z_i^3 + 2\right) + 3z_i^3 + 1}{\left(6x_i^4z_i^2 + x_i^3\left(6z_i^3 + 1\right) + 3x_i^2\left(z_i^4 + z_i\right) + 3x_i^5z_i + x_i^6 + 3x_iz_i^2 + 1\right)^2} \tag{9}$$

Thus,

$$\frac{\partial \phi_S}{\partial x}(z, x) = \begin{pmatrix} \left(x_1^3 - 1\right)\left(z_1^3 - 1\right) GX_1(z, x) & 0 \\ 0 & \left(x_2^3 - 1\right)\left(z_2^3 - 1\right) GX_2(z, x) \end{pmatrix}.$$

Let us now obtain the character of the fixed points (S_i, S_i), for $i = 1, \ldots, 4$.

- For the point associated to S_1, the related matrices are

$$\frac{\partial \phi_S}{\partial z}(S_1, S_1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\frac{\partial \phi_S}{\partial x}(S_1, S_1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

From Theorem 2, we conclude that the fixed point is super attracting point.

- For the point associated to S_2, we obtain

$$\frac{\partial \phi_S}{\partial z}(S_2, S_2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\frac{\partial \phi_S}{\partial x}(S_2, S_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

By applying Theorem 2, the eigenvalues of this point are the values λ satisfying

$$0 = \det(\lambda I - O_S'(x, z)) \quad = \det\left(\lambda^2 I - \lambda \frac{\partial \rho}{\partial x} - \frac{\partial \rho}{\partial z}\right) \\ = \lambda^2(\lambda - 1)^2. \tag{10}$$

It follows that the eigenvalues are 0 and 1, so we cannot conclude anything about the character of this strange fixed point as it is not hyperbolic.

- For the fixed point associated to S_3, the matrices are

$$\frac{\partial \phi_S}{\partial z}(S_3, S_3) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\frac{\partial \phi_S}{\partial x}(S_3, S_3) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The eigenvalues associated with this fixed point are those λ values that satisfy

$$0 = \det(\lambda I - O_S'(x, z)) \quad = \det\left(\lambda^2 I - \lambda \frac{\partial \rho}{\partial x} - \frac{\partial \rho}{\partial z}\right) \\ = \lambda^3(\lambda - 1). \tag{11}$$

It follows that the eigenvalues are 0 and 1, so again the point is not hyperbolic.

- Finally, let us study the character of the fixed point associated with S_4. The matrices for this fixed point are

$$\frac{\partial \phi_S}{\partial z}(S_4, S_4) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\frac{\partial \phi_S}{\partial x}(S_4, S_4) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

So the eigenvalues of the fixed point associated with S_4 are the values that satisfy

$$\begin{aligned}0 = \det(\lambda I - O'_S(x,z)) \quad &= \det\left(\lambda^2 I - \lambda\frac{\partial\rho}{\partial x} - \frac{\partial\rho}{\partial z}\right)\\ &= \lambda^3(\lambda - 1).\end{aligned}\tag{12}$$

It follows that the eigenvalues are 0 and 1, so again the point is not hyperbolic.
□

Now, let us calculate the critical points.

Theorem 7. *The critical points of operator $O_S(z, x)$ are vectors $z = (z_1, z_2)$ and $x = (x_1, x_2)$, which satisfy that they are of one of the following 16 types, which we denote by $CS_i(z, x)$ for $i = 1, \ldots, 16$. Table 2 is a summary of the different types of critical points we obtain.*

Table 2. Types of points of Steffensen's scheme that are critical points for $p(x) = 0$.

	$x_1 = 1$	$z_1 = -\frac{1}{2}x_1$	$z_1 = \frac{-3x_1^2 + \sqrt{-3x_1^4 - 24x_1}}{6x_1}$	$z_1 = \frac{-3x_1^2 - \sqrt{-3x_1^4 - 24x_1}}{6x_1}$
$x_2 = 1$	$CS_1(z, x)$	$CS_5(z, x)$	$CS_9(z, x)$	$CS_{13}(z, x)$
$z_2 = -\frac{1}{2}x_2$	$CS_2(z, x)$	$CS_6(z, x)$	$CS_{10}(z, x)$	$CS_{14}(z, x)$
$z_2 = \frac{-3x_2^2 + \sqrt{-3x_2^4 - 24x_2}}{6x_2}$	$CS_3(z, x)$	$CS_7(z, x)$	$CS_{11}(z, x)$	$CS_{15}(z, x)$
$z_2 = \frac{-3x_2^2 - \sqrt{-3x_2^4 - 24x_2}}{6x_2}$	$CS_4(z, x)$	$CS_8(z, x)$	$CS_{12}(z, x)$	$CS_{16}(z, x)$

We are working with systems of equations with real variables, so it is assumed that the critical points have real numbers as their components.

Proof. If we define $DGX_i(z, x)$, $i = 1, 2$, as

$$DGX_i(z, x) = \lambda^2 - \lambda(x_1^3 - 1)(z_1^3 - 1)GX_i(z, x) - GX_i(z, x),$$

then, we can check that

$$\begin{aligned}\det(\lambda I - O'_S(x,z)) \quad &= \det\left(\lambda^2 I - \lambda\frac{\partial\phi_S}{\partial x} - \frac{\partial\phi_S}{\partial z}\right)\\ &= \det\begin{pmatrix}DGX_1(z, x) & 0\\ 0 & DGX_2(z, x)\end{pmatrix}.\end{aligned}\tag{13}$$

Additionally, it follows that all eigenvalues are zero if the point (z, x) has one of the forms given in Table 3. □

As we can see on Table 3, there is a certain symmetry relation between the following $CS_i(z, x)$.

Table 3. Symmetry relation between $CS_i(z, x)$.

|	
$CS_2(z, x)$	$CS_5(z, x)$
$CS_3(z, x)$	$CS_9(z, x)$
$CS_4(z, x)$	$CS_{13}(z, x)$
$CS_7(z, x)$	$CS_{10}(z, x)$
$CS_8(z, x)$	$CS_{14}(z, x)$
$CS_{12}(z, x)$	$CS_{15}(z, x)$

For that reason, and because the operator also satisfies certain symmetry with the components, we only study the behaviour of certain types of points that are critical points.

- The asymptotic behaviour of the critical points $CS_2(z, x)$ is analysed with the following plane where the convergence to the fixed points is shown in different colours. In this case, if the distance from the iteration to the fixed point is less than 10^{-3}, we say that the iteration is in the basin of attraction of the fixed point. In this case, it is painted orange if the critical point converges to (S_1, S_1), blue if it converges to the strange fixed point (S_3, S_3), red if it converges to the strange fixed point (S_4, S_4) and green if it converges to the point (S_2, S_2). If the points are painted black, they have not converged to any of the fixed points in less than 40 iterations. In this case, we have a fixed value x_1, and the value z_2 depends on x_2, so the variables of the axes are x_2 and z_1 as shown in Figure 4.
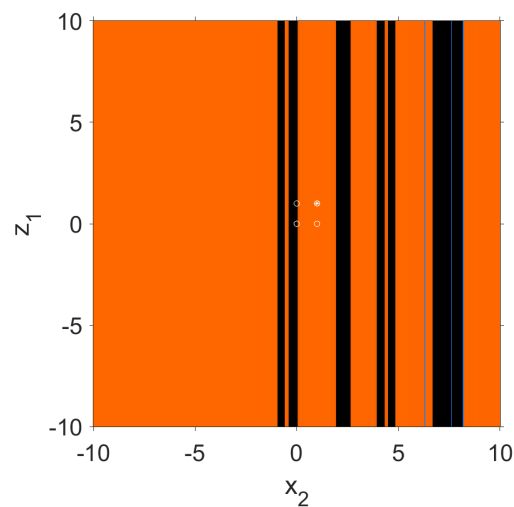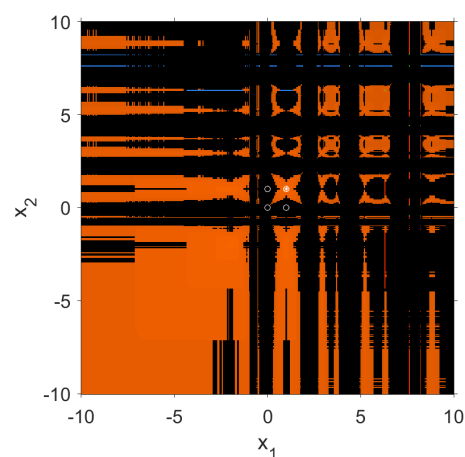

Figure 4. Convergence of the critical points of type $CS_2(z, x)$.

- In a similar way to the previous case, we study the convergence of the critical points of type $CS_3(z, x)$ and of type $CS_4(z, x)$. In these cases, the value x_1 is also fixed and the value z_2 depends on x_2; for this reason, the variables of the axes are x_2 and z_1 as in the previous cases and as can be seen in Figure 5. In this case, we have that the behaviour of both types of critical points is the same; for that reason, we only show one dynamical plane.
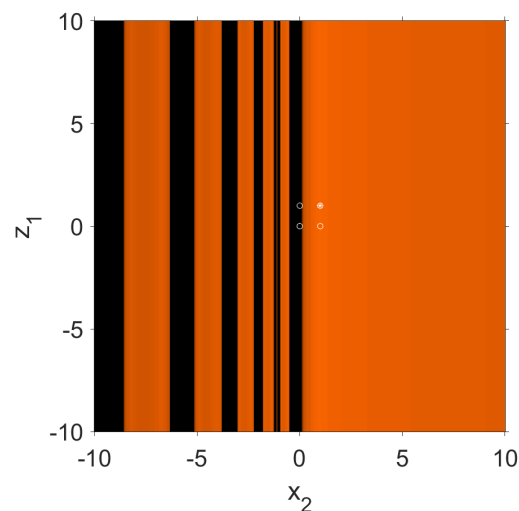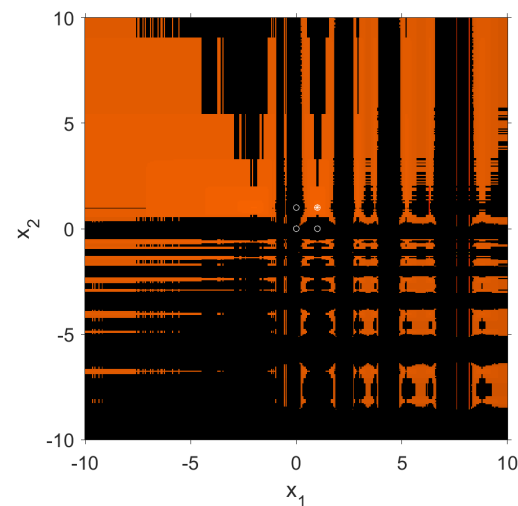

Figure 5. Convergence of the critical points of type $CS_3(z, x)$ and $CS_4(z, x)$.

- For the critical points of type $CS_6(z, x)$, the convergence study is similar to the previous ones, but in this case none of the variables are fixed, and it is z_1 and z_2 that depend on x_1 and x_2, respectively; for this reason, the dynamical plane has as axis variables the values of x_1 and x_2, as shown in Figure 6.
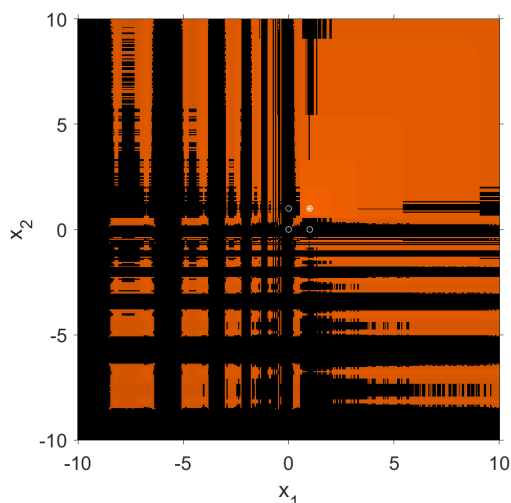

Figure 6. Convergence of the critical points of type $CS_6(z, x)$.

- For the critical points of type $CS_7(z, x)$ and $CS_8(z, x)$, we also have as variables on the axes the values of x_1 and x_2, as shown in Figure 7. In this case, we have decided to show only one dynamical plane because the behaviour of both types of critical points is the same.

Figure 7. Convergence of the critical points of type $CS_7(z, x)$ and $CS_8(z, x)$.

- For the critical points of type $CS_{11}(z, x)$, $CS_{12}(z, x)$ and $CS_{16}(z, x)$, we also have as variables on the axes the values of x_1 and x_2. In this case, we have that the behaviour of these 3 types of critical points is the same; for that reason, we only show one dynamical plane (Figure 8).

Figure 8. Convergence of the critical points of type $CS_{11}(z, x)$, $CS_{12}(z, x)$ and $CS_{16}(z, x)$.

3.2. A Coupled Second-Order System

Now, we are going to solve other system that has a more complicated aspect since the variables cannot be separated, that is to say, we do not have that the first component of the operator only depends on the first components of the variables of x and z and the same with the second component; instead, in this case, we have that both components of the operator depend on both components of the vectors. The next system we solve, denoted by $q(x) = 0$, is

$$\begin{cases} x_1 x_2 + x_1 - x_2 - 1 = 0, & (14) \\ x_1 x_2 - x_1 + x_2 - 1 = 0, & (15) \end{cases}$$

where $(x_1, x_2) \in \mathbb{R}^2$. The real roots of this system are $(-1, -1)$ and $(1, 1)$.

3.2.1. Kurchatov's Scheme

If we apply the Kurchatov's scheme to the proposed system, we obtain the following operator:

$$\varphi_K(z, x) = \begin{pmatrix} \frac{1 - x_2 z_1 + x_1 (x_2 + z_2)}{2x_1 - z_1 + z_2} \\ \frac{1 - x_2 z_1 + x_1 (x_2 + z_2)}{2x_1 - z_1 + z_2} \end{pmatrix}. \tag{16}$$

Theorem 8. *The only fixed points of the operator* $O_K(z, x) = (z, \varphi_K(z, x))$ *are* $z = x = (-1, -1)$ *and* $z = x = (1, 1)$, *and both have superattractor character.*

Proof. Now, we calculate the matrices $\dfrac{\partial \varphi_K}{\partial z}$ and $\dfrac{\partial \varphi_K}{\partial x}$ to obtain the character of these fixed points.

$$\frac{\partial \varphi_K}{\partial z}(z, x) = \begin{pmatrix} \frac{x_1(z_2 - x_2) - x_2 z_2 + 1}{(2x_1 - z_1 + z_2)^2} & \frac{2x_1^2 - x_1(x_2 + z_1) + x_2 z_1 - 1}{(2x_1 - z_1 + z_2)^2} \\ \frac{x_1(z_2 - x_2) - x_2 z_2 + 1}{(2x_1 - z_1 + z_2)^2} & \frac{2x_1^2 - x_1(x_2 + z_1) + x_2 z_1 - 1}{(2x_1 - z_1 + z_2)^2} \end{pmatrix}$$

and

$$\frac{\partial \varphi_K}{\partial x}(z, x) = \begin{pmatrix} \frac{x_2(z_1 + z_2) - z_1 z_2 + z_2^2 - 2}{(2x_1 - z_1 + z_2)^2} & \frac{x_1 - z_1}{2x_1 - z_1 + z_2} \\ \frac{x_2(z_1 + z_2) - z_1 z_2 + z_2^2 - 2}{(2x_1 - z_1 + z_2)^2} & \frac{x_1 - z_1}{2x_1 - z_1 + z_2} \end{pmatrix}.$$

If we evaluate the previous matrices in the fixed points, we obtain in both cases that both are the matrix that all the components are 0. So, by Theorem 2, both eigenvalues are 0 for all the fixed points. For that reason, both fixed points are super attracting points. □

Theorem 9. *Operator $O_K(z, x) = (z, \varphi_K(z, x))$ has two types of points that are critical points. These points have one of the following two structures:*

- $C_+(z, x) = (z_1, z_2, x_1, x_2)$ *where* $z_2 = -x_1 + \sqrt{2 - x_1^2} + z_1$.
- $C_-(z, x) = (z_1, z_2, x_1, x_2)$ *where* $z_2 = -x_1 - \sqrt{2 - x_1^2} + z_1$.

Proof. As we can see,

$$\det(\lambda I - O_K'(z, x)) = \det\left(\lambda^2 I - \lambda \frac{\partial \varphi_K}{\partial x} - \frac{\partial \varphi_K}{\partial x}\right). \tag{17}$$

By the form of $\frac{\partial \varphi_K}{\partial x}$ and $\frac{\partial \varphi_K}{\partial x}$, we can see that all the eigenvalues are zero if

$$x_1 - x_2 = 0 \quad \text{and} \quad (z_2 - z_1 + x_1)^2 = 2 - x_1^2.$$

□

Let us draw the orbit of these critical points. In this case, we draw on the abscissa axis the values of x_1, which is the same value as x_2, and we draw on the other axis the value of z_1 since z_2 is obtained from x_1 and z_2.

To generate these convergence planes of the points of type $C_+(z, x)$, we are going to see which of these points belong to the basins of attraction of the attractor fixed points, that is, which of these points converge to the attractor fixed points.

To do this, we make a mesh of 400×400 points of the set $[-2, 2] \times [-2, 2]$. We made sure that increasing the set did not alter the behaviour. On one of the axes, we have the variable x_1, and on the other, the variable z_1, and with these variables we construct our points of type $C_+(z, x)$. We take each of these points of type $C_+(z, x)$, and we apply our operator φ on them.

If this initial point converges to $(1, 1)$, we paint it in orange, and if converges to $(-1, -1)$, we paint it in blue. As convergence criteria, we have that the distance from the iteration to the fixed point is less than 10^{-3} in less than 40 iterations. If this is not verified, we paint it in black.

Figure 9 shows the plane of convergence for the points of type $C_+(z, x)$.
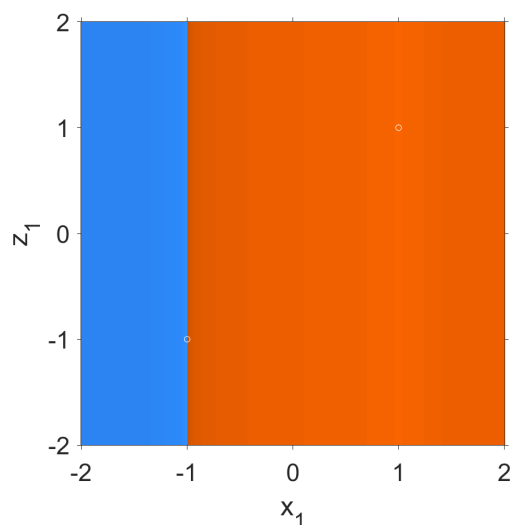
Figure 9. Convergence of the critical points of type $C_+(z, x)$.

In the same way that the plane of convergence of the points of type $C_+(z, x)$ is generated, we generate the plane of convergence of the points of type $C_-(z, x)$, which is shown in Figure 10.
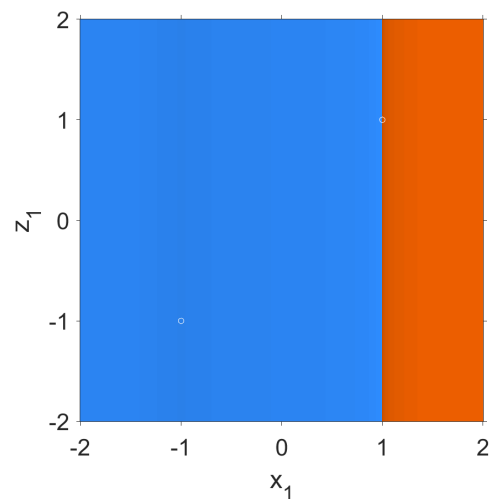
Figure 10. Convergence of the critical points of type $C_-(z, x)$.

We observe in this planes of convergence, Figures 9 and 10, that global convergence to the roots of the system exists.

To conclude the dynamical study of the Kurchatov method for this system, we draw some dynamical planes in order to see the behaviour of the points in general. To draw these planes, given that we have an operator with 4 variables, what we have done is to select a parameter a, so that $z = x - (a, a)$. Thus, our variables would be x_1 and x_2, and the variables z are a variation of these.

To make the dynamical planes, we have chosen a mesh of 400×400 points. If the initial point converges to the point $(1, 1)$, it is painted orange; if it converges to point $(-1, -1)$, it is painted blue; and if it does not converge to any point, is painted black.

We have tested with different values of a over a wide range and obtained that there are similar dynamical planes for different values of a; for that reason, we only show Figure 11. As we can see on this figure, we have that all initial estimation converge to the roots of the polynomial.
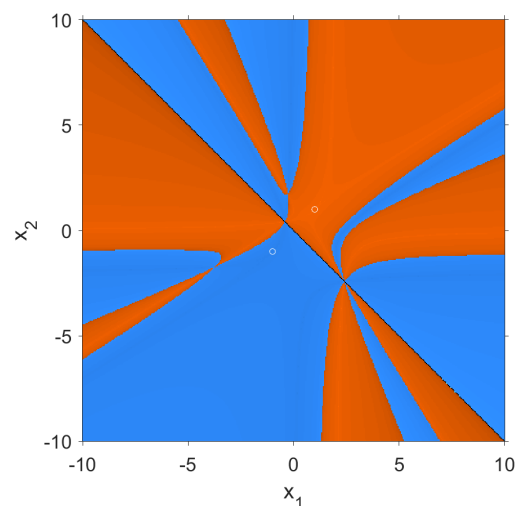
Figure 11. Dynamical plane of Kurchatov's scheme with $a = 0.1$.

3.2.2. Steffensen's Scheme

If we apply Steffensen's scheme with memory to system $q(x) = 0$, we obtain the following operator

$$\theta_S(z, x) = \begin{pmatrix} \frac{2x_1^2 x_2 + x_1 x_2 z_2 + x_1 - x_2 + z_2}{2x_1^2 + x_1(x_2 + z_2) + x_2 z_2 - 1} \\ \frac{2x_1^2 x_2 + x_1 x_2 z_2 + x_1 - x_2 + z_2}{2x_1^2 + x_1(x_2 + z_2) + x_2 z_2 - 1} \end{pmatrix}. \tag{18}$$

Theorem 10. *The operator $O(z, x) = (x, \theta_S(z, x))$ has three fixed points, that is,*

- $z = x = (-1, -1)$*, which is a superattractor point.*
- $z = x = (1, 1)$*, which is a superattractor point.*
- $z = x = (0, 0)$*, which is a strange fixed point and not hyperbolic.*

Proof. Let us calculate the matrices $\dfrac{\partial \theta_S}{\partial z}$ and $\dfrac{\partial \theta_S}{\partial x}$ to obtain the character of the fixed points.

$$\frac{\partial \theta_S}{\partial z}(z, x) = \begin{pmatrix} 0 & -\frac{(x_1^2-1)(x_2^2-1)}{(2x_1^2+x_1(x_2+z_2)+x_2 z_2-1)^2} \\ 0 & -\frac{(x_1^2-1)(x_2^2-1)}{(2x_1^2+x_1(x_2+z_2)+x_2 z_2-1)^2} \end{pmatrix}, \tag{19}$$

$$\frac{\partial \theta_S}{\partial x}(z, x) = \begin{pmatrix} \frac{(x_2^2-1)(2x_1^2+4x_1 z_2+z_2^2+1)}{(2x_1^2+x_1(x_2+z_2)+x_2 z_2-1)^2} & \frac{(x_1^2-1)(4x_1^2+4x_1 z_2+z_2^2-1)}{(2x_1^2+x_1(x_2+z_2)+x_2 z_2-1)^2} \\ \frac{(x_2^2-1)(2x_1^2+4x_1 z_2+z_2^2+1)}{(2x_1^2+x_1(x_2+z_2)+x_2 z_2-1)^2} & \frac{(x_1^2-1)(4x_1^2+4x_1 z_2+z_2^2-1)}{(2x_1^2+x_1(x_2+z_2)+x_2 z_2-1)^2} \end{pmatrix}. \tag{20}$$

For the fixed points associated with the roots, both matrices are the zero matrix. So, by Theorem 2, both eigenvalues are 0. Then, the fixed points associated with the roots are super attracting points. Let see what happens to the strange fixed point. The matrices are

$$\frac{\partial \theta_S}{\partial z}((0, 0), (0, 0)) = \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix}, \tag{21}$$

$$\frac{\partial \theta_S}{\partial x}((0, 0), (0, 0)) = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}. \tag{22}$$

By Theorem 2, the eigenvalues for that strange fixed point are the values λ that satisfy the following equation:

$$\det(\lambda I - O'_S(z, x)) = \det\left(\lambda^2 I - \lambda \frac{\partial \theta_S}{\partial x} - \frac{\partial \theta_S}{\partial z}\right) = \lambda^2(\lambda^2 + 1) = 0. \tag{23}$$

So, the eigenvalues are $\pm i$ and 0. We cannot determine the character of that strange fixed-point non-hyperbolic. □

Theorem 11. *Operator $O_S(z, x) = (x, \theta_S(z, x))$ has six types of critical points. These types of points are:*

All these points are preimages of one of the fixed points.

Proof. Since

$$\det(\lambda I - O'_S(z, x)) = \lambda^4 + \lambda^2 \frac{(-1 + x_1^2)(-1 + x_2^2) - \lambda x_2^2}{(-1 + 2x_1^2 + x_2 z_2 + x_1(x_2 + z_2))^2} \\ + \lambda^3 \frac{x_1^2(7 - 4x_1^2 - 2x_2^2) - 4x_1(-2 + x_1^2 + x_2^2)z_2 - (-2 + x_1^2 + x_2^2)z_2^2}{(-1 + 2x_1^2 + x_2 z_2 + x_1(x_2 + z_2))^2} \tag{24}$$

Then, all the eigenvalues are zero, if (z, x) has one of the types shown in Table 4.

1. Since θ evaluated at the $CP_2(z,x)$, $CP_4(z,x)$ and $CP_6(z,x)$ points is $(-1,-1)$, then those types of critical points belongs to the basin of attraction of $(-1,-1)$.
2. Since θ evaluated at the $CP_1(z,x)$, $CP_3(z,x)$ and $CP_5(z,x)$ points is $(1,1)$, then those types of critical points belongs to the basin of attraction of $(1,1)$.

□

Table 4. Types of points of Steffensen's scheme that are critical points for $q(x) = 0$.

	$x_1 = 1$	$x_1 = -1$	$z_2 = -x_2 - 2x_1$
$x_2 = 1$	$CP_1(z,x)$ if $z_2 \neq -3$		$CP_5(z,x)$ if $|x_1| \neq 1$
$x_2 = -1$		$CP_2(z,x)$ if $z_2 \neq 3$	$CP_6(z,x)$ if $|x_1| \neq 1$
$z_2 = -1 - 2x_1$	$CP_3(z,x)$ if $x_2 \neq -1$		
$z_2 = 1 - 2x_1$		$CP_4(z,x)$ if $x_2 \neq 1$	

Below, we draw a dynamical plane for the Steffensen's scheme with memory in the same way as was done for the Kurchatov's scheme. In Figure 12, we can see a black region; that is because we have slow convergence to the roots in that region since there are no critical points outside the basins of attraction of the roots, so there cannot be convergence to any point other than the roots. Here, we also tried different values for the parameter a, and similar results were obtained, although the larger the parameter was, the more slowly convergence zone increased.
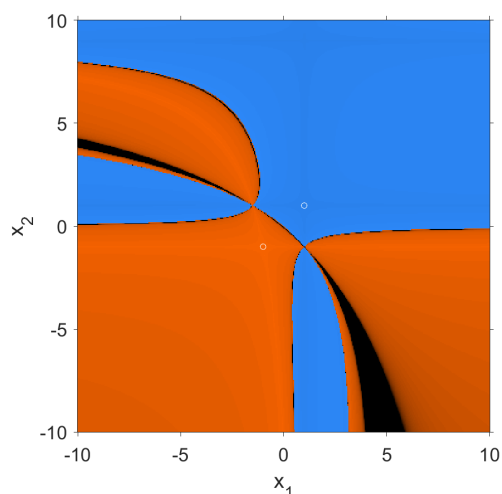
Figure 12. Dynamical plane of Steffensen's method with $a = -0.1$.

4. Conclusions

The design of new vectorial iterative schemes with memory for solving nonlinear problems is a developing area of numerical analysis that has expanded in recent years. These methods can be numerically checked, but, until now, there were no possibility of analyzing their qualitative performance as all the existing techniques, including both complex and real discrete dynamics, were not defined to overcome the high dimensionality of the rational functions involved. This proposed procedure has been tested with the analysis of the performance of Kurchatov' and Steffensen's multidimensional methods on coupled and non-coupled nonlinear polynomial systems. The results obtained show the applicability of this technique and present many possibilites for future research.

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