ON THE RANDOM WAVE EQUATION WITHIN THE MEAN SQUARE CONTEXT

Julia Calatayud

Departament de Matemàtiques,
Universitat Jaume I,
12071 Castellón, Spain

Juan Carlos Cortés*

Instituto Universitario de Matemática Multidisciplinar,
Universitat Politècnica de València,
Camino de Vera s/n, 46022, Valencia, Spain

Marc Jornet

Departament de Matemàtiques,
Universitat Jaume I,
12071 Castellón, Spain

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Abstract. This paper deals with the random wave equation on a bounded domain with Dirichlet boundary conditions. Randomness arises from the velocity wave, which is a positive random variable, and the two initial conditions, which are regular stochastic processes. The aleatory nature of the inputs is mainly justified from data errors when modeling the motion of a vibrating string. Uncertainty is propagated from these inputs to the output, so that the solution becomes a smooth random field. We focus on the mean square contextualization of the problem. Existence and uniqueness of the exact series solution, based upon the classical method of separation of variables, are rigorously established. Exact series for the mean and the variance of the solution process are obtained, which converge at polynomial rate. Some numerical examples illustrate these facts.

1. **Introduction.** The use of models based on partial differential equations is ubiquitous in science [1]. These models depend upon coefficients, boundary values, initial conditions, etc. that must be set from physical interpretation or from data. Data are inherently uncertain, due to ignorance of the process under study and measurement errors. Such uncertainty is transmitted to the model parameters; therefore, randomness should be incorporated from the beginning into the model. A random partial differential equation problem considers the coefficients, boundary values, initial conditions, etc. of the deterministic analogue as random variables, regular stochastic processes or regular random fields. The solution is a regular random field. Its sample-paths are not the main concern, rather its statistical content is the main interest (uncertainty quantification) [2, 3, 4, 5].

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* Corresponding author: Juan Carlos Cortés.

Differential equations are based on calculus, which is constructed through the notion of limit. When the differential equation is considered in a stochastic sense, one should take into account that, in Probability Theory, there are different notions of limit: limit almost surely (a.s.), limit in probability, limit in Lebesgue spaces (statistical moments), etc. A great deal of research has been devoted to random ordinary and partial differential equations in the Lebesgue sense. Recall that, given a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the Lebesgue space $\mathrm{L}^p(\Omega)$, $1 \leq p < \infty$, is the set of random variables $U : \Omega \to \mathbb{R}$ such that $\|U\|_p = (\mathbb{E}[|U|^p])^{1/p} < \infty$. Here $\mathbb{E}$ denotes the expectation operator. When $p = \infty$, $\|U\|_\infty = \inf\{C > 0 : |U| \leq C \text{ a.s.}\}$. These spaces are Banach. Given a stochastic process, its continuity, differentiability, Riemann integrability, etc., can be defined in the sense of $(\mathrm{L}^p(\Omega), \|\cdot\|_p)$, by considering the corresponding limits. This approach leads to a new random calculus. Of particular importance is the case $p = 2$, which gives rise to a Hilbert space with the inner product $(U, V) \mapsto \mathbb{E}[UV]$. The random variables in $\mathrm{L}^2(\Omega)$ have well-defined expectation $\mathbb{E}$ and variance $\mathbb{V}$. The calculus in $\mathrm{L}^2(\Omega)$ is called mean square (m.s.) calculus [2, 6].

In this paper, we deal with the random wave equation on a bounded spatial domain, $[0, L]$, with Dirichlet boundary conditions:

$$\begin{cases} u_{tt}(x,t) = \alpha^2 u_{xx}(x,t), & 0 < x < L,\ t > 0, \\ u(0,t) = u(L,t) = 0, & t \geq 0, \\ u(x,0) = f(x), & 0 \leq x \leq L, \\ u_t(x,0) = g(x), & 0 \leq x \leq L. \end{cases} \tag{1}$$

The parameter α^2 is assumed to be a positive random variable, and it equals the ratio of the tension and the linear density of the string; physically, α represents the velocity wave. The boundary conditions in (1) reflect the fact that the string is held fixed at the endpoints $x = 0$ and $x = L$. The initial conditions, $f = f(x)$ and $g = g(x)$, represent the initial displacement and the initial velocity of each point on the string. These are assumed to be regular stochastic processes. The space $C^p([0, L])$, which shall be used later, denotes the existence of continuous derivatives on $[0, L]$ up to order p. The solution $u(x, t)$ is a smooth random field. The stochastic nature of the inputs α^2, $f(x)$ and $g(x)$, is mainly justified from measurement errors when modeling the motion of a vibrating string using real data. Here we treat (1) in the m.s. sense, namely the partial derivatives are defined by m.s. limits.

Our context should not be confused with the Itô approach for stochastic (partial) differential equations. In the Itô sense, randomness arises from a white noise perturbation (formal derivative of Brownian motion), which gives rise to a random field solution that is continuous but nowhere differentiable. The formalization of these ideas is made through Itô calculus [7, 8, 9].

Problem (1) is a mixed problem whose formal solution can be constructed through Fourier series, by the classical method of separation of variables [10, 11, 12]. In the deterministic setting, extensions to (1) have been tackled: time-dependent wave velocity [13], telegraph equation [14], and retarded wave model [15]. However, in the random m.s. sense from [2], there is more work to be done. The random heat equation has been investigated in [16]. To our knowledge, the m.s. treatment of the method of separation of variables for the wave model (1) has not been conducted yet.

The goal of this paper is to rigorously study the stochastic problem (1), by using m.s. calculus. To this end, in Section 2 we take advantage of the series representation of the solution, which can be obtained using the classical method of separation of variables, and then we provide sufficient conditions in order to guarantee that the solution converges in the m.s. sense. As Section 3 demonstrates, m.s. convergence is useful to approximate the expectation and the variance of the solution. Two numerical examples illustrate this fact in Section 4. Finally, Section 5 draws the main conclusions.

2. Main result: sufficient conditions for the existence and uniqueness of a m.s. solution.

According to the deterministic theory, the method of separation variables can be applied to obtain a formal series solution to the initial-boundary value problem (1). Solutions of the form $u(x,t) = X(x)T(t)$ are then sought, where $X(x)$ and $T(t)$ are unknown functions. Afterward, linear superposition generates the candidate series solution

$$
\begin{aligned}
u(x,t) = & \sum_{n=1}^{\infty} \frac{2}{L} \left(\int_0^L f(y) \sin\left(\frac{n\pi y}{L}\right) \mathrm{d}y \right) \cos\left(\frac{n\pi\alpha}{L} t\right) \sin\left(\frac{n\pi x}{L}\right) \\
& + \sum_{n=1}^{\infty} \frac{2}{n\pi\alpha} \left(\int_0^L g(y) \sin\left(\frac{n\pi y}{L}\right) \mathrm{d}y \right) \sin\left(\frac{n\pi\alpha}{L} t\right) \sin\left(\frac{n\pi x}{L}\right). \qquad (2)
\end{aligned}
$$

When α is a random variable and f and g are stochastic processes, such $u(x,t)$ is a random field. This work is devoted to finding conditions under which $u(x,t)$ defined by (2) is a m.s. solution to (1) [2]. To do so, the usual operations carried out in classical calculus need to be rigorously verified using the m.s. calculus.

The following result provides two sets of sufficient conditions so that existence and uniqueness of m.s. solution to (1) can be guaranteed. It is interesting to compare both sets of conditions. In the first set of hypotheses (see H1 and H6 down below), it is assumed that the velocity of the wave, $\alpha(\omega)$, is bounded above and bounded away from zero, and the stochastic process $g(x)$, describing the initial velocity, belongs to $C^3([0,L])$ (see H3). While in the second set of hypotheses, we remove the condition that $\alpha(\omega) \geq \alpha_0$ a.s., at the expense of assuming the stronger condition $g \in C^4([0,L])$ (see H2').

Theorem 2.1. *Consider the random wave equation (1). Let $f = f(x)$ and $g = g(x)$ be two stochastic processes on $x \in [0,L]$, and let α be a random variable that is positive a.s. Consider the following two sets of hypotheses:*

- **H1:** $\alpha(\omega) \geq \alpha_0 > 0$ *a.s., where α_0 is constant;*
- **H2:** $f \in C^4([0,L])$ *in the m.s. sense;*
- **H3:** $g \in C^3([0,L])$ *in the m.s. sense;*
- **H4:** $f(0) = f(L) = 0$, $f''(0) = f''(L) = 0$ *a.s.;*
- **H5:** $g(0) = g(L) = 0$, $g''(0) = g''(L) = 0$ *a.s.;*
- **H6:** α *is bounded a.s.;*

and

- **H1':** $f \in C^4([0,L])$ *in the m.s. sense;*
- **H2':** $g \in C^4([0,L])$ *in the m.s. sense;*
- **H3':** $f(0) = f(L) = 0$, $f''(0) = f''(L) = 0$ *a.s.;*
- **H4':** $g(0) = g(L) = 0$, $g''(0) = g''(L) = 0$ *a.s.;*
- **H5':** α *is bounded a.s.*

If H1–H6 or H1'–H5' hold, then u, given by (2), is $C^2([0,L]\times[0,\infty))$ *in the m.s. sense and the unique m.s. solution to (1). The rate of m.s. convergence of (2), truncated at N terms, is* $\mathcal{O}(N^{-3})$*. The constant corresponding to* $\mathcal{O}$ *depends on the magnitude of L, on* $\max_{x\in[0,L]}\|f^{(iv)}(x)\|_2$, $\max_{x\in[0,L]}\|g'''(x)\|_2$ *(if H1–H6 are assumed), and* $\max_{x\in[0,L]}\|g^{(iv)}(x)\|_2$ *(if H1'–H5' are assumed).*

Proof. Let us first suppose H1–H6. We need to prove that the series $u(t,x)$ is m.s. convergent. Since $\|\cos U\|_\infty \le 1$ and $\|\sin U\|_\infty \le 1$ for every random variable U, the convergence of the series depends on the decay of the Fourier coefficients. Integration by parts in the m.s. sense [2, p. 104], together with H2 and H4, render

$$\int_0^L f(y)\sin\left(\frac{n\pi y}{L}\right)\mathrm{d}y = \frac{L^4}{n^4\pi^4}\int_0^L f^{(iv)}(y)\sin\left(\frac{n\pi y}{L}\right)\mathrm{d}y. \tag{3}$$

By [2, p. 102] and H4,

$$\begin{aligned}\left\|\int_0^L f(y)\sin\left(\frac{n\pi y}{L}\right)\mathrm{d}y\right\|_2 &\le \frac{L^4}{n^4\pi^4}\int_0^L \|f^{(iv)}(y)\|_2\mathrm{d}y \\ &\le \frac{L^5}{n^4\pi^4}\max_{x\in[0,L]}\|f^{(iv)}(x)\|_2 = \mathcal{O}(n^{-4}).\end{aligned} \tag{4}$$

Analogously, H3 and H5 yield

$$\left\|\int_0^L g(y)\sin\left(\frac{n\pi y}{L}\right)\mathrm{d}y\right\|_2 \le \frac{L^4}{n^3\pi^3}\max_{x\in[0,L]}\|g'''(x)\|_2 = \mathcal{O}(n^{-3}). \tag{5}$$

Notice also that, by H1,

$$\left\|\frac{1}{\alpha}\right\|_\infty \le \frac{1}{\alpha_0} < \infty.$$

Thus, the n-th term of (2) is dominated, in 2-norm, as $\mathcal{O}(n^{-4})$. Convergence of $\sum_{n=1}^\infty n^{-4}$ implies that the series (2) converges in the m.s. sense. Note that H6 has not been used yet.

To differentiate $u(x,t)$ in the m.s. sense, [16, Theorem 3.1] allows for interchanging differentiation and series. Notice that $\cos(n\pi\alpha t/L)$, $\sin(n\pi x/L)$ and $\sin(n\pi\alpha t/L)$ are m.s. differentiable, by [6, Theorem 3.19] (m.s. chain rule). The series of the partial derivatives of $u(x,t)$ converge uniformly in the m.s. sense, by Weierstrass M-test. For example, if $u_{tt}(x,t)$ is formally written, then using (4) and (5), the 2-norm of the corresponding series is bounded (except a constant) by

$$\begin{aligned}&\sum_{n=1}^\infty n^2\|\alpha\|_\infty^2\left\|\int_0^L f(y)\sin\left(\frac{n\pi y}{L}\right)\mathrm{d}y\right\|_2 + \sum_{n=1}^\infty n\|\alpha\|_\infty\left\|\int_0^L g(y)\sin\left(\frac{n\pi y}{L}\right)\mathrm{d}y\right\|_2 \\ &= \mathcal{O}\left(\sum_{n=1}^\infty n^{-2}\right) < \infty.\end{aligned}$$

If $u_{xx}(x,t)$ is formally written, the 2-norm of the corresponding series is bounded (except a constant) by

$$\begin{aligned}&\sum_{n=1}^\infty n^2\left\|\int_0^L f(y)\sin\left(\frac{n\pi y}{L}\right)\mathrm{d}y\right\|_2 + \sum_{n=1}^\infty n\left\|\int_0^L g(y)\sin\left(\frac{n\pi y}{L}\right)\mathrm{d}y\right\|_2 \\ &= \mathcal{O}\left(\sum_{n=1}^\infty n^{-2}\right) < \infty.\end{aligned}$$

These uniform convergences permit differentiating under the summation sign. Notice that H6 has been used.

Under H1'–H5', the same bound (4) holds, by H1' and H3' (which are the same as H2 and H4). Now (5) becomes

$$\left\| \int_0^L g(y) \sin\left(\frac{n\pi y}{L}\right) \mathrm{d}y \right\|_2 \leq \frac{L^5}{n^4\pi^4} \max_{x\in[0,L]} \|g^{(iv)}(x)\|_2 = \mathcal{O}(n^{-4}),$$

by H2' and H4'. From $|\sin y| \leq |y|$ for all y,

$$\left| \frac{1}{n\pi\alpha} \sin\left(\frac{n\pi\alpha}{L} t\right) \right| \leq \frac{t}{L}, \tag{6}$$

for every n. The same reasoning as H1–H6 applies. The n-th term of (2) is dominated, in 2-norm, as $\mathcal{O}(n^{-4})$. Convergence of $\sum_{n=1}^{\infty} n^{-4}$ implies that the series (2) converges in the m.s. sense. Note that H5' has not been used yet. This hypothesis is used to differentiate $u(x,t)$ in the m.s. sense, by [16, Theorem 3.1].

For uniqueness, a similar argument to the energy method from [17, Theorem 3.1] is used. It is only assumed that α is bounded a.s. Suppose that u_1 and u_2 are two smooth solutions to (1). Let $v = u_1 - u_2$, which satisfies (1) with $f = 0$ and $g = 0$, i.e.,

$$\begin{cases} v_{tt}(x,t) = \alpha^2 v_{xx}(x,t), & 0 < x < L,\ t > 0, \\ v(0,t) = v(L,t) = 0, & t \geq 0, \\ v(x,0) = 0, & 0 \leq x \leq L, \\ v_t(x,0) = 0, & 0 \leq x \leq L. \end{cases} \tag{7}$$

We shall prove that $v = 0$ a.s. Let

$$I(t) = \frac{1}{2} \int_0^L \mathbb{E}\left[\alpha^2 v_x^2(x,t) + v_t^2(x,t)\right] \mathrm{d}x. \tag{8}$$

Let us first observe that differentiating the first initial condition with respect to x, $v(x,0) = 0$, leads to $v_x(x,0) = 0$. So, this fact and the second initial condition, $v_t(x,0) = 0$, entail that $I(0) = 0$. This conclusion will be used later on.

The expectation $\mathbb{E}[\cdot]$ in expression (8) is well-defined and continuous (hence integrable), because α is bounded and v_x, v_t exist and are continuous in the m.s. sense. By differentiating,

$$\begin{aligned} I'(t) &= \frac{1}{2} \int_0^L \frac{\partial}{\partial t} \mathbb{E}\left[\alpha^2 v_x^2(x,t) + v_t^2(x,t)\right] \mathrm{d}x \\ &= \int_0^L \mathbb{E}\left[\alpha^2 v_x(x,t) v_{xt}(x,t) + v_t(x,t) v_{tt}(x,t)\right] \mathrm{d}x. \end{aligned}$$

Notice that the derivative and the expectation can be interchanged, because the limit in the derivative is considered in $\mathrm{L}^1(\Omega)$. We have, by integration in $\mathrm{L}^1(\Omega)$ and integration by parts (taking here into account the last initial condition in (7),

$v_t(x,0)=0$, at $x=0$ and $x=L$),

$$\int_0^L \mathbb{E}\left[\alpha^2 v_x(x,t)v_{xt}(x,t)\right]\mathrm{d}x = \mathbb{E}\left[\alpha^2\int_0^L v_x(x,t)v_{xt}(x,t)\mathrm{d}x\right]$$
$$= -\,\mathbb{E}\left[\alpha^2\int_0^L v_t(x,t)v_{xx}(x,t)\mathrm{d}x\right]$$
$$= -\int_0^L \mathbb{E}\left[\alpha^2 v_t(x,t)v_{xx}(x,t)\right]\mathrm{d}x.$$

As a consequence, using that $v(x,t)$ solves (7), one gets

$$I'(t)=\int_0^L \mathbb{E}\left[v_t(x,t)\left(v_{tt}(x,t)-\alpha^2 v_{xx}(x,t)\right)\right]\mathrm{d}x = 0.$$

This implies that $I(t)$ is constant, and since $I(0)=0$, we conclude that $I(t)=0$ for all t. Therefore, from (8), $v_x=v_t=0$ a.s. By the m.s. Barrow's rule [2, p. 104],

$$v(x,t)=\int_0^x v_x(y,t)\mathrm{d}y+\int_0^t v_t(0,s)\mathrm{d}s+v(0,0)=0,$$

as wanted.

Finally, the rate of convergence of (2) is deduced from the fact that the n-th term of (2) is $\mathcal{O}(n^{-4})$, uniformly on n. We have that

$$\sum_{n=N}^{\infty} n^{-4}\le\sum_{n=N}^{\infty}\int_{n-1}^{n}x^{-4}\mathrm{d}x=\int_{N-1}^{\infty}x^{-4}\mathrm{d}x=\frac{1}{3(N-1)^3},$$

so the series (2) converges at rate $\mathcal{O}(N^{-3})$. □

Remark 1. The following observations concern the hypotheses of Theorem 2.1.

1. No independence between the inputs α, f and g has been assumed, which confers more generality to Theorem 2.1. This is an important point, since most of the results on random systems presume independence between the input random parameters [4, chapter 4].
2. The conditions $f(0)=f(L)=0$ and $g(0)=g(L)=0$, assumed in hypotheses H5 and H4', respectively, have been employed when applying for the first time the m.s. integration by parts formula in (3) and (5), respectively. But in addition, it must be noticed that these are also required for the initial and boundary conditions to be consistent.
3. Let us compare H1–H6 and H1'–H5'. Under H3, the bound in (5) is $\mathcal{O}(n^{-3})$, instead of $\mathcal{O}(n^{-4})$ as in (4). The terms $1/(n\alpha)$ in (2) make the complete n-th terms within the second series of (2) bounded as $\mathcal{O}(n^{-4})$ in m.s. It is used H1 to avoid α be near 0. In H1'–H5', α may be near 0 and $1/(n\alpha)$ be large towards infinity as $\alpha\to 0$ for a sequence of realizations. Then, the best we can do is to use the bound (6). To make the complete n-th terms within the second series of (2) bounded as $\mathcal{O}(n^{-4})$ in 2-norm, we need H2' instead of H3. To summarize, there should be a balance between $(1/(n\alpha))\sin(n\pi\alpha t/L)$ and $\int_0^L g(y)\sin(n\pi y/L)\mathrm{d}y$ to achieve the m.s. decay $\mathcal{O}(n^{-4})$. The m.s. decay $\mathcal{O}(n^{-4})$ for the complete n-th terms within the two series of (2) is necessary to differentiate twice in m.s. with respect to t and x.

4. For uniqueness, it is assumed that there are two m.s. solutions u_1 and u_2, and it is proved that $v = u_1 - u_2$ is 0 a.s. The only hypothesis required is the a.s. boundedness of α (since it is already supposed the existence of u_1 and u_2).

3. Approximations of the mean and variance of the solution.

So far, we have established sufficient conditions in order to guarantee that the series (2), which defines the m.s. solution to problem (1), converges in the m.s. sense. Now, we will take advantage of the following key properties of this strong stochastic convergence type to compute reliable approximations of the mean and the variance of the solution by truncating the series (2).

Lemma 3.1. [2, Theorem 4.3.1] *If $\{U_N : N \geq 0\}$ is a sequence of random variables in $\mathrm{L}^2(\Omega)$ such that $U_N \to U \in \mathrm{L}^2(\Omega)$ in the m.s. sense, then*

$$\mathbb{E}[U_N] \xrightarrow[N\to\infty]{} \mathbb{E}[U] \quad \text{and} \quad \mathbb{E}[(U_N)^2] \xrightarrow[N\to\infty]{} \mathbb{E}[U^2]. \tag{9}$$

Since the variance can be written as $\mathbb{V}[U_N] = \mathbb{E}[(U_N)^2] - (\mathbb{E}[U_N])^2$, one also gets

$$\mathbb{V}\left[U_N\right] \xrightarrow[N\to\infty]{} \mathbb{V}\left[U\right]. \tag{10}$$

Proposition 1. [2, p. 104]. *Let $\{X(y) : -\infty \leq a \leq y \leq b \leq +\infty\}$ be a m.s. integrable stochastic process and $h(y)$ a Riemann integrable deterministic function on $y \in (a, b)$. Then*

$$\mathbb{E}\left[\int_a^b h(y)X(y)\mathrm{d}y\right] = \int_a^b h(y)\mathbb{E}[X(y)]\mathrm{d}y,$$

where $\int_a^b h(y)X(y)\mathrm{d}y$ is a m.s. Riemann integral and $\int_a^b h(y)\mathbb{E}[X(y)]\mathrm{d}y$ is an ordinary Riemann integral.

Remark 2. The result given in Proposition 1 can easily be generalized to dimension n. Later, we will only use the bidimensional case:

$$\mathbb{E}\left[\int_a^b\int_c^d h_1(y)h_2(z)X_1(y)X_2(z)\mathrm{d}y\,\mathrm{d}z\right] = \int_a^b\int_c^d h_1(y)h_2(z)\mathbb{E}[X_1(y)X_2(z)]\mathrm{d}y\,\mathrm{d}z.$$

So, let us denote by $u_N(x,t)$ the finite sum defining the truncation of order N of the series solution (2),

$$\begin{aligned} u_N(x,t) = {} & \sum_{n=1}^{N} \frac{2}{L}\left(\int_0^L f(y)\sin\left(\frac{n\pi y}{L}\right)\mathrm{d}y\right)\cos\left(\frac{n\pi\alpha}{L}t\right)\sin\left(\frac{n\pi x}{L}\right) \\ & + \sum_{n=1}^{N}\frac{2}{n\pi\alpha}\left(\int_0^L g(y)\sin\left(\frac{n\pi y}{L}\right)\mathrm{d}y\right)\sin\left(\frac{n\pi\alpha}{L}t\right)\sin\left(\frac{n\pi x}{L}\right). \end{aligned} \tag{11}$$

Let us assume that (f, g) and α are independent. This is done here for convenience of notation and easiness of manipulation and calculation [4, chapter 4]. An important property of independence is the following:

$$\mathbb{E}[H_1(f(y))H_2(\alpha)] = \mathbb{E}[H_1(f(y))]\mathbb{E}[H_2(\alpha)],$$

$$\mathbb{E}[H_1(g(y))H_2(\alpha)] = \mathbb{E}[H_1(g(y))]\mathbb{E}[H_2(\alpha)],$$

for any two Borel measurable real functions H_1 and H_2 [18, p. 92]. That is, expectations are split. As a consequence, one first obtains

$$\mathbb{E}\left[\left(\int_0^L f(y)\sin\left(\frac{n\pi y}{L}\right)\mathrm{d}y\right)\cos\left(\frac{n\pi\alpha}{L}t\right)\right]$$
$$=\mathbb{E}\left[\left(\int_0^L f(y)\sin\left(\frac{n\pi y}{L}\right)\mathrm{d}y\right)\right]\mathbb{E}\left[\cos\left(\frac{n\pi\alpha}{L}t\right)\right],$$

$$\mathbb{E}\left[\frac{1}{\alpha}\left(\int_0^L g(y)\sin\left(\frac{n\pi y}{L}\right)\mathrm{d}y\right)\sin\left(\frac{n\pi\alpha}{L}t\right)\right]$$
$$=\mathbb{E}\left[\left(\int_0^L g(y)\sin\left(\frac{n\pi y}{L}\right)\mathrm{d}y\right)\right]\mathbb{E}\left[\frac{1}{\alpha}\sin\left(\frac{n\pi\alpha}{L}t\right)\right].$$

Next, we apply Proposition 1 with $a=0$, $b=L$, $h(y)=\sin\left(\frac{n\pi y}{L}\right)$ and $X(y)=f(y)$ ($X(y)=g(y)$) to express the above expectations of the resulting integrals as follows:

$$\mathbb{E}\left[\left(\int_0^L f(y)\sin\left(\frac{n\pi y}{L}\right)\mathrm{d}y\right)\cos\left(\frac{n\pi\alpha}{L}t\right)\right]$$
$$=\left(\int_0^L \mathbb{E}[f(y)]\sin\left(\frac{n\pi y}{L}\right)\mathrm{d}y\right)\mathbb{E}\left[\cos\left(\frac{n\pi\alpha}{L}t\right)\right],$$

$$\mathbb{E}\left[\frac{1}{\alpha}\left(\int_0^L g(y)\sin\left(\frac{n\pi y}{L}\right)\mathrm{d}y\right)\sin\left(\frac{n\pi\alpha}{L}t\right)\right]$$
$$=\left(\int_0^L \mathbb{E}[g(y)]\sin\left(\frac{n\pi y}{L}\right)\mathrm{d}y\right)\mathbb{E}\left[\frac{1}{\alpha}\sin\left(\frac{n\pi\alpha}{L}t\right)\right]. \tag{12}$$

Then, taking the expectation operator in expression (11), using its linearity together with (12), one finally obtains the expectation of the approximation of order N, $u_N(x,t)$,

$$\mathbb{E}[u_N(x,t)]=\sum_{n=1}^{N}\frac{2}{L}\left(\int_0^L \mathbb{E}[f(y)]\sin\left(\frac{n\pi y}{L}\right)\mathrm{d}y\right)\mathbb{E}\left[\cos\left(\frac{n\pi\alpha}{L}t\right)\right]\sin\left(\frac{n\pi x}{L}\right)$$
$$+\sum_{n=1}^{N}\frac{2}{n\pi}\left(\int_0^L \mathbb{E}[g(y)]\sin\left(\frac{n\pi y}{L}\right)\mathrm{d}y\right)\mathbb{E}\left[\frac{1}{\alpha}\sin\left(\frac{n\pi\alpha}{L}t\right)\right]\sin\left(\frac{n\pi x}{L}\right). \tag{13}$$

On the other hand, simple algebra yields

$$
\begin{aligned}
u_N^2(x,t) = & \sum_{n,m=1}^{N} \frac{4}{L^2} \left(\int_0^L \int_0^L f(y) f(z) \sin\left(\frac{n\pi y}{L}\right) \sin\left(\frac{m\pi z}{L}\right) \mathrm{d}y \, \mathrm{d}z \right) \\
& \times \cos\left(\frac{n\pi\alpha}{L} t\right) \cos\left(\frac{m\pi\alpha}{L} t\right) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) \\
& + \sum_{n,m=1}^{N} \frac{4}{nm\pi^2\alpha^2} \left(\int_0^L \int_0^L g(y) g(z) \sin\left(\frac{n\pi y}{L}\right) \sin\left(\frac{m\pi z}{L}\right) \mathrm{d}y \, \mathrm{d}z \right) \\
& \times \sin\left(\frac{n\pi\alpha}{L} t\right) \sin\left(\frac{m\pi\alpha}{L} t\right) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) \\
& + 2 \sum_{n,m=1}^{N} \frac{4}{Lm\pi\alpha} \left(\int_0^L \int_0^L f(y) g(z) \sin\left(\frac{n\pi y}{L}\right) \sin\left(\frac{m\pi z}{L}\right) \mathrm{d}y \, \mathrm{d}z \right) \\
& \times \cos\left(\frac{n\pi\alpha}{L} t\right) \sin\left(\frac{m\pi\alpha}{L} t\right) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right).
\end{aligned}
$$

Similarly as we have shown for the expectation of $u_N(x,t)$ and using the Remark 2, we obtain

$$
\begin{aligned}
\mathbb{E}[u_N^2(x,t)] = & \sum_{n,m=1}^{N} \frac{4}{L^2} \left(\int_0^L \int_0^L \mathbb{E}[f(y) f(z)] \sin\left(\frac{n\pi y}{L}\right) \sin\left(\frac{m\pi z}{L}\right) \mathrm{d}y \, \mathrm{d}z \right) \\
& \times \mathbb{E}\left[\cos\left(\frac{n\pi\alpha}{L} t\right) \cos\left(\frac{m\pi\alpha}{L} t\right)\right] \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) \\
& + \sum_{n,m=1}^{N} \frac{4}{n\pi^2 m} \left(\int_0^L \int_0^L \mathbb{E}[g(y) g(z)] \sin\left(\frac{n\pi y}{L}\right) \sin\left(\frac{m\pi z}{L}\right) \mathrm{d}y \, \mathrm{d}z \right) \\
& \times \mathbb{E}\left[\frac{1}{\alpha^2} \sin\left(\frac{n\pi\alpha}{L} t\right) \sin\left(\frac{m\pi\alpha}{L} t\right)\right] \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) \\
& + 2 \sum_{n,m=1}^{N} \frac{4}{Lm\pi} \left(\int_0^L \int_0^L \mathbb{E}[f(y) g(z)] \sin\left(\frac{n\pi y}{L}\right) \sin\left(\frac{m\pi z}{L}\right) \mathrm{d}y \, \mathrm{d}z \right) \\
& \times \mathbb{E}\left[\frac{1}{\alpha} \cos\left(\frac{n\pi\alpha}{L} t\right) \sin\left(\frac{m\pi\alpha}{L} t\right)\right] \sin\left(\frac{n\pi}{L} x\right) \sin\left(\frac{m\pi x}{L}\right).
\end{aligned} \tag{14}
$$

For every (x,t), $0 \le x \le L$ and $t > 0$, in Theorem 2.1 we have shown that $u_N(x,t)$ is m.s. convergent to $u(x,t)$ given by (2). Then applying Lemma 3.1 it is guaranteed that

$$
\mathbb{E}[u_N(x,t)] \xrightarrow[n\to\infty]{} \mathbb{E}[u(x,t)] \quad \text{and} \quad \mathbb{E}[u_N^2(x,t)] \xrightarrow[N\to\infty]{} \mathbb{E}[u^2(x,t)],
$$

where $\mathbb{E}[u_N(x,t)]$ and $\mathbb{E}[u_N^2(x,t)]$ are defined by (13) and (14), respectively, and $\mathbb{E}[u(x,t)]$ and $\mathbb{E}[u^2(x,t)]$ are given, respectively, by

$$
\begin{aligned}
\mathbb{E}[u(x,t)] = & \sum_{n=1}^{\infty} \frac{2}{L} \left(\int_0^L \mathbb{E}[f(y)] \sin\left(\frac{n\pi y}{L}\right) \mathrm{d}y \right) \mathbb{E}\left[\cos\left(\frac{n\pi\alpha}{L} t\right)\right] \sin\left(\frac{n\pi x}{L}\right) \\
& + \sum_{n=1}^{\infty} \frac{2}{n\pi} \left(\int_0^L \mathbb{E}[g(y)] \sin\left(\frac{n\pi y}{L}\right) \mathrm{d}y \right) \mathbb{E}\left[\frac{1}{\alpha} \sin\left(\frac{n\pi\alpha}{L} t\right)\right] \sin\left(\frac{n\pi x}{L}\right).
\end{aligned} \tag{15}
$$

and

$$\begin{aligned}\mathbb{E}[u^2(x,t)] = &\sum_{n,m=1}^{\infty} \frac{4}{L^2}\left(\int_0^L \int_0^L \mathbb{E}[f(y)f(z)] \sin\left(\frac{n\pi y}{L}\right) \sin\left(\frac{m\pi z}{L}\right) \mathrm{d}y\, \mathrm{d}z\right) \\ &\times \mathbb{E}\left[\cos\left(\frac{n\pi\alpha}{L}t\right)\cos\left(\frac{m\pi\alpha}{L}t\right)\right] \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) \\ &+ \sum_{n,m=1}^{\infty} \frac{4}{n\pi^2 m}\left(\int_0^L \int_0^L \mathbb{E}[g(y)g(z)] \sin\left(\frac{n\pi y}{L}\right) \sin\left(\frac{m\pi z}{L}\right) \mathrm{d}y\, \mathrm{d}z\right) \\ &\times \mathbb{E}\left[\frac{1}{\alpha^2}\sin\left(\frac{n\pi\alpha}{L}t\right)\sin\left(\frac{m\pi\alpha}{L}t\right)\right] \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) \\ &+2 \sum_{n,m=1}^{\infty} \frac{4}{Lm\pi}\left(\int_0^L \int_0^L \mathbb{E}[f(y)g(z)] \sin\left(\frac{n\pi y}{L}\right) \sin\left(\frac{m\pi z}{L}\right) \mathrm{d}y\, \mathrm{d}z\right) \\ &\times \mathbb{E}\left[\frac{1}{\alpha}\cos\left(\frac{n\pi\alpha}{L}t\right)\sin\left(\frac{m\pi\alpha}{L}t\right)\right] \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi x}{L}\right). \end{aligned} \tag{16}$$

The approximation of the variance is obtained using (10),

$$\mathbb{V}[u_N(x,t)] = \mathbb{E}[u_N^2(x,t)] - (\mathbb{E}[u_N(x,t)])^2 \xrightarrow[N\to\infty]{} \mathbb{E}[u^2(x,t)] - (\mathbb{E}[u(x,t)])^2 = \mathbb{V}[u(x,t)]. \tag{17}$$

Now, we summarize the main result obtained throughout our previous development:

Theorem 3.2. *Consider the random wave equation* (1)*. Let f and g be two stochastic processes on $[0, L]$, and let α be a random variable that is positive a.s. Assume that (f, g) and α are independent. Suppose that H1–H6 or H1'–H5' hold. Then the expectation and the variance of the m.s. solution $u(x, t)$, given by* (2)*, are* (15) *and* (16)*–*(17)*, respectively. Given the truncation $u_N(x, t)$* (11)*, the convergence of $\mathbb{E}[u_N(x, t)]$ and $\mathbb{V}[u_N(x, t)]$ towards $\mathbb{E}[u(x, t)]$ and $\mathbb{V}[u(x, t)]$ is guaranteed. Besides, the rate of convergence of these statistics is $\mathcal{O}(N^{-3})$.*

Proof. The convergence of $\mathbb{E}[u_N(x, t)]$ and $\mathbb{V}[u_N(x, t)]$ towards $\mathbb{E}[u(x, t)]$ and $\mathbb{V}[u(x, t)]$ is guaranteed by the m.s. convergence; check the development before the theorem. For the rate of convergence, it suffices to note that the rapidity at which the expectation and the variance converge is inherited by the rapidity of m.s. convergence (see the part at the end of the proof of Theorem 2.1):

$$\begin{aligned}|\mathbb{E}[u_N(x,t)] - \mathbb{E}[u(x,t)]| &= |\mathbb{E}[u_N(x,t) - u(x,t)]| \le \mathbb{E}[|u_N(x,t) - u(x,t)|] \\ &\le \|u_N(x,t) - u(x,t)\|_2 = \mathcal{O}(N^{-3}),\end{aligned}$$

and

$$
\begin{aligned}
&|\mathbb{V}[u_N(x,t)] - \mathbb{V}[u(x,t)]| \\
&= |\mathbb{E}[u_N^2(x,t)] - (\mathbb{E}[u_N(x,t)])^2 - \mathbb{E}[u^2(x,t)] + (\mathbb{E}[u(x,t)])^2| \\
&\leq \mathbb{E}[|u_N^2(x,t) - u^2(x,t)|] + |(\mathbb{E}[u(x,t)])^2 - (\mathbb{E}[u(x,t)])^2| \\
&= \mathbb{E}[|u_N(x,t) - u(x,t)||u_N(x,t) + u(x,t)|] \\
&\quad + |\mathbb{E}[u_N(x,t)] - \mathbb{E}[u(x,t)]||\mathbb{E}[u_N(x,t)] + \mathbb{E}[u(x,t)]| \\
&\leq \|u_N(x,t) - u(x,t)\|_2 \|u_N(x,t) + u(x,t)\|_2 \\
&\quad + |\mathbb{E}[u_N(x,t)] - \mathbb{E}[u(x,t)]|(|\mathbb{E}[u_N(x,t)]| + |\mathbb{E}[u(x,t)]|) \\
&\leq \|u_N(x,t) - u(x,t)\|_2 (\|u_N(x,t)\|_2 + \|u(x,t)\|_2) \\
&\quad + |\mathbb{E}[u_N(x,t)] - \mathbb{E}[u(x,t)]|(|\mathbb{E}[u_N(x,t)]| + |\mathbb{E}[u(x,t)]|) \\
&= \mathcal{O}(N^{-3}).
\end{aligned}
$$

The triangular, Jensen's and Cauchy-Schwarz inequalities have been utilized. □

Remark 3. For simplicity of formulation, reasoning and calculation, it was assumed that (f, g) and α are independent. This assumption, which is usually used in the study of random systems, permitted splitting the expectation of expressions involving (f, g, α). Let us point out here the necessary modifications when no independence holds. The expressions in (12) become

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_0^L f(y) \sin\left(\frac{n\pi y}{L}\right) \mathrm{d}y\right) \cos\left(\frac{n\pi\alpha}{L}t\right)\right] \\
&= \left(\int_0^L \mathbb{E}\left[f(y) \cos\left(\frac{n\pi\alpha}{L}t\right)\right] \sin\left(\frac{n\pi y}{L}\right) \mathrm{d}y\right),
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\left[\frac{1}{\alpha}\left(\int_0^L g(y) \sin\left(\frac{n\pi y}{L}\right) \mathrm{d}y\right) \sin\left(\frac{n\pi\alpha}{L}t\right)\right] \\
&= \left(\int_0^L \mathbb{E}\left[g(y) \frac{1}{\alpha} \sin\left(\frac{n\pi\alpha}{L}t\right)\right] \sin\left(\frac{n\pi y}{L}\right) \mathrm{d}y\right).
\end{aligned}
$$

Then (13) becomes

$$
\begin{aligned}
\mathbb{E}[u_N(x,t)] = &\sum_{n=1}^N \frac{2}{L} \left(\int_0^L \mathbb{E}\left[f(y) \cos\left(\frac{n\pi\alpha}{L}t\right)\right] \sin\left(\frac{n\pi y}{L}\right) \mathrm{d}y\right) \sin\left(\frac{n\pi x}{L}\right) \\
&+ \sum_{n=1}^N \frac{2}{n\pi} \left(\int_0^L \mathbb{E}\left[g(y) \frac{1}{\alpha} \sin\left(\frac{n\pi\alpha}{L}t\right)\right] \sin\left(\frac{n\pi y}{L}\right) \mathrm{d}y\right) \sin\left(\frac{n\pi x}{L}\right).
\end{aligned}
$$

Similar changes occur for the statistical moment of order two. There is convergence as $N \to \infty$. Though the theoretical study of the expectation and the variance can be performed under no independence, numerical calculations may get severely affected, since the joint distribution of (f, g, α) is needed. The complexity could be partially lowered if f and α, or g and α, depend on the same random variables ξ, as the joint distribution of (f, α), or (g, α), would be related to the distribution of ξ.

Remark 4. When uncertainties are involved, randomness must be incorporated into the model from the beginning. It is not convenient to fix average values for the input parameters and then solve the problem deterministically, since the expectation of the stochastic process solution (15) does not coincide with the deterministic solution

$$\sum_{n=1}^{\infty} \frac{2}{L} \left(\int_0^L \mathbb{E}[f(y)] \sin\left(\frac{n\pi y}{L}\right) \mathrm{d}y \right) \cos\left(\frac{n\pi \mathbb{E}[\alpha]}{L} t\right) \sin\left(\frac{n\pi x}{L}\right)$$
$$+ \sum_{n=1}^{\infty} \frac{2}{n\pi} \left(\int_0^L \mathbb{E}[g(y)] \sin\left(\frac{n\pi y}{L}\right) \mathrm{d}y \right) \frac{1}{\mathbb{E}[\alpha]} \sin\left(\frac{n\pi \mathbb{E}[\alpha]}{L} t\right) \sin\left(\frac{n\pi x}{L}\right).$$

The method that consists in substituting each random input in a random differential equation by its mean value is sometimes referred to as *dishonest method*. It was first introduced by Keller, [19, Sec. 4.3.2]. A number of contributions have shown that it does not provide, in general, reliable approximations [20, 21]. It may provide sufficient approximations only when the uncertainties involved are not high.

The dishonest method is based on the approximation

$$\mathbb{E}[\Lambda(\xi)] \approx \Lambda(\mathbb{E}[\xi]),$$

where ξ is a random variable (the uncertainty) and Λ is a deterministic function. From the Taylor's expansion of Λ around the point $\mathbb{E}[\xi]$, it is obtained

$$\mathbb{E}[\Lambda(\xi)] = \Lambda(\mathbb{E}[\xi]) + \frac{\Lambda''(\mathbb{E}[\xi])}{2} \mathbb{V}[\xi] + \mathcal{O}(\mathbb{S}[\xi]),$$

where $\mathbb{S}$ is the skewness. Thus, the approximation presumed by the dishonest method needs $\xi \approx \mathbb{E}[\xi]$ with high probability.

4. Numerical examples.

Example 1. Fix the string length $L = 1$. Let f and g have the form $f(x) = ax^4(1-x)^3$ and $g(x) = bx^4(1-x)^3$. The variables α, a and b are independent random quantities; α is uniform on $[0,1]$, a is uniform on $[-0.1, 0]$, and b is triangular with endpoints $[1, 1.5]$ and mode 1.25. Figure 1 illustrates the approximations of the expectation and the variance of $u(x,t)$, through (13), (14) and (17). It is observed that, as the number of series terms N increases, the approximations become indistinguishable at the scale of the figure.

Figure 2 shows the rate of convergence, in log-log scale. Let $\mathrm{NP}(N)$ be the result of a numerical procedure NP with N terms, which tends to the exact solution e_0 as $N \to \infty$. Let p be the order of $\mathrm{NP}(N)$ (i.e. $|\mathrm{NP}(N) - e_0| = \mathcal{O}(N^{-p})$). Then

$$\lim_{N\to\infty} \frac{|\mathrm{NP}(N) - e_0|}{|\mathrm{NP}(N) - \mathrm{NP}(2N)|} = \frac{2^p}{2^p - 1}.$$

This is easily proved by expanding $\mathrm{NP}(N) = e_0 + e_1 N^{-p} + e_2 N^{-(p+1)} + \ldots$. Thus, for large N, the error $\mathrm{NP}(N) - e_0$ can be examined through the difference $\mathrm{NP}(N) - \mathrm{NP}(2N)$.

Example 2. Fix the string length $L = 1$. Let $f(x) = 2\sum_{n=1}^{\infty} \xi_n \sin(n\pi x)$, $x \in [0,1]$, where $\xi_n = U_n/n^{10}$, and U_n is exponentially distributed of mean value 1. Due to the rapid decrease of ξ_n with n, the process $f(x)$ satisfies the regularity requirements of the theorem. Notice that ξ_n corresponds to the Fourier coefficient $\int_0^1 f(y)\sin(n\pi y)\mathrm{d}y$, and the series $2\sum_{n=1}^{\infty} \xi_n \sin(n\pi x)$ is a Karhunen-Loève expansion. The Karhunen-Loève expansion is one of the most widely used techniques to

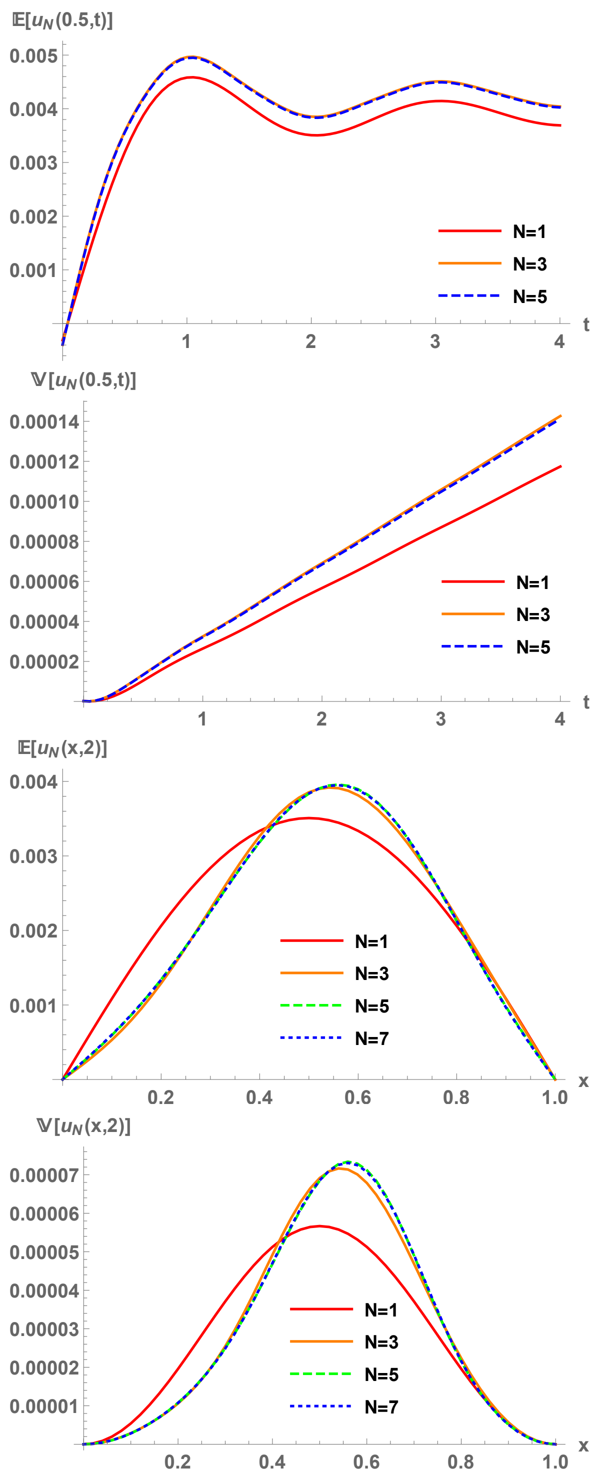

FIGURE 1. Expectation and variance of the solution $u(x,t)$ to (1), for different space-time points and orders of truncation N of the series (2). This figure corresponds to Example 1.

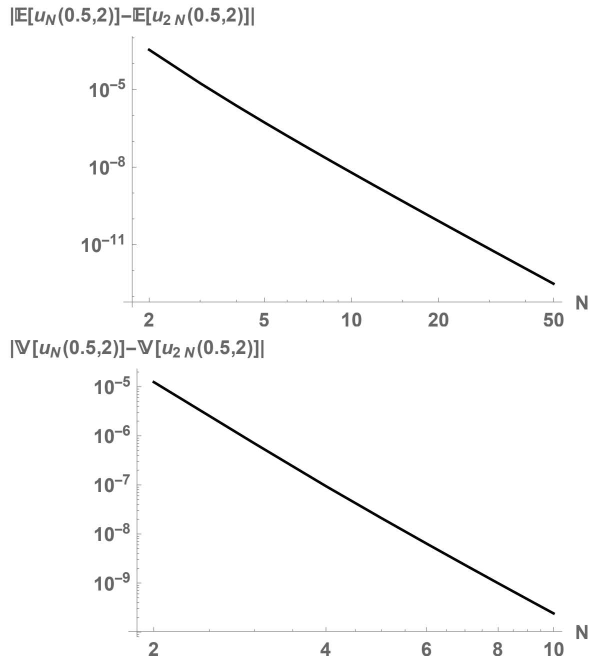

FIGURE 2. Rate of convergence of $\mathbb{E}[u_N(0.5,2)]$ and $\mathbb{V}[u_N(0.5,2)]$ with N, where $u_N(x,t)$ is the truncation (11) of $u(x,t)$ (2). This figure corresponds to Example 1.

represent infinite dimensional random processes in random systems [4]. Here we use it for illustration of our theoretical development. Let $g(x) = bx^4(1-x)^3$ and α be defined as in the previous example; the variables α and b are uniform on $[0,1]$ and triangular with endpoints $[1,1.5]$ and mode 1.25, respectively. All the random quantities, $U_1, U_2, \ldots$, α and b, are assumed to be independent. Figure 3 illustrates the approximations of the expectation and the variance of $u(x,t)$, through (13), (14) and (17). Figure 4 reports the rate of convergence.

5. **Conclusion.** We have successfully solved the random wave problem (1) in the m.s. sense. The approach consists in setting appropriate probabilistic assumptions on the input terms, so that the formal infinite series solution constructed by the method of separation of variables is a rigorous m.s. solution. The m.s. convergence of the series makes it possible to approximate the expectation and the variance of the solution, at polynomial convergence rate. Our strategy could be applied to further extensions of (1), namely problems with retarded or fractional terms. Also of interest would be the study of probability densities.

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Conflict of Interest Statement. The authors declare that there is no conflict of interests regarding the publication of this article.

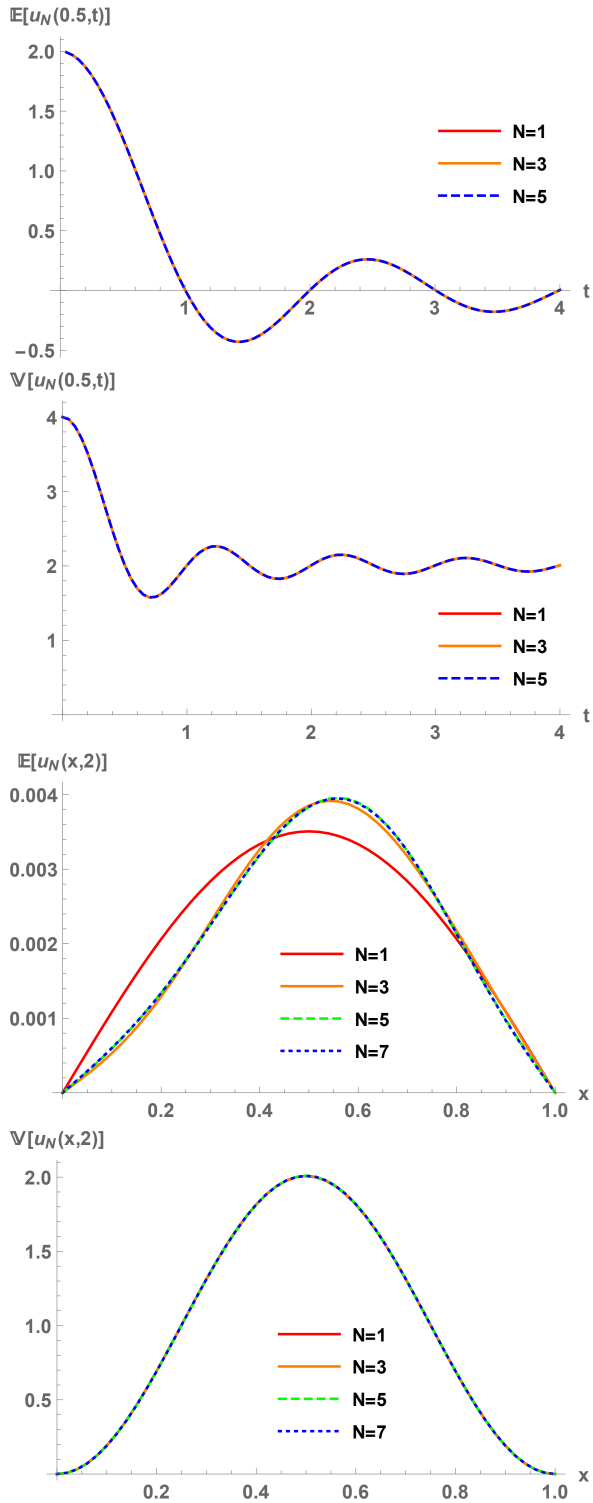

FIGURE 3. Expectation and variance of the solution $u(x, t)$ to (1), for different space-time points and orders of truncation N of the series (2). This figure corresponds to Example 2.

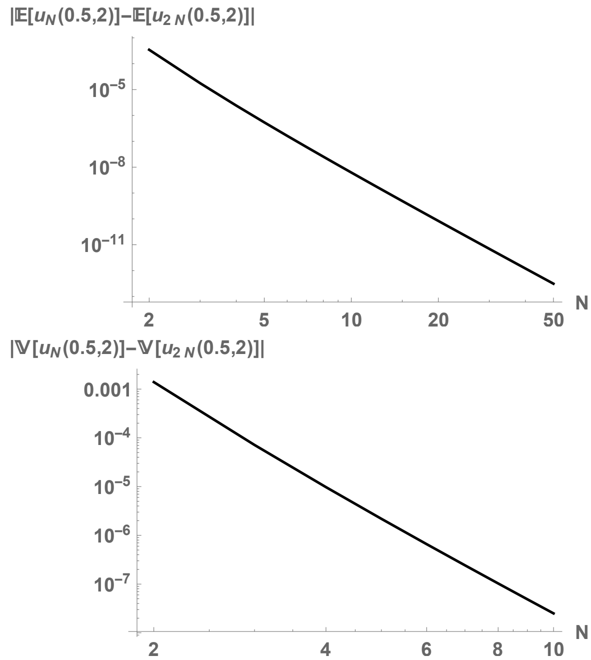

FIGURE 4. Rate of convergence of $\mathbb{E}[u_N(0.5, 2)]$ and $\mathbb{V}[u_N(0.5, 2)]$ with N, where $u_N(x, t)$ is the truncation (11) of $u(x, t)$ (2). This figure corresponds to Example 2.

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E-mail address: calatayj@uji.es
E-mail address: jccortes@imm.upv.es
E-mail address: jornet@uji.es