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ORIGINAL PAPER

On 2MP-, MP2- and C2MP-inverses for rectangular matrices

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Abstract
This paper introduces 2MP-inverses, MP2-inverses, and C2MP-inverses, for rectangular matrices following a different approach to that used in the recent literature. These new inverses generalize some classical inverses in the literature. Instead of considering a system of matrix equations as usually, in order to define 2MP-inverses and MP2-inverses, we consider a construction from oblique projectors represented by means of outer generalized inverses. We use an adequate equivalence relation, and then we pass to the quotient set in order to get the most simple canonical representative. An interesting advantage of our extension of CMP inverses from square to rectangular matrices is that we do not need any auxiliary weight matrix, but we are using the own matrix A for doing it. In addition, some properties and representations of 2MP-, MP2-, and C2MP-inverses are given.

Keywords Outer inverse · Moore-Penrose inverse · Matrix equation · Partial order · Quotient set

Mathematics subject classification 15A09

1 Introduction and preliminary results

Let $\mathbb{C}^{m\times n}$ be the set of $m \times n$ complex matrices. For $A \in \mathbb{C}^{m\times n}$, let A^*, A^{-1}, $\mathrm{rk}(A)$, $\mathcal{R}(A)$, and $\mathcal{N}(A)$ denote the conjugate transpose, the inverse (when $m = n$), the rank, the range space, and the null space of A, respectively. As usual, I and 0 stand for the identity matrix and the zero matrix of adequate size.

For a given $A \in \mathbb{C}^{m\times n}$, we consider the following sets of matrices:

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- $\mathcal{A}\{1\} = \{X \in \mathbb{C}^{n\times m} : AXA = A\}$; an element of this set is called a $\{1\}$-*inverse* or *inner inverse* of A, and is denoted by A^-.
- $\mathcal{A}\{2\} = \{X \in \mathbb{C}^{n\times m} : XAX = X\}$; an element of this set is called a $\{2\}$-*inverse* or *outer inverse* of A, and is denoted by A^{2-}.

The *Moore-Penrose inverse* of $A \in \mathbb{C}^{m\times n}$ is the unique matrix $X \in \mathbb{C}^{n\times m}$ that satisfies

$$AXA = A,\ XAX = X,\ (AX)^* = AX,\ (XA)^* = XA,$$

and is denoted by $A^\dagger$. The *group inverse* of $A \in \mathbb{C}^{n\times n}$ is the unique matrix $X \in \mathbb{C}^{n\times n}$ that satisfies

$$AXA = A,\ XAX = X,\ AX = XA,$$

and, when it exists, is denoted by $A^\#$. It is well known that $A^\#$ exists if and only if $ind(A) \leq 1$, where $ind(A)$ denotes the smallest nonnegative integer k such that $\mathrm{rk}(A^{k+1}) = \mathrm{rk}(A^k)$. The *Drazin inverse* of $A \in \mathbb{C}^{n\times n}$ is the unique matrix $X \in \mathbb{C}^{n\times n}$ that satisfies

$$XAX = X,\ AX = XA,\ A^{k+1}X = A^k,$$

always exists and is denoted by A^D. It is well known that $A^D = A^\#$ when $ind(A) \leq 1$.

A detailed analysis of all these generalized inverses can be found, for example, in [2].

The following result is used later.

Theorem 1.1 [2] *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$, and $C \in \mathbb{C}^{m\times q}$. The matrix equation $AXB = C$ has a solution if and only if there exist $\{1\}$-inverses A^- and B^- of A and B, respectively, such that $AA^-C = C$ and $CB^-B = C$. In this case, the general solution is $X = A^-CB^- + Y - A^-AYBB^-$, for arbitrary $Y \in \mathbb{C}^{n\times p}$.*

For any matrix $A \in \mathbb{C}^{m\times n}$ with $\mathrm{rk}(A) = a > 0$, a singular value decomposition (SVD, for short) [2] is given by

$$A = U \begin{pmatrix} D_a & 0 \\ 0 & 0 \end{pmatrix} V^*, \tag{1.1}$$

where $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary matrices, and $D_a \in \mathbb{C}^{a\times a}$ is a positive definite diagonal matrix.

For $A \in \mathbb{C}^{m\times n}$, written as in (1.1), it is well known that the Moore-Penrose inverse of A is given by

$$A^\dagger = V \begin{pmatrix} D_a^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^*. \tag{1.2}$$

The set $\mathcal{A}\{2\}$ can be characterized as follows.

Lemma 1.2 *Let $A \in \mathbb{C}^{m\times n}$ be written as is (1.1), with $\mathrm{rk}(A) = a > 0$. The set of all outer inverses of A is given by*

$$\mathcal{A}\{2\} = \left\{ V \begin{pmatrix} M & MY_{12} \\ Y_{21}M & Y_{21}MY_{12} \end{pmatrix} U^* : MD_aM = M,\ \text{for arbitrary } Y_{12}, Y_{21} \right\}.$$

For a matrix $A \in \mathbb{C}^{n\times n}$ of index at most 1, two types of (unique) hybrid generalized inverses, namely $A^{\textcircled{\#}} = A^\#AA^\dagger$ and $A_{\textcircled{\#}} = A^\dagger AA^\#$ were defined in [22, p.97] and called as the *core* and the *dual core* inverse of A, respectively. These inverses were rediscovered by Baksalary and Trenkler in [1] and, since then, they were a key point of the study of Generalized

Inverses Theory. In [25, Theorem 2.1], Wang and Liu proved that if $ind(A) \leq 1$, then the core inverse of A is the unique matrix $X \in \mathbb{C}^{n \times n}$ satisfying the following three equations:

$$AXA = A, \quad AX^2 = X, \quad \text{and} \quad (AX)^* = AX.$$

These inverses were generalized for matrices of arbitrary index by Malik and Thome in [12]. They introduced the *DMP-inverse* and its dual, for a matrix $A \in \mathbb{C}^{n \times n}$ of index k, by $A^{D,\dagger} = A^D AA^\dagger$ and $A^{\dagger,D} = A^\dagger AA^D$, respectively. It was also proved that the matrix $A^{D,\dagger}$ is the unique solution of the matrix equations system

$$XAX = X, \quad XA = A^D A, \quad \text{and} \quad A^k X = A^k A^\dagger.$$

On the other hand, *CMP-inverses* were defined by Mehdipour and Salemi in [13] for a square matrix A by $A^{c,\dagger} = A^\dagger A_1 A^\dagger$, with $A_1 = AA^D A$, by the unique solution of the matrix equations system

$$XAX = X, \quad AXA = A_1 \quad AX = A_1 A^\dagger, \quad \text{and} \quad XA = A^\dagger A_1.$$

Some results, applications, and extensions of these generalized inverses can be found in the following references. For example, in order to mention only a few of them, characterizations using arbitrary index in [6, 7, 12, 19, 28], problems related to quaternion matrix equations in [8], extensions to finite potent endomorphisms in [20], extensions to outer inverses in [16], applications to partial orders in [5], characterizations and representations of generalized inverses in rings in [3, 21, 26], generalizations to operator theory in [15], and applications to neural networks in [27]. For some recent papers related to generalized inverses, we refer the reader to [9–11, 16, 18, 23, 24].

The common fact that all of aforementioned inverses share is that all of them were defined as the unique solution to a system of suitable matrix equations.

The main goal of this paper is to introduce new generalized (hybrid) inverses and their duals, namely 2MP- and MP2-inverses, as well as C2MP-inverses, and investigate many properties of them. These new classes of inverses provide not only a generalization of the core inverse to matrices of arbitrary index but also a generalization to rectangular matrices. Sect. 2 is devoted to the analysis of 2MP-inverses. In Sect. 3, we study MP2-inverses as the dual concept of 2MP-inverses. Finally, Sect. 4 introduces C2MP-inverses and investigates their properties.

2 2MP-inverses

Let $A \in \mathbb{C}^{m \times n}$ be written as in (1.1). By Lemma 1.2, the general form for {2}-inverses of A is given by

$$A^{2-} = V \begin{pmatrix} M & MY_{12} \\ Y_{21}M & Y_{21}MY_{12} \end{pmatrix} U^*, \tag{2.1}$$

partitioned according to the partition of A, where $M \in \mathbb{C}^{a \times a}$ satisfies $MD_aM = M$, and Y_{12}, Y_{21} are arbitrary matrices of adequate sizes. Then, it is easy to see that

$$A^{2-}A = V \begin{pmatrix} MD_a & 0 \\ Y_{21}MD_a & 0 \end{pmatrix} V^*.$$

We are interested on finding under which conditions two of these projectors (of type $A^{2-}A$) are different each other by ranging $A^{2-} \in \mathcal{A}\{2\}$. In order to do that, an equivalence relation is defined on the set $\mathcal{A}\{2\}$.

Let $A \in \mathbb{C}^{m\times n}$. We define the binary relation $\sim_\ell$ on the set $\mathcal{A}\{2\}$ as follows. For $A^{2-}, A^{2=} \in \mathcal{A}\{2\}$, we say that

$$A^{2-} \sim_\ell A^{2=} \quad \text{if and only if} \quad A^{2-}A = A^{2=}A.$$

It is easy to see that $\sim_\ell$ is an equivalence relation on $\mathcal{A}\{2\}$. Let $A^{2-} \in \mathcal{A}\{2\}$. Then, there exist M, Y_{12}, Y_{21} of adequate sizes such that (2.1) holds. The equivalence class of $A^{2-} \in \mathcal{A}\{2\}$ is given by $[A^{2-}]_{\sim_\ell} = \{A^{2=} \in \mathcal{A}\{2\} : A^{2=}A = A^{2-}A\}$. It can be shown that

$$A^{2=} = V\begin{pmatrix} M' & M'Y'_{12} \\ Y'_{21}M' & Y'_{21}M'Y'_{12} \end{pmatrix}U^* \in [A^{2-}]_{\sim_\ell} \quad \text{if and only if} \quad M' = M,\ Y'_{21}M' = Y_{21}M.$$

That is,

$$[A^{2-}]_{\sim_\ell} = \left\{ V\begin{pmatrix} M & MY'_{12} \\ Y_{21}M & Y_{21}MY'_{12} \end{pmatrix}U^* \in \mathcal{A}\{2\} :\ MD_aM = M,\ \text{for arbitrary } Y'_{12} \right\}. \tag{2.2}$$

A complete set of representatives of the partition on $\mathcal{A}\{2\}$ induced by $\sim_\ell$ is given by

$$\mathcal{R}_{\sim_\ell} := \left\{ V\begin{pmatrix} M & 0 \\ Y_{21}M & 0 \end{pmatrix}U^* : MD_aM = M,\ Y_{21} \in \mathbb{C}^{(n-a)\times a} \right\}.$$

By observing that any element in $\mathcal{R}_{\sim_\ell}$ can be factorized as $A^{2-}AA^\dagger$, we can state the following definition, which introduces a new class of generalized inverses.

Definition 2.1 Let $A \in \mathbb{C}^{m\times n}$. For each $A^{2-} \in \mathcal{A}\{2\}$, the matrix

$$A^{2MP} := A^{2-}AA^\dagger \in \mathbb{C}^{n\times m}$$

is called a 2MP-inverse of A. That is, A^{2MP} is defined as the most simple representative of the equivalence class (2.2) of A^{2-} by $\sim_\ell$.

The symbol $\mathcal{A}\{2MP\}$ stands for the set of all 2MP-inverses of A, that is,

$$\mathcal{A}\{2MP\} = \{A^{2-}AA^\dagger : A^{2-} \in \mathcal{A}\{2\}\}.$$

Clearly, since $A^\dagger$ is an element of this set, $\mathcal{A}\{2MP\} \neq \emptyset$. The following result provides a matrix representation of elements of the set $\mathcal{A}\{2MP\}$.

Lemma 2.2 *Let $A \in \mathbb{C}^{m\times n}$ be written as in (1.1) and A^{2-} be written as in (2.1). Then*

$$\mathcal{A}\{2MP\} = \left\{ V\begin{pmatrix} M & 0 \\ Y_{21}M & 0 \end{pmatrix}U^* : MD_aM = M,\ Y_{21} \in \mathbb{C}^{(n-a)\times a} \right\}. \tag{2.3}$$

This lemma states a canonical form for every 2MP-inverse of A.

The existence of $\{2\}$-inverses and the Moore-Penrose inverse of A guarantees that 2MP-inverses of A always exist. It is clear that $\mathcal{A}\{2MP\} = \{A^{-1}\}$ whenever $A \in \mathbb{C}^{n\times n}$ is nonsingular and, moreover, the 2MP-inverse of the zero matrix is itself. In general, 2MP-inverses are not unique.

We focus our attention on the interesting case: the one determined by matrices $A^{2-}, A^{2=} \in \mathcal{A}\{2\}$ such that $A^{2-}A \neq A^{2=}A$; otherwise, both A^{2-} and $A^{2=}$ provide the same 2MP-inverse: $A^{2-}AA^\dagger = A^{2=}AA^\dagger$. This fact is shown in the next result, where the symbol $M \simeq N$ indicates that there exists a bijection between the sets M and N.

Proposition 2.3 *Let $A \in \mathbb{C}^{m\times n}$ of rank $a > 0$ be written as in (1.1). Then*

$$\mathcal{A}\{2\}/\sim_\ell \;\simeq\; \mathcal{A}\{2MP\}.$$

Proof Let $\varphi : \mathcal{A}\{2\}/\sim_\ell \to \mathcal{A}\{2MP\}$ be the function defined by $\varphi([A^{2-}]_{\sim_\ell}) = A^{2-}AA^\dagger$. Clearly, φ is well-defined. Let $[A^{2-}]_{\sim_\ell}$ and $[A^{2=}]_{\sim_\ell}$ be in $\mathcal{A}\{2\}/\sim_\ell$ such that $\varphi([A^{2-}]_{\sim_\ell}) = \varphi([A^{2=}]_{\sim_\ell})$. Then $A^{2-}AA^\dagger = A^{2=}AA^\dagger$, so $A^{2-}AA^\dagger A = A^{2=}AA^\dagger A$. Hence, $A^{2-}A = A^{2=}A$, i.e., $[A^{2-}]_{\sim_\ell} = [A^{2=}]_{\sim_\ell}$, from where φ is injective. If $Y \in \mathcal{A}\{2MP\}$, there exists $A^{2-} \in \mathcal{A}\{2\}$ such that $Y = A^{2-}AA^\dagger$. Thus, $\varphi([A^{2-}]_{\sim_\ell}) = A^{2-}AA^\dagger = Y$. Hence, φ is surjective. In consequence, φ is a one-to-one correspondence between the sets $\mathcal{A}\{2\}/\sim_\ell$ and $\mathcal{A}\{2MP\}$. $\square$

If we solve (by applying Theorem 1.1) the matrix equation $A^{2-}A = A^{2=}A$ (in $A^{2=}$), its solution set is given by

$$[A^{2-}]_{\sim_\ell} = \{A^{2=} \in \mathcal{A}\{2\} : A^{2=} = A^{2-}AA^\dagger + Y(I - AA^\dagger), \text{ for arbitrary } Y \in \mathbb{C}^{n\times m}\},$$

which allows us to express the solution set as a 1-parametrized set.

In what follows, we show that 2MP-inverses are different from the known inverses in the literature and we state that 2MP-inverses are a generalization of the well-known inverses.

Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. If we consider $A^{2-} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix} \in \mathcal{A}\{2\}$, we obtain that $A^\dagger = \begin{pmatrix} 1/2 & 0 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $A^{2MP} = A^{2-}AA^\dagger = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix} \in \mathcal{A}\{2MP\}$. Clearly, $A^{2MP} \neq A^\dagger$. Moreover, $A^{2MP} \neq A^{D,\dagger}$ and $A^{2MP} \neq A^{\textcircled{\#}}$, since $ind(A) = 1$ and $AA^{2MP} \neq AA^\dagger$.

Remark 2.4 For any $A \in \mathbb{C}^{m\times n}$, by setting adequate matrices $A^{2-} \in \mathcal{A}\{2\}$, the inverse matrix $A^{2MP} \in \mathbb{C}^{n\times m}$ recovers, as particular cases, the Moore-Penrose, the DMP inverse, the core inverse. In fact, it immediately follows by setting $A^{2-} = A^\dagger$, $A^{2-} = A^D$ (the Drazin inverse of A), and $A^{2-} = A^\#$ (when $m = n$ and $ind(A) \leq 1$).

The next results give some properties of the 2MP-inverses.

Note that $A^{2MP}A = A^{2-}A$. Then, $AA^{2MP}A = AA^{2-}A$. Following a similar idea to that studied in [12], for $A \in \mathbb{C}^{m\times n}$ and each $A^{2-} \in \mathcal{A}\{2\}$, by the expression

$$C_2^A := AA^{2MP}A = AA^{2-}A \tag{2.4}$$

we denote a 2MP core-part of A.

Theorem 2.5 *Let $A \in \mathbb{C}^{m\times n}$. For each $A^{2-} \in \mathcal{A}\{2\}$, the matrix A^{2MP} is the unique that satisfies the following equations system (in the unknown X):*

$$(i)\; XAX = X,\; (ii)\; XA = A^{2-}A,\; (iii)\; C_2^A X = C_2^A A^\dagger. \tag{2.5}$$

Proof It is immediate that A^{2MP} satisfies equations (i), (ii) and (iii). To prove uniqueness, suppose that there exist $X_1, X_2 \in \mathbb{C}^{n\times m}$ which satisfy conditions (i), (ii) and (iii). Then $X_1 = X_1AX_1 = A^{2-}AX_1 = A^{2-}AA^{2-}AX_1 = A^{2-}C_2^A X_1 = A^{2-}C_2^A X_2 = A^{2-}AA^{2-}AX_2 = A^{2-}AX_2 = X_2AX_2 = X_2$. $\square$

Theorem 2.6 *Let $A \in \mathbb{C}^{m\times n}$. For a given $A^{2-} \in \mathcal{A}\{2\}$, the matrix $A^{2MP} \in \mathbb{C}^{n\times m}$ satisfies the following properties:*

(a) AA^{2MP} *is an oblique projector onto* $\mathcal{R}(C_2^A)$ *along* $\mathcal{N}(A^{2MP})$*. Moreover,* $\mathcal{R}(A^{2MP}) \subseteq \mathcal{R}(A^{2-})$.

(b) $A^{2MP}A$ *is an oblique projector onto* $\mathcal{R}(A^{2-}A) = \mathcal{R}(A^{2MP}) = \mathcal{R}(A^{2-})$ *along* $\mathcal{N}(A^{2-}A) = \mathcal{N}(C_2^A)$.

Proof (a) Taking into account that $A^{2MP}A = A^{2-}A$ is satisfied, we have

$$(AA^{2MP})(AA^{2MP}) = A(A^{2MP}A)A^{2-}AA^{\dagger} = AA^{2-}AA^{2-}AA^{\dagger} = AA^{2-}AA^{\dagger} = AA^{2MP}.$$

Moreover, $\mathcal{R}(AA^{2MP}) = \mathcal{R}((AA^{2-})AA^{\dagger}) = AA^{2-}\mathcal{R}(AA^{\dagger}) = AA^{2-}\mathcal{R}(A) = \mathcal{R}(AA^{2-}A) = \mathcal{R}(C_2^A)$.

Clearly, $\mathcal{N}(AA^{2MP}) = \mathcal{N}(AA^{2-}AA^{\dagger}) = \mathcal{N}(A^{2-}AA^{\dagger}) = \mathcal{N}(A^{2MP})$.

In addition, $\mathcal{R}(A^{2MP}) = \mathcal{R}(A^{2-}AA^{\dagger}) \subseteq \mathcal{R}(A^{2-})$.

(b) Since $A^{2MP}A = A^{2-}A$, we have that $A^{2MP}A$ is clearly idempotent. Moreover, $\mathcal{R}(A^{2MP}A) = \mathcal{R}(A^{2-}A) = A^{2-}\mathcal{R}(A) = A^{2-}\mathcal{R}(AA^{\dagger}) = \mathcal{R}(A^{2-}AA^{\dagger}) = \mathcal{R}(A^{2MP})$.

On the other hand, $\mathcal{R}(A^{2MP}A) = \mathcal{R}(A^{2-}A) \subseteq \mathcal{R}(A^{2-}) = \mathcal{R}(A^{2-}AA^{2-}) \subseteq \mathcal{R}(A^{2-}A)$. Finally,

$\mathcal{N}(A^{2MP}A) = \mathcal{N}(A^{2-}A) = \mathcal{N}(AA^{2-}A) = \mathcal{N}(C_2^A)$. □

Theorem 2.7 *Let* $A \in \mathbb{C}^{m\times n}$*. For each* $A^{2-} \in \mathcal{A}\{2\}$*, the matrix* A^{2MP} *is the unique that satisfies the following properties:*

$$(i)\ AX = P_{\mathcal{R}(C_2^A),\mathcal{N}(X)},\ (ii)\ \mathcal{R}(X) \subseteq \mathcal{R}(A^{2-}). \tag{2.6}$$

Proof From Theorem 2.6 (a), it is clear that A^{2MP} satisfies conditions (i) and (ii).

To prove uniqueness, suppose that there exist $X_1, X_2 \in \mathbb{C}^{n\times m}$ that satisfy conditions (i) and (ii). Then $AX_1 = P_{\mathcal{R}(C_2^A)\mathcal{N}(C_2^A A^{\dagger})} = AX_2$, from where $A(X_1 - X_2) = 0$, hence $\mathcal{R}(X_1 - X_2) \subseteq \mathcal{N}(A)$. Moreover, from $\mathcal{R}(X_i) \subseteq \mathcal{R}(A^{2-})$ for $i \in \{1, 2\}$ we have $\mathcal{R}(X_1 - X_2) \subseteq \mathcal{R}(A^{2-})$. Then $\mathcal{R}(X_1 - X_2) \subseteq \mathcal{R}(A^{2-}) \cap \mathcal{N}(A)$. Since $\mathcal{N}(A) \subseteq \mathcal{N}(A^{2-}A)$ and $\mathcal{R}(A^{2-}) = \mathcal{R}(A^{2-}A)$, we obtain $\mathcal{R}(X_1 - X_2) \subseteq \mathcal{R}(A^{2-}A) \cap \mathcal{N}(A^{2-}A) = \{0\}$ because $A^{2-}A$ is a projector. Therefore $X_1 = X_2$. □

From Theorem 2.5 and Theorem 2.7 we can conclude that, for a fixed matrix $A^{2-} \in \mathcal{A}\{2\}$, the matrix A^{2MP} is the unique that satisfies equations (2.5) and also is the unique matrix that satisfies relations (2.6). So, each approach is equivalent to the other one.

3 MP2-inverses

This section is devoted to present dual inverses of the 2MP-inverses introduced and characterized in the previous section. Since the development of MP2-inverses is analogous to 2MP-inverses, we provide the results without proofs.

Proceeding as in Sect. 2, if $A \in \mathbb{C}^{m\times n}$ is written as in (1.1) and the general form for {2}-inverses of A is given by (2.1), we obtain

$$AA^{2-} = U\begin{pmatrix} D_a M & D_a M Y_{12} \\ 0 & 0 \end{pmatrix} U^*.$$

Now, by defining the equivalence relation: for $A^{2-}, A^{2=} \in \mathcal{A}\{2\}$,

$$A^{2-} \sim_r A^{2=} \quad \text{if and only if} \quad AA^{2-} = AA^{2=},$$

we get

$$[A^{2-}]_{\sim_r} = \left\{ V \begin{pmatrix} M & MY_{12} \\ Y'_{21}M & Y'_{21}MY_{12} \end{pmatrix} U^* \in \mathcal{A}\{2\} : \; MD_aM = M, \; \text{for arbitrary } Y'_{21} \right\}.$$

A complete set of representatives of the partition on $\mathcal{A}\{2\}$ induced by $\sim_r$ is given by

$$\mathcal{R}_{\sim_r} := \left\{ V \begin{pmatrix} M & MY_{12} \\ 0 & 0 \end{pmatrix} U^* : \; MD_aM = M, Y_{12} \in \mathbb{C}^{a \times (m-a)} \right\}.$$

Now, we observe that any element in $\mathcal{R}_{\sim_r}$ can be factorized as $A^\dagger AA^{2-}$. So, we introduce a new class of generalized inverses, which is the dual of the 2MP-inverses.

Definition 3.1 Let $A \in \mathbb{C}^{m \times n}$. For each $A^{2-} \in \mathcal{A}\{2\}$, the MP2-inverse of A, denoted by A^{MP2}, is the $n \times m$ matrix

$$A^{MP2} := A^\dagger AA^{2-}.$$

The symbol $\mathcal{A}\{MP2\}$ stands for the set of all MP2-inverses of A; clearly $A^\dagger$ is an element of this set, thus $\mathcal{A}\{MP2\} \neq \emptyset$. Hence, $\mathcal{A}\{MP2\} = \{A^\dagger AA^{2-} : \; A^{2-} \in \mathcal{A}\{2\}\}$. Therefore, MP2-inverses of A always exist; in general, they are not unique.

We notice that, if $A \in \mathbb{C}^{m \times n}$, for each $A^{2-} \in \mathcal{A}\{2\}$, the matrix $A^{MP2} \in \mathcal{A}\{2\}$.

A 1-parametrized formula for MP2-inverses can be also established. Let $A \in \mathbb{C}^{m \times n}$. Then $Z \in \mathcal{A}\{MP2\}$ if and only if there exists $A^{2-} \in \mathcal{A}\{2\}$ such that $Z = A^\dagger AA^{2-} + (I - A^\dagger A)Y$, for arbitrary Y of suitable size.

Theorem 3.2 *Let $A \in \mathbb{C}^{m \times n}$. For each $A^{2-} \in \mathcal{A}\{2\}$, the matrix A^{MP2} is the unique that satisfies the following equations system (in the unknown X):*

$$(i)\; XAX = X, \; (ii)\; AX = AA^{2-}, \; (iii)\; XC_2^A = A^\dagger C_2^A. \tag{3.1}$$

Theorem 3.3 *Let $A \in \mathbb{C}^{m \times n}$. For a given $A^{2-} \in \mathcal{A}\{2\}$, the matrix $A^{MP2} \in \mathbb{C}^{n \times m}$ satisfies the following properties:*

(a) *$A^{MP2}A$ is an oblique projector onto $\mathcal{R}(A^{MP2})$ along $\mathcal{N}(C_2^A)$. Moreover, $\mathcal{R}(A^{MP2}) \subseteq \mathcal{R}(A^\dagger)$.*
(b) *AA^{MP2} is an oblique projector onto $\mathcal{R}(AA^{2-}) = \mathcal{R}(C_2^A)$ along $\mathcal{N}(AA^{2-}) = \mathcal{N}(A^{2-})$.*

Theorem 3.4 *Let $A \in \mathbb{C}^{m \times n}$. For each $A^{2-} \in \mathcal{A}\{2\}$, the matrix A^{MP2} is the unique that satisfies the following properties:*

$$(i)\; XA = P_{\mathcal{R}(X)\mathcal{N}(C_2^A)}, \; (ii)\; \mathcal{R}(X) \subseteq \mathcal{R}(A^\dagger). \tag{3.2}$$

We would like to highlight that the method followed in this paper could be used in future research to introduce new kind of inverses by considering adequate equivalence relations.

4 C2MP-inverses

In this section, we give a natural generalization for rectangular matrices of CMP-inverses defined by Mehdipour and Salemi in [13] for a square matrix A by $A^{c,\dagger} = A^\dagger A_1 A^\dagger$, with $A_1 = AA^DA$.

Our extension of the CMP-inverses is different from the one presented by D. Mosić in [14], based on a weight matrix W, where the W-weighted Drazin inverse is used instead of the Drazin inverse. That is, in [14], the author used an auxiliary weight matrix W in order to define the W-weighted CMP inverse as $A^{c,\dagger,W} = A^\dagger AWA^{D,W}WAA^\dagger$, where $A^{D,W}$ denotes the W-weighted Drazin inverse introduced by Cline and Greville in [4]. In this paper, we use an outer inverse of a given rectangular matrix A instead of the W-weighted Drazin of A. Specifically, we consider a 2MP core-part of A, that is, for each $A^{2-} \in \mathcal{A}\{2\}$, we take advantage of C_2^A, defined as in (2.4). The most important novelty in our approach is that we do not need an auxiliary weight matrix but are using the matrix A itself for doing the extension. This fact represents an interesting advantage with respect to the previous approaches used for extending generalized inverses from square to rectangular matrices.

Definition 4.1 Let $A \in \mathbb{C}^{m\times n}$. For each $A^{2-} \in \mathcal{A}\{2\}$, the matrix

$$A^{C2MP} := A^\dagger C_2^A A^\dagger \in \mathbb{C}^{n\times m}$$

is called a C2MP-inverse of A, where C_2^A is defined as in (2.4).

Clearly, these inverses are not unique but always exist, because for $A^{2-} = A^\dagger$, we get $A^{C2MP} = A^\dagger$. By the symbol $\mathcal{A}\{C2MP\}$ we denote the set of all C2MP-inverses of A, that is,

$$\mathcal{A}\{C2MP\} = \{A^\dagger C_2^A A^\dagger : C_2^A = AA^{2-}A, \text{ for each } A^{2-} \in \mathcal{A}\{2\}\}.$$

It is clear that, for square matrices, if we take $A^{2-} = A^D$ then $A^{C2MP} = A^{CMP}$.

The following example shows that the Weighted CMP inverse of a matrix A defined by Mosić in [14] as $A^{c,\dagger,W} = A^\dagger AWA^{D,W}WAA^\dagger$ is, in general, different from a C2MP-inverse of A analyzed in this paper.

Example 4.2 We consider

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad A^{2-} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

It is easy to see that

$$A^\dagger = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad A^{2-} \in \mathcal{A}\{2\}.$$

By definition we get $A^{D,W} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$, $C_2^A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$, $A^{c,\dagger,W} = W$, and $A^{C2MP} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Hence, $A^{C2MP} \neq A^{c,\dagger,W}$.

Next result presents some properties of $C2MP$-inverses and relationships with $2MP$- and $MP2$-inverses.

Proposition 4.3 *Let $A \in \mathbb{C}^{m\times n}$, $A^{2-} \in \mathcal{A}\{2\}$, C_2^A defined as in (2.4), and $A^{C2MP} := A^\dagger C_2^A A^\dagger$. Then the following properties are satisfied:*

(a) $A^{C2MP} = A^{MP2}AA^{2MP}$.
(b) $A^{C2MP} = A^\dagger AA^{2MP}AA^\dagger$.

(c) $A^{C2MP} \in \mathcal{A}\{2\}$.
(d) $AA^{C2MP}A = C_2^A$.
(e) $AA^{C2MP} = C_2^A A^\dagger = AA^{2MP}$.
(f) $A^{C2MP}A = A^\dagger C_2^A = A^{MP2}A$.

Proof In order to simplify the notation, in this proof we set $X := A^{C2MP}$. □

(a) $X = A^\dagger C_2^A A^\dagger = A^\dagger AA^{2-}AA^\dagger = A^\dagger AA^{2-}AA^{2-}AA^\dagger = A^{MP2}AA^{2MP}$.
(b) $X = A^\dagger C_2^A A^\dagger = A^\dagger AA^{2MP}AA^\dagger$.
(c) $XAX = A^\dagger C_2^A A^\dagger AA^\dagger C_2^A A^\dagger = A^\dagger AA^{2-}AA^\dagger AA^{2-}AA^\dagger = A^\dagger AA^{2-}AA^\dagger = A^\dagger C_2^A A^\dagger = X$.
(d) $AXA = AA^\dagger C_2^A A^\dagger A = AA^\dagger AA^{2-}AA^\dagger A = AA^{2-}A = C_2^A$.
(e) $AX = AA^\dagger C_2^A A^\dagger = AA^\dagger AA^{2-}AA^\dagger = AA^{2-}AA^\dagger$. Now, $AX = (AA^{2-}A)A^\dagger = C_2^A A^\dagger$ and $AX = A(A^{2-}AA^\dagger) = AA^{2MP}$.
(f) $XA = A^\dagger C_2^A A^\dagger A = A^\dagger AA^{2-}AA^\dagger A = A^\dagger AA^{2-}A$. Now, $XA = A^\dagger(AA^{2-}A) = A^\dagger C_2^A$ and $XA = (A^\dagger AA^{2-})A = A^{MP2}A$.

Theorem 4.4 *Let $A \in \mathbb{C}^{m\times n}$. For each $A^{2-} \in \mathcal{A}\{2\}$, the matrix A^{C2MP} is the unique that satisfies the following equations system:*

$$(i)\, XAX = X,\ (ii)\, AX = C_2^A A^\dagger,\ (iii)\, XA = A^\dagger C_2^A. \tag{4.1}$$

Proof By Proposition 4.3, it is clear that A^{C2MP} satisfies equations in (4.1). For uniqueness, suppose that X_1 and X_2 are both solutions of the system (4.1). Then $X_1 = X_1AX_1 = X_1AX_2 = X_2AX_2 = X_2$. □

The following example shows that, in general, a C2MP-inverse of A is not an inner inverse of A.

Example 4.5 Let A as in Example 4.2. If we consider $A^{2-} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \in \mathcal{A}\{2\}$, we obtain $C_2^A = \begin{pmatrix} 2 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $A^{C2MP} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}$. It is easy to see that $AA^{C2MP}A \neq A$.

Next, we give more properties of C2MP-inverses. For short, we write $P_A := AA^\dagger$ and $Q_A := A^\dagger A$.

Proposition 4.6 *Let $A \in \mathbb{C}^{m\times n}$. For each $A^{2-} \in \mathcal{A}\{2\}$ the following properties are satisfied:*

(a) $A^{C2MP} \in A\{1\}$ *if and only if* $A^{2-} \in \mathcal{A}\{1\}$.
(b) $A^{C2MP} = Q_A A^{2-} P_A = Q_A A^{2MP} P_A$.
(c) A^{C2MP} *is a* $\{1,2\}$*-inverse of* C_2^A.
(d) $C_2^A A^{C2MP} = AA^{C2MP}$.
(e) $A^{C2MP} C_2^A = A^{C2MP}A$.
(f) $P_A C_2^A Q_A = C_2^A$.

Proof By using that $AA^\dagger A = A$, it is evident that $A(A^\dagger AA^{2-}AA^\dagger)A = A \Leftrightarrow AA^{2-}A = A$ holds, which shows item (a).

Item (b) follows from Proposition 4.3 and Definition of $C2MP$-inverse.

Now, $C_2^A A^{C2MP} C_2^A = AA^{2-}AA^\dagger A(A^{2-}AA^\dagger AA^{2-}A) = AA^{2-}AA^{2-}A = C_2^A$. On the other hand, $A^{C2MP}C_2^A A^{C2MP} = A^\dagger AA^{2-}AA^\dagger AA^{2-}AA^\dagger AA^{2-}AA^\dagger = A^\dagger AA^{2-}AA^\dagger = A^\dagger C_2^A A^\dagger = A^{C2MP}$, hence A^{C2MP} is a $\{1,2\}$-inverse of C_2^A, which proves (c).

From $C_2^A A^{C2MP} = AA^{2-}AA^\dagger AA^{2-}AA^\dagger = AA^{2-}AA^\dagger = C_2^A A^\dagger = AA^{C2MP}$ we have that (d) holds.

In order to show (e), $A^{C2MP}C_2^A = A^\dagger AA^{2-}AA^\dagger AA^{2-}A = A^\dagger AA^{2-}A = A^\dagger C_2^A = A^{C2MP}A$.

Finally, $P_A C_2^A Q_A = AA^\dagger AA^{2-}AA^\dagger A = AA^{2-}A = C_2^A$, which proves (f). $\square$

Next lemma states canonical forms for every C2MP-inverse and 2MP core-part of A.

Lemma 4.7 *If $A \in \mathbb{C}^{m\times n}$ is written as in (1.1) and A^{2-} is written as in (2.1) then*

$$C_2^A = U\begin{pmatrix} D_aMD_a & 0 \\ 0 & 0\end{pmatrix}V^* \qquad \text{and} \qquad A^{C2MP} = V\begin{pmatrix} M & 0 \\ 0 & 0\end{pmatrix}U^*, \text{ with } MD_aM = M.$$

Proof The proof follows immediately by taking into account the expression (1.2) for $A^\dagger$. $\square$

Theorem 4.8 *Let $A \in \mathbb{C}^{m\times n}$. For each $A^{2-} \in \mathcal{A}\{2\}$, the following conditions are equivalent:*

(a) $A^{C2MP} = A^\dagger$,
(b) $C_2^A = A$,
(c) $A^{2-} \in \mathcal{A}\{1\}$,
(d) $A^{2-} \in \mathcal{A}\{1,2\}$,
(e) $M = D_a^{-1}$, *for A being as in (1.1) and A^{2-} as in (2.1),*
(f) $A^{C2MP} \in \mathcal{A}\{1\}$.

Proof (a) $\Rightarrow$ (b) If $A^{C2MP} = A^\dagger$, then pre and post multiplying by A we have $P_A C_2^A Q_A = A$. By Proposition 4.6 (f), $C_2^A = A$ holds.

(b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (d) are trivial.

(d) $\Rightarrow$ (a) Since $A^{2-} \in \mathcal{A}\{1\}$, it is clear that $C_2^A = A$. Then, $A^{C2MP} = A^\dagger C_2^A A^\dagger = A^\dagger AA^\dagger = A^\dagger$.

(e) $\Longleftrightarrow$ (a) is immediately from Lemma 4.7.

(c) $\Longleftrightarrow$ (f) follows directly from Proposition 4.6 (a). $\square$

Remark 4.9 If we assume that $A^{2-} = A^\dagger$ then clearly $C_2^A = A$ and $A^{C2MP} = A^\dagger$. However, the converse is not true in general. Let A as in Example 4.2. Consider

$$A^{2-} = \begin{pmatrix} 1 & -1 & c \\ 0 & 1 & d\end{pmatrix} \in \mathcal{A}\{1,2\} \text{ with } c \neq 0 \text{ or } d \neq 0.$$

So, $C_2^A = A$ and $A^{C2MP} = A^\dagger$ but $A^{2-} \neq A^\dagger$.

Theorem 4.10 *Let $A \in \mathbb{C}^{m\times n}$ written as in (1.1). For each $A^{2-} \in \mathcal{A}\{2\}$ written as in (2.1), the following properties are satisfied:*

(a) $(A^{C2MP})^\dagger = U\begin{pmatrix} M^\dagger & 0 \\ 0 & 0\end{pmatrix}V^*$, *where M has to satisfy $MD_aM = M$.*
(b) $(A^\dagger)^{C2MP} = U\begin{pmatrix} T & 0 \\ 0 & 0\end{pmatrix}V^*$, *where T has to satisfy $TD_a^{-1}T = T$.*
(c) $(A^{C2MP})^\dagger = (A^\dagger)^{C2MP}$ *if and only if $M^\dagger = T$, $MD_aM = M$, and $TD_a^{-1}T = T$.*
(d) $A^{C2MP} = A^*$ *if and only if $M = D_a$.*

(e) $A^{C2MP} = 0$ *if and only if* $A^{2-} = 0$ *if and only if* $C_2^A = 0$.

Proof (a) is trivial.

(b) If A is written as in (1.1) then

$$(A^\dagger)^{2-} = U \begin{pmatrix} T & TZ_{12} \\ Z_{21}T & Z_{21}TZ_{12} \end{pmatrix} V^*, \quad \text{where } T \text{ has to satisfy } TD_a^{-1}T = T.$$

By computing $(A^\dagger)^{C2MP} = AA^\dagger(A^\dagger)^{2-}A^\dagger A$, we get the result.

(c) Follows immediately from (a) and (b). Items (c), (d), and (e) follow immediately from the expressions and straightforward computations. □

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References

1. Baksalary, O.M., Trenkler, G.: Core inverse of matrices. Linear Multilinear Algebra **58**(6), 681–697 (2010)
2. Ben-Israel, A., Greville, T.N.E.: Generalized Inverses: Theory and Applications, 2nd edn. Springer-Verlag, New York (2003)
3. Chen, J.L., Zhu, H.H., Patrício, P., Zhang, Y.L.: Characterizations and representations of core and dual core inverses. Can. Math. Bull. **60**(2), 269–282 (2017)
4. Cline, R.E., Greville, T.N.E.: A Drazin inverse for rectangular matrices. Linear Algebra Appl. **29**, 53–62 (1980)
5. Cvetković-Ilić, D.S., Mosić, D., Wei, Y.: Partial orders on $B(H)$. Linear Algebra Appl. **481**, 115–130 (2015)
6. Ferreyra, D.E., Levis, F.E., Thome, N.: Characterizations of k-commutative equalities for some outer generalized inverses. Linear Multilinear Algebra **68**(1), 177–192 (2020)
7. Ferreyra, D.E., Orquera, V., Thome, N.: A weak group inverse for rectangular matrices. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, 113:3727-3740, 2019
8. Kyrchei, I.I., Mosić, D., Stanimirović, P.: MPD-DMP-solutions to quaternion two-sided restricted matrix equations. Comput. Appl. Math. **40**, 177 (2021)
9. Ma, H.: Perturbation bounds for the core inverse of matrices. Computational and Applied Mathematics, 41:3, Paper No. 101, 14 p., 2022
10. Ma, H.: Displacement structure of the core inverse. Linear Multilinear Algebra **70**(2), 203–214 (2022)
11. Ma, H., Li, T.: Characterizations and representations of the core inverse and its applications. Linear Multilinear Algebra **69**(1), 93–103 (2021)
12. Malik, S.B., Thome, N.: On a new generalized inverse for matrices of an arbitrary index. Appl. Math. Comput. **226**, 575–580 (2014)
13. Mehdipour, M., Salemi, A.: On a new generalized inverse of matrices. Linear Multilinear Algebra **66**(5), 1046–1053 (2018)
14. Mosić, D.: The CMP inverse for rectangular matrices. Aequationes Math. **92**, 649–659 (2018)

15. Mosić, D., Dolinar, G., Marovt, J., EP-quasinilpotent decomposition and its generalizations. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, 115:188, 2021
16. Mosić, D., Stanimirović, P.S.: Composite outer inverses for rectangular matrices. Quaest. Math. **44**(1), 45–72 (2021)
17. Mosić, D., Stanimirović, P.S., Katsikis, V.N.: Properties of the CMP inverse and its computation. Comput. Appl. Math. **41**, 4 (2022)
18. Mosić, D., Stanimirović, P.S., Ma, H., Generalization of core-EP inverse for rectangular matrices. Journal of Mathematical Analysis and Applications, 500, No. 1, Article ID 125101, 19 p., 2021
19. Manjunatha Prasad, K., Mohana, K.S.: Core EP inverse. Linear Multilinear Algebra **62**(6), 792–802 (2014)
20. Pablos Romo, F.: On Drazin-Moore-Penrose inverses of finite potent endomorphisms. Linear Multilinear Algebra **69**(4), 627–647 (2021)
21. Rakić, D.S., Dinčić, N.C., Djordjević, D.S.: Group, Moore-Penrose, core and dual core inverse in rings with involution. Linear Algebra Appl. **463**, 115–133 (2014)
22. Rao, C.R., Mitra, S.K.: Generalizaed inverse of matrices and its applications. John Wiley & Sons Inc, New York (1971)
23. Stanimirović, P.S., Mosić, D., Ma, H.: New classes of more general weighted outer inverses. Linear Multilinear Algebra **70**(1), 122–147 (2022)
24. Stanimirović, P.S., Mosić, D., Wei, Y.: Generalizations of composite inverses with certain image and/or kernel. Appl. Math. Comput. **428**, 16 (2022)
25. Wang, H., Liu, X.: Characterizations of the core inverse and the partial ordering. Linear Multilinear Algebra **63**(9), 1829–1836 (2015)
26. Zhu, H.H., Patrício, P.: Several types of one-sided partial orders in rings. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, 113, 3177-3184, 2019
27. Zhou, M.M., Chen, J., Stanimirovic, P.S., Katsikis, V.N., Ma, H.: Complex varying-parameter Zhang neural networks for computing core and core-EP inverse. Neural Process. Lett. **51**(2), 1299–1329 (2020)
28. Zuo, K., Cvetković-Ilić, D., Cheng, Y.: Different characterizations of DMP-inverse of matrices. Linear Multilinear Algebra (2020). https://doi.org/10.1080/03081087.2020.1729084