PROBABILISTIC ANALYSIS OF A CLASS OF IMPULSIVE LINEAR RANDOM DIFFERENTIAL EQUATIONS FORCED BY STOCHASTIC PROCESSES ADMITTING KARHUNEN-LOÈVE EXPANSIONS

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ABSTRACT. We study a full randomization of the complete linear differential equation subject to an infinite train of Dirac's delta functions applied at different time instants. The initial condition and coefficients of the differential equation are assumed to be absolutely continuous random variables, while the external or forcing term is a stochastic process. We first approximate the forcing term using the Karhunen-Loève expansion, and then we take advantage of the Random Variable Transformation method to construct a formal approximation of the first probability density function (1-p.d.f.) of the solution. By imposing mild conditions on the model parameters, we prove the convergence of the aforementioned approximation to the exact 1-p.d.f. of the solution. All the theoretical findings are illustrated by means of two examples, where different types of probability distributions are assumed to model parameters.

1. **Introduction.** Initially, the classical theory of differential equations with discontinuous right-hand side, also termed impulsive differential equations, has been to a great extend stimulated by the problems coming from Mechanics, Automatic Control, Electrical Engineering, etc., [18, 23], later applications have been extended in other areas such as Medicine and Epidemiology [24, 26]. In such types of problems the right-hand-side function contains discontinuities in the form of finite or infinite jumps. Depending on the phenomenon under study, jumps may appear at fixed time instants, such as chemotherapy in cancer treatments [4], or triggered by an event, for example, in epidemiology [9]. Some of them are related to control problems in models of species food chain, predator-prey, pest control, diabetes control, and many others [16, 19, 27]. An excellent recent reference for the study of impulsive systems is [25]. When these models are applied to real-world problems, their input data (initial/boundary conditions, forcing/control term and coefficients) need to be set from experimental data collected via experiments, surveys, etc. These data contain

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uncertainties coming from error measurements (epistemic uncertainty). When the problem under analysis is very complex, to this source of uncertainty one must add the one due to the lack of knowledge of certain aspects of the problem (aleatoric uncertainty). This latter happens when addressing epidemiological models since the *exact* measurement of contagion or recovering coefficients is simply unaffordable. All these facts motivate the study of differential equations with discontinuous right-hand side with uncertainty.

To introduce and rigorously handling uncertainty in the context of differential equations, one mainly distinguishes two approaches, Stochastic Differential Equations (SDEs) and Random Differential Equations (RDEs).

In the setting of SDEs, uncertainty is included by means of a specific stochastic process with irregular sample behavior. For example, in the case of Itô-type SDEs, uncertainty is driven by White Noise, that is a Gaussian stochastic process defined as the generalized derivative (in the sense of the theory of distributions) of the standard Wiener process (also termed Brownian motion). The rigorous handling of SDEs requires a special stochastic calculus, called Itô calculus, whose cornerstone tool is the Itô lemma, that plays the role of chain rule for differentiating stochastic diffusion processes. The solutions of these SDEs are stochastic processes with irregular sample paths due to the fact that the trajectories of the Wiener process are nowhere differentiable [17]. Similarly as it happens with deterministic differential equations, solving SDEs in an exact way is exceptional [32], so their mathematical treatment mainly rely on numerical methods [22]. SDEs have been successfully applied to modelling problems in many areas, particularly in Finance and Biology [1, 3], and, particularly, impulsive SDEs are applied to solve stochastic control/stabilization problems in different areas as Engineering and Biology [30, 33].

In the framework of RDEs, uncertainty is directly represented through the input data of the differential equation and, as a consequence, the initial/boundary conditions, source/control term and/or coefficients are assumed to be random variables or stochastic process. It must be pointed out that, in contrast to SDEs, the stochastic processes involved in the formulation of the RDE are assumed to have certain sample regularity properties. For example, strongly irregular stochastic processes, as the White noise, can not be treated as input data when dealing with RDEs, but the standard Wiener process is allowed [12, 31]. As it shall be seen later, in this paper we deal with a RDE whose right-hand side is very irregular, since it involves impulses, via the Dirac delta distribution, and the standard Wiener process. Finally, it must be underlined that a major advantage of RDEs over SDEs is that they permit considering different probabilistic distributions for their input parameters rather than assuming a common pattern, like the Gaussian one, implicitly assumed in the formulation of the aforementioned Itô-type SDEs. This fact may be crucial to success when applying RDEs to modelling real-world as an alternative to SDEs. This feature has been recently underlined in different relevant contributions [2, 20]. To the best of our knowledge, in the setting of impulsive RDEs, the contributions are still scarce and they have mainly focused on theoretical questions about existence and uniqueness of solutions. For example, in [35] one studies existence, uniqueness and stability via continuous dependence of weak solutions of neutral partial differential equations using the fixed point theory. In [37], sufficient conditions for p-moment boundedness of nonlinear impulsive RDEs are presented. Afterwards, necessary and sufficient conditions for oscillation in mean, p-moment stability and

p-moment boundedness for second-order linear differential system with random impulses are obtained, by the same authors, in [36]. Additionally, the obtained results are compared with the corresponding ones for the nonimpulsive differential equation. In [38] the same type of impulse RDEs is analyzed but assuming that impulses are driven by the Erlang distribution.

Hereinafter, $(\Omega, \mathcal{F}_\Omega, \mathbb{P})$ will denote a complete probability space. The aim of this paper is to continue contributing the study of impulsive RDEs, but from another viewpoint, namely, by determining the main probabilistic properties of the solution stochastic process, say $x(t,\omega)$, under mild hypotheses. In particular, as it shall be motivated later, the computation of the so called first probability density function (1-p.d.f.), $f_1(x,t)$, of the solution stochastic process is of great interest, since from this deterministic function one can obtain, by integration, all one-dimensional moments of the solution,

$$\mathbb{E}[(x(t))^k] = \int_{-\infty}^{\infty} x^k f_1(x,t)\,\mathrm{d}x, \quad k = 1, 2, \ldots,$$

where $\mathbb{E}[\cdot]$ denotes the expectation operator. In particular, for $k = 1$ one obtains the mean, $\mu_x(t) = \mathbb{E}[x(t)]$, and for $k = 2$, the second-order moment, and hence the standard deviation $\sigma_x(t) = \sqrt{\mathbb{E}[(x(t))^2] - (\mathbb{E}[x(t)])^2}$. Besides, fixed $\alpha \in (0,1)$, the 1-p.d.f. permits constructing $(1-\alpha)$-confidence intervals, $[\mu_x(\hat{t}) - k\sigma_x(\hat{t}), \mu_x(\hat{t}) + k\sigma_x(\hat{t})]$, at every time instant, say $\hat{t}$, by determining $k > 0$ such that

$$\begin{aligned}&\mathbb{P}\left[\{\omega \in \Omega : x(\hat{t};\omega) \in [\mu_x(\hat{t}) - k\sigma_x(\hat{t}), \mu_x(\hat{t}) + k\sigma_x(\hat{t})]\}\right] \\ &\qquad = \int_{\mu_x(\hat{t})-k\sigma_x(\hat{t})}^{\mu_x(\hat{t})+k\sigma_x(\hat{t})} f_1(x,\hat{t})\,\mathrm{d}x = 1-\alpha.\end{aligned}$$

This avoids the use of possible inadequate approximations, such as the 2σ and 3σ-rules that, guarantee the construction of confidence intervals at 95% ($\alpha = 0.05$) and 99% ($\alpha = 0.01$) confidence intervals only when the solution is Gaussian. The 1-p.d.f. also allows us to determine the probability that the solution lies in any interval of specific interest, say $[\hat{x}_1, \hat{x}_2]$, at any time instant $\hat{t}$,

$$\mathbb{P}\left[\{\omega \in \Omega : x(\hat{t};\omega) \in [\hat{x}_1, \hat{x}_2]\}\right] = \int_{\hat{x}_1}^{\hat{x}_2} f_1(x,\hat{t})\,\mathrm{d}x.$$

In the setting of RDEs, the computation of the 1-p.d.f. of the solution has been addressed for different classes of equations including equations with delay and fractional derivatives [5, 6, 8, 13, 14, 15, 21]. In recent contributions, we have extended the study to a class of first-order linear impulsive RDEs with finite [11] and infinite jumps [10]. In both cases, the impulses represent controls and they are evenly applied over time via the Heaviside and Dirac's delta functions, respectively. Moreover, in both papers it is assumed that all model parameters are random variables.

In this paper, we study a generalization of the homogeneous equation addressed in [10], because we deal with its non-homogeneous formulation by considering that the source term is a stochastic process (belonging to a space that will be introduced later), and we then address the computation of the 1-p.d.f. of the solution. As it shall be seen in the numerical experiments, we allow the source term can be, for example, the standard Wiener process or Brownian motion, so resulting an impulsive RDE whose right-hand side has very irregular trajectories, since the Dirac delta distribution and the Brownian motion are both involved. Specifically, we shall study the following impulsive first-order linear RDE with initial condition

$$\left.\begin{aligned} \frac{\mathrm{d}x(t,\omega)}{\mathrm{d}t} &= \alpha(\omega)x(t,\omega)-\gamma(\omega)x(t,\omega)\sum_{i=1}^{\infty}\delta(t-T_i)+b(t,\omega), \\ x(0,\omega) &= x_0(\omega), \end{aligned}\right\} \quad \omega\in\Omega. \quad (1)$$

We will assume that the initial condition, $x_0(\omega)$, the growing rate, $\alpha(\omega)>0$, and the intensity of the impulsive effect, $\gamma(\omega)>0$, are absolutely continuous random variables with finite variance (usually referred to as second-order random variables [34, Ch. 4]) defined on the common complete probability space $(\Omega,\mathcal{F}_\Omega,\mathbb{P})$. For the sake of generality, we will further assume that they have a joint p.d.f., $f_{x_0,\alpha,\gamma}(x_0,\alpha,\gamma)$. So, independence is not assumed thereafter. Besides, $\sum_{i=1}^{\infty}\delta(t-T_i)$ represents the infinite train of impulses via the Dirac distribution at the time instants T_i, $i=1,2,\dots$, and, $b(t,\omega)$ is a stochastic process that admits a Karhunen-Loève expansion (KLE), [29].

For the sake of completeness, we now recall the main definitions and results that will be required with regard to the stochastic process, $b(t,\omega)$ and its KLE. It is assumed that this process belongs to the Hilbert space $\left(\mathrm{L}^2(\Omega,\mathrm{L}^2(\mathcal{T})),||\cdot||_{\mathrm{L}^2(\Omega,\mathrm{L}^2(\mathcal{T}))}\right)$, of square integrable stochastic processes, say $x:\mathcal{T}\times\Omega\longrightarrow\mathrm{L}^2(\mathcal{T})$, which are defined in the finite interval $\mathcal{T}=[t_0,T]$, $T>0$, where the norm is defined by

$$||\, x\,||_{\mathrm{L}^2(\Omega,\mathrm{L}^2(\mathcal{T}))}=\left(\int_{\mathcal{T}}\mathbb{E}\left[|x(t,\omega)|^2\right]\mathrm{d}t\right)^{1/2}<\infty. \quad (2)$$

It is assumed that $\mathbb{E}\left[|x(\cdot,\omega)|^2\right]<\infty$ for every $t\in\mathcal{T}$, i.e., the stochastic process is such that for every $t\in[t_0,T]$, it becomes a second-order random variable, in other words, with finite variance. The following result states that every square integrable stochastic process, $x(t,\omega)$, can be expanded in terms of a set of eigenpairs associated with an homogeneous Fredholm integral equation of the second kind involving the correlation function of the $x(t,\omega)$.

Theorem 1.1. [29, p. 202] *Consider a mean square integrable continuous time stochastic process $x\equiv\{x(t,\omega):t\in\mathcal{T},\omega\in\Omega\}$, i.e. $x\in\mathrm{L}^2(\Omega,\mathrm{L}^2(\mathcal{T}))$, being $\mu_x(t)$ and $c_x(s,t)$ its mean and covariance functions, respectively. Then,*

$$x(t,\omega)=\mu_X(t)+\sum_{j=1}^{\infty}\sqrt{\nu_j}\phi_j(t)\xi_j(\omega),\quad \omega\in\Omega, \quad (3)$$

converges in $\mathrm{L}^2(\Omega,\mathrm{L}^2(\mathcal{T}))$, being

$$\xi_j(\omega):=\frac{1}{\sqrt{\nu_j}}\left\langle x(t,\omega)-\mu_X(t),\phi_j(t)\right\rangle_{\mathrm{L}^2(\mathcal{T})},$$

where $\langle u,v\rangle_{\mathrm{L}^2(\mathcal{T})}:=\int_{\mathcal{T}}u(x)v(x)\mathrm{d}x$, and $\{(\nu_j,\phi_j(t)):j\geq 1\}$ denote, respectively, the eigenvalues, with $\nu_1\geq\nu_2\geq\ldots\geq 0$, and the eigenfunctions of the following Fredholm integral operator

$$\mathcal{C}(t):=\int_{\mathcal{T}}c_x(s,t)f(s)\mathrm{d}s,\quad f\in\mathrm{L}^2(\mathcal{T}),$$

associated to the covariance function $c_x(s,t)$. The random variables $\xi_j(\omega)$ are independent and identically distributed according to a standard Gaussian distribution, $\xi_j(\omega)\sim\mathrm{N}(0,1)$.

In our subsequent development, the role of the KLE is fundamental because it will allow us to compute the solution of model (1) by requesting that the eigenfunctions $\phi_j(t)$ have a Laplace transform. This procedure can be applied to any stochastic process, playing the role of the forcing or source term, $b(t,\omega)$, with a KLE in terms of a set of eigenfunctions admitting a Laplace transform and whose mean or expectation of $b(t,\omega)$ is also Laplace transformable. However, as it has been previously emphasized, thoroughly analysis of any RDE means not only to compute its exact or approximate solution stochastic process, but also to determine its 1-p.d.f. To this end, the following result plays a key role in our subsequent study.

Theorem 1.2. *(Random Variable Transformation (RVT) method,* [34, pp. 24–25]*). Let* $\mathbf{u} = (u_1, \ldots, u_n)$ *and* $\mathbf{v} = (v_1, \ldots, v_n)$ *be two n-dimensional absolutely continuous random vectors. Let* $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$ *be a one-to-one deterministic transformation of* $\mathbf{u}$ *into* $\mathbf{v}$*, i.e.,* $\mathbf{v} = \mathbf{g}(\mathbf{u})$*. Assume that* $\mathbf{g}$ *is continuous in* $\mathbf{u}$ *and has continuous partial derivatives with respect to* $\mathbf{u}$*. Then, if* $f_{\mathbf{u}}(\mathbf{u})$ *denotes the joint probability density function of vector* $\mathbf{u}$*, and* $\mathbf{h} = \mathbf{g}^{-1} = (h_1(v_1, \ldots, v_n), \ldots, h_n(v_1, \ldots, v_n))$ *represents the inverse mapping of* $\mathbf{g} = (g_1(u_1, \ldots, u_n), \ldots, g_n(u_1, \ldots, u_n))$*, the joint probability density function of vector* $\mathbf{v}$ *is given by*

$$f_{\mathbf{v}}(\mathbf{v}) = f_{\mathbf{u}}(\mathbf{h}(\mathbf{v}))|J|,$$

where $|J|$ *is the absolute value of the Jacobian of mapping* $\mathbf{h}$*, defined by the determinant*

$$J = \det\left(\frac{\partial \mathbf{h}}{\partial \mathbf{v}}\right) = \begin{pmatrix} \frac{\partial h_1(v_1,\ldots,v_n)}{\partial v_1} & \cdots & \frac{\partial h_n(v_1,\ldots,v_n)}{\partial v_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_1(v_1,\ldots,v_n)}{\partial v_n} & \cdots & \frac{\partial h_n(v_1,\ldots,v_n)}{\partial v_n} \end{pmatrix}.$$

Following this approach previously described, the main goal of this paper is to compute reliable approximations of the 1-p.d.f. of the solution stochastic process to the random IVP (1). To achieve this objective, we will combine the KLE and the Random Variable Transformation (RVT) method.

The paper is organized as follows. In Section 2, we construct an approximation of the 1-p.d.f. of the solution stochastic process of the random initial value problem (IVP) formulated in (1). For the sake of clarity, this section is divided into two parts. In Subsection 2.1, we will first take advantage of the KLE of the non-homogeneous term, $b(t)$, and, secondly, we will apply the Laplace transform, to obtain an explicit solution of the IVP (1). In Subsection 2.2, we truncate the KLE by retaining N terms, and we then apply the RVT method to determine an approximation, $f_1^N(x,t)$, to the exact 1-p.d.f., $f_1(x,t)$, of the solution stochastic process. In Section 3, we establish sufficient conditions on the data so that $f_1^N(x,t) \to f_1(x,t)$ as $N \to \infty$ for each (x,t). Finally, some illustrative examples are shown in Section 4. Conclusions are drawn in Section 5.

2. Computing the solution stochastic process and the 1-p.d.f. via the KLE.

This section is divided into two subsections. In Subsection 2.1, we solve the random IVP (1). To this end, we first expand the forcing term, $b(t,\omega)$, via the KLE, and then we apply the Laplace integral transform. The solution is then expressed in terms of an infinite sum. In Subsection 2.2, we truncate the above mentioned infinite sum and, afterwards, apply the RVT method to obtain an approximation of the 1-p.d.f.

2.1. **Solving the model.** Let us consider the random IVP (1) and assume that the non-homogeneous term $b(t,\omega)$ is a mean square integrable and continuous stochastic process, being $\mu_b(t)$ and $c_b(s,t)$, its mean and covariance functions, respectively. By Th. 1.1, $b(t,\omega)$ admits a KLE, so the random IVP (1) can be formally represented as

$$\left.\begin{aligned} \frac{\mathrm{d}x(t,\omega)}{\mathrm{d}t} &= \alpha(\omega)x(t,\omega) - \gamma(\omega)x(t,\omega)\sum_{i=1}^{\infty}\delta(t-T_i) + \mu_b(t) + \sum_{j=1}^{\infty}\sqrt{\nu_j}\phi_j(t)\xi_j(\omega), \\ x(0,\omega) &= x_0(\omega), \end{aligned}\right\} \tag{4}$$

where the functions $\phi_j(t)$ are described in Th 1.1. The IVP (4) can be solved, using the Laplace transform, provided $\phi_j(t)$, $j=1,2,\ldots$, and the expectation of $b(t,\omega)$, $\mu_b(t)$, admit Laplace transform. As the KLE converges uniformly with respect to t [29], when taking the Laplace transform in the impulsive RDE of (4), this integral operator can be interchanged with the infinite sum to solve for $X(s,\omega)=\mathcal{L}\{x(t,\omega)\}(s)$, being s the Laplace transform parameter and $\omega\in\Omega$. Here, $\mathcal{L}\{x(t,\omega)\}(s)$ or $X(s,\omega)$ denote the Laplace transform of $x(t,\omega)$, for each $\omega\in\Omega$. Therefore,

$$\begin{aligned} \mathcal{L}\left\{\frac{\mathrm{d}x(t,\omega)}{\mathrm{d}t}\right\}(s) \quad = \quad & \alpha(\omega)\mathcal{L}\left\{x(t,\omega)\right\}(s) - \gamma(\omega)\sum_{i=1}^{\infty}\mathcal{L}\left\{x(t,\omega)\delta(t-T_i)\right\}(s) \\ & +\mathcal{L}\left\{\mu_b(t)\right\}(s) + \sum_{j=1}^{\infty}\sqrt{\nu_j}\xi_j(\omega)\mathcal{L}\left\{\phi_j(t)\right\}(s), \end{aligned}$$

i.e., using the properties of the Laplace transform,

$$sX(s,\omega)-x_0(\omega) = \alpha(\omega)X(s,\omega)-\gamma(\omega)\sum_{i=1}^{\infty}\mathrm{e}^{-T_i s}x(T_i,\omega)+U_b(s)+\sum_{j=1}^{\infty}\sqrt{\nu_j}\xi_j(\omega)\Phi_j(s),$$

where $U_b(s)$ and $\Phi_j(s)$ denote the Laplace transform of the deterministic functions $\mu_b(t)$ and $\phi_j(t)$, respectively. Now, we solve for $X(s,\omega)$

$$\begin{aligned} X(s,\omega) \quad = \quad & \frac{x_0(\omega)}{s-\alpha(\omega)} - \frac{\gamma(\omega)}{s-\alpha(\omega)}\sum_{i=1}^{\infty}\mathrm{e}^{-T_i s}x(T_i,\omega) \\ & +\frac{1}{s-\alpha(\omega)}U_b(s) + \sum_{j=1}^{\infty}\sqrt{\nu_j}\Phi_j(s)\frac{1}{s-\alpha(\omega)}\xi_j(\omega). \end{aligned}$$

The solution is finally obtained by taking the inverse Laplace transform,

$$\begin{aligned} x(t,\omega) \quad = \quad & x_0(\omega)\mathrm{e}^{\alpha(\omega)t} - \gamma(\omega)\sum_{i=1}^{\infty}x(T_i,\omega)\mathrm{e}^{\alpha(\omega)(t-T_i)}H(t-T_i) \\ & +(\mu_b * g)(t) + \sum_{j=1}^{\infty}\sqrt{\nu_j}(\phi_j * g)(t)\xi_j(\omega) \end{aligned} \tag{5}$$

where, $g(t,\omega)=\mathcal{L}^{-1}\{\frac{1}{s-\alpha(\omega)}\}=\mathrm{e}^{\alpha(\omega)t}$, $H(\cdot)$ denotes the Heaviside function and the star notation $*$ stands for the convolution of two functions $(f*g)(t)=\int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau=\int_0^t f(t-\tau)g(\tau)\mathrm{d}\tau$. To complete this solution is necessary to compute the values of $x(T_i,\omega)$, for $i=1,2,\ldots$, which is achieved by evaluating the solution recursively at $t=T_i$, for $i=1,2,\ldots$. To simplify the notation of this evaluation, let us first introduce the following notation

$$C_{\mathrm{KL}}(t,\omega) := (\mu_b * g)(t,\omega) + \sum_{j=1}^{\infty} \sqrt{\nu_j}(\phi_j * g)(t,\omega)\xi_j(\omega). \tag{6}$$

Evaluating (5) at $t = T_1$

$$x(T_1,\omega) = x_0(\omega)\mathrm{e}^{\alpha(\omega)T_1} - \gamma(\omega)x(T_1,\omega) + C_{\mathrm{KL}}(T_1,\omega),$$

and solving for $x(T_1,\omega)$, we obtain,

$$x(T_1,\omega) = \frac{x_0(\omega)\mathrm{e}^{\alpha(\omega)T_1}}{1+\gamma(\omega)} + \frac{C_{\mathrm{KL}}(T_1,\omega)}{1+\gamma(\omega)}.$$

When evaluating (5) at $t = T_2 > T_1$,

$$x(T_2,\omega) = x_0(\omega)\mathrm{e}^{\alpha(\omega)T_2} - \gamma(\omega)x(T_1,\omega)\mathrm{e}^{\alpha(\omega)(T_2-T_1)} - \gamma(\omega)x(T_2,\omega) + C_{\mathrm{KL}}(T_2,\omega),$$

and solving for $x(T_2,\omega)$, one obtains

$$x(T_2,\omega) = \frac{x_0(\omega)\mathrm{e}^{\alpha(\omega)T_2}}{(1+\gamma(\omega))^2} - \frac{\gamma(\omega)C_{\mathrm{KL}}(T_1,\omega)\mathrm{e}^{\alpha(\omega)(T_2-T_1)}}{(1+\gamma(\omega))^2} + \frac{C_{\mathrm{KL}}(T_2,\omega)}{1+\gamma(\omega)}.$$

In general, we can evaluate (5) at $t = T_i$, and solve for $x(T_i,\omega)$ in terms of the previous computed values, i.e.

$$x(T_i,\omega) = \frac{x_0(\omega)\mathrm{e}^{\alpha(\omega)T_i}}{(1+\gamma(\omega))^i} - \sum_{j=1}^{i-1} \frac{\gamma(\omega)C_{\mathrm{KL}}(T_j,\omega)\mathrm{e}^{\alpha(\omega)(T_i-T_j)}}{(1+\gamma(\omega))^{i-j+1}} + \frac{C_{\mathrm{KL}}(T_i,\omega)}{1+\gamma(\omega)},\ i = 1,2,3,\dots$$

After substituting the last expression for the $x(T_i,\omega)$ in (5), we obtain the solution stochastic process,

$$\begin{aligned} x(t,\omega) &= x_0(\omega)\mathrm{e}^{\alpha(\omega)t}\left(1 - \gamma(\omega)\sum_{i=1}^{\infty}\frac{H(t-T_i)}{(1+\gamma(\omega))^i}\right) \\ &\quad +(\gamma(\omega))^2 \sum_{i=1}^{\infty}\sum_{j=1}^{i-1} \frac{C_{KL}(T_j,\omega)\mathrm{e}^{\alpha(\omega)(t-T_j)}}{(1+\gamma(\omega))^{i-j+1}} H(t-T_i) \\ &\quad -\frac{\gamma(\omega)}{1+\gamma(\omega)}\sum_{i=1}^{\infty} C_{\mathrm{KL}}(T_i,\omega)\mathrm{e}^{\alpha(\omega)(t-T_i)}H(t-T_i) + C_{\mathrm{KL}}(t,\omega). \end{aligned} \tag{7}$$

Summarizing, the following result has been established.

Proposition 1. *Consider random IVP (1) and assume that $b(t,\omega)$ is a mean square integrable continuous stochastic process, $b(t,\omega) \in \mathrm{L}^2(\Omega,\mathrm{L}^2(\mathcal{T}))$, $\mathcal{T} \subset \mathbb{R}$, such that the eigenfunctions associated to its Karhunen-Loève expansion and the expectation of $b(t,\omega)$ are Laplace transformable. Then, its solution is given by (7), where the term $C_{KL}(t,\omega)$ is defined by (6) and $H(t-T_i)$ denotes the Heaviside function.*

2.2. **Approximation of the 1-p.d.f.** In this section we will take advantage of the RVT method to obtain an approximate expression for the 1-p.d.f. of the solution stochastic process of the random IVP (1) after truncating the KLE given in (7).

Let us consider the first N terms of the series $C_{\mathrm{KL}}(t,\omega)$ defined in (6), within the expression given in (7), and let us denote by $x^N(t,\omega) = x^N(t;x_0(\omega),\alpha(\omega),\gamma(\omega),\boldsymbol{\xi}^N(\omega))$, where $\boldsymbol{\xi}^N(\omega) = (\xi_1(\omega), \xi_2(\omega), \ldots, \xi_N(\omega))$ the resulting truncated expression of $C_{\mathrm{KL}}(t,\omega)$. Then, the formal approximation of the solution (7) can be written as,

$$\begin{aligned} x^N(t,\omega) \quad = \quad & x_0(\omega)\mathrm{e}^{\alpha(\omega)t}\left(1-\gamma(\omega)\sum_{i=1}^{\infty}\frac{H(t-T_i)}{(1+\gamma(\omega))^i}\right) \\ & +D^N(t;\alpha(\omega),\gamma(\omega),\boldsymbol{\xi}^N(\omega))+C_{\mathrm{KL}}^N(t;\alpha(\omega),\boldsymbol{\xi}^N(\omega)), \end{aligned} \tag{8}$$

where we have shorten the above expression introducing the following notation,

$$\begin{aligned} D^N(t;\alpha(\omega),\gamma(\omega),\boldsymbol{\xi}^N(\omega)) = & (\gamma(\omega))^2\sum_{i=1}^{\infty}\sum_{j=1}^{i-1}\frac{C_{\mathrm{KL}}^N(T_j;\alpha(\omega),\boldsymbol{\xi}^N(\omega))\mathrm{e}^{\alpha(\omega)(t-T_j)}}{(1+\gamma(\omega))^{i-j+1}}H(t-T_i) \\ & -\frac{\gamma(\omega)}{1+\gamma(\omega)}\sum_{i=1}^{\infty}C_{\mathrm{KL}}^N(T_i;\alpha(\omega),\boldsymbol{\xi}^N(\omega))\mathrm{e}^{\alpha(\omega)(t-T_i)}H(t-T_i), \end{aligned} \tag{9}$$

and

$$\begin{aligned} C_{\mathrm{KL}}^N(t;\alpha(\omega),\boldsymbol{\xi}^N(\omega)) \quad & = \quad (\mu_b * g)(t,\omega)+\sum_{j=1}^{N}\sqrt{\nu_j}(\phi_j * g)(t,\omega)\xi_j(\omega) \\ & = \quad (\mu_b * g)(t,\omega)+\int_0^t \mathrm{e}^{\alpha(\omega)(t-\tau)}\sum_{j=1}^{N}\sqrt{\nu_j}\phi_j(\tau)\xi_j(\omega)\,\mathrm{d}\tau, \end{aligned} \tag{10}$$

where in the last step we have applied the definition of the convolution taking into account that $g(t,\omega) = \mathrm{e}^{\alpha(\omega)t}$.

Let us recall that the KLE is defined in a time interval $\mathcal{T} = [t_0, T]$ (in our case with $t_0 = 0$), and within this interval, we apply Dirac's delta impulses at the times $T_i \in \mathcal{T}$, $i = 1, 2, 3, \ldots$ To apply the RVT method, we fix $t > 0$ in a time interval $[T_n, T_{n+1}) \subset \mathcal{T}$. Then, $x^N(t,\omega)$ is expressed as

$$\begin{aligned} x^N(t,\omega) \quad = \quad & x_0(\omega)\mathrm{e}^{\alpha(\omega)t}\left(1-\gamma(\omega)\sum_{i=1}^{n}\frac{H(t-T_i)}{(1+\gamma(\omega))^i}\right) \\ & +D^N(t;\alpha(\omega),\gamma(\omega),\boldsymbol{\xi}^N(\omega))+C_{\mathrm{KL}}^N(t;\alpha(\omega),\boldsymbol{\xi}^N(\omega)), \end{aligned}$$

because the Heaviside functions become zero after $t > T_{n+1}$. If we substitute the Heaviside functions and perform the finite sum, this expression can be written as

$$x^N(t,\omega) = \frac{x_0(\omega)\mathrm{e}^{\alpha(\omega)t}}{(1+\gamma(\omega))^n}+D^N(t;\alpha(\omega),\gamma(\omega),\boldsymbol{\xi}^N(\omega))+C_{\mathrm{KL}}^N(t;\alpha(\omega),\boldsymbol{\xi}^N(\omega)), \tag{11}$$

where the expression for D^N, given in (9), becomes

$$\begin{aligned} D^N(t;\alpha(\omega),\gamma(\omega),\boldsymbol{\xi}^N(\omega)) \quad = \quad & (\gamma(\omega))^2\sum_{i=1}^{n}\sum_{j=1}^{i-1}\frac{C_{\mathrm{KL}}^N(T_j;\alpha(\omega),\boldsymbol{\xi}^N(\omega))\mathrm{e}^{\alpha(\omega)(t-T_j)}}{(1+\gamma(\omega))^{i-j+1}} \\ & -\frac{\gamma(\omega)}{1+\gamma(\omega)}\sum_{i=1}^{n}C_{\mathrm{KL}}^N(T_i;\alpha(\omega),\boldsymbol{\xi}^N(\omega))\mathrm{e}^{\alpha(\omega)(t-T_i)}. \end{aligned} \tag{12}$$

Now, we will apply the RVT technique, stated in Th. 1.2, to the approximate solution of the random IVP (4), which is given by (11), (12) and (10), to obtain its 1-p.d.f.

Theorem 2.1. *Let $t > 0$ be fixed and the joint p.d.f. of x_0, $\alpha(\omega) > 0$, $\gamma(\omega) > 0$ and $\boldsymbol{\xi}^N(\omega)$, $f_{x_0,\alpha,\gamma,\boldsymbol{\xi}^N}(x_0, \alpha, \gamma, \boldsymbol{\xi}^N)$. Assume that the hypotheses of Proposition 1 are satisfied. Then, the 1-p.d.f. the stochastic process $x^N(t, \omega)$, given by (11), is*

$$f_1^N(x,t) = \int_{\mathbb{R}^{N+2}} f_{x_0,\alpha,\gamma,\boldsymbol{\xi}^N}\left((x - D^N(t;\alpha,\gamma,\boldsymbol{\xi}^N) - C^N_{KL}(t;\alpha,\boldsymbol{\xi}^N)) \frac{(1+\gamma)^n}{\mathrm{e}^{\alpha t}}, \alpha, \gamma, \boldsymbol{\xi}^N \right) \times \frac{(1+\gamma)^n}{\mathrm{e}^{\alpha t}} \,\mathrm{d}\alpha \,\mathrm{d}\gamma \,\mathrm{d}\boldsymbol{\xi}^N, \tag{13}$$

where $C_{KL}(t; \alpha, \boldsymbol{\xi}^N)$ and $D^N(t; \alpha, \gamma, \boldsymbol{\xi}^N)$ are given in (10) and (12), respectively.

Proof. Let us define the following map $\mathbf{g} : \mathbb{R}^{N+3} \to \mathbb{R}^{N+3}$, $\mathbf{v} = \mathbf{g}(\mathbf{u})$, $\mathbf{u} = (x_0, \alpha, \gamma, \boldsymbol{\xi}^N) \equiv (x_0, \alpha, \gamma, \xi_1, \ldots, \xi_N)$, i.e. explicitly,

$$\begin{array}{rcll}
v_1 &=& g_1(x_0,\alpha,\gamma,\xi_1,\ldots,\xi_N) &= \dfrac{x_0 \mathrm{e}^{\alpha t}}{(1+\gamma)^n} + D^N(t;\alpha,\gamma,\boldsymbol{\xi}^N) + C^N_{\mathrm{KL}}(t;\alpha,\boldsymbol{\xi}^N),\\
v_2 &=& g_2(x_0,\alpha,\gamma,\xi_1,\ldots,\xi_N) &= \alpha,\\
v_3 &=& g_3(x_0,\alpha,\gamma,\xi_1,\ldots,\xi_N) &= \gamma,\\
v_4 &=& g_4(x_0,\alpha,\gamma,\xi_1,\ldots,\xi_N) &= \xi_1,\\
\vdots & \vdots & \vdots & \vdots \ \vdots\\
v_{N+3} &=& g_{N+3}(x_0,\alpha,\gamma,\xi_1,\ldots,\xi_N) &= \xi_N.
\end{array}$$

Notice that $v_1 = x$ is the solution stochastic process. The inverse mapping of $\mathbf{g}$, $\mathbf{g} = \mathbf{h}^{-1}$, is obtained by solving for $\mathbf{u}$,

$$\begin{array}{rcll}
x_0 &=& h_1(v_1,v_2,\ldots,v_{N+3}) &= [v_1 - D^N(t;\alpha,\gamma,\boldsymbol{\xi}^N) - C^N_{\mathrm{KL}}(t;\alpha,\boldsymbol{\xi}^N)] \dfrac{(1+\gamma)^n}{\mathrm{e}^{\alpha t}},\\
\alpha &=& h_2(v_1,v_2,\ldots,v_{N+3}) &= v_2,\\
\gamma &=& h_3(v_1,v_2,\ldots,v_{N+3}) &= v_3,\\
\xi_1 &=& h_4(v_1,v_2,\ldots,v_{N+3}) &= v_4,\\
\vdots & \vdots & \vdots & \vdots \ \vdots\\
\xi_N &=& h_{N+3}(v_1,v_2,\ldots,v_{N+3}) &= v_{N+3}.
\end{array}$$

The Jacobian of this mapping is $\mathcal{J} = \frac{\partial x_0}{\partial v_1} = (1+v_3)^n \mathrm{e}^{-v_2 t} \neq 0$ with probability 1 (w.p. 1), since $v_3 = \gamma > 0$ (notice that here γ denotes the realizations of the absolutely continuous random variable $\gamma(\omega)$, $\omega \in \Omega$). The application of the RVT method, stated in Th. 1.2, allows us to determine the joint p.d.f. of the random vector $\mathbf{v}$,

$$\begin{array}{rcl}
f_{\mathbf{v}}(v) &=& f_{\mathbf{u}}((x - D^N(t;v_2,v_3,\mathbf{v}_4) - C^N_{\mathrm{KL}}(t;v_2,\mathbf{v}_4))(1+v_3)^n \mathrm{e}^{-v_2 t}, v_2, v_3, \mathbf{v}_4)\\
&& \times (1+v_3)^n \mathrm{e}^{-v_2 t},
\end{array}$$

where, for the consistency with the notation, we have introduced the notation $\mathbf{v}_4 := (v_4, \ldots, v_{N+3}) = \boldsymbol{\xi}^N$.

Finally, by marginalizing this expression with respect to α, γ, and $\boldsymbol{\xi}^N$, one obtains the 1-p.d.f. of the stochastic process, $x^N(t, \omega)$, given in (13). □

Remark 1. In the particular case that $x_0(\omega)$ is independent of $\alpha(\omega)$, $\gamma(\omega)$ and $\boldsymbol{\xi}^N(\omega)$, and that $f_{x_0}(x_0)$ and $f_{\alpha,\gamma,\boldsymbol{\xi}^N}(\alpha, \gamma, \boldsymbol{\xi}^N)$, denote their respective p.d.f.s, the 1-p.d.f. (13) can be expressed as

$$f_1^N(x,t) = \mathbb{E}_{\alpha,\gamma,\boldsymbol{\xi}^N}\left[f_{x_0}\left((x - D^N(t;\alpha,\gamma,\boldsymbol{\xi}^N) - C^N_{\mathrm{KL}}(t;\alpha,\boldsymbol{\xi}^N))(1+\gamma)^n \mathrm{e}^{-\alpha t} \right) (1+\gamma)^n \mathrm{e}^{-\alpha t} \right], \tag{14}$$

where $\mathbb{E}_{\alpha,\gamma,\boldsymbol{\xi}^N}[\cdot]$ denotes the expectation operator with respect to the random vector $(\alpha(\omega), \gamma(\omega), \boldsymbol{\xi}^N(\omega))$.

With regard to the integral expression (13), it is important to remark that the domain of integration is the corresponding subset of $\mathbb{R}^{N+2}$, where the random vector $(\alpha(\omega), \gamma(\omega), \boldsymbol{\xi}^N(\omega)) \equiv (\alpha(\omega), \gamma(\omega), \xi_1(\omega), \xi_2(\omega), \ldots, \xi_N(\omega))$ takes values for all ω in Ω. From a computational view point, expression (14), which is given in terms of the expectation operator, is very useful to calculate the 1-p.d.f. via Monte Carlo simulations. This key fact will be illustrated in the numerical examples.

3. Convergence of approximations.

In this section, we provide sufficient conditions to guarantee the sequence of approximations, $f_1^N(x,t)$, given in Th. 2.1 to the exact 1-p.d.f., $f_1(x,t)$, of the solution stochastic process of the random IVP (4) is a Cauchy sequence, so convergent. For the sake of clarity in the presentation, we first summarize the hypotheses that will be used throughout our subsequent presentation as well some results that will be required.

- **H1**: $x_0(\omega)$, $\alpha(\omega) > 0$, $\gamma(\omega) > 0$ are absolutely continuous random variables belonging to $\mathrm{L}^2(\Omega)$ and $b(t,\omega)$ is a mean square integrable continuous stochastic process, i.e, $\mathrm{L}^2(\Omega, \mathrm{L}^2(\mathcal{T}))$. Notice that, according to Th. 1.1, this entails the stochastic process $b(t,\omega)$ admits a KLE.
- **H2**: The p.d.f., $f_{x_0}(x)$, of $x_0(\omega)$ is Lipschitz in $\mathbb{R}$, i.e. there exists $L > 0$ such that $\mid f_{x_0}(x_1) - f_{x_0}(x_2) \mid \leq L \mid x_1 - x_2 \mid, \forall x_1, x_2 \in \mathbb{R}$.
- **H3**: The mean $\mu_b(t)$ as well as the eigenfunctions, $\phi_j(t)$, $\forall j = 1, 2, 3, \ldots$, associated to the KLE of the stochastic process, $b(t,\omega)$, see Th. 1.1, admit Laplace transform.
- **H4**: The moment generating function of the positive random variable $\gamma(\omega) > 0$, $\psi_\gamma(p) = \mathbb{E}[\mathrm{e}^{p\gamma}]$, exists and is finite in a neighbourhood of $p = 0$. Notice that $\psi_\gamma(p)$ is just the Laplace transform of its p.d.f., $f_\gamma(\gamma)$. Recall, this guarantees the existence of the moments with respect to the origin, $\mathbb{E}[\gamma(\omega)^m] < \infty, \forall m \in \mathbb{N}$.
- **H5**: $x_0(\omega)$ is independent of the random vector $(\alpha(\omega), \gamma(\omega), \boldsymbol{\xi}^N(\omega))$, where $\boldsymbol{\xi}^N(\omega) = (\xi_1, \ldots, \xi_N)$ denotes the vector of random variables used in the truncated KLE of the forcing term $b(t,\omega)$. Their p.d.f. will be denoted by $f_{x_0}(x_0)$ and $f_{\alpha,\gamma,\boldsymbol{\xi}^N}(\alpha, \gamma, \boldsymbol{\xi}^N)$, respectively.

Remark 2. Notice that the condition $b(t,\omega) \in \mathrm{L}^2(\Omega, \mathrm{L}^2(\mathcal{T}))$ in hypotheses H1 has been required to apply Th. 1.1; hypothesis H3 is obviously strongly related to hypothesis H1 and it has been used to apply the Laplace transform when solving the random IVP (4). The hypothesis H5 has been applied to obtain the representation of $f_1^N(x,t)$ given in Th. 2.1. The usefulness of the rest of hypotheses will be apparent later.

3.1. Technical results.

In the following, we state some useful lemmas that will be applied in the proof of the main result. The first lemma is a classical result called the c_s- inequality.

Lemma 3.1. [28, p. 157] *Let $y(\omega)$ and $z(\omega)$ be random variables and $s \in \mathbb{R}$. Then,*

$$\mathbb{E}[|y(\omega) + z(\omega)|^s] \leq c_s(\mathbb{E}[|y(\omega)|^s] + \mathbb{E}[|z(\omega)|^s]), \quad c_s = \begin{cases} 1, & \text{if} \quad s \leq 1, \\ 2^{s-1}, & \text{if} \quad s \geq 1, \end{cases}$$

provided the following expectations $\mathbb{E}[|y(\omega)|^s]$ and $\mathbb{E}[|z(\omega)|^s]$ exist.

The following two technical lemmas establish useful inequalities for the difference between two terms, say N and M, first for the sequence, $C_{\mathrm{KL}}^N(\cdot)$, given in (10), and later for the sequence, $D^N_{\mathrm{KL}}(\cdot)$, given in (12), in terms of the corresponding difference of the truncated KLE, $K_N(\cdot)$, of $b(t,\omega)$. To avoid using a cumbersome notation, we will not write the full dependence in terms of the specific random variables involved in each term, but only the ω-notation, i.e., we will write, for example, $C^N_{\mathrm{KL}}(t;\alpha(\omega),\boldsymbol{\xi}^N(\omega)) \equiv C^N_{\mathrm{KL}}(t,\omega)$, when denoting a random quantity, while $C^N_{\mathrm{KL}}(t)$ will denote its realization.

Lemma 3.2. *Let $\alpha(\omega)>0$ be a random variable and assume that $b(t,\omega)$ fulfills hypothesis H1. Let $K_N(t;\boldsymbol{\xi}^N(\omega)) \equiv \mu_b(t)+\sum_{j=1}^N \sqrt{\nu_j}\phi_j(t)\xi_j(\omega)$ be the truncated KLE of the stochastic process $b(t,\omega)\in \mathrm{L}^2(\Omega,\mathrm{L}^2(\mathcal{T}))$, $t\in\mathcal{T}=[0,T]$, defined according to Th. 1.1. Then,*

$$|C^N_{KL}(t,\omega)-C^M_{KL}(t,\omega)| \le \mathrm{e}^{\alpha(\omega)t}\int_0^t \left|K_N(\tau;\boldsymbol{\xi}^N(\omega))-K_M(\tau;\boldsymbol{\xi}^M(\omega))\right|\mathrm{d}\tau, \qquad (15)$$

for $N>M>0$ positive integers, and each $\omega\in\Omega$.

Proof. Using the expression of $C^N_{\mathrm{KL}}(t,\omega)$, given in (10), and that $\alpha(\omega)>0$, $\omega\le\omega$, one obtains

$$\begin{aligned}|C^N_{\mathrm{KL}}(t,\omega)-C^M_{\mathrm{KL}}(t,\omega)| =& \left|\int_0^t\left(\mathrm{e}^{\alpha(\omega)(t-\tau)}\sum_{j=M+1}^N \sqrt{\nu_j}\phi_j(\tau)\xi_j(\omega)\right)\mathrm{d}\tau\right|\\ \le& \int_0^t \mathrm{e}^{\alpha(\omega)(t-\tau)}\left|K_N(\tau;\boldsymbol{\xi}^N(\omega))-K_M(\tau;\boldsymbol{\xi}^M(\omega))\right|\mathrm{d}\tau\\ \le& \int_0^t \mathrm{e}^{\alpha(\omega)t}\left|K_N(\tau;\boldsymbol{\xi}^N(\omega))-K_M(\tau;\boldsymbol{\xi}^M(\omega))\right|\mathrm{d}\tau\\ =& \,\mathrm{e}^{\alpha(\omega)t}\int_0^t \left|K_N(\tau;\boldsymbol{\xi}^N(\omega))-K_M(\tau;\boldsymbol{\xi}^M(\omega))\right|\mathrm{d}\tau.\end{aligned}$$

□

Next, we present the analogous bound for $D^N(t,\omega)-D^M(t,\omega)$ in terms of the difference, $K_N(t,\omega)-K_M(t,\omega)$, of the KLE of $b(t,\omega)$, for each $\omega\in\Omega$.

Lemma 3.3. *Let $\gamma(\omega)>0$ be a random variable and assume the same hypotheses as in Lemma 3.2. Then, for $D^N(t,\omega)$, defined in (12), one satisfies*

$$\begin{aligned}|D^N(t,\omega)-D^M(t,\omega)| \le& (\gamma(\omega))^2\sum_{i=1}^n\sum_{j=1}^{i-1}\frac{\mathrm{e}^{\alpha(\omega)(2t-T_j)}\int_0^{T_j}\left|K_N(\tau;\boldsymbol{\xi}^N(\omega))-K_M(\tau;\boldsymbol{\xi}^M(\omega))\right|\mathrm{d}\tau}{(1+\gamma(\omega))^{i-j+1}}\\ &+\frac{\gamma(\omega)}{1+\gamma(\omega)}\sum_{i=1}^n \mathrm{e}^{\alpha(\omega)(2t-T_i)}\int_0^{T_i}\left|K_N(\tau;\boldsymbol{\xi}^N(\omega))-K_M(\tau;\boldsymbol{\xi}^M(\omega))\right|\mathrm{d}\tau,\end{aligned}$$

for $T_i<T$, $\omega\in\Omega$ and $t\in[0,T]$.

Proof. From the definition of $D^N(t,\omega)$, given in (12), and the result established in Lemma 3.2, one deduces

$$\begin{aligned}|D^N(t,\omega)-D^M(t,\omega)| \le& (\gamma(\omega))^2\sum_{i=1}^n\sum_{j=1}^{i-1}\frac{\mathrm{e}^{\alpha(\omega)(t-T_j)}\left|C^N_{\mathrm{KL}}(T_j,\omega)-C^M_{\mathrm{KL}}(T_j,\omega)\right|}{(1+\gamma(\omega))^{i-j+1}}\\ &+\frac{\gamma(\omega)}{1+\gamma(\omega)}\sum_{i=1}^n \mathrm{e}^{\alpha(\omega)(t-T_i)}\left|C^N_{\mathrm{KL}}(T_i,\omega)-C^M_{\mathrm{KL}}(T_i,\omega)\right|,\end{aligned}$$

$$\leq (\gamma(\omega))^2 \sum_{i=1}^{n} \sum_{j=1}^{i-1} \frac{\mathrm{e}^{\alpha(\omega)(2t-T_j)} \int_0^{T_j} \left| K_N(\tau;\boldsymbol{\xi}^N(\omega)) - K_M(\tau;\boldsymbol{\xi}^M(\omega)) \right| \mathrm{d}\tau}{(1+\gamma(\omega))^{i-j+1}}$$
$$+ \frac{\gamma(\omega)}{1+\gamma(\omega)} \sum_{i=1}^{n} \mathrm{e}^{\alpha(\omega)(2t-T_i)} \int_0^{T_i} \left| K_N(\tau;\boldsymbol{\xi}^N(\omega)) - K_M(\tau;\boldsymbol{\xi}^M(\omega)) \right| \mathrm{d}\tau.$$

□

The next result establishes a useful bound for an expectation that will appear later, in terms of the norm of the space $\mathrm{L}^2(\Omega, \mathrm{L}^2(\mathcal{T}))$, where the KLE of $b(t,\omega)$ is assumed to be convergent (see hypothesis H1).

Lemma 3.4. *Let us assume the hypotheses and notation in Lemma 3.2, then*

$$\left(\mathbb{E}\left[\left(\int_0^t \left| K_N(\tau;\boldsymbol{\xi}^N(\omega)) - K_M(\tau;\boldsymbol{\xi}^M(\omega)) \right| \mathrm{d}\tau \right)^2 \right] \right)^{\frac{1}{2}}$$
$$\leq T^{\frac{1}{2}} \| K_N(\tau;\boldsymbol{\xi}^N(\omega)) - K_M(\tau;\boldsymbol{\xi}^M)(\omega) \|_{\mathrm{L}^2(\Omega),\mathrm{L}^2(\mathcal{T}))}, \quad t \in \mathcal{T} = [0,T].$$

Proof. We will first apply the Cauchy-Schwarz inequality for integrals. Next, we use the fact that $0 \leq t \leq T$ and the monotonicity of the expectation operator. After, we interchange the expectation operator and the integral, to finally recognize the definition of the norm of the space $\mathrm{L}^2(\Omega), \mathrm{L}^2(\mathcal{T}))$ (see (2) with $\mathcal{T} = [t_0, T] = [0, T]$),

$$\mathbb{E}\left[\left(\int_0^t \left| K_N(\tau;\boldsymbol{\xi}^N(\omega)) - K_M(\tau;\boldsymbol{\xi}^M(\omega)) \right| \mathrm{d}\tau \right)^2 \right]^{\frac{1}{2}}$$
$$\leq \left(\mathbb{E}\left[\left(\int_0^t 1^2 \mathrm{d}\tau \right) \left(\int_0^t \left| K_N(\tau;\boldsymbol{\xi}^N(\omega)) - K_M(\tau;\boldsymbol{\xi}^M(\omega)) \right|^2 \mathrm{d}\tau \right) \right] \right)^{\frac{1}{2}}$$
$$= \left(\mathbb{E}\left[t \int_0^t \left| K_N(\tau;\boldsymbol{\xi}^N(\omega)) - K_M(\tau;\boldsymbol{\xi}^M(\omega)) \right|^2 \mathrm{d}\tau \right] \right)^{\frac{1}{2}}$$
$$\leq T^{\frac{1}{2}} \left(\mathbb{E}\left[\int_0^T \left| K_N(\tau;\boldsymbol{\xi}^N(\omega)) - K_M(\tau;\boldsymbol{\xi}^M(\omega)) \right|^2 \mathrm{d}\tau \right] \right)^{\frac{1}{2}}$$
$$\leq T^{\frac{1}{2}} \left(\int_0^T \mathbb{E}\left[\left| K_N(\tau;\boldsymbol{\xi}^N) - K_M(\tau;\boldsymbol{\xi}^M) \right|^2 \right] \mathrm{d}\tau \right)^{\frac{1}{2}}$$
$$= T^{\frac{1}{2}} \| K_N(\tau;\boldsymbol{\xi}^N(\omega)) - K_M(\tau;\boldsymbol{\xi}^M(\omega)) \|_{\mathrm{L}^2(\Omega),\mathrm{L}^2(\mathcal{T}))}.$$

□

This result will be used in Lemma 3.5 and Lemma 3.6 to construct bounds for the expectations of a product of random variables, which will appear later in the main result.

Lemma 3.5. *Let us assume the hypotheses of Lemma 3.2 and Lemma 3.3, together with hypothesis H4. Then, for $t > 0$,*

$$\mathbb{E}[|C_{KL}^N(t,\omega) - C_{KL}^M(t,\omega)|(1+\gamma(\omega))^{2n} \mathrm{e}^{-2\alpha(\omega)t}]$$
$$\leq 2^{(4n-1)/2} T^{1/2} \| K_N(\tau;\boldsymbol{\xi}^N(\omega)) - K_M(\tau;\boldsymbol{\xi}^M(\omega)) \|_{\mathrm{L}^2(\Omega),\mathrm{L}^2(\mathcal{T}))} (1 + \mathbb{E}[(\gamma(\omega))^{4n}]).$$

Proof. Let us take the expectation operator on both sides of the inequality (15). The inequality is proved by noticing that $\alpha(\omega) > 0$ and $\gamma(\omega) > 0$, and applying the Schwarz inequality,

$$\begin{aligned}
&\mathbb{E}[|C_{\mathrm{KL}}^{N}(t,\omega) - C_{\mathrm{KL}}^{M}(t,\omega)(1+\gamma(\omega))^{2n}\mathrm{e}^{-2\alpha(\omega)t}] \\
&\leq \mathbb{E}\left[\int_0^t \left|K_N(\tau;\boldsymbol{\xi}^N(\omega)) - K_M(\tau;\boldsymbol{\xi}^M(\omega))\right| \mathrm{d}\tau (1+\gamma(\omega))^{2n}\mathrm{e}^{-\alpha(\omega)t}\right] \\
&\leq \mathbb{E}\left[\int_0^t \left|K_N(\tau;\boldsymbol{\xi}^N(\omega)) - K_M(\tau;\boldsymbol{\xi}^M(\omega))\right| \mathrm{d}\tau (1+\gamma(\omega))^{2n}\right] \\
&\leq \mathbb{E}\left[\left(\int_0^t \left|K_N(\tau;\boldsymbol{\xi}^N(\omega)) - K_M(\tau;\boldsymbol{\xi}^M(\omega))\right| \mathrm{d}\tau\right)^2\right]^{\frac{1}{2}} \mathbb{E}\left[(1+\gamma(\omega))^{4n}\right]^{\frac{1}{2}}.
\end{aligned}$$

Applying the Lemma 3.4, the previous inequality became,

$$\begin{aligned}
&\mathbb{E}[|C_{\mathrm{KL}}^{N}(t,\omega) - C_{\mathrm{KL}}^{M}(t,\omega)|(1+\gamma(\omega))^{2n}\mathrm{e}^{-2\alpha(\omega)t}] \\
&\leq T^{\frac{1}{2}}\|K_N(\tau;\boldsymbol{\xi}^N(\omega)) - K_M(\tau;\boldsymbol{\xi}^M(\omega))\|\mathbb{E}\left[(1+\gamma(\omega))^{4n}\right]^{\frac{1}{2}}.
\end{aligned} \tag{16}$$

Now, observe that

$$\mathbb{E}\left[(1+\gamma(\omega))^{4n}\right] \leq 2^{4n-1}(1+\mathbb{E}[(\gamma(\omega))^{4n}]) < \infty, \tag{17}$$

where we have applied Lemma 3.1 with $s = 4n \geq 1$ and hypothesis H4 to guarantee the finiteness of the right-hand side term. By combining inequalities (16) and (17), one gets the result. □

Now, we establish a similar result for the expectation of the difference of $D^N(t,\omega)$ and $D^M(t,\omega)$.

Lemma 3.6. *Under the hypotheses of Lemma 3.5, one satisfies*

$$\begin{aligned}
&\mathbb{E}[\left|D^N(t,\omega) - D^M(t,\omega)\right|(1+\gamma(\omega))^{2n}\mathrm{e}^{-2\alpha(\omega)t}] \\
&\leq \sum_{i=1}^{n}\sum_{j=1}^{i-1} T^{\frac{1}{2}}\|K_N(\tau;\boldsymbol{\xi}^N(\omega)) - K_M(\tau;\boldsymbol{\xi}^M(\omega))\|_{\mathrm{L}^2(\Omega,\mathrm{L}^2(\mathcal{T}))}\mathbb{E}\left[(\gamma(\omega))^4(1+\gamma(\omega))^{4n-2i+2j-2}\right]^{\frac{1}{2}} \\
&+\sum_{i=1}^{n} T^{\frac{1}{2}}\|K_N(\tau;\boldsymbol{\xi}^N(\omega)) - K_M(\tau;\boldsymbol{\xi}^M(\omega))\|_{\mathrm{L}^2(\Omega,\mathrm{L}^2(\mathcal{T}))}\mathbb{E}\left[(\gamma(\omega))^2(1+\gamma(\omega))^{4n-2}\right]^{\frac{1}{2}}.
\end{aligned}$$

Proof. To prove this inequality, we apply Lemma 3.3. Recall that random variable $\alpha(\omega) > 0$, so $\mathrm{e}^{-\alpha(\omega)t} < 1$, $t > 0$. Now, let us observe that, by the Schwarz's inequality for expectations, one gets

$$\begin{aligned}
&\mathbb{E}\left[\left|D^N(t,\omega) - D^M(t,\omega)\right|(1+\gamma(\omega))^{2n}\mathrm{e}^{-2\alpha(\omega)t}\right] \\
&\leq \mathbb{E}\left[\left((\gamma(\omega))^2\sum_{i=1}^{n}\sum_{j=1}^{i-1}\frac{\mathrm{e}^{\alpha(\omega)(2t-T_j)}\int_0^{T_j}\left|K_N(\tau;\boldsymbol{\xi}^N(\omega)) - K_M(\tau;\boldsymbol{\xi}^M(\omega))\right|\mathrm{d}\tau}{(1+\gamma(\omega))^{i-j+1}}\right.\right. \\
&\left.\left.+\ \frac{\gamma(\omega)}{1+\gamma(\omega)}\sum_{i=1}^{n}\mathrm{e}^{\alpha(\omega)(2t-T_i)}\int_0^{T_i}\left|K_N(\tau;\boldsymbol{\xi}^N(\omega)) - K_M(\tau;\boldsymbol{\xi}^M(\omega))\right|\mathrm{d}\tau\right)(1+\gamma(\omega))^{2n}\mathrm{e}^{-2\alpha(\omega)t}\right] \\
&=\sum_{i=1}^{n}\sum_{j=1}^{i-1}\mathbb{E}\left[\int_0^{T_j}\left|K_N(\tau;\boldsymbol{\xi}^N(\omega)) - K_M(\tau;\boldsymbol{\xi}^M(\omega))\right|\mathrm{d}\,\tau\,(\gamma(\omega))^2(1+\gamma(\omega))^{2n-i+j-1}\mathrm{e}^{-\alpha(\omega)T_j}\right] \\
&\quad+\sum_{i=1}^{n}\mathbb{E}\left[\int_0^{T_i}\left|K_N(\tau;\boldsymbol{\xi}^N(\omega)) - K_M(\tau;\boldsymbol{\xi}^M(\omega))\right|\mathrm{d}\tau\,\gamma(\omega)(1+\gamma(\omega))^{2n-1}\mathrm{e}^{-\alpha(\omega)T_i}\right]
\end{aligned}$$

$$\leq \sum_{i=1}^{n}\sum_{j=1}^{i-1}\mathbb{E}\left[\int_0^{T_j}\left|K_N(\tau;\boldsymbol{\xi}^N(\omega))-K_M(\tau;\boldsymbol{\xi}^M(\omega))\right|\mathrm{d}\tau\,(\gamma(\omega))^2(1+\gamma(\omega))^{2n-i+j-1}\right]$$
$$+\sum_{i=1}^{n}\mathbb{E}\left[\int_0^{T_i}\left|K_N(\tau;\boldsymbol{\xi}^N(\omega))-K_M(\tau;\boldsymbol{\xi}^M(\omega))\right|\mathrm{d}\tau\,\gamma(\omega)(1+\gamma(\omega))^{2n-1}\right]$$
$$\leq \sum_{i=1}^{n}\sum_{j=1}^{i-1}\mathbb{E}\left[\left(\int_0^{T_j}\left|K_N(\tau;\boldsymbol{\xi}^N(\omega))-K_M(\tau;\boldsymbol{\xi}^M(\omega))\right|\mathrm{d}\tau\right)^2\right]^{\frac{1}{2}}\mathbb{E}\left[(\gamma(\omega))^4(1+\gamma(\omega))^{4n-2i+2j-2}\right]^{\frac{1}{2}}$$
$$+\sum_{i=1}^{n}\mathbb{E}\left[\left(\int_0^{T_i}\left|K_N(\tau;\boldsymbol{\xi}^N(\omega))-K_M(\tau;\boldsymbol{\xi}^M(\omega))\right|\mathrm{d}\tau\right)^2\right]^{\frac{1}{2}}\mathbb{E}\left[(\gamma(\omega))^2(1+\gamma(\omega))^{4n-2}\right]^{\frac{1}{2}}.$$

Notice that to legitimate the application of Schwarz's inequality, we have implicitly used hypothesis H4 for $\gamma(\omega)$. Applying the Lemma 3.4, we obtain

$$\mathbb{E}\left[\left|D^N(t,\omega)-D^M(t,\omega)\right|(1+\gamma(\omega))^{2n}\mathrm{e}^{-2\alpha(\omega)t}\right]$$
$$\leq \sum_{i=1}^{n}\sum_{j=1}^{i-1}T^{\frac{1}{2}}\|K_N(\tau;\boldsymbol{\xi}^N(\omega))-K_M(\tau;\boldsymbol{\xi}^M(\omega))\|_{\mathrm{L}^2(\Omega,\mathrm{L}^2(\mathcal{T}))}\mathbb{E}\left[(\gamma(\omega))^4(1+\gamma(\omega))^{4n-2i+2j-2}\right]^{\frac{1}{2}}$$
$$\sum_{i=1}^{n}T^{\frac{1}{2}}\|K_N(\tau;\boldsymbol{\xi}^N(\omega))-K_M(\tau;\boldsymbol{\xi}^M(\omega))\|_{\mathrm{L}^2(\Omega,\mathrm{L}^2(\mathcal{T}))}\mathbb{E}\left[(\gamma(\omega))^2(1+\gamma(\omega))^{4n-2}\right]^{\frac{1}{2}}.$$

□

3.2. **Convergence of the sequence of approximations of the 1-p.d.f.** In this section we prove that, under hypotheses H1-H5, the sequence of approximations, $f_1^N(x,t)$, given in (14), is a Cauchy sequence, so convergent. Approximation of the exact 1-p.d.f., $f_1(x,t)$, of the solution stochastic process of the random IVP (4) will be then obtained by means of the $f_1^N(x,t)$, with N large enough, so that the differences between two consecutive approximations are very similar in order to achieve convergence. In the examples presented in the next section, this will be graphically illustrated.

Theorem 3.7. *Assume hypotheses H1-H5, $\epsilon>0$, $\mathcal{J}\subset\mathbb{R}$ bounded, $(x,t)\in\mathcal{J}\times[t_0,T]$ an arbitrary point and $N>M$ positive integers. Then, the sequence of 1-p.d.f., $f_1^N(x,t)$, given by (14) is a Cauchy sequence.*

Proof. To shorten the expressions, we will omit the arguments of C_{KL}^N and D^N. Let $\epsilon>0$ and $N>M>0$ integers. Expressing the expectation (14) as an integral, let us consider the following difference,

$$|f_1^N(x,t)-f_1^M(x,t)|=$$
$$\left|\int_{\mathbb{R}^{N+2}}f_{x_0}\left((x-D^N-C_{\mathrm{KL}}^N)(1+\gamma)^n\mathrm{e}^{-\alpha t}\right)f_{\alpha,\gamma,\boldsymbol{\xi}^N}(\alpha,\gamma,\boldsymbol{\xi}^N)\frac{(1+\gamma)^n}{\mathrm{e}^{\alpha t}}\mathrm{d}\alpha\,\mathrm{d}\gamma\,\mathrm{d}\boldsymbol{\xi}^N\right.$$
$$\left.-\int_{\mathbb{R}^{M+2}}f_{x_0}\left((x-D^M-C_{\mathrm{KL}}^M)(1+\gamma)^n\mathrm{e}^{-\alpha t}\right)f_{\alpha,\gamma,\boldsymbol{\xi}^M}(\alpha,\gamma,\boldsymbol{\xi}^M)\frac{(1+\gamma)^n}{\mathrm{e}^{\alpha t}}\mathrm{d}\alpha\,\mathrm{d}\gamma\,\mathrm{d}\boldsymbol{\xi}^M\right|.$$

Notice that, according to Remark 2, here we have implicitly used hypotheses H1, H3 and H5. In order to write this expression using a common integral, let us first notice that,

$$f_{\alpha,\gamma,\boldsymbol{\xi}^M}(\alpha,\gamma,\boldsymbol{\xi}^M)=\int_{\mathbb{R}^{N-M}}f_{\alpha,\gamma,\boldsymbol{\xi}^N}(\alpha,\gamma,\boldsymbol{\xi}^N)\,\mathrm{d}\xi_{M+1}\cdots\mathrm{d}\xi_N,$$

and we then substitute it into the previous expression,

$$|f_1^N(x,t)-f_1^M(x,t)|$$

$$= \Big| \int_{\mathbb{R}^{N+2}} f_{x_0}\left((x - D^N - C^N_{\mathrm{KL}})(1+\gamma)^n \mathrm{e}^{-\alpha t} \right) f_{\alpha,\gamma,\boldsymbol{\xi}^N}(\alpha,\gamma,\boldsymbol{\xi}^N)(1+\gamma)^n \mathrm{e}^{-\alpha t}\, \mathrm{d}\alpha\, \mathrm{d}\gamma\, \mathrm{d}\boldsymbol{\xi}^N$$
$$- \int_{\mathbb{R}^{M+2}} f_{x_0}\left((x - D^M - C^M_{\mathrm{KL}})(1+\gamma)^n \mathrm{e}^{-\alpha t} \right) \left(\int_{\mathbb{R}^{N-M}} f_{\alpha,\gamma,\boldsymbol{\xi}^N}(\alpha,\gamma,\boldsymbol{\xi}^N)\, \mathrm{d}\xi_{M+1} \cdots \mathrm{d}\xi_N \right)$$
$$(1+\gamma)^n \mathrm{e}^{-\alpha t} \mathrm{d}\alpha\, \mathrm{d}\gamma\, \mathrm{d}\boldsymbol{\xi}^M \Big|$$
$$= \Big| \int_{\mathbb{R}^{N+2}} \left(f_{x_0}\left((x - D^N - C^N_{\mathrm{KL}})(1+\gamma)^n \mathrm{e}^{-\alpha t} \right) - f_{x_0}\left((x - D^M - C^M_{\mathrm{KL}})(1+\gamma)^n \mathrm{e}^{-\alpha t} \right) \right)$$
$$f_{\alpha,\gamma,\boldsymbol{\xi}^N}(\alpha,\gamma,\boldsymbol{\xi}^N)(1+\gamma)^n \mathrm{e}^{-\alpha t}\, \mathrm{d}\alpha\, \mathrm{d}\gamma\, \mathrm{d}\boldsymbol{\xi}^N \Big|.$$

Now, we apply the hypothesis H2, to bound the above integral expression in terms of expectation operator. This permits obtaining

$$|f_1^N(x,t) - f_1^M(x,t)|$$
$$\leq L \int_{\mathbb{R}^{N+2}} \left| -D^N + D^M - C^N_{\mathrm{KL}} + C^M_{\mathrm{KL}} \right| (1+\gamma)^n \mathrm{e}^{-\alpha t} f_{\alpha,\gamma,\boldsymbol{\xi}^N}(\alpha,\gamma,\boldsymbol{\xi}^N)$$
$$(1+\gamma)^n \mathrm{e}^{-\alpha t}\, \mathrm{d}\alpha\, \mathrm{d}\gamma\, \mathrm{d}\boldsymbol{\xi}^N$$
$$= L\mathbb{E}\left[\left| -D^N(t,\omega) + D^M(t,\omega) - C^N_{\mathrm{KL}}(t,\omega) + C^M_{\mathrm{KL}}(t,\omega)) \right| (1+\gamma(\omega))^{2n} \mathrm{e}^{-2\alpha(\omega)t} \right].$$

Now, we utilize the triangular inequality, the monotonicity and linearity of the expectation operator, and we then obtain

$$|f_1^N(x,t) - f_1^M(x,t)|$$
$$\leq \underbrace{L\,\mathbb{E}\left[\left| -D^N + D^M \right| (1+\gamma(\omega))^{2n} \mathrm{e}^{-2\alpha(\omega)t} \right]}_{\text{(I)}} + \underbrace{L\,\mathbb{E}\left[\left| -C^N_{\mathrm{KL}} + C^M_{\mathrm{KL}} \right| (1+\gamma(\omega))^{2n} \mathrm{e}^{-2\alpha(\omega)t} \right]}_{\text{(II)}}.$$

Now, we apply Lemma 3.5 for expression (II), and Lemma 3.6 for expression (I) to obtain

$$|f_1^N(x,t) - f_1^M(x,t)|$$
$$\leq L \sum_{i=1}^{n} \sum_{j=1}^{i-1} T^{\frac{1}{2}} \|K_N(\tau;\boldsymbol{\xi}^N(\omega)) - K_M(\tau;\boldsymbol{\xi}^M(\omega))\|_{\mathrm{L}^2(\Omega,\mathrm{L}^2(\mathbb{T}))} \underbrace{\mathbb{E}\left[(\gamma(\omega))^4 (1+\gamma(\omega))^{4n-2i+2j-2} \right]^{\frac{1}{2}}}_{\text{(III)}}$$
$$+ L \sum_{i=1}^{n} T^{\frac{1}{2}} \|K_N(\tau;\boldsymbol{\xi}^N(\omega)) - K_M(\tau;\boldsymbol{\xi}^M(\omega))\|_{\mathrm{L}^2(\Omega,L^2(\mathbb{T}))} \underbrace{\mathbb{E}\left[(\gamma(\omega))^2 (1+\gamma(\omega))^{4n-2} \right]^{\frac{1}{2}}}_{\text{(IV)}}$$
$$+ 2^{(4n-1)/2} L T^{\frac{1}{2}} \|K_N(\tau;\boldsymbol{\xi}^N(\omega)) - K_M(\tau;\boldsymbol{\xi}^M(\omega))\|_{\mathrm{L}^2(\Omega,L^2(\mathbb{T}))} (1 + \mathbb{E}[(\gamma(\omega))^{4n}]). \quad (18)$$

Now, we bound term (III) of the above inequality applying first the Schwarz's inequality for expectations, and secondly, Lemma 3.1 with $s = 8n - 4i + 4j - 4$, we then obtain,

$$\mathbb{E}[(\gamma(\omega))^4 (1+\gamma(\omega))^{4n-2i+2j-2}]^{\frac{1}{2}} \leq \mathbb{E}[(\gamma(\omega))^8]^{1/4} \mathbb{E}[(1+\gamma(\omega))^{8n-4i+4j-4}]^{1/4}$$
$$\leq 2^{8n-4i+4j-5} \mathbb{E}[(\gamma(\omega))^8]^{1/4} (1 + \mathbb{E}[(\gamma(\omega))^{8n-4i+4j-4}]) < \infty.$$

Similarly, we can use both the Schwarz's inequality and Lemma 3.1 with $s = 8n - 4$ to bound expression (IV),

$$\mathbb{E}\left[(\gamma(\omega))^2 (1+\gamma(\omega))^{4n-2} \right]^{\frac{1}{2}} \leq \mathbb{E}\left[(\gamma(\omega))^4 \right]^{1/4} \mathbb{E}\left[(1+\gamma(\omega))^{8n-4} \right]^{\frac{1}{4}}$$
$$\leq \mathbb{E}[(\gamma(\omega))^4]^{1/4} (2^{8n-5}(1 + \mathbb{E}[(\gamma(\omega))^{8n-4}]))^{1/4} < \infty.$$

Notice that expressions (III) and (IV) are indeed finite because, by hypothesis H4, the random variable $\gamma(\omega)$ has moments with respect to the origin of any order. Notice that it also legitimates the previous applications of Schwarz's inequality.

Then, each addend of the right-hand side in (18) tends to zero, since, by hypothesis H4, is the product of a bounded quantity by another term that tends to zero, $\|K_N(\tau;\boldsymbol{\xi}^N(\omega)) - K_M(\tau;\boldsymbol{\xi}^M(\omega))\|_{\mathrm{L}^2(\Omega,\mathrm{L}^2(\mathbb{T}))} \to 0$, since $K_N(\tau;\boldsymbol{\xi}^N(\omega))$ is the KLE of $b(t,\omega))$. This indicates that $f_1^N(x,t)$ is a Cauchy sequence, so the convergence of the sequence of approximations for the 1-p.d.f. of the solution stochastic process of the random IVP (1) is proved. □

Remark 3. Notice that in Th. 2.1, the approximation $f_1^N(x,t)$ is given in terms of the joint p.d.f., $f_{x_0,\alpha,\gamma,\boldsymbol{\xi}^N}(x_0,\alpha,\gamma,\boldsymbol{\xi}^N)$ of model parameters, while in Th. 3.7 the p.d.f. $f_1^N(x,t)$ is expressed in terms of the product of $f_{x_0}(x_0)$ by $f_{\alpha,\gamma,\boldsymbol{\xi}^N}(\alpha,\gamma,\boldsymbol{\xi}^N)$. This is legitimated because of hypothesis H5 to represent, according to Remark 1, $f_1^N(x,t)$ via an expectation. Observe that this representation plays a key role in the proof of Th. 3.7.

4. **Numerical examples.** In this section, we illustrate the theoretical results established in the previous sections by considering two numerical examples. For both of them, we will compute and plot 2D and 3D graphical representations of the approximation to the 1-p.d.f., $f_1(x,t)$, of the solution stochastic process of the random IVP (1) given in (13), via $f_1^N(x,t)$. To better illustrate the flexibility of the obtained results, we will carry out computations assuming different probability distributions for the parameters of the IVP (1), i.e. $x_0(\omega)$, $\alpha(\omega)$ and $\gamma(\omega)$. For the source term, $b(t,\omega) \equiv W(t,\omega)$, $t \in \mathcal{T} = [t_0,T]$, we will take, in both examples, a standard Wiener process or Brownian motion, $W(t)$. Remember that the mean and the covariance functions of $W(t,\omega)$ are, $\mu_W(t) = 0$ and $c_W(t) = \min(s,t)$, $(s,t) \in \mathcal{T}\times\mathcal{T}$, respectively [1]. We will take $\mathcal{T} = [0,T] = [0,1]$ So, the standard deviation function is $\sigma_W(t) = t$. For the KLE of the standard Wiener process, it is known (see [29, Ch. 5]) that $\xi_j(\omega)$ are pairwise uncorrelated standard Gaussian random variables, $\xi_j(\omega) \sim \mathrm{N}(0;1)$, and the eigenvalues, ν_j, and eigenfunctions, $\phi_j(t)$, are given by

$$\nu_j = \frac{4T^2}{\pi^2(2j-1)^2}, \quad \phi_j(t) = \sqrt{\frac{2}{T}}\sin(k_j t), \; k_j = \frac{(2j-1)\pi}{2T}, \quad j = 1,2,\ldots, \tag{19}$$

respectively. Since $\mu_b(t,\omega) = 0$, it clearly admits a Laplace transform as well as the eigenfunctions $\phi_j(t)$ of $b(t,\omega)$. So, hypothesis H3 is satisfied.

To approximate the 1-p.d.f., $f_1^N(x,t)$, given in (13), we first need to compute $C_{\mathrm{KL}}(t;\alpha(\omega),\boldsymbol{\xi}^N(\omega))$ and $D^N(t;\alpha(\omega),\gamma(\omega),\boldsymbol{\xi}^N(\omega))$, given in (10) and (12), respectively. Likewise, to determine the former term, we need to calculate the convolution of functions $\phi_j(t)$, $j = 1,2,\ldots,N$, and $g(t,\omega) = \mathrm{e}^{\alpha(\omega)t}$, i.e.,

$$\begin{aligned}(\phi_j * g)(t,\omega) &= \int_0^t \phi_j(\tau) g(t-\tau,\omega)\,\mathrm{d}\tau = \sqrt{\frac{2}{T}}\int_0^t \sin(k_j\tau)\mathrm{e}^{\alpha(\omega)(t-\tau)}\,\mathrm{d}\tau \\ &= \sqrt{\frac{2}{T}}\left[A_j(\omega)\mathrm{e}^{\alpha(\omega)t} + B_j(\omega)\cos(k_j t) + C_j(\omega)\sin(k_j t)\right],\end{aligned}$$

where

$$A_j(\omega) = \frac{k_j}{(\alpha(\omega))^2 + k_j^2}, \quad B_j(\omega) = \frac{-k_j}{(\alpha(\omega))^2 + k_j^2}, \quad C_j(\omega) = \frac{-\alpha(\omega)}{(\alpha(\omega))^2 + k_j^2},$$

and the parameters k_j are defined in (19).

Then,

$$
\begin{aligned}
C_{\mathrm{KL}}^{N}(t;\alpha(\omega),\boldsymbol{\xi}^{N}(\omega)) &= (\mu_b * g)(t,\omega) + \sum_{j=1}^{N} \sqrt{\nu_j}(\phi_j * g)(t,\omega)\xi_j(\omega) \\
&= \sqrt{\frac{2}{T}} \sum_{j=1}^{N} \sqrt{\nu_j} \left(A_j(\omega)\mathrm{e}^{\alpha(\omega)t} + B_j(\omega)\cos(k_j t) + C_j(\omega)\sin(k_j t) \right) \xi_j(\omega).
\end{aligned}
$$

This expression is evaluated at $t = T_i$, $i = 1, 2, \ldots, n$, and then substituted into (12) to obtain the $D^N(t;\alpha(\omega),\gamma(\omega),\boldsymbol{\xi}^N(\omega))$.

In the two examples, we will assume that $x_0(\omega)$, $\alpha(\omega)$, $\gamma(\omega)$ and, the random variables $\xi_i(\omega)$, $i = 1, \ldots, N$, utilized to represent the standard Wiener process $b(t,\omega) := W(t,\omega)$ via its KLE, are independent (indeed, we will take $\xi_i \sim \mathrm{N}(0;1)$ uncorrelated as usual), so hypothesis H5 is guaranteed. Later, when the probability distributions of $x_0(\omega)$, $\alpha(\omega)$ and $\gamma(\omega)$ had been specified, we will check that they also satisfy hypothesis H1 and, in the case of $\gamma(\omega)$, also the hypothesis H4.

In the examples, we will present the results graphically to better compare the approximations of the 1-p.d.f., $f_1^N(x,t)$, given by (14), as well as the mean and the standard deviation, for different orders of the truncation N. For these statistical moments, we have used that $\mu_X^N(t) = \mathbb{E}[x^N(t,\omega)] = \int_{-\infty}^{\infty} x f_1^N(x,t)\,\mathrm{d}x$ and $\sigma_X^N(t) = \sqrt{\mathbb{V}[x^N(t,\omega)]} = \left(\int_{-\infty}^{\infty} x^2 f_1^N(x,t)\,\mathrm{d}x - (\mu_X^N(t))^2\right)^{1/2}$, respectively. With these statistics, we also construct confidence intervals using the 2σ-rule. We point out that the integrals involved in these formulas have been computed using the Monte Carlo method with samples of size 10^6 for each random variable.

Example 1. We will assume that the Dirac's delta impulses are applied at the following evenly time instants, $T_i^\delta = i\Delta T$, $i = 1, 2, 3$, with $\Delta T = 1/3$. Furthermore, we will assume that $x_0(\omega)$, $\alpha(\omega)$ and $\gamma(\omega)$ are independent Gaussian distributed random variables, $x_0(\omega) \sim \mathrm{N}(0.5, 0.0625^2)$, $\alpha(\omega) \sim \mathrm{N}_{[0.375,1.625]}(1; 0.0625^2)$ and $\gamma(\omega) \sim \mathrm{N}_{[0.375,1.625]}(1; 0.0625^2)$. Now we check that with this choice of the distributions for $x_0(\omega)$, $\alpha(\omega)$ and $\gamma(\omega)$ hypotheses H1, H2 and H4 fulfil (remember that we have previously justified hypotheses H3 and H5 hold). The hypothesis H1 is satisfied since $x_0(\omega)$, $\alpha(\omega)$ and $\gamma(\omega)$ are Gaussian, so having a p.d.f., i.e. absolutely continuous and with finite variance. Moreover, it is clear that $\alpha(\omega)$ and $\gamma(\omega)$ are both positive because of the interval of truncation where they are defined. At this point, it is interesting to point out that the interval of truncation has been defined using the so called $k\sigma$–rule with $k = 10$ that, according to the Bienaymé–Chebyshev inequality [7], guarantees the truncated distribution contains at least 99% of the probability density regardless the specific type of the original distribution, however in our case that we do know the original distribution, it can be directly checked that the interval of truncation contains 99.999999% of the probability density. As it has been previously indicated, $b(t,\omega)$ is the standard Wiener process, so it belongs to $\mathrm{L}^2(\Omega, \mathrm{L}^2(\mathcal{T}))$. The first derivative of the p.d.f. of $x_0(\omega)$ is clearly bounded on the whole real line, hence applying the mean value theorem it is deduced that it is Lipschitz, and hence hypothesis H2 holds. The moment generating function, $\phi_\gamma(p) := \mathbb{E}[\mathrm{e}^{p\gamma}]$, of $\gamma(\omega) \sim \mathrm{N}_{[0.375,1.625]}(1; 0.0625^2)$ exists and is finite because γ is a bounded random variable, therefore hypothesis H4 is satisfied too.

In Figure 1, we show the approximations of the 1-p.d.f., $f_1^N(x,t)$, using as orders of truncation $N = 1, 2, 3$. They have been computed by expression (14). It is clearly

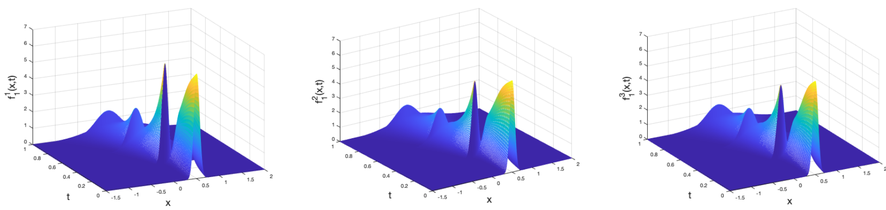

FIGURE 1. 3D-graphical representation of the approximate 1-p.d.f, $f_1^N(x,t)$, of the solution stochastic process of the random IVP (1) for different orders of truncation $N = 1, 2, 3$, and impulse applications at $T_i^\delta = i\Delta T$, $i = 1, 2, 3$, with $\Delta T = 1/3$. Example 1.

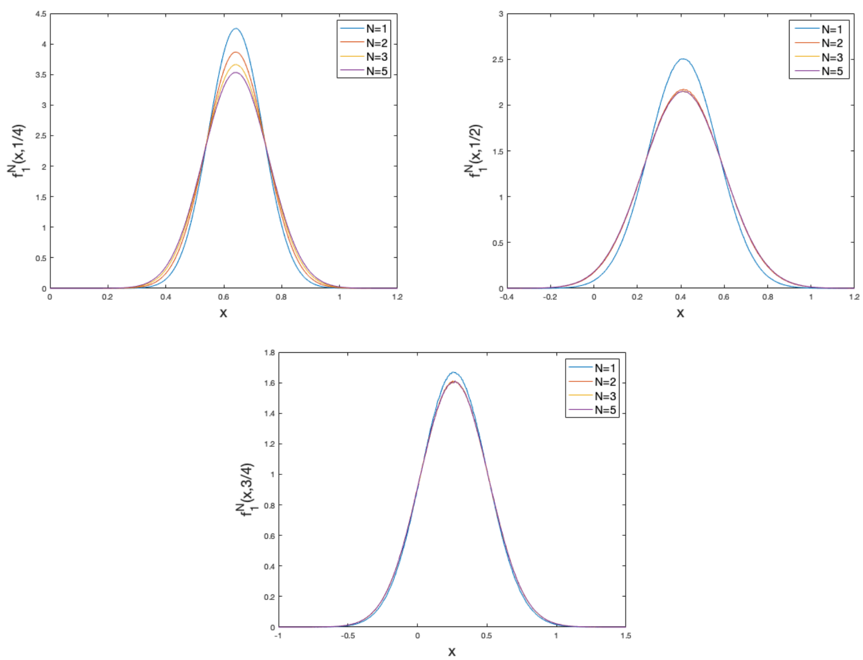

FIGURE 2. 2D-graphical representation of the approximate 1-p.d.f., $f_1^N(x,t)$, of the solution stochastic process of the random IVP (4) for different orders of truncation, $N = 1, 2, 3, 5$, and impulse applications at $T_i^\delta = i\Delta T$, $i = 1, 2, 3$, with $\Delta T = 1/3$ at the times instants: $t = 1/4$, $1/2$, $3/4$. Example 1.

visualized the effect of the three impulses. Since plots corresponding to $N = 2$ (central panel) and $N = 3$ (right panel) are very similar, we deduce that even using such small values of N the approximations are very good. It is also interesting to observe that immediately after each one of the three impulse applications, the approximation of the 1-p.d.f. becomes leptokurtic about the mean, so reducing the uncertainty (variability) to later increasing again the uncertainty after the next impulse, as time goes on. This shows the role of impulses as a type of control. In the long term, the approximations of the 1.p.d.f. flattens, meaning an increase in the variance.

Figure 2 allows us to better magnify graphically differences between the approximations of the 1-p.d.f. for different orders of truncation, $N = 1, 2, 3, 5$. The graphical representations have been plotted at the times, $t = 1/4$ (left panel), $t = 1/2$ (central panel), and $t = 3/4$ (right panel). On the left panel, note that the densities differ from each other more notoriously. However, as time increases, those differences narrow, as observed in the central and right panels. Also, as time increases, the densities are very similar for truncation orders starting at $N = 2$. This is in full agreement with the results observed in Figure 1.

In Figure 3, we show graphically comparisons among the approximations for different orders of truncation, N, of the expectation, $\mu_x^N(t)$ (left panel) and standard deviation, $\sigma_x^N(t)$ (central panel). From the plot corresponding to the expectation (left panel), we can observe that it does not practically change as N increases from 1 to 3, so showing that a good approximation of the mean is obtained even using $N = 1$. Differences are more notorious with regard to the standard deviation when comparing the approximations of order $N = 1$ against $N = 2, 3$, being these two last ones very similar. This shows that reliable approximations of the standard deviation are obtained with $N = 3$. In the right panel, we show confidence intervals constructed according to the 2σ-rule, taking $N = 3$, $[\mu_x^3 - 2\sigma_x^3]$, when the approximations of the mean and standard deviations are reliable.

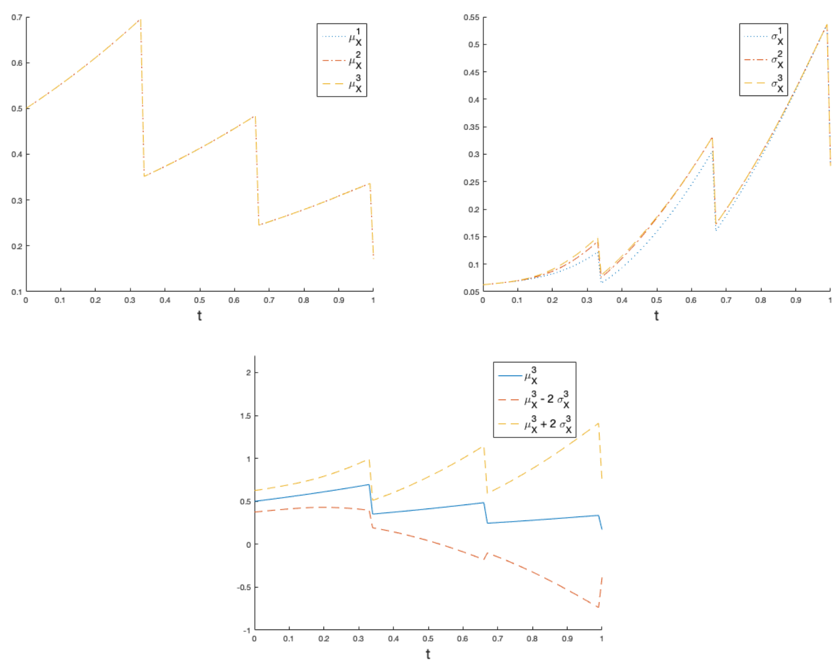

FIGURE 3. Comparison of the approximations of expectation, μ_x^N (left panel) and standard deviation, σ_x^N (central panel), of the solution stochastic process using different orders of truncation, $N = 1, 2, 3$, for $t \in [0, 1]$. Confidence intervals constructed according to the 2σ-rule are shown taking $N = 3$, $[\mu_x^3 - 2\sigma_x^3, \mu_x^3 + 2\sigma_x^3]$ (right panel). Example 1.

Example 2. In this example, we use the same conditions as in Example 1, but changing the probability distributions of random variables by $x_0(\omega) \sim \mathrm{N}(0.5;$

$\sqrt{0.025^2}$) (Gaussian), $\alpha(\omega) \sim \mathrm{B}(2,8)$ (Beta) and $\gamma(\omega) \sim \mathrm{Exp}(1)$ (Exponential). Notice that the positiveness of $\alpha(\omega) \sim \mathrm{B}(2,8)$ (Beta) and $\gamma(\omega) \sim \mathrm{Exp}(1)$ is guaranteed by definition, and hence hypothesis H1 holds. Hypotheses H2 and H4 fulfil using the same reasoning exhibited in Example 1. In Figure 4, we show 3D-graphical representations of the approximate 1-p.d.f., $f_1^N(x,t)$, using as orders of truncation $N = 1$ (left panel), $N = 2$ (central panel) and $N = 3$ (right panel). Comparing the plots shown in Figures 1 and 4, we can observe that uncertainty increases faster as the time goes on with the distributions chosen in Example 1. From the plots we can observe that the 1-p.d.f.s are very similar, so the approximation retained by the truncation of order $N = 1$ seems to be quite good. However, differences between different orders of truncation are better magnified in Figure 5, where 2D-graphical representations of the approximations are shown at the time instants $t = 1/4$ (left panel), $t = 1/2$ (central panel) and $t = 3/4$ (right panel). From these plots we can see that approximations are closer as t increases, but being sensitively different for $t = 1/4$.

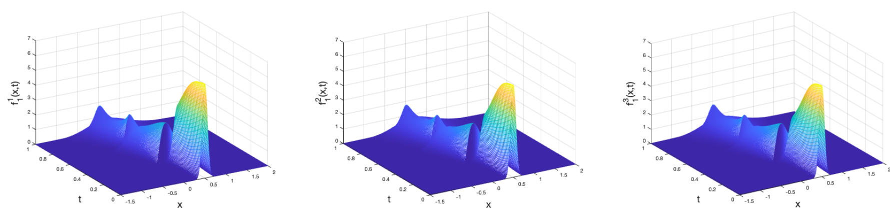

FIGURE 4. 3D-graphical representation of the approximate 1-p.d.f, $f_1^N(x,t)$, of the solution stochastic process of the random IVP (4) for different orders of truncation $N = 1, 2, 3$, and impulses at $T_i^\delta = i\Delta T$, $i = 1, 2, 3$, with $\Delta T = 1/3$. Example 2.

In Figure 6, we show comparisons among the approximations of the expectation, μ_x^N (left panel) and standard deviation, σ_x^N (central panel) of the solution stochastic process using different orders of truncation $N = 1, 2, 3$. We can observe that the approximations of the mean are very good even taking $N = 1$, while differences appear for the approximations of the standard deviation for the order of truncation $N = 1$ and for $N = 2, 3$. On the right panel, confidence intervals for $N = 3$ (when the approximations of the mean and the variance are very good), $[\mu_x^3 - 2\sigma_x^3, \mu_x^3 + 2\sigma_x^3]$, are shown.

5. **Conclusions.** In this paper we have studied a full randomization of a class of impulsive random differential equations by assuming that the initial condition and the coefficients of the model are random variables, while the source/external term is a stochastic process admitting a Karhunen-Loève expansion satisfying mild hypotheses. Impulses are given by means of an infinite train of Dirac's delta functions applied at specific time instants. Our probabilistic analysis is very general in the sense that under mild hypotheses on the model inputs, the obtained results permit considering for the external input even stochastic processes whose trajectories are very irregular including the standard Wiener process or Brownian motion. A key point of our approach is that we provide reliable approximations of the first probability density function of solution stochastic process instead of computing, as is

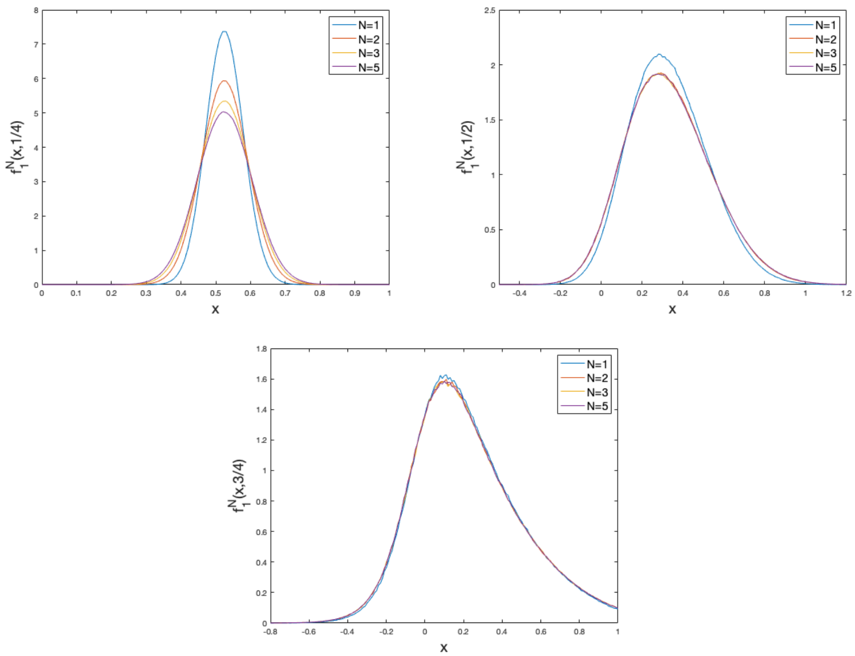

FIGURE 5. 2D-graphical representation of the approximate 1-p.d.f., $f_1^N(x,t)$, of the solution stochastic process of the random IVP (4) for different orders of truncation, $N = 1, 2, 3, 5$, and impulses at $T_i^\delta = i\Delta T$, $i = 1, 2, 3$, with $\Delta T = 1/3$ at the times instants $t = 1/4,\ 1/2,\ 3/4$. Example 2.

more usual, approximations of the first moments (mean and variance) of the solution. We think that the ideas exhibited in the paper may be useful to study more complex impulsive random differential equations in future works, and then they can contribute to open new avenues in the study of random dynamical systems in the theory of control.

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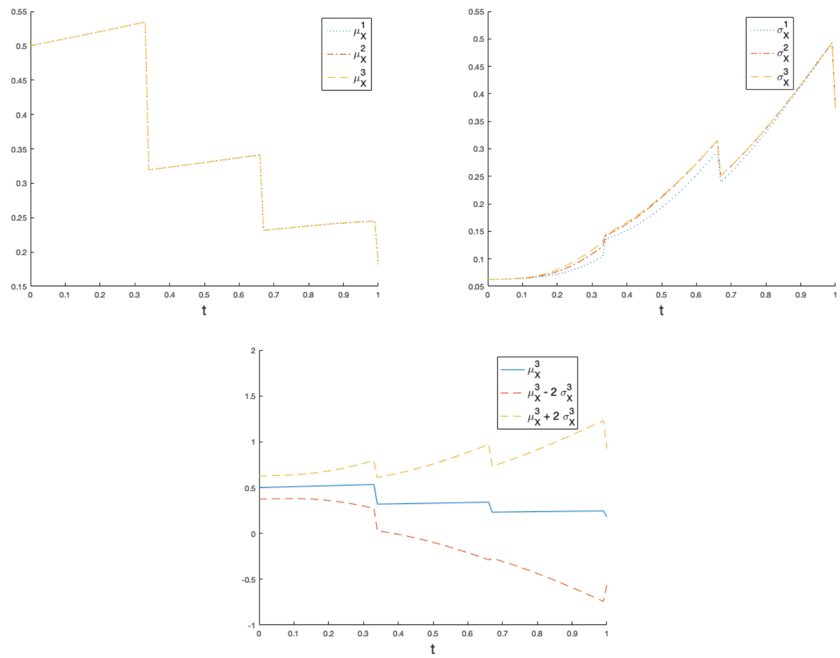

FIGURE 6. Comparison of the approximations of expectation, μ_x^N (left panel) and standard deviation, σ_x^N (central panel), of the solution stochastic process using different orders of truncation $N = 1, 2, 3$, for $t \in [0, 1]$. Confidence intervals, constructed according to the 2σ-rule, are shown taking $N = 3$, $[\mu_x^3 - 2\sigma_x^3, \mu_x^3 + 2\sigma_x^3]$ (right panel). Example 2.

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