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ON THE CHANGE OF THE WEYR CHARACTERISTICS OF MATRIX PENCILS AFTER RANK-ONE PERTURBATIONS*

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Abstract. The change of the Kronecker structure of a matrix pencil perturbed by another pencil of rank one has been characterized in terms of the homogeneous invariant factors and the chains of column and row minimal indices of the initial and the perturbed pencils. We obtain here a new characterization in terms of the homogeneous invariant factors and the conjugate partitions of the corresponding chains of column and row minimal indices of both pencils. We also define the generalized Weyr characteristic of an arbitrary matrix pencil and obtain bounds for the change of it when the pencil is perturbed by another pencil of rank one. The results improve known results on the problem and hold for arbitrary perturbation pencils of rank one and for any algebraically closed field.

Key words. matrix pencil, Jordan chain, rank perturbations

AMS subject classifications. 15A22, 47A55, 15A18

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1. Introduction.

Much has been said about perturbations of matrix operators. In particular, changes of the Jordan structure of a square matrix or of the Weierstrass structure of a regular pencil under bounded rank perturbations have been studied, for example, in [6, 7, 15, 17, 18] from a generic point of view and in [1, 3, 19, 20, 21] for general perturbations. Results on perturbations of arbitrary pencils can be found in [2, 5, 9]. See also the references therein.

Recently, there have been obtained bounds for the changes of the generalized Weyr characteristic of a complex square matrix pencil (see Remark 3.8 below) perturbed by another pencil of the form $w(su^* - v^*)$ ([13, Theorem 7.8]). This has been done relating the Jordan chains of a square pencil with those of a linear relation. In this paper we extend the notion of a Jordan chain to possibly nonsquare matrix pencils and express the generalized Weyr characteristic of a pencil in terms of its Kronecker structure.

Observe that complex pencils of rank one can also be of the form $(su - v)w^*$. We also obtain bounds for the change of the generalized Weyr characteristic of a matrix pencil perturbed by another arbitrary matrix pencil of rank one over an algebraically closed field, improving the bounds of [13, Theorem 7.8].

Important to the present work is the characterization in [2, Theorem 5.1] of the changes of the Kronecker structure of a pencil perturbed by another pencil of rank one (see also [9]). We express here this characterization in terms of the conjugate partitions of the corresponding chains of column and row minimal indices of the pencils involved (Theorem 4.8 below). Although Theorem 5.1 in [2] and the current

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result in Theorem 4.8 hold for pencils over arbitrary fields, to simplify the analysis we only state here both (Theorem 5.1 in [2] and Theorem 4.8) for algebraically closed fields.

To achieve our results we have introduced two types of sequences of integers: partitions and chains. The partitions can be finite or infinite. We will identify two partitions if they only differ in the number of zeros. The chains are of fixed length. This distinction has to be kept in mind throughout the paper.

The paper is organized as follows: in section 2 we present the notation and some preliminary results, including [2, Theorem 5.1]. In section 3 we recall the definition of a Jordan chain, extend it to arbitrary matrix pencils, define the generalized Weyr characteristic of a pencil, and express it in terms of the Kronecker invariants. Section 4 is devoted to translating [2, Theorem 5.1] into terms of the conjugate partitions of the minimal indices of the pencils involved. The main result of the paper is presented in subsection 5.2. It requires several technical results which appear in subsection 5.1. Finally, the paper ends with a conclusion section.

2. Preliminaries. Let $\mathbb{F}$ be an algebraically closed field. $\mathbb{F}[s]$ denotes the ring of polynomials in the indeterminate s with coefficients in $\mathbb{F}$, and $\mathbb{F}[s,t]$ denotes the ring of polynomials in two variables s,t with coefficients in $\mathbb{F}$. We denote by $\mathbb{F}^{p\times q}$, $\mathbb{F}[s]^{p\times q}$, and $\mathbb{F}[s,t]^{p\times q}$ the vector spaces of $p\times q$ matrices with elements in $\mathbb{F}$, $\mathbb{F}[s]$, and $\mathbb{F}[s,t]$, respectively. $\mathrm{Gl}_p(\mathbb{F})$ will be the general linear group of invertible matrices in $\mathbb{F}^{p\times p}$.

A *matrix pencil* is a polynomial matrix $A(s)\in\mathbb{F}[s]^{p\times q}$ of degree at most one ($A(s)=A_0+sA_1$ with $A_0,A_1\in\mathbb{F}^{p\times q}$). The *normal rank* of $A(s)$, denoted by $\mathrm{rank}(A(s))$, is the order of the largest nonidentically zero minor of $A(s)$; i.e., it is the rank of $A(s)$ considered as a matrix on the field of fractions of $\mathbb{F}[s]$. The pencil is *regular* if $p=q=\mathrm{rank}(A(s))$. Otherwise it is *singular*.

Two matrix pencils $A(s)=A_0+sA_1, B(s)=B_0+sB_1\in\mathbb{F}[s]^{p\times q}$ are *strictly equivalent* ($A(s)\overset{s.e.}{\sim}B(s)$) if there exist invertible matrices $P\in\mathrm{Gl}_p(\mathbb{F})$, $Q\in\mathrm{Gl}_q(\mathbb{F})$ such that $A(s)=PB(s)Q$. By $A(s)\overset{s.e.}{\not\sim}B(s)$ we will understand that $A(s)$ and $B(s)$ are not strictly equivalent.

Given the pencil $A(s)=A_0+sA_1\in\mathbb{F}[s]^{p\times q}$ of $\mathrm{rank}\,A(s)=\rho$, a complete system of invariants for the strict equivalence is formed by a chain of homogeneous polynomials $\phi_1(s,t)\mid\cdots\mid\phi_\rho(s,t),\ \phi_i(s,t)\in\mathbb{F}[s,t],\ 1\le i\le\rho$, called the *homogeneous invariant factors*, and two collections of nonnegative integers $c_1\ge\cdots\ge c_{q-\rho}$ and $u_1\ge\cdots\ge u_{p-\rho}$, called the *column and row minimal indices* of the pencil, respectively. In turn, the homogeneous invariant factors are determined by a chain of polynomials $\alpha_1(s)\mid\ldots\mid\alpha_\rho(s)$ in $\mathbb{F}[s]$, called the *invariant factors*, and a chain of polynomials $t^{k_1}\mid\ldots\mid t^{k_\rho}$ in $\mathbb{F}[t]$, called the *infinite elementary divisors* (see [10, Chapter 2] or [11, Chapter 12]). In fact, we can write

$$\phi_i(s,t)=t^{k_i}t^{\deg(\alpha_i(s))}\alpha_i\left(\frac{s}{t}\right),\ 1\le i\le\rho.$$

We will refer to the complete system of invariants for the strict equivalence as the *Kronecker structure* of the pencil.

The sum of the degrees of the homogeneous invariant factors plus the sum of the minimal indices is equal to the rank of the pencil. Also, if $B(s)=A(s)^T$, then $A(s)$ and $B(s)$ share the homogeneous invariant factors and have interchanged minimal indices; i.e., the column (row) minimal indices of $B(s)$ are the row (column) minimal indices of $A(s)$. If $A(s)\in\mathbb{F}[s]^{p\times q}$ and $\mathrm{rank}(A(s))=p$ ($\mathrm{rank}(A(s))=q$), then $A(s)$

does not have row (column) minimal indices. As a consequence, the invariants for the strict equivalence of regular matrix pencils are reduced to the homogeneous invariant factors.

Denote by $\overline{\mathbb{F}} = \mathbb{F} \cup \{\infty\}$. The *spectrum* of $A(s) = A_0 + sA_1 \in \mathbb{F}[s]^{p\times q}$ is defined as $\Lambda(A(s)) = \{\lambda \in \overline{\mathbb{F}} \,:\, \operatorname{rank}(A(\lambda)) < \operatorname{rank}(A(s))\}$, where we agree that $A(\infty) = A_1$. The elements $\lambda \in \Lambda(A(s))$ are the *eigenvalues* of $A(s)$.

Let $\alpha_1(s) \mid \cdots \mid \alpha_\rho(s)$ and $\phi_1(s,t) \mid \cdots \mid \phi_\rho(s,t)$, $\rho = \operatorname{rank} A(s)$, be the invariant factors and the homogeneous invariant factors of $A(s) = A_0 + sA_1 \in \mathbb{F}[s]^{p\times q}$, respectively. Factorizing both the invariant factors and the homogeneous invariant factors we can write

$$\alpha_{\rho-i+1}(s) = \prod_{\lambda\in\Lambda(A(s))\setminus\{\infty\}} (s-\lambda)^{n_i(\lambda,A(s))}, \quad 1 \le i \le \rho,$$

$$\phi_{\rho-i+1}(s,t) = t^{n_i(\infty,A(s))} \prod_{\lambda\in\Lambda(A(s))\setminus\{\infty\}} (s-\lambda t)^{n_i(\lambda,A(s))}, \quad 1 \le i \le \rho.$$

The integers $n_1(\lambda, A(s)) \ge \cdots \ge n_\rho(\lambda, A(s))$ are called the *partial multiplicities* of λ in $A(s)$. For $\lambda \in \overline{\mathbb{F}} \setminus \Lambda(A(s))$ we take $n_1(\lambda, A(s)) = \cdots = n_\rho(\lambda, A(s)) = 0$. We agree that $n_i(\lambda, A(s)) = +\infty$ for $i < 1$ and $n_i(\lambda, A(s)) = 0$ for $i > \rho$, for $\lambda \in \overline{\mathbb{F}}$. We also agree that $\alpha_i(s) = \phi_i(s,t) = 1$ for $i < 1$ and $\alpha_i(s) = \phi_i(s,t) = 0$ for $i > \rho$.

A canonical form for the strict equivalence of matrix pencils is the Kronecker canonical form. It is a matrix pencil of the form

$$\begin{bmatrix} J(s) & 0 & 0 & 0 & 0 \\ 0 & N(s) & 0 & 0 & 0 \\ 0 & 0 & L(s) & 0 & 0 \\ 0 & 0 & 0 & R(s) & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{F}[s]^{p\times q},$$

where $J(s)$ is a diagonal of Jordan blocks (here $J_{\lambda_0,k}(s)$ corresponds to the elementary divisor $(s-\lambda_0)^k$),

$$J_{\lambda_0,k}(s) = \begin{bmatrix} s-\lambda_0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & s-\lambda_0 \end{bmatrix} \in \mathbb{F}[s]^{k\times k}, \tag{2.1}$$

$N(s) = \operatorname{diag}(N_{k_1}(s), \ldots, N_{k_\rho}(s))$, where $t^{k_1} \mid \cdots \mid t^{k_\rho}$ are the infinite elementary divisors and (the block will be empty if $k = 0$)

$$N_k(s) = \begin{bmatrix} 1 & s & & \\ & \ddots & \ddots & \\ & & \ddots & s \\ & & & 1 \end{bmatrix} \in \mathbb{F}[s]^{k\times k}, \tag{2.2}$$

$L(s) = \operatorname{diag}(L_{c_1}(s), \ldots, L_{c_r}(s))$, where $c_1 \ge \cdots \ge c_r > 0 = c_{r+1} = \cdots = c_{q-\rho}$ are the column minimal indices and

$$L_k(s) = \begin{bmatrix} s & 1 & & \\ & \ddots & \ddots & \\ & & s & 1 \end{bmatrix} \in \mathbb{F}[s]^{k\times(k+1)}, \tag{2.3}$$

$R(s) = \operatorname{diag}(R_{u_1}(s), \ldots, R_{u_{r'}}(s))$, where $u_1 \ge \cdots \ge u_{r'} > 0 = u_{r'+1} = \cdots = u_{p-\rho}$ are the row minimal indices and $R_k(s) = L_k(s)^T \in \mathbb{F}[s]^{(k+1)\times k}$, understanding that the nonspecified components are zero. For details see [10, Chapter 2] or [11, Chapter 12] for infinite fields and [16, Chapter 2] for arbitrary fields.

The results in this paper are strongly linked to some properties of collections of nonnegative integers. We will distinguish two notions.

We call a *partition* of a positive integer n a finite or infinite sequence of nonnegative integers $\mathbf{a} = (a_1, a_2, \ldots)$, almost all being zero, such that $a_1 \geq a_2 \geq \cdots$ and $a_1 + a_2 + \cdots = n$. The number of nonzero components of $\mathbf{a}$ is the *length* of $\mathbf{a}$ (denoted $\ell(\mathbf{a})$). Notice that $\ell(\mathbf{a}) \leq n$. Given a finite partition $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, if necessary, we consider $a_i = 0$ if $i > n$. We identify two partitions that differ only in the number of components equal to zero. Given two partitions $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$, $\mathbf{a}$ is *majorized* by $\mathbf{b}$ (denoted $\mathbf{a} \prec \mathbf{b}$) if $\sum_{i=1}^k a_i \leq \sum_{i=1}^k b_i$ for $1 \leq k \leq n-1$ and $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$.

The *conjugate partition* of $\mathbf{a}$, $\overline{\mathbf{a}} = (\bar{a}_1, \bar{a}_2, \ldots)$, is defined as $\bar{a}_k := \#\{i : a_i \geq k\}$, $k \geq 1$. We define $\mathbf{a} \cup \mathbf{b}$ to be the partition whose components are those of $\mathbf{a}$ and $\mathbf{b}$ arranged in decreasing order (perhaps not strictly), and $\mathbf{a} + \mathbf{b}$ will be the partition whose components are the sums of the corresponding components of $\mathbf{a}$ and $\mathbf{b}$. The following properties are satisfied: $\mathbf{a} \prec \mathbf{b} \Leftrightarrow \overline{\mathbf{b}} \prec \overline{\mathbf{a}}$ and $\overline{\mathbf{a} \cup \mathbf{b}} = \overline{\mathbf{a}} + \overline{\mathbf{b}}$.

We call a *chain* a finite sequence of integers $\mathsf{c} = (c_1, c_2, \ldots, c_m)$ such that $c_1 \geq c_2 \geq \cdots \geq c_m$. When necessary, we will consider $c_i = +\infty$ if $i < 1$ and $c_i = -\infty$ when $i > m$. We remark that a chain has a fixed number of integer components.

Definition 2.1 (1step-generalized majorization).[1] *Given two chains of integers* $\mathsf{c} = (c_1, \ldots, c_m)$ *and* $\mathsf{d} = (d_1, \ldots, d_{m+1})$, *we say that* d *is* 1step-majorized *by* c *(denoted* $\mathsf{d} \prec' \mathsf{c}$*) if*

$$c_i = d_{i+1}, \quad h \leq i \leq m,$$

where $h = \min\{i : c_i < d_i\}$ *(*$c_{m+1} = -\infty$*).*

All throughout this paper, the chains involved have nonnegative components. We define the *conjugate of a chain* of nonnegative integers $\mathsf{c} = (c_1, \ldots, c_m)$ as the partition $\bar{\mathsf{c}} = (\bar{c}_1, \ldots)$, where $\bar{c}_k := \#\{i : c_i \geq k\}$, $k \geq 1$. When necessary, we will consider the term $\bar{c}_0 = \#\{i : c_i \geq 0\} = m$. Notice that if $\mathbf{c}$ is the partition $\mathbf{c} = (c_1, \ldots, c_m, 0, \ldots)$, then $\bar{\mathsf{c}} = \bar{\mathbf{c}}$ and $\bar{\bar{\mathsf{c}}} = \mathbf{c}$.

In the next theorem the change of the Kronecker structure of a pencil perturbed by another pencil of rank one is characterized. The result was independently obtained in [9] and [2].

Theorem 2.2 ([2, Theorem 5.1, Corollary 5.4]). *Let* $A(s), B(s) \in \mathbb{F}[s]^{p \times q}$ *be matrix pencils such that* $A(s) \overset{s.e.}{\not\sim} B(s)$. *Let* $\operatorname{rank} A(s) = \rho_1$, $\operatorname{rank} B(s) = \rho_2$, *and let* $\phi_1(s,t) \mid \cdots \mid \phi_{\rho_1}(s,t)$, $c_1 \geq \cdots \geq c_{q-\rho_1} \geq 0$, *and* $u_1 \geq \cdots \geq u_{p-\rho_1} \geq 0$ *be the homogeneous invariant factors, column minimal indices, and row minimal indices, respectively, of* $A(s)$, *and let* $\psi_1(s,t) \mid \cdots \mid \psi_{\rho_2}(s,t)$, $d_1 \geq \cdots \geq d_{q-\rho_2} \geq 0$, *and* $v_1 \geq \cdots \geq v_{p-\rho_2} \geq 0$ *be the homogeneous invariant factors, column minimal indices, and row minimal indices, respectively, of* $B(s)$. *Let* $\mathsf{c} = (c_1, \ldots, c_{q-\rho_1})$, $\mathsf{d} = (d_1, \ldots, d_{q-\rho_2})$, $\mathsf{u} = (u_1, \ldots, u_{p-\rho_1})$, $\mathsf{v} = (v_1, \ldots, v_{p-\rho_2})$ *(*$c_0 = d_0 = u_0 = v_0 = +\infty$*), and* $\rho = \min\{\rho_1, \rho_2\}$.

1. *If* $\mathsf{c} = \mathsf{d}$, $\mathsf{u} = \mathsf{v}$, *then there exists a pencil* $P(s) \in \mathbb{F}[s]^{p \times q}$ *of* $\operatorname{rank}(P(s)) = 1$ *such that* $A(s) + P(s) \overset{s.e.}{\sim} B(s)$ *if and only if*

$$\psi_{i-1}(s,t) \mid \phi_i(s,t) \mid \psi_{i+1}(s,t), \quad 1 \leq i \leq \rho. \tag{2.4}$$

[1] Particular case of generalized majorization [8, Definition 2].

2. *If* $\mathbf{c} \neq \mathbf{d}$, $\mathbf{u} = \mathbf{v}$, *let*

$$\ell = \max\{i \;:\; c_i \neq d_i\},$$

$$f = \max\{i \in \{1, \dots, \ell\} \;:\; c_i < d_{i-1}\}, \quad f' = \max\{i \in \{1, \dots, \ell\} \;:\; d_i < c_{i-1}\},$$

$$G = \rho - 1 - \sum_{i=1}^{\rho-1} \deg(\gcd(\phi_{i+1}(s,t), \psi_{i+1}(s,t))) - \sum_{i=1}^{p-\rho} u_i.$$

Then there exists a pencil $P(s) \in \mathbb{F}[s]^{p\times q}$ *of* $\operatorname{rank}(P(s)) = 1$ *such that* $A(s) + P(s) \overset{s.e.}{\sim} B(s)$ *if and only if* (2.4) *holds and*

$$G \leq \sum_{i=1}^{q-\rho} \min\{c_i, d_i\} + \max\{c_f, d_{f'}\}.$$

3. *If* $\mathbf{c} = \mathbf{d}$, $\mathbf{u} \neq \mathbf{v}$, *let*

$$\bar{\ell} = \max\{i \;:\; u_i \neq v_i\},$$

$$\bar{f} = \max\{i \in \{1, \dots, \bar{\ell}\} \;:\; u_i < v_{i-1}\}, \quad \bar{f}' = \max\{i \in \{1, \dots, \bar{\ell}\} \;:\; v_i < u_{i-1}\},$$

$$\bar{G} = \rho - 1 - \sum_{i=1}^{\rho-1} \deg(\gcd(\phi_{i+1}(s,t), \psi_{i+1}(s,t))) - \sum_{i=1}^{q-\rho} c_i.$$

Then there exists a pencil $P(s) \in \mathbb{F}[s]^{p\times q}$ *of* $\operatorname{rank}(P(s)) = 1$ *such that* $A(s) + P(s) \overset{s.e.}{\sim} B(s)$ *if and only if* (2.4) *holds and*

$$\bar{G} \leq \sum_{i=1}^{p-\rho} \min\{u_i, v_i\} + \max\{u_{\bar{f}}, v_{\bar{f}'}\}.$$

4. *If* $\mathbf{c} \neq \mathbf{d}$, $\mathbf{u} \neq \mathbf{v}$, *then there exists a pencil* $P(s) \in \mathbb{F}[s]^{p\times q}$ *of* $\operatorname{rank}(P(s)) = 1$ *such that* $A(s)+P(s) \overset{s.e.}{\sim} B(s)$ *if and only if* (2.4) *and one of the four following conditions hold:*
   (a)

$$\mathbf{c} \prec' \mathbf{d}, \quad \mathbf{u} \prec' \mathbf{v}, \tag{2.5}$$

$$\sum_{i=1}^{\rho} \deg(\operatorname{lcm}(\phi_i(s,t), \psi_i(s,t))) \leq x \leq \sum_{i=1}^{\rho} \deg(\gcd(\phi_{i+1}(s,t), \psi_{i+1}(s,t))), \tag{2.6}$$

   where $x = \rho - \sum_{i=1}^{q-\rho_1} c_i - \sum_{i=1}^{p-\rho_2} v_i$.
   (b)

$$\mathbf{d} \prec' \mathbf{c}, \quad \mathbf{v} \prec' \mathbf{u}, \tag{2.7}$$

$$\sum_{i=1}^{\rho} \deg(\operatorname{lcm}(\phi_i(s,t), \psi_i(s,t))) \leq y \leq \sum_{i=1}^{\rho} \deg(\gcd(\phi_{i+1}(s,t), \psi_{i+1}(s,t))), \tag{2.8}$$

   where $y = \rho - \sum_{i=1}^{q-\rho_2} d_i - \sum_{i=1}^{p-\rho_1} u_i$.
   (c) (2.5) *and* (2.8) *hold.*
   (d) (2.7) *and* (2.6) *hold.*

3. Jordan chains of matrix pencils. The definition of Jordan chain can be found in [12, section 1.4] for square regular matrix polynomials and in [13, Definition 7.1] for square matrix pencils (both regular and singular) over $\mathbb{C}$. On the other hand, the notion of Weyr characteristic of an eigenvalue of a pencil was introduced in [4] as the conjugate partition of that of the partial multiplicities of the eigenvalue.

The target of this section is to generalize to arbitrary matrix pencils the notions of Jordan chains and Weyr characteristics of pencils and to express the generalized Weyr characteristic in terms of the Kronecker invariants of the pencil. The results hold over arbitrary fields.

DEFINITION 3.1. *Given a matrix pencil $A(s) = A_0 + sA_1 \in \mathbb{F}[s]^{p\times q}$, an ordered set $(x_k, \dots, x_0)$ in $\mathbb{F}^q$ is a* right Jordan chain *of $A(s)$ at $\lambda \in \overline{\mathbb{F}}$, of length $k+1$, if $x_0 \neq 0$ and*

$$\begin{aligned} \lambda \in \mathbb{F}: &\quad A(\lambda)x_0 = 0, \quad A(\lambda)x_i = -A_1x_{i-1}, \quad 1 \leq i \leq k,\\ \lambda = \infty: &\quad A_1x_0 = 0, \quad A_1x_i = -A_0x_{i-1}, \quad 1 \leq i \leq k. \end{aligned}$$

The set $(x_k, \dots, x_0)$ in $\mathbb{F}^p$ is a *left Jordan chain* of $A(s)$ at $\lambda \in \overline{\mathbb{F}}$, of length $k+1$, if it is a right Jordan chain of $A(s)^T$ at λ.

In what follows we will deal with right Jordan chains, and we will refer to them just as Jordan chains, omitting the term "right." It can be easily seen that the results obtained for right Jordan chains hold for left Jordan chains by transposition.

Following [13], we denote by $\mathcal{L}_\lambda^\ell(A(s))$ the subspace spanned by the vectors of the Jordan chains at $\lambda \in \overline{\mathbb{F}}$, up to length $\ell \geq 1$. We agree that $\mathcal{L}_\lambda^0(A(s)) = \{0\}$. If $\operatorname{rank}(A(\lambda)) = q$, then there is no Jordan chain at λ for $A(s)$, and we take $\mathcal{L}_\lambda^\ell(A(s)) = \{0\}$ for $\ell \geq 0$. Observe that, for $\lambda \in \overline{\mathbb{F}}$, $\mathcal{L}_\lambda^{i-1}(A(s)) \subseteq \mathcal{L}_\lambda^i(A(s))$ for $i \geq 1$ and $\mathcal{L}_\lambda^{i-1}(A(s)) = \mathcal{L}_\lambda^i(A(s))$ for $i > q$. Again as in [13], we denote by $w_i(\lambda, A(s))$ the dimension of the quotient space $\frac{\mathcal{L}_\lambda^i(A(s))}{\mathcal{L}_\lambda^{i-1}(A(s))}$; i.e.,

$$w_i(\lambda, A(s)) = \dim \mathcal{L}_\lambda^i(A(s)) - \dim \mathcal{L}_\lambda^{i-1}(A(s)), \quad 1 \leq i \leq q.$$

The following theorem was obtained in [13].

THEOREM 3.2 (see [13, Theorem 7.8]). *Given a matrix pencil $A(s) \in \mathbb{C}[s]^{n\times n}$, let $P(s) \in \mathbb{F}[s]^{n\times n}$ be a rank-one matrix pencil of the form*

$$P(s) = w(su^* + v^*), \quad u, v, w \in \mathbb{C}^n, \quad (u, v) \neq (0,0), \quad w \neq 0.$$

For $\lambda \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and $i \geq 1$, the following statements hold:

(i) *If both pencils $A(s)$ and $A(s) + P(s)$ are regular, then*

$$| \, w_i(\lambda, A(s) + P(s)) - w_i(\lambda, A(s)) \, | \leq 1.$$

(ii) *If $A(s)$ is regular but $A(s) + P(s)$ is singular, then*

$$-i \leq w_i(\lambda, A(s) + P(s)) - w_i(\lambda, A(s)) \leq 1.$$

(iii) *If $A(s)$ is singular and $A(s) + P(s)$ is regular, then*

$$-1 \leq w_i(\lambda, A(s) + P(s)) - w_i(\lambda, A(s)) \leq i.$$

(iv) *If both $A(s)$ and $A(s) + P(s)$ are singular, then*

$$| \, w_i(\lambda, A(s) + P(s)) - w_i(\lambda, A(s)) \, | \leq i.$$

In this paper we obtain bounds for $w_i(\lambda, A(s)+P(s)) - w_i(\lambda, A(s))$ for arbitrary matrix pencils $A(s)$ and arbitrary rank-one perturbations $P(s)$.

Since the relation between the Kronecker invariants of a pencil and those of the pencil obtained by a rank-one perturbation of it is known (see Theorem 2.2), we are going to relate the values of $w_i(\lambda, A(s))$ with the Kronecker invariants of $A(s)$. This relation can also be obtained from [13, Theorem 8.1], but to improve the readability of the paper, a short proof of it is included here.

First, in the next lemma we state that to compute $\dim \mathcal{L}^i_\lambda(A(s))$ we can substitute $A(s)$ by a strictly equivalent pencil.

LEMMA 3.3. *Let $A(s), \bar{A}(s) \in \mathbb{F}[s]^{p\times q}$ be matrix pencils, and let $\lambda \in \overline{\mathbb{F}}$. If $A(s)$ and $\bar{A}(s)$ are strictly equivalent, then*

$$\dim \mathcal{L}^i_\lambda(A(s)) = \dim \mathcal{L}^i_\lambda(\bar{A}(s)), \quad i \geq 0.$$

Proof. The proof is straightforward. ∎

In the next proposition we analyze $\dim \mathcal{L}^i_\lambda(A(s))$ under certain structures of $A(s)$: when $A(s)$ has a diagonal decomposition and when it has zero columns or zero rows.

PROPOSITION 3.4.

1. *Let $C(s) \in \mathbb{F}[s]^{p_1\times q_1}$, $D(s) \in \mathbb{F}[s]^{p_2\times q_2}$ be matrix pencils, $A(s) = \begin{bmatrix} C(s) & 0 \\ 0 & D(s) \end{bmatrix}$, and $\lambda \in \overline{\mathbb{F}}$. Then*

$$\dim \mathcal{L}^i_\lambda(A(s)) = \dim \mathcal{L}^i_\lambda(C(s)) + \dim \mathcal{L}^i_\lambda(D(s)), \quad i \geq 0.$$

2. *Let $C(s) \in \mathbb{F}[s]^{p\times q_1}$ be a matrix pencil, $A(s) = \begin{bmatrix} C(s) & 0 \end{bmatrix} \in \mathbb{F}[s]^{p\times (q_1+q_2)}$, and $\lambda \in \overline{\mathbb{F}}$. Then*

$$\dim \mathcal{L}^i_\lambda(A(s)) = \dim \mathcal{L}^i_\lambda(C(s)) + q_2, \quad i \geq 1.$$

3. *Let $C(s) \in \mathbb{F}[s]^{p_1\times q}$ be a matrix pencil, $A(s) = \begin{bmatrix} C(s) \\ 0 \end{bmatrix} \in \mathbb{F}[s]^{(p_1+p_2)\times q}$, and $\lambda \in \overline{\mathbb{F}}$. Then*

$$\dim \mathcal{L}^i_\lambda(A(s)) = \dim \mathcal{L}^i_\lambda(C(s)), \quad i \geq 0.$$

Proof.

1. The set $(\begin{bmatrix} x_k \\ y_k \end{bmatrix}, \dots, \begin{bmatrix} x_0 \\ y_0 \end{bmatrix})$ is a Jordan chain of $A(s)$ at λ if and only if $(x_k, \dots, x_0)$ is a Jordan chain of $C(s)$ at λ and $(y_k, \dots, y_0)$ is a Jordan chain of $D(s)$ at λ. Therefore, for $i \geq 0$, $\mathcal{L}^i_\lambda(A(s)) = (\mathcal{L}^i_\lambda(C(s)) \times \{0\}) \oplus (\{0\} \times \mathcal{L}^i_\lambda(D(s)))$.
2. The set $(\begin{bmatrix} x_k \\ y_k \end{bmatrix}, \dots, \begin{bmatrix} x_0 \\ y_0 \end{bmatrix})$ is a Jordan chain of $A(s)$ at λ if and only if $(x_k, \dots, x_0)$ is a Jordan chain of $C(s)$ at λ. Therefore, for $i \geq 1$, $\mathcal{L}^i_\lambda(A(s)) = \mathcal{L}^i_\lambda(C(s)) \times \mathbb{F}^{q_2}$.
3. The set $(x_k, \dots, x_0)$ is a Jordan chain of $A(s)$ at λ if and only if $(x_k, \dots, x_0)$ is a Jordan chain of $C(s)$ at λ. Therefore, for $i \geq 0$, $\mathcal{L}^i_\lambda(A(s)) = \mathcal{L}^i_\lambda(C(s))$. ∎

The next step is to compute $w_i(\lambda, A(s))$ when $A(s)$ is a block component of a pencil in Kronecker canonical form.

PROPOSITION 3.5.

1. *Let $\lambda_0 \in \mathbb{F}$ and $\lambda \in \overline{\mathbb{F}} \setminus \{\lambda_0\}$. Let $J_{\lambda_0,k}(s)$ be the pencil defined in* (2.1). *Then*

$$w_i(\lambda_0, J_{\lambda_0,k}(s)) = 1, \quad 1 \leq i \leq k,$$
$$w_i(\lambda, J_{\lambda_0,k}(s)) = 0, \quad 1 \leq i \leq k.$$

2. *Let* $\lambda \in \mathbb{F}$. *Let* $N_k(s)$ *be the pencil defined in* (2.2). *Then*

$$w_i(\infty, N_k(s)) = 1, \quad 1 \le i \le k,$$
$$w_i(\lambda, N_k(s)) = 0, \quad 1 \le i \le k.$$

3. *Let* $\lambda \in \overline{\mathbb{F}}$. *Let* $L_{k-1}(s)$ *be the pencil defined in* (2.3). *Then*

$$w_i(\lambda, L_{k-1}(s)) = 1, \quad 1 \le i \le k.$$

4. *Let* $\lambda \in \overline{\mathbb{F}}$, *and let* $R_{k-1}(s) = L_{k-1}(s)^T$. *Then*

$$w_i(\lambda, R_{k-1}(s)) = 0, \quad 1 \le i \le k-1.$$

Proof. Let $e_1, \dots, e_k$ be the columns of I_k.

1. For $1 \le i \le k$, $\mathcal{L}^i_{\lambda_0}(J_{\lambda_0,k}(s)) = \operatorname{span}\{e_1, \dots, e_i\}$; hence $\dim \mathcal{L}^i_{\lambda_0}(J_{\lambda_0,k}(s)) = i$, from where we obtain $w_i(\lambda_0, J_{\lambda_0,k}(s)) = 1$. If $\lambda \in \mathbb{F} \setminus \{\lambda_0\}$, rank $J_{\lambda_0,k}(\lambda) = k$; hence $w_i(\lambda, J_{\lambda_0,k}(s)) = 0$ for $1 \le i \le k$. Analogously, for $1 \le i \le k$, we obtain $w_i(\infty, J_{\lambda_0,k}(s)) = 0$.
2. As in the previous case, for $1 \le i \le k$, $\mathcal{L}^i_\infty(N_k(s)) = \operatorname{span}\{e_1, \dots, e_i\}$, and rank $N_k(\lambda) = k$ for $\lambda \in \mathbb{F}$.
3. We have $\mathcal{L}^i_\infty(L_{k-1}(s)) = \operatorname{span}\{e_k, e_{k-1}, \dots, e_{k-i+1}\}$ for $1 \le i \le k$. If $\lambda \in \mathbb{F}$, then

$$\mathcal{L}^1_\lambda(L_{k-1}(s)) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -\lambda \\ \lambda^2 \\ \\ \vdots \\ (-\lambda)^{k-1} \end{bmatrix} \right\},$$

and for $2 \le i \le k$,

$$\mathcal{L}^i_\lambda(L_{k-1}(s)) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -\lambda \\ \lambda^2 \\ \vdots \\ (-\lambda)^{i-1} \\ \vdots \\ (-\lambda)^{k-1} \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ x_{32} \\ \vdots \\ x_{i-1,2} \\ x_{i2} \\ \vdots \\ x_{k2} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ x_{i-1,3} \\ x_{i3} \\ \vdots \\ x_{k3} \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ (-1)^{i-1} \\ \vdots \\ x_{ki} \end{bmatrix} \right\},$$

where x_{uj} are recursively defined as

$$x_{uj} = -x_{u-1,j-1} - \lambda x_{u-1,j}, \quad 3 \le u \le k, \ 2 \le j \le u-1.$$

Therefore, $\dim \mathcal{L}^i_\lambda(L_{k-1}(s)) = i$ for $0 \le i \le k$.
4. For $\lambda \in \mathbb{F}$, we have $\operatorname{rank}(R_{k-1}(\lambda)) = k-1$. Hence, $w_i(\lambda, R_{k-1}(s)) = 0$ for $1 \le i \le k$. Analogously, we obtain $w_i(\infty, R_{k-1}(s)) = 0$ for $1 \le i \le k-1$. ∎

Corollary 3.6. *With the notation of Proposition* 3.5, *do the following:*

1. *Let* $\lambda_0 \in \mathbb{F}$ *and* $\lambda \in \overline{\mathbb{F}} \setminus \{\lambda_0\}$. *Then*

$$\overline{(k)} = (w_1(\lambda_0, J_{\lambda_0,k}(s)), \dots, w_k(\lambda_0, J_{\lambda_0,k}(s))),$$
$$\overline{(0)} = (w_1(\lambda, J_{\lambda_0,k}(s)), \dots, w_k(\lambda, J_{\lambda_0,k}(s))).$$

2. *Let* $\lambda \in \mathbb{F}$. *Then*

$$\overline{(k)} = (w_1(\infty, N_k(s)), \dots, w_k(\infty, N_k(s))),$$
$$\overline{(0)} = (w_1(\lambda, N_k(s)), \dots, w_k(\lambda, N_k(s))).$$

3. *Let* $\lambda \in \overline{\mathbb{F}}$. *Then*

$$\overline{(k)} = (w_1(\lambda, L_{k-1}(s)), \dots, w_k(\lambda, L_{k-1}(s))).$$

4. *Let* $\lambda \in \overline{\mathbb{F}}$. *Then*

$$\overline{(0)} = (w_1(\lambda, R_{k-1}(s)), \dots, w_{k-1}(\lambda, R_{k-1}(s))).$$

The next theorem relates the values of $w_i(\lambda, A(s))$ with the Kronecker invariants of $A(s)$.

THEOREM 3.7. *Let* $A(s) \in \mathbb{F}[s]^{p\times q}$ *be a matrix pencil such that* $\operatorname{rank}(A(s)) = \rho$, *and* $\lambda \in \overline{\mathbb{F}}$. *Let* $n_1(\lambda, A(s)) \geq \cdots \geq n_\rho(\lambda, A(s)) \geq 0$ *be the partial multiplicities of* λ *in* $A(s)$, *and let* $c_1 \geq \cdots \geq c_{q-\rho}$ *and* $u_1 \geq \cdots \geq u_{p-\rho}$ *be the column and row minimal indices of* $A(s)$, *respectively. Let* $(w_1^R(\lambda, A(s)), \dots, w_q^R(\lambda, A(s))) = \overline{(n_1(\lambda, A(s)), \dots, n_\rho(\lambda, A(s))}$, $(r_1, \dots, r_q) = \overline{(c_1, \dots, c_{q-\rho})}$, *and* $r_0 = q - \rho$. *Then*

$$w_i(\lambda, A(s)) = w_i^R(\lambda, A(s)) + r_{i-1}, \quad 1 \leq i \leq q.$$

Proof. By Lemma 3.3, we can assume that $A(s)$ is in Kronecker canonical form. By Propositions 3.4 and 3.5, for $\lambda \in \overline{\mathbb{F}}$,

$$\sum_{i=1}^{\rho} \overline{(n_i(\lambda, A(s))} + \sum_{i=1}^{r_1} \overline{(c_i + 1)} + (q - \rho - r_1, 0, \dots, 0) = (w_1(\lambda, A(s)), \dots, w_q(\lambda, A(s))).$$

Since

$$\begin{aligned} \textstyle\sum_{i=1}^{\rho} \overline{(n_i(\lambda, A(s))} = {} & \overline{\cup_{i=1}^{\rho}(n_i(\lambda, A(s))} = \overline{(n_1(\lambda, A(s)), \dots, n_\rho(\lambda, A(s))} \\ = {} & (w_1^R(\lambda, A(s)), \dots, w_q^R(\lambda, A(s))) \end{aligned}$$

and $(q - \rho - r_1, 0, \dots, 0) = \sum_{i=r_1+1}^{q-\rho} 1 = \sum_{i=r_1+1}^{q-\rho} \overline{(c_i + 1)}$, we obtain

$$\begin{aligned} \textstyle\sum_{i=1}^{r_1} \overline{(c_i + 1)} + (q - \rho - r_1, 0, \dots, 0) = {} & \textstyle\sum_{i=1}^{q-\rho} \overline{(c_i + 1)} = \overline{\cup_{i=1}^{q-\rho}(c_i + 1)} \\ = \overline{(c_1, \dots, c_{q-\rho}) + (1, \overset{(q-\rho)}{\dots}, 1)} = {} & \overline{(c_1, \dots, c_{q-\rho})} \cup \overline{(1, \overset{(q-\rho)}{\dots}, 1)} \\ = (r_1, \dots, r_q) \cup (q - \rho) = {} & (q - \rho, r_1, \dots, r_q). \end{aligned}$$

∎

Remark 3.8.

- Let $A(s) \in \mathbb{F}[s]^{p\times q}$ be a matrix pencil and $\lambda \in \overline{\mathbb{F}}$. Then, by Theorem 3.7, we have
$$w_1(\lambda, A(s)) \geq w_2(\lambda, A(s)) \cdots \geq w_q(\lambda, A(s)).$$
We call the partition
$$\mathbf{w}(\lambda, A(s)) = (w_1(\lambda, A(s)), w_2(\lambda, A(s)), \dots, w_q(\lambda, A(s)))$$
the *generalized Weyr characteristic* of the pencil $A(s)$ at λ.
- Notice that, by Theorem 3.7,
$$\mathbf{w}(\lambda, A(s)) = (w_1^R(\lambda, A(s)), \dots, w_q^R(\lambda, A(s))) + (r_0, r_1, \dots, r_{q-1}), \quad \lambda \in \overline{\mathbb{F}},$$

where $(w_1^R(\lambda, A(s)), \dots, w_q^R(\lambda, A(s)))$ is the Weyr characteristic at λ of the regular part of $A(s)$ and $(r_0, r_1, \dots, r_{q-1})$ is that of the column minimal indices block. Observe that the row minimal indices do not play any role in the calculation of $\mathbf{w}(\lambda, A(s))$.

In [4], $(w_1^R(\lambda, A(s)), \dots, w_q^R(\lambda, A(s)))$ is called the *Weyr characteristic* of the pencil for $\lambda \in \overline{\mathbb{C}}$, and $(r_1, \dots, r_{q-1})$ is called the *partition of the r-numbers* of the pencil. Notice also that, if the pencil does not have column minimal indices, the Weyr characteristic and the generalized Weyr characteristic coincide.

In the next example we compute the Weyr and the generalized Weyr characteristics of a pencil.

Example 3.9. The spectrum of the pencil

$$A(s) = \left[\begin{array}{ccc|ccc|c} 1 & s & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ \hline & & & s & 1 & & \\ & & & & s & 1 & \\ \hline & & & & & & 0 \end{array}\right] \in \mathbb{C}[s]^{6\times 7}$$

is $\Lambda(A(s)) = \{\infty\}$. The partial multiplicities of ∞ in $A(s)$ are $n_1(\infty, A(s)) = 2 \geq n_2(\infty, A(s)) = 1$. The column and row minimal indices of $A(s)$ are $c_1 = 2 \geq c_2 = 0$ and $u_1 = 0$, respectively.

In [4], the Weyr characteristic of $A(s)$ for ∞ is

$$(w_1^R(\infty, A(s)), \dots) = \overline{(2,1)} = (2, 1, 0, \dots).$$

The partitions of the r-numbers and of the s-numbers of $A(s)$ are, respectively,

$$(r_1, \dots) = \overline{(2,0)} = (1, 1, 0, \dots), \quad (s_1, \dots) = \overline{(0)} = (0, \dots).$$

The generalized Weyr characteristic of $A(s)$ at ∞ is

$$\begin{aligned} \mathbf{w}(\infty, A(s)) &= (w_1^R(\infty, A(s)), \dots) + (r_0, r_1, \dots) \\ &= (2, 1, 0, \dots) + (2, 1, 1, 0, \dots) = (4, 2, 1, 0, \dots). \end{aligned}$$

4. Theorem 2.2 in terms of the conjugate partitions. The target of this section is to rewrite the characterizations stated in Theorem 2.2 in terms of the conjugate partitions of the corresponding chains of column and row minimal indices of the pencils involved. To achieve it we previously prove some technical results.

We start with the introduction of a new majorization between partitions of nonnegative integers.

Definition 4.1. *Given two partitions of nonnegative integers* $\mathbf{r} = (r_0, r_1, \dots)$ *and* $\mathbf{s} = (s_0, s_1, \dots)$ *such that* $r_0 \geq r_1 \geq \cdots$ *and* $s_0 \geq s_1 \geq \cdots$, *we say that* $\mathbf{s}$ *is* conjugate majorized *by* $\mathbf{r}$ *(denoted* $\mathbf{s} \angle \mathbf{r}$*) if* $r_0 = s_0 + 1$ *and*

$$r_i = s_i + 1, \quad 0 \leq i \leq g,$$

where $g = \max\{i : r_i > s_i\}$.

Remark 4.2. Let $\mathbf{a} = (a_1, \dots)$ be a partition of nonnegative integers, and let $(r_1, \dots) = \overline{(a_1, \dots)}$ be its conjugate partition. In what follows we will frequently use the following properties:

- $r_j = i$ for $a_{i+1} < j \leq a_i$, $i \geq 1$. If $a_{i+1} = a_i$ there are no j such that $r_j = i$.
- For $i \in \{1, \dots, \ell(\mathbf{a})\}$ we have $r_{a_i} \geq i$, and if $j > a_i$, then $r_j < i$. Recall that $\ell(\mathbf{a})$ is the length of the partition $\mathbf{a}$, i.e., the number of nonzero elements of $\mathbf{a}$. In other words, $\ell(\mathbf{a}) = r_1$.

The proof of the next lemma is analogous to that of [14, Lemma 3.2].

LEMMA 4.3. *Let* $(a_1, \dots)$ *and* $(b_1, \dots)$ *be partitions of nonnegative integers. Let* $\mathbf{p} = (p_1, \dots) = \overline{(a_1, \dots)}$ *and* $\mathbf{q} = (q_1, \dots) = \overline{(b_1, \dots)}$ *be the conjugate partitions. Let* $k \geq 0$ *be an integer. Then*

$$a_j \geq b_{j+k} \quad \text{for every} \quad j \geq 1$$

if and only if

$$p_j \geq q_j - k \quad \text{for every} \quad j \geq 1.$$

LEMMA 4.4. *Given two chains of nonnegative integers* $\mathbf{c} = (c_1, \dots, c_{m+1})$ *and* $\mathbf{d} = (d_1, \dots, d_m)$*, let* $\mathbf{r} = (r_1, \dots) = \overline{(c_1, \dots, c_{m+1})}$*,* $\mathbf{s} = (s_1, \dots) = \overline{(d_1, \dots, d_m)}$*, and* $r_0 = m + 1 = s_0 + 1$*. Let*

$$g = \max\{i : r_i > s_i\}, \quad h = \min\{i : d_i < c_i\}.$$

Then

$$g = c_h \tag{4.1}$$

and

$$\sum_{j=1}^{g} (r_j - s_j - 1) = \sum_{j=h}^{m} (c_{j+1} - d_j). \tag{4.2}$$

Proof. As $d_h < c_h$, we have $s_{c_h} < h \leq r_{c_h}$; hence $g \geq c_h$. Observe that $g \leq \ell(\mathbf{r})$. If $\ell(\mathbf{r}) \geq i > c_h$, then $c_{r_i} \geq i > c_h$; hence $r_i < h$. By the definition of h, we have $d_{r_i} \geq c_{r_i} \geq i$; hence $\#\{j : d_j \geq i\} \geq r_i$; i.e., $s_i \geq r_i$. Therefore, (4.1) holds. Then,

$$\sum_{j=1}^{g} r_j = \sum_{j=1}^{c_h} r_j = (m+1)c_{m+1} + \sum_{j=h}^{m} j(c_j - c_{j+1}) = hc_h + \sum_{j=h}^{m} c_{j+1},$$

and, bearing in mind that $d_h < c_h \leq c_{h-1} \leq d_{h-1}$,

$$\begin{aligned} \sum_{j=1}^{g} s_j = \sum_{j=1}^{d_h} s_j + \sum_{j=d_h+1}^{c_h} s_j &= md_m + (c_h - d_h)(h-1) + \sum_{j=h}^{m-1} j(d_j - d_{j+1}) \\ &= hd_h + (c_h - d_h)(h-1) + \sum_{j=h}^{m-1} d_{j+1} = (h-1)c_h + \sum_{j=h}^{m} d_j. \end{aligned}$$

Therefore,

$$\sum_{j=1}^{g} (r_j - s_j - 1) = c_h - g + \sum_{j=h}^{m} (c_{j+1} - d_j) = \sum_{j=h}^{m} (c_{j+1} - d_j). \qquad \square$$

PROPOSITION 4.5. *Given two chains of nonnegative integers* $\mathbf{c} = (c_1, \dots, c_{m+1})$ *and* $\mathbf{d} = (d_1, \dots, d_m)$*, let* $(r_1, \dots) = \overline{(c_1, \dots, c_{m+1})}$*,* $(s_1, \dots) = \overline{(d_1, \dots, d_m)}$ *be the conjugate partitions,* $r_0 = m+1 = s_0+1$*, and* $\mathbf{r} = (r_0, r_1, \dots)$*,* $\mathbf{s} = (s_0, s_1, \dots)$*. Then* $\mathbf{c} \prec' \mathbf{d}$ *if and only if* $\mathbf{s} \angle \mathbf{r}$*.*

Proof. Let $g = \max\{i : r_i > s_i\}$ and $h = \min\{i : d_i < c_i\}$. Then, by Lemma 4.4, (4.1) and (4.2) hold. Moreover, $r_i \le s_i < s_i + 1$ for $i > g$ and $d_i \ge c_i \ge c_{i+1}$ for $1 \le i < h$.

Assume that $\mathbf{c} \prec' \mathbf{d}$; then $d_i \ge c_{i+1}$ for $1 \le i \le m$ and $\sum_{j=h}^{m}(c_{j+1} - d_j) = 0$. From Lemma 4.3 we have $r_i \le s_i + 1$ for $i \ge 1$, and then from (4.2) we derive that $r_i = s_i + 1$ for $1 \le i \le g$; i.e., $\mathbf{s} \angle \mathbf{r}$.

Conversely, assume that $\mathbf{s} \angle \mathbf{r}$; then $r_i \le s_i+1$ for $i \ge 1$ and $\sum_{j=1}^{g}(r_j - s_j - 1) = 0$. From Lemma 4.3 we have $d_i \ge c_{i+1}$ for $1 \le i \le m$, and then from (4.2) we derive that $d_i = c_{i+1}$ for $h \le i \le m$; i.e., $\mathbf{c} \prec' \mathbf{d}$. ∎

LEMMA 4.6. *Given two chains of nonnegative integers* $\mathbf{c} = (c_1, \dots, c_m)$ *and* $\mathbf{d} = (d_1, \dots, d_m)$ $(d_0 = c_0 = +\infty)$ *such that* $\mathbf{c} \ne \mathbf{d}$*, let*

$$\ell = \max\{i \ : \ c_i \ne d_i\},$$

$$f = \max\{i \in \{1, \dots, \ell\} \ : \ c_i < d_{i-1}\}, \quad f' = \max\{i \in \{1, \dots, \ell\} \ : \ d_i < c_{i-1}\}.$$

Let $(r_1, \dots) = \overline{(c_1, \dots, c_m)}$*,* $(s_1, \dots) = \overline{(d_1, \dots, d_m)}$*,* $r_0 = s_0 = m$*,*

$$x = \min\{i \ : \ r_i \ne s_i\},$$

$$e = \min\{i \ge x - 1 : \ s_{i+1} \ge r_{i+1}\}, \quad e' = \min\{i \ge x - 1 : \ r_{i+1} \ge s_{i+1}\}.$$

Then

$$e = c_f, \quad e' = d_{f'}.$$

Proof. Assume that $d_\ell < c_\ell$ (if $c_\ell < d_\ell$ the proof is analogous). As $d_\ell < c_\ell \le c_{\ell-1}$, we have $f' = \ell$.

From the definition of ℓ, $c_i = d_i$, $\ell + 1 \le i \le m$; then $r_i = s_i$, $1 \le i \le d_\ell$. From $d_\ell < c_\ell$ we also derive that $s_{d_\ell+1} < r_{d_\ell+1}$; hence $x = d_\ell + 1$ and, from the definition of e', $e' = d_\ell$.

From the definition of f, $c_f < d_{f-1}$. If $f < \ell$, then $d_f \le c_{f+1} \le c_f < d_{f-1}$, and if $f = \ell$, then $d_f = d_\ell < c_\ell = c_f < d_{f-1}$. Hence, $r_{c_f+1} \le f - 1 = s_{c_f+1}$ and $x - 1 = d_\ell \le e \le c_f$. Moreover, by definition, $c_i \ge d_{i-1}$ for $f + 1 \le i \le \ell$.

We prove next that for $d_\ell \le i \le c_f - 1$, $r_{i+1} > s_{i+1}$ holds. Let i be such that $c_f \ge i > d_\ell$. If there exists $j \in \{f, \dots, \ell\}$ such that $c_j \ge i > c_{j+1} \ge d_j$, then $r_i = j > j - 1 \ge s_i$. Otherwise, $c_\ell \ge i > d_\ell$; then $r_i > s_i$. Hence $e = c_f$. ∎

LEMMA 4.7. *Given two chains of nonnegative integers* $\mathbf{c} = (c_1, \dots, c_m)$ *and* $\mathbf{d} = (d_1, \dots, d_m)$*, let* $x_i = \min\{c_i, d_i\}$*,* $1 \le i \le m$*. Let* $(r_1, \dots) = \overline{(c_1, \dots, c_m)}$*,* $(s_1, \dots) = \overline{(d_1, \dots, d_m)}$*, and* $y_i = \min\{r_i, s_i\}$*,* $i \ge 1$*. Then*

$$(y_1, \dots) = \overline{(x_1, \dots, x_m)}.$$

Proof. For $i \ge 1$, let $C_i = \{j \ : \ c_j \ge i\}$, $D_i = \{j \ : \ d_j \ge i\}$, $X_i = \{j \ : \ x_j \ge i\}$. Then $X_i = C_i \cap D_i$, $r_i = \#C_i$, $s_i = \#D_i$, $i \ge 1$. We must prove that $y_i = \#X_i$ for $i \ge 1$. Let $i \ge 1$. If $y_i = r_i$, then $C_i \subseteq D_i$ and $X_i = C_i$; hence $\#X_i = r_i = y_i$. Analogously, if $y_i = s_i$, then $\#X_i = s_i = y_i$. ∎

The result of the next theorem is the target of the section.

THEOREM 4.8. *Let* $A(s), B(s) \in \mathbb{F}[s]^{p\times q}$ *be matrix pencils such that* $A(s) \overset{s.e.}{\not\sim} B(s)$. *Let* $\operatorname{rank} A(s) = \rho_1$, $\operatorname{rank} B(s) = \rho_2$, *and let* $\phi_1(s,t) \mid \cdots \mid \phi_{\rho_1}(s,t)$, $c_1 \geq \cdots \geq c_{q-\rho_1} \geq 0$, *and* $u_1 \geq \cdots \geq u_{p-\rho_1} \geq 0$ *be the homogeneous invariant factors, column minimal indices, and row minimal indices of* $A(s)$, *respectively, and let* $\psi_1(s,t) \mid \cdots \mid \psi_{\rho_2}(s,t)$, $d_1 \geq \cdots \geq d_{q-\rho_2} \geq 0$, *and* $v_1 \geq \cdots \geq v_{p-\rho_2} \geq 0$ *be the homogeneous invariant factors, column minimal indices, and row minimal indices of* $B(s)$, *respectively.*

Let $\rho = \min\{\rho_1, \rho_2\}$, $\rho' = \max\{\rho_1, \rho_2\}$, $\boldsymbol{c} = (c_1, \ldots, c_{q-\rho_1})$, $\boldsymbol{d} = (d_1, \ldots, d_{q-\rho_2})$, $\boldsymbol{u} = (u_1, \ldots, u_{p-\rho_1})$, $\boldsymbol{v} = (v_1, \ldots, v_{p-\rho_2})$, $(r_1, \ldots) = \overline{(c_1, \ldots, c_{q-\rho_1})}$, $(s_1, \ldots) = \overline{(d_1, \ldots, d_{q-\rho_2})}$, $(r'_1, \ldots) = \overline{(u_1, \ldots, u_{p-\rho_1})}$, $(s'_1, \ldots) = \overline{(v_1, \ldots, v_{p-\rho_2})}$, $r_0 = q - \rho_1$, $s_0 = q-\rho_2$, $r'_0 = p-\rho_1$, $s'_0 = p-\rho_2$, $\mathbf{r} = (r_0, r_1, \ldots)$, $\mathbf{s} = (s_0, s_1, \ldots)$, $\mathbf{r}' = (r'_0, r'_1, \ldots)$, *and* $\mathbf{s}' = (s'_0, s'_1, \ldots)$.

1. *If* $\mathbf{r} = \mathbf{s}$, $\mathbf{r}' = \mathbf{s}'$, *then there exists a pencil* $P(s) \in \mathbb{F}[s]^{p\times q}$ *of* $\operatorname{rank}(P(s)) = 1$ *such that* $A(s) + P(s) \overset{s.e.}{\sim} B(s)$ *if and only if* (2.4) *holds.*
2. *If* $\mathbf{r} \neq \mathbf{s}$, $\mathbf{r}' = \mathbf{s}'$, *let*

$$x = \min\{i \ : \ r_i \neq s_i\}, \tag{4.3}$$

(4.4)
$$e = \min\{i \geq x - 1 \ : \ s_{i+1} \geq r_{i+1}\}, \quad e' = \min\{i \geq x - 1 \ : \ r_{i+1} \geq s_{i+1}\},$$

$$G = \rho - 1 - \sum_{i=1}^{\rho-1} \deg(\gcd(\phi_{i+1}(s,t), \psi_{i+1}(s,t))) - \sum_{i=1}^{\rho} r'_i.$$

Then, there exists a pencil $P(s) \in \mathbb{F}[s]^{p\times q}$ *of* $\operatorname{rank}(P(s)) = 1$ *such that* $A(s)+P(s) \overset{s.e.}{\sim} B(s)$ *if and only if* (2.4) *holds and*

$$G \leq \sum_{i=1}^{\rho} \min\{r_i, s_i\} + \max\{e, e'\}. \tag{4.5}$$

3. *If* $\mathbf{r} = \mathbf{s}$, $\mathbf{r}' \neq \mathbf{s}'$, *let*

$$\bar{x} = \min\{i \ : \ r'_i \neq s'_i\},$$

$$\bar{e} = \min\{i \geq \bar{x} - 1 \ : \ s'_{i+1} \geq r'_{i+1}\}, \quad \bar{e}' = \min\{i \geq \bar{x} - 1 \ : \ r'_{i+1} \geq s'_{i+1}\},$$

$$\bar{G} = \rho - 1 - \sum_{i=1}^{\rho-1} \deg(\gcd(\phi_{i+1}(s,t), \psi_{i+1}(s,t))) - \sum_{i=1}^{\rho} r_i.$$

Then, there exists a pencil $P(s) \in \mathbb{F}[s]^{p\times q}$ *of* $\operatorname{rank}(P(s)) = 1$ *such that* $A(s)+P(s) \overset{s.e.}{\sim} B(s)$ *if and only if* (2.4) *holds and*

$$\bar{G} \leq \sum_{i=1}^{\rho} \min\{r'_i, s'_i\} + \max\{\bar{e}, \bar{e}'\}.$$

4. *If* $\mathbf{r} \neq \mathbf{s}$, $\mathbf{r}' \neq \mathbf{s}'$, *then there exists a pencil* $P(s) \in \mathbb{F}[s]^{p\times q}$ *of* $\operatorname{rank}(P(s)) = 1$ *such that* $A(s)+P(s) \overset{s.e.}{\sim} B(s)$ *if and only if* (2.4) *and one of the four following conditions hold:*
   (a)

$$\mathbf{s} \angle \mathbf{r}, \quad \mathbf{s}' \angle \mathbf{r}', \tag{4.6}$$

and

$$\sum_{i=1}^{\rho} \deg(\mathrm{lcm}(\phi_i(s,t),\psi_i(s,t))) \leq x \leq \sum_{i=1}^{\rho} \deg(\gcd(\phi_{i+1}(s,t),\psi_{i+1}(s,t))), \tag{4.7}$$

where $x = \rho - \sum_{i=1}^{\rho'} r_i - \sum_{i=1}^{\rho'} s'_i$.

(b)

$$\mathbf{r} \angle \mathbf{s}, \quad \mathbf{r}' \angle \mathbf{s}', \tag{4.8}$$

and

$$\sum_{i=1}^{\rho} \deg(\mathrm{lcm}(\phi_i(s,t),\psi_i(s,t))) \leq y \leq \sum_{i=1}^{\rho} \deg(\gcd(\phi_{i+1}(s,t),\psi_{i+1}(s,t))), \tag{4.9}$$

where $y = \rho - \sum_{i=1}^{\rho'} s_i - \sum_{i=1}^{\rho'} r'_i$.

(c) (4.6) *and* (4.9) *hold.*

(d) (4.8) *and* (4.7) *hold.*

Proof. The proof is a consequence of Theorem 2.2, Lemmas 4.6 and 4.7, and Proposition 4.5. ∎

5. Bounds.

Given a pencil $A(s)$ and a perturbation of it, $A(s) + P(s)$, where $P(s)$ is a pencil of rank one, and given $\lambda \in \overline{\mathbb{F}}$, in this section we obtain bounds for the differences between the generalized Weyr characteristics of $A(s)$ and $A(s) + P(s)$ at λ. We include some technical lemmas in subsection 5.1 and prove the main result in subsection 5.2.

The notation used in this section corresponds to that introduced in Theorem 4.8. In particular, when $\mathbf{r} \neq \mathbf{s}$ and $\mathbf{r}' = \mathbf{s}'$, the values of x, e, and e' are defined in (4.3) and (4.4) (they also appear in Lemma 4.6). Notice that either $e = x - 1 < e'$ or $e' = x - 1 < e$; therefore $e \neq e'$.

5.1. Technical lemmas.

LEMMA 5.1. *Assume that* $\mathbf{r} \neq \mathbf{s}$, $\mathbf{r}' = \mathbf{s}'$, *and* (4.5) *holds.*

1. *Case* $e > e'$.

   (a) *Let* $i \in \{x, \dots, e\}$*; then*

   $$-x - 1 \leq s_i - r_i \leq -1. \tag{5.1}$$

   Moreover, if $s_i - r_i = -x - 1$, *then*

   $$\phi_j(s,t) \mid \psi_j(s,t), \quad 1 \leq j \leq \rho. \tag{5.2}$$

   (b) *Let* $i > e$*; then*

   $$-x \leq s_i - r_i \leq e + 1. \tag{5.3}$$

   Moreover, if $s_i - r_i = e + 1$, *then*

   $$\psi_j(s,t) \mid \phi_j(s,t), \quad 1 \leq j \leq \rho. \tag{5.4}$$

2. *Case* $e' > e$.

(a) *Let* $i \in \{x, \ldots, e'\}$; *then*

$$-x-1 \le r_i - s_i \le -1.$$

If $r_i - s_i = -x-1$, *then* (5.4) *holds.*

(b) *Let* $i > e'$; *then*

$$-x \le r_i - s_i \le e'+1.$$

If $r_i - s_i = e'+1$, *then* (5.2) *holds.*

Proof. If $\mathbf{r} \neq \mathbf{s}$ and $\mathbf{r}' = \mathbf{s}'$, then $\operatorname{rank} A(s) = \operatorname{rank} B(s) = \rho$, $r_0 = s_0 = q - \rho$, and $\min\{e, e'\} = x - 1 < \max\{e, e'\}$.

As $\mathbf{r} \neq \mathbf{s}$, we have $\sum_{i=1}^{\rho} r_i \neq 0$ or $\sum_{i=1}^{\rho} s_i \neq 0$; hence $\phi_1(s,t) = 1$ or $\psi_1(s,t) = 1$, $\deg(\gcd(\phi_1(s,t), \psi_1(s,t))) = 1$, and

$$\sum_{i=1}^{\rho-1} \deg(\gcd(\phi_{i+1}(s,t), \psi_{i+1}(s,t))) = \sum_{i=1}^{\rho} \deg(\gcd(\phi_i(s,t), \psi_i(s,t))).$$

Bearing in mind that $\mathbf{r}' = \mathbf{s}'$ and

$$\sum_{i=1}^{\rho} r_i + \sum_{i=1}^{\rho} r_i' + \sum_{i=1}^{\rho} \deg(\phi_i(s,t)) = \sum_{i=1}^{\rho} s_i + \sum_{i=1}^{\rho} s_i' + \sum_{i=1}^{\rho} \deg(\psi_i(s,t)) = \rho,$$

we have

$$G = \sum_{i=1}^{\rho} r_i + \sum_{i=1}^{\rho} X_i - 1 = \sum_{i=1}^{\rho} s_i + \sum_{i=1}^{\rho} Y_i - 1,$$

where for $1 \le i \le \rho$, $X_i = \deg(\phi_i(s,t)) - \deg(\gcd(\phi_i(s,t), \psi_i(s,t)))$ and $Y_i = \deg(\psi_i(s,t)) - \deg(\gcd(\phi_i(s,t), \psi_i(s,t)))$. Hence condition (4.5) is equivalent to

$$\sum_{i=1}^{\rho} (r_i - \min\{r_i, s_i\}) + \sum_{i=1}^{\rho} X_i \le \max\{e, e'\} + 1 \tag{5.5}$$

and to

$$\sum_{i=1}^{\rho} (s_i - \min\{r_i, s_i\}) + \sum_{i=1}^{\rho} Y_i \le \max\{e, e'\} + 1. \tag{5.6}$$

1. Assume that $e > e'$. Then $e' = x - 1 < e = \max\{e, e'\} \le \rho$.
   (a) Let $i \in \{x, \ldots, e\}$. Then $r_i > s_i$ and the upper bound of (5.1) holds. Moreover,

$$\begin{aligned} \textstyle\sum_{j=1}^{i} (r_j - s_j) &= \textstyle\sum_{j=1}^{e} (r_j - s_j) - \sum_{j=i+1}^{e} (r_j - s_j) \\ &\le \textstyle\sum_{j=1}^{e} (r_j - s_j) - (e - i) \\ &\le \textstyle\sum_{j=1}^{\rho} (r_j - \min\{r_j, s_j\}) - (e - i). \end{aligned}$$

   From (5.5) we derive

$$\begin{aligned} \textstyle\sum_{j=1}^{i} (r_j - s_j) &\le \textstyle\sum_{j=1}^{\rho} (r_j - \min\{r_j, s_j\}) + \sum_{j=1}^{\rho} X_j - (e - i) \\ &\le (e+1) - (e-i) = i + 1; \end{aligned} \tag{5.7}$$

then

$$r_i - s_i = \sum_{j=1}^{i}(r_j - s_j) - \sum_{j=x}^{i-1}(r_j - s_j) \leq (i+1) - (i-x) = x+1,$$

and the lower bound of (5.1) holds.

In the case that $s_i - r_i = -x-1$, from (5.7),

$$i+1 \geq \sum_{j=1}^{i}(r_j - s_j) = \sum_{j=x}^{i-1}(r_j - s_j) + x + 1 \geq (i-x) + x + 1 = i+1.$$

Then, again from (5.7), we have

$$(i+1) + \sum_{j=i+1}^{e}(r_j - s_j) + \sum_{j=e+1}^{\rho}(r_j - \min\{r_j, s_j\}) + \sum_{j=1}^{\rho} X_j - (e-i) \leq i+1;$$

i.e.,

$$\sum_{j=i+1}^{e}(r_j - s_j) - (e-i) + \sum_{j=e+1}^{\rho}(r_j - \min\{r_j, s_j\}) + \sum_{j=1}^{\rho} X_j = 0.$$

From this equation we conclude that $X_j = 0,\ 1 \leq j \leq \rho$, which is equivalent to condition (5.2).

(b) Let $i \in \{e+1, \ldots, \rho\}$. Then, from (5.6),

$$s_i - r_i \leq s_i - \min\{r_i, s_i\} \leq \sum_{j=1}^{\rho}(s_j - \min\{r_j, s_j\}) + \sum_{i=1}^{\rho} Y_j \leq e+1, \tag{5.8}$$

and, from (5.5),

$$\begin{aligned} r_i - s_i \leq \ & r_i - \min\{r_i, s_i\} \leq \textstyle\sum_{j=e+1}^{\rho}(r_j - \min\{r_j, s_j\}) \\ = \ & \textstyle\sum_{j=1}^{\rho}(r_j - \min\{r_j, s_j\}) - \sum_{j=x}^{e}(r_j - s_j) \\ \leq \ & \textstyle\sum_{j=1}^{\rho}(r_j - \min\{r_j, s_j\}) + \sum_{i=1}^{\rho} X_j - (e-x+1) \\ \leq \ & (e+1) - (e-x+1) = x. \end{aligned}$$

Therefore (5.3) holds.

In the case that $s_i - r_i = e+1$, from (5.8) we obtain that $Y_j = 0,\ 1 \leq j \leq \rho$, which is equivalent to (5.4).

2. The proof is analogous to that of case 1. ∎

As a consequence of Lemma 5.1 we obtain the following result.

LEMMA 5.2. *Assume that* $\mathbf{r} \neq \mathbf{s}$, $\mathbf{r}' = \mathbf{s}'$, *and* (4.5) *holds. Let* $a' = \min\{e, e'\} = x-1$ *and* $b' = \max\{e, e'\}$. *Then*

$$\begin{array}{rl} s_i - r_i = 0, & 0 \leq i \leq a', \\ -(a'+2) \leq s_i - r_i \leq a'+2, & a'+1 \leq i \leq b', \\ -(b'+1) \leq s_i - r_i \leq b'+1, & i \geq b'+1. \end{array}$$

Additionally the following holds:

If $s_i - r_i = -(a'+2)$ *for some* $i \in \{a'+1, \ldots, b'\}$ *or* $s_i - r_i = -(b'+1)$ *for some* $i > b'$, *then* (5.2) *holds.*

If $s_i - r_i = (a'+2)$ *for some* $i \in \{a'+1, \ldots, b'\}$ *or* $s_i - r_i = b'+1$ *for some* $i > b'$, *then* (5.4) *holds.*

LEMMA 5.3.

1. *Assume that* $\operatorname{rank}(A(s)) = \rho$ *and* $\operatorname{rank}(B(s)) = \rho + 1$ *and that* (4.6) *holds. Let* $g = \max\{i : r_i > s_i\}$ *and* $i \in \{g+1, \dots, \rho+1\}$.
   (a) *If* (4.9) *holds, then* $0 \le s_i - r_i \le g$. *In addition, if* $s_i - r_i = g$, *then* (5.4) *is satisfied.*
   (b) *If* (4.7) *holds, then* $0 \le s_i - r_i \le g + 1$. *In addition, if* $s_i - r_i = g + 1$, *then* (5.4) *is satisfied.*
2. *Assume that* $\operatorname{rank}(A(s)) = \rho + 1$ *and* $\operatorname{rank}(B(s)) = \rho$ *and that* (4.8) *holds. Let* $g = \max\{i : s_i > r_i\}$ *and* $i \in \{g+1, \dots, \rho+1\}$.
   (a) *If* (4.7) *holds, then* $0 \le r_i - s_i \le g$. *In addition, if* $r_i - s_i = g$, *then* (5.2) *is satisfied.*
   (b) *If* (4.9) *holds, then* $0 \le r_i - s_i \le g + 1$. *In addition, if* $r_i - s_i = g + 1$, *then* (5.2) *is satisfied.*

Proof.

1. Notice that, from the definition of g, we have $s_i - r_i \ge 0$.
   (a) From (4.9) we derive that

$$\sum_{j=1}^{\rho} \deg(\operatorname{lcm}(\phi_j(s,t), \psi_j(s,t))) \le \sum_{j=1}^{\rho} \deg(\phi_j(s,t)) + \sum_{j=1}^{\rho+1} (r_j - s_j).$$

As a consequence,

$$s_i - r_i \le \sum_{j=g+1}^{\rho+1} (s_j - r_j) \le \sum_{j=1}^{g} (r_j - s_j) + \sum_{j=1}^{\rho} X'_j \le \sum_{j=1}^{g} (r_j - s_j) = g,$$

where for $1 \le j \le \rho$, $X'_j = \deg(\phi_j(s,t)) - \deg(\operatorname{lcm}(\phi_j(s,t), \psi_j(s,t)))$. If $s_i - r_i = g$, then $\sum_{j=1}^{\rho} X'_j = 0$, and (5.4) is satified.

   (b) From (4.7) we derive that

$$\sum_{j=1}^{\rho+1} \deg(\psi_j(s,t)) + \sum_{j=1}^{\rho+1} (s_j - r_j) - 1 \le \sum_{i=1}^{\rho} \deg(\gcd(\phi_{i+1}(s,t), \psi_{i+1}(s,t))). \tag{5.9}$$

   - If $\deg(\gcd(\phi_1(s,t), \psi_1(s,t))) = 1$, then $s_i = r_i = 0$ and $g = 0$.
   - If $\deg(\gcd(\phi_1(s,t), \psi_1(s,t))) = 0$, then, from (5.9),

$$\begin{aligned} s_i - r_i &\le \textstyle\sum_{j=g+1}^{\rho+1} (s_j - r_j) \le \sum_{j=1}^{g} (r_j - s_j) + \sum_{j=1}^{\rho+1} Y_j + 1 \\ &\le \textstyle\sum_{j=1}^{g} (r_j - s_j) + 1 = g + 1, \end{aligned}$$

where $Y_j = \deg(\gcd(\phi_j(s,t), \psi_j(s,t)) - \deg(\psi_j(s,t))$, $1 \le j \le \rho + 1$. If $s_i - r_i = g + 1$, then $\sum_{j=1}^{\rho+1} Y_j = 0$, and (5.4) is satisfied.

2. The proof is analogous to that of case 1. ∎

5.2. Main theorem. We now prove the main theorem of the paper. Recall that we use the notation of Theorem 4.8.

THEOREM 5.4. *Given a matrix pencil* $A(s) \in \mathbb{F}[s]^{p \times q}$, *let* $P(s) \in \mathbb{F}[s]^{p \times q}$ *be a matrix pencil of rank one. For* $\lambda \in \overline{\mathbb{F}}$, *the following statements hold:*

(i) *If both pencils* $A(s)$ *and* $A(s) + P(s)$ *are regular, then*

$$-1 \le w_i(\lambda, A(s) + P(s)) - w_i(\lambda, A(s)) \le 1, \quad i \ge 1.$$

(ii) *If $A(s)$ is regular and $A(s)+P(s)$ is singular, then, taking $a=d_1+1$, we have*

$$\begin{array}{ll} -1 \leq w_i(\lambda, A(s)+P(s)) - w_i(\lambda, A(s)) \leq 1, & 1 \leq i \leq a, \\ -2 \leq w_i(\lambda, A(s)+P(s)) - w_i(\lambda, A(s)) \leq 0, & i \geq a+1. \end{array}$$

(iii) *If $A(s)$ is singular and $A(s)+P(s)$ is regular, then, taking $a=c_1+1$, we have*

$$\begin{array}{ll} -1 \leq w_1(\lambda, A(s)+P(s)) - w_1(\lambda, A(s)) \leq 1, & 1 \leq i \leq a, \\ 0 \leq w_i(\lambda, A(s)+P(s)) - w_i(\lambda, A(s)) \leq 2, & i \geq a+1. \end{array}$$

(iv) *If both $A(s)$ and $A(s)+P(s)$ are singular the following holds:*

- *Assume that* $\operatorname{rank}(A(s)) = \operatorname{rank}(A(s)+P(s))$.
  - *If $A(s)$ and $A(s)+P(s)$ have the same column minimal indices, then*

    $$-1 \leq w_i(\lambda, A(s)+P(s)) - w_i(\lambda, A(s)) \leq 1, \quad i \geq 1.$$

  - *If $A(s)$ and $A(s)+P(s)$ have different column minimal indices, then, taking $a=x$ and $b=\max\{e,e'\}+1$ (notice that $b>a\geq 1$), we have*

    (5.10)
    $$\begin{array}{ll} -1 \leq w_i(\lambda, A(s)+P(s)) - w_i(\lambda, A(s)) \leq 1, & 1 \leq i \leq a, \\ -(a+1) \leq w_i(\lambda, A(s)+P(s)) - w_i(\lambda, A(s)) \leq a+1, & a+1 \leq i \leq b, \\ -b \leq w_i(\lambda, A(s)+P(s)) - w_i(\lambda, A(s)) \leq b, & i \geq b+1. \end{array}$$

- *Assume that* $\operatorname{rank}(A(s)) < \operatorname{rank}(A(s)+P(s))$. *Let* $a=\max\{i : r_i > s_i\}+1$. *Then*

  $$\begin{array}{ll} -1 \leq w_i(\lambda, A(s)+P(s)) - w_i(\lambda, A(s)) \leq 1, & 1 \leq i \leq a, \\ 0 \leq w_i(\lambda, A(s)+P(s)) - w_i(\lambda, A(s)) \leq a+1, & i \geq a+1. \end{array}$$

- *Assume that* $\operatorname{rank}(A(s)) > \operatorname{rank}(A(s)+P(s))$. *Let* $a=\max\{i : s_i > r_i\}+1$. *Then*

  $$\begin{array}{ll} -1 \leq w_i(\lambda, A(s)+P(s)) - w_i(\lambda, A(s)) \leq 1, & 1 \leq i \leq a, \\ -(a+1) \leq w_i(\lambda, A(s)+P(s)) - w_i(\lambda, A(s)) \leq 0, & i \geq a+1. \end{array}$$

Proof. We take $B(s)=A(s)+P(s)$. Notice that $-1 \leq \rho_2 - \rho_1 \leq 1$. Let

$$(w_1^R(\lambda, A(s)), \dots) = \overline{(n_1(\lambda, A(s)), \dots, n_{\rho_1}(\lambda, A(s))},$$

$$(w_1^R(\lambda, B(s)), \dots) = \overline{(n_1(\lambda, B(s)), \dots, n_{\rho_2}(\lambda, B(s))}.$$

By Theorem 3.7, for $i \geq 1$,

$$w_i(\lambda, B(s)) - w_i(\lambda, A(s)) = w_i^R(\lambda, B(s)) - w_i^R(\lambda, A(s)) + s_{i-1} - r_{i-1}. \tag{5.11}$$

By Theorem 4.8 condition (2.4) holds; then

$$n_{i+\rho_2-\rho_1+1}(\lambda, B(s)) \leq n_i(\lambda, A(s)) \leq n_{i+\rho_2-\rho_1-1}(\lambda, B(s)), \quad i \geq 1,$$

and, by Lemma 4.3, this is equivalent to

$$\rho_2 - \rho_1 - 1 \leq w_i^R(\lambda, B(s)) - w_i^R(\lambda, A(s)) \leq \rho_2 - \rho_1 + 1, \quad i \geq 1. \tag{5.12}$$

Additionally, if (5.4) holds, then

$$\rho_2 - \rho_1 - 1 \le w_i^R(\lambda, B(s)) - w_i^R(\lambda, A(s)) \le \rho_2 - \rho_1, \quad i \ge 1, \tag{5.13}$$

and if (5.2) holds, then

$$\rho_2 - \rho_1 \le w_i^R(\lambda, B(s)) - w_i^R(\lambda, A(s)) \le \rho_2 - \rho_1 + 1, \quad i \ge 1. \tag{5.14}$$

We analyze now the different cases enumerated in the statement of the theorem.

(i) If $A(s)$ and $A(s)+P(s)$ are regular, then $p = q = \rho_1 = \rho_2$. The result follows from (5.11) and (5.12). This result was proven in [1, Corollary 4.7].

(ii) If $A(s)$ is regular and $A(s)+P(s)$ is singular, then $p = q = \rho_1 = \rho_2 + 1$, $r_0 = 0$, $(s_1, \dots) = \overline{(d_1)} = (1, \overset{(d_1)}{\dots}, 1, 0, \dots)$, and $s_0 = 1$. Taking into account (5.11) and (5.12), we get

$$-1 \le w_i(\lambda, B(s)) - w_i(\lambda, A(s)) \le 1, \quad 1 \le i \le d_1 + 1,$$

$$-2 \le w_i(\lambda, B(s)) - w_i(\lambda, A(s)) \le 0, \quad i \ge d_1 + 2.$$

(iii) If $A(s)$ is singular and $A(s)+P(s)$ is regular the proof is analogous to (ii), exchanging the roles of $A(s)$ and $B(s)$.

(iv) If $A(s)$ and $A(s)+P(s)$ are singular, there are three possibilities: $\rho_1 = \rho_2$, $\rho_2 = \rho_1 + 1$, or $\rho_1 = \rho_2 + 1$.

- Let $\rho_1 = \rho_2$. As $s_0 - r_0 = 0$, from Theorem 4.8, necessarily $\mathbf{r} = \mathbf{s}$ or $\mathbf{r}' = \mathbf{s}'$ (notice that if $\mathbf{s} \angle \mathbf{r}$, then $r_0 = s_0 + 1$).
  - If $\mathbf{r} = \mathbf{s}$, then from conditions (5.11) and (5.12) we get

    $$-1 \le w_i(\lambda, B(s)) - w_i(\lambda, A(s)) \le 1, \quad i \ge 1.$$

  - If $\mathbf{r} \ne \mathbf{s}$, then $\mathbf{r}' = \mathbf{s}'$, and (4.5) holds. By Lemma 5.2, we have

    $$\begin{array}{ll} s_{i-1} - r_{i-1} = 0, & 1 \le i \le a, \\ -(a+1) \le s_{i-1} - r_{i-1} \le a+1, & a+1 \le i \le b, \\ -b \le s_{i-1} - r_{i-1} \le b, & i \ge b+1. \end{array} \tag{5.15}$$

    Moreover, if (5.2) does not hold, then

    $$\begin{array}{ll} -a \le s_{i-1} - r_{i-1} \le a+1, & a+1 \le i \le b, \\ -b+1 \le s_{i-1} - r_{i-1} \le b, & i \ge b+1, \end{array} \tag{5.16}$$

    and, if (5.4) does not hold, then

    $$\begin{array}{ll} -(a+1) \le s_{i-1} - r_{i-1} \le a, & a+1 \le i \le b, \\ -b \le s_{i-1} - r_{i-1} \le b-1, & i \ge b+1. \end{array} \tag{5.17}$$

    We obtain different results depending on the relation between the homogeneous invariant factors of $A(s)$ and $B(s)$.
    * If $\phi_i(s,t) = \psi_i(s,t)$, $1 \le i \le \rho$, then by (5.11), (5.13), (5.14), and (5.15) we obtain

      $$\begin{array}{ll} w_i(\lambda, B(s)) - w_i(\lambda, A(s)) = 0, & 1 \le i \le a, \\ -(a+1) \le w_i(\lambda, B(s)) - w_i(\lambda, A(s)) \le a+1, & a+1 \le i \le b, \\ -b \le w_i(\lambda, B(s)) - w_i(\lambda, A(s)) \le b, & i \ge b+1. \end{array}$$

  * If (5.4) holds and (5.2) does not hold, then by (5.11), (5.13), and (5.16) we obtain

$$\begin{aligned} -1 \le w_i(\lambda, B(s)) - w_i(\lambda, A(s)) &\le 0, && 1 \le i \le a, \\ -(a+1) \le w_i(\lambda, B(s)) - w_i(\lambda, A(s)) &\le a+1, && a+1 \le i \le b, \\ -b \le w_i(\lambda, B(s)) - w_i(\lambda, A(s)) &\le b, && i \ge b+1. \end{aligned}$$

  * If (5.2) holds and (5.4) does not hold, then analogously by (5.11), (5.14), and (5.17) we obtain

$$\begin{aligned} 0 \le w_i(\lambda, B(s)) - w_i(\lambda, A(s)) &\le 1, && 1 \le i \le a, \\ -(a+1) \le w_i(\lambda, B(s)) - w_i(\lambda, A(s)) &\le a+1, && a+1 \le i \le b, \\ -b \le w_i(\lambda, B(s)) - w_i(\lambda, A(s)) &\le b, && i \ge b+1. \end{aligned}$$

  * If neither (5.2) nor (5.4) is satisfied, then, by (5.11), (5.12), (5.16), and (5.17), we obtain that (5.10) holds. □

- If $\rho_2 = \rho_1 + 1$, we have $r_0 = s_0 + 1$, and by Theorem 4.8, (4.6), and (4.7) or (4.9), hold. Let $g = \max\{i : r_i > s_i\}$; then

$$s_{i-1} - r_{i-1} = -1, \quad 1 \le i \le g+1.$$

  - If (4.9) holds, or (5.4) does not hold, then by Lemma 5.3,

$$0 \le s_{i-1} - r_{i-1} \le g, \quad i \ge g+2.$$

    Therefore, from (5.11) and (5.12) we obtain

$$\begin{aligned} -1 \le w_i(\lambda, B(s)) - w_i(\lambda, A(s)) &\le 1, && 1 \le i \le g+1, \\ 0 \le w_i(\lambda, B(s)) - w_i(\lambda, A(s)) &\le g+2, && i \ge g+2. \end{aligned}$$

  - Alternatively, if (4.7) and (5.4) hold, then, by Lemma 5.3,

$$0 \le s_{i-1} - r_{i-1} \le g+1, \quad i \ge g+2.$$

    Therefore, from (5.11) and (5.13) we obtain

$$\begin{aligned} -1 \le w_i(\lambda, B(s)) - w_i(\lambda, A(s)) &\le 0, && 1 \le i \le g+1, \\ 0 \le w_i(\lambda, B(s)) - w_i(\lambda, A(s)) &\le g+2, && i \ge g+2. \end{aligned}$$

  As $a = g + 1$, the result follows.
- If $\rho_1 = \rho_2 + 1$, taking $g = \max\{i : s_i > r_i\}$, the proof is analogous:
  - If (4.7) holds, or (5.2) does not hold, then

$$\begin{aligned} -1 \le w_i(\lambda, B(s)) - w_i(\lambda, A(s)) &\le 1, && 1 \le i \le g+1, \\ -g-2 \le w_i(\lambda, B(s)) - w_i(\lambda, A(s)) &\le 0, && i \ge g+2. \end{aligned}$$

  - Alternatively, if (4.9) and (5.2) hold, then

$$\begin{aligned} 0 \le w_i(\lambda, B(s)) - w_i(\lambda, A(s)) &\le 1, && 1 \le i \le g+1, \\ -g-2 \le w_i(\lambda, B(s)) - w_i(\lambda, A(s)) &\le 0, && i \ge g+2. \end{aligned}$$

Remark 5.5. We would like to point out that the bounds obtained in Theorem 5.4 are sharp; i.e., there are examples showing that the bounds are attained. Concerning the sufficiency, it is proven in [1, Corollary 4.14] that if $A(s)$ and $A(s) + P(s)$ are regular, the conditions are sufficient, in the sense that if the bounds are satisfied for some numbers w_i', then there exists a rank-one perturbation $P(s)$ of $A(s)$ such that $w_i(\lambda, A(s) + P(s)) = w_i'$. This property is immediately extended to the case where $\mathbf{r} = \mathbf{s}$. On the other hand, there are examples showing that in the general case the conditions are not sufficient.

6. Conclusion. We have generalized the notion of a Weyr characteristic of an eigenvalue of a pencil (see [4]) and have extended the definition of a Jordan chain of square pencils ([13]) to arbitrary pencils. Out of them, we have obtained bounds for the changes of the generalized Weyr characteristic of a matrix pencil perturbed by another pencil of rank one. The results in this paper improve the bounds obtained in [13, Theorem 7.8]. The bounds in Theorem 5.4 cases (ii) and (iii) are clearly sharper than the corresponding ones in [13, Theorem 7.8]. Concerning the case (iv), at the cost of splitting the range of indices into different parts, we obtain significantly better bounds.

It must be remarked that our results hold for any algebraically closed field and for arbitrary rank-one perturbations.

Additionally, we have translated the characterization obtained in [2, Theorem 5.1] of the changes of the Kronecker structure of a pencil perturbed by another pencil of rank one into terms of the conjugate partitions of the corresponding chains of column and row minimal indices of the pencils involved.

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