MODELLING FOR ENGINEERING & HUMAN BEHAVIOUR 2022

PROCEEDINGS

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Modelling for Engineering & Human Behaviour 2022

València, July 14th-16th, 2022

This book includes the extended abstracts of papers presented at XXIV Edition of the Mathematical Modelling Conference Series at the Institute for Multidisciplinary Mathematics *Mathematical Modelling in Engineering & Human Behaviour.*

I.S.B.N.: 978-84-09-47037-2

November 30th, 2022
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Contents

The Relativistic Anharmonic Oscillator within a Double-Well Potential

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1 Introduction

Non-harmonic oscillations described by a double-well potential allow to model various physical phenomena and provide insight into their properties without overly simplifying the problem, thereby extending the more elementary harmonic case. Important applications of the anharmonic oscillator range from classical instantons and field theory to quantum theory and particle physics [1–3].

This presentation focuses on the treatment of the anharmonic oscillator in connection with the double-well potential including the non-negligible relativistic effects of a strong and uniform gravitational field. It extends previous work of the relativistic harmonic oscillator to the anharmonic model [4].

Our analysis parts from a relativistic action principle, where the dynamics is described as usual by the corresponding Euler-Lagrange equations. However, complications arise on how to handle the relativistic potential of the strong gravitational field. We proceed by deriving an integral equation for this potential following an approach—a method introduced by Goldstein & Bender for the brachistochrone problem [5] and more recently applied to the pendulum [6, 7].

Finally, a numerical simulation of the relativistic results is carried out to examine the dynamics of the model and compare it with the purely classical calculations. A detailed study of the phase space for both models will enable us to detect very distinct features among them, and thereby confirms the necessity for including relativistic corrections to yield reliable predictions for the strong-gravity case.

2 Methods

2.1 Lagrangian approach

Lagrangian mechanics is based on a Hamilton's principle (see e.g. Ref. [8]). The Lagrangian $L = T - V$ consists of the kinetic energy T and the potential energy V. In this model, the potential

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energy consists of the standard gravitational potential and also the double-well potential given by

$$V_{\text{double-well}} = \beta x^4 - \alpha x^2, \tag{1}$$

where α and β are positive parameters to control the typical double-well shape of the potential. In particular, the ratio α/β determines the shape of the double-well. The behaviour around the origin is shown in Figure 1, where the red curve represents a high α/β-ratio, the blue curve a low ratio, and the green curve a balanced ratio, respectively.

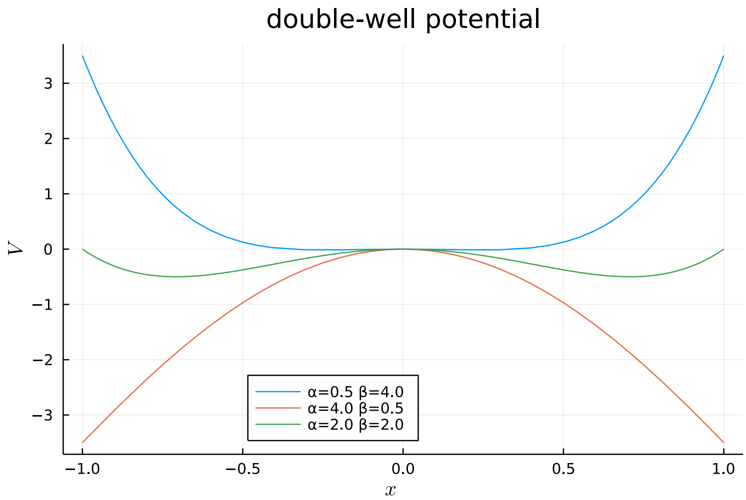

Figure 1: Double-well potential, *viz.* Eq. (1), for different parameters $\alpha, \beta > 0$.

Substituting for the potential Eq. (1), the deterministic equations of motion will result from the following variational principle by varying over all possible paths $x(t)$, while keeping the two end points fixed:

$$\delta \int dt\, L(x, \dot{x}) = \delta \int dt \left[\frac{1}{2} m_0 \dot{x}^2 - m_0 g x - \beta x^4 + \alpha x^2\right] = 0. \tag{2}$$

Here, as usual, g is the gravitational constant appearing in the gravitational potential, $V_g(x) = m_0 g x$, with inertial mass m_0. Thus, we consider a one-dimensional dynamical system, and the corresponding Euler-Lagrange equation are derived from

$$\left(\frac{d}{dt}\frac{\partial}{\partial \dot{x}} - \frac{\partial}{\partial x}\right) L = 0, \tag{3}$$

which then leads to the equations of motion for the classical case:

$$m_0 \ddot{x} + m_0 g + 4\beta x^3 - 2\alpha x = 0. \tag{4}$$

2.2 Relativistic generalization

For the generalization of Eq. (2) to special relativity, it is essential to regard the mass as a relativistic quantity, which thus will depend on the actual velocity of the body in motion $\dot{x}(t)$. The relation between rest mass, m_0, and the relativistic mass, m, is expressed by $m = \gamma m_0$, with the relativistic factor $\gamma = 1/\sqrt{1 - \dot{x}^2/c^2}$ and the speed of light c. Hence, we postulate the following variational principle for the relativistic Lagrangian:

$$\delta \int dt \left[-m_0 c^2 \sqrt{1 - \frac{\dot{x}^2}{c^2}} - \beta x^4 + \alpha x^2 - V_g(x)\right] = 0, \tag{5}$$

where the relativistically corrected gravitational potential, V_g, is still to be determined. The Lagrangian, integrand of Eq. (5), does not explicitly depend on time, so that total energy is conserved and serves as one of the integrators of the equation of motion.

According to the procedure first established in Ref. [5] and also explained in Ref. [4], we are able to obtain a integral equation for V_g and solve it analytically, thus yielding

$$\begin{aligned} V_g(x) = \left(1 - e^{-\frac{g}{c^2}x}\right)\left[m_0c^2 - 24\beta\left(\frac{c^2}{g}\right)^4 + 2\alpha\left(\frac{c^2}{g}\right)^2\right] \\ +24\beta x\left(\frac{c^2}{g}\right)^3 - 12\beta x^2\left(\frac{c^2}{g}\right)^2 + 2x\left[2\beta x^2 - \alpha\right]\left(\frac{c^2}{g}\right) - \beta x^4 + \alpha x^2. \end{aligned} \tag{6}$$

Note that taking $\alpha = -k/2$ and $\beta = 0$ reproduces the result for the harmonic oscillator calculated in Ref. [4]. This completes all the required data necessary for entirely describing the relativistic motion of the anharmonic oscillator in the gravitational field. The governing equation resulting from Eq. (5) is then provided by the second-order differential equation

$$\frac{m_0\ddot{x}}{\left(1 - \frac{\dot{x}^2}{c^2}\right)^{\frac{3}{2}}} + 4\beta x^3 - 2\alpha x + V_g'(x) = 0, \tag{7}$$

where the exact derivative of Eq. (6) is readily obtained, the analytic integration of Eq. (7), however, is impossible to carry out. Note that the direct integration of the equation of motion for the classical case, Eq. (4), is considerably complicated, producing elliptical functions and beyond. So computing a result in fully symbolic form for the more involved relativistic case, *viz.* Eq. (7), is hopeless.

3 Results

3.1 Numerical integration

For accurate estimates with precise predictions, we require to numerically integrate the differential equation in Eq. (7). For a state-of-the-art approach we employ the Julia programming language with the library `DifferentialEquations.jl`, see Refs. [9, 10].

Observe that for a forthright coding Eq. (7) can be recast into the convenient form

$$A(x)\ddot{x} + B(x)g = 0, \tag{8}$$

where the factors $A(x)$ and $B(x)$ are lengthy expressions containing simple polynomials in the variable x and also the gravitational potential $V_g(x)$ given by Eq. (6). The natural choice for the boundary conditions is $x(0) = 0$ and $\dot{x}(0) = 0$. For the integration using Julia, we adopt the default first-order interpolation algorithm, with at least a stepsize of $\Delta t = 0.005\,\mathrm{s}$ for the time domain, and a relative tolerance of 10^{-12} for high accuracy.

3.2 Simulation of phase space

Figure 2 depicts the phase-space trajectories for the classical case (red curve) and the relativistic case (blue curve) of the anharmonic oscillator within a gravitational field and a double-well potential. For mechanical systems, the phase space represents all physically possible values of the position and velocity variables (or alternatively momentum). For illustration purposes, we have chosen $\alpha = 2.0$ [kg/s^2] and $\beta = 0.5$ [kg/m^{2}s^2] for the shape parameters in the double-well potential, *viz.* Eq. (1). Obviously both oscillatory phenomena are periodic, which is made manifest by the closed trajectories. As is well-known, the phase space of the plain oscillator (not suspended in a gravitational field) acquires a perfectly elliptical form. However, the asymmetric shape of the red curve in Figure 2 is caused by gravity, which introduces a directional preference. Moreover, incorporating relativistic effects, further distorts the pear-shaped classical trajectory into a bullet shaped trajectory—clearly shown in Figure 2 for this extreme case. The pronounced elongation of the blue, relativistic prediction in x-direction is due to the dynamic quality of the relativistic mass—a variable mass instead of the fixed inertial mass in the classical phase-space simulation. A heavier mass certainly will be more affected by gravity. Additionally, the slimmer shape of the blue curve in $\dot{x}$-direction results from the increasing inertia resisting further acceleration with increasing velocity. This effect is purely relativistic and absent in any classical model. The reduced velocity range for the blue curve, taking values for $\dot{x}$ in the interval $[-1, 1]$ instead of $[-3, 3]$, is thus explained by the physical upper speed limit of relativistic dynamics.

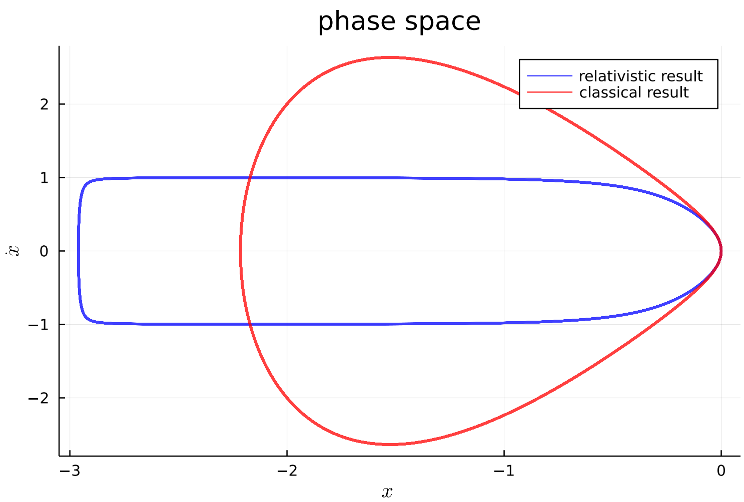

Figure 2: Phase-space diagram of the anharmonic oscillator in a uniform gravitational field, corresponding to the classical and relativistic case for the double-well potential, Eq. (1), with $\alpha = 2.0$ [kg/s^2] and $\beta = 0.5$ [kg/m^{2}s^2].

4 Conclusions

Starting from a Lagrangian approach for a classical anharmonic oscillator in a gravitational field and with a double-well potential, we generalized the discussion to include relativistic effects. By neglecting insignificant tidal effects on a small scale, it was not necessary to employ the fully developed general relativistic framework, but it sufficed to resort to special relativity.

The governing equations of motion for the classical model and its corresponding relativistic extension had to be integrated numerically. We used the Julia programming language as a state-of-the-art tool for obtaining numerical estimates for both models with high accuracy and precision. This allowed us to compute and visualize the phase space for the classical and relativistic case, and then compare both models.

A careful analysis of the phase-space trajectories showed that the differences between the predictions for the classical and the relativistic model are significant. More importantly, these differences are not merely quantitative but show that their origin is of fundamental nature—an immediate consequence of the very different concept of mass in the classical and relativistic approach. This reflects itself in the noticeably distinct shapes of the classical and relativistic trajectories taken in phase space.

In summary, our observations confirm the necessity for including relativistic corrections in our model to produce reliable predictions for the strong-gravity case or when velocities comparatively close to the speed of light are reached.

Acknowledgements

This work has been supported by the Vicerrectorado de Investigación de la Universitat Politècnica de València under Grant PAID-11-21.

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