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Point displacements during classical measurements – a practical approach to pseudo epochs between measurements

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ABSTRACT

Various measurement techniques and data processing are applied to determine point displacements and deformation of geodetic networks or buildings. Considering classical measurements and analysis of the network deformation, we should realize that the measurements are not "immediate." The question arises: what happens if a point (or some points) displaces between particular measurements within one epoch. In such a case, the observation set would consist of the observations before and after point displacement, and such hypothetical observation groups can be regarded as related to two (or more) pseudo epochs. The paper's main objective is to examine some estimation methods that would probably deal with such a problem, namely M_{split} estimation (in two variants, the squared and the absolute M_{split} estimation) and chosen robust methods, namely Huber's method (example M-estimation) and the Hodges-Lehmann weighted estimation (basic R-estimation). The first approach can provide two (or more) variants of the network point coordinates (here, before and after point movements), providing information about two (or more) states of the network during measurements. In contrast, the robust methods can only decrease the influence of the outliers on the computed network point coordinates. Thus, estimation results would concern only one network state in such a case. The presented empirical analyses show that the better and more realistic results are obtained by applying M_{split} estimation. Huber's method can also provide acceptable results (describing the network state at the epoch beginning) only if the number of observations conducted after the point displacements is not too high.

I. INTRODUCTION

The displacement of the network point is determined between at least two measurement epochs. Thus, the deformation analysis is conducted between two moments in time. However, geodetic observations, namely measurements of angles, distances, or height differences, are not "immediate," and sometimes they require some time. Such a fact concerns traditional measurements and, in some cases also, other techniques like GPS measurements. Generally, we assume that the network points are stable during the measurements of each epoch. The question arises, what if a point (or some points) displaces between measurements within one epoch. It is evident that such displacements would affect the observation set, in which some observations relate to the stage before displacements, whereas the other observations to the stage after displacements. Since all observations belong to one epoch, thus we can call such subsets observations at the first and second pseudo epoch; however, we do not know the assignment of each observation to either of the pseudo epochs in practice. This problem was mentioned in (Wiśniewski *et al.*, 2019). It is also no doubt that the existence of the pseudo epoch would also affect further computations, such as the deformation analysis between two epochs. That would indeed happen if one applied the usual approach based on the least squares method, without any statistical tests or analyses detecting "outlying" observations (in the problem considered here, the observations of one of the pseudo epochs could be regarded as outliers). An alternative would be the application of robust estimation or other methods which can deal with such an observation set. In the first case, we can apply any robust M-estimation (Caspary *et al.*, 1990; Hekimoğlu, 1999; Xu, 2005; Nowel, 2015), or R-estimation (Hodges and Lehmann, 1963; Duchnowski, 2013; Wyszkowska and Duchnowski, 2018). A method called M_{split} estimation was created to estimate the location parameters (more generally, the parameters of the split functional models) when an observation set consists of the subsets mentioned before. The method has already been applied in deformation analysis (Wiśniewski, 2009; Zienkiewicz *et al.*, 2017; Wyszkowska and Duchnowski, 2019).

The paper aims to compare different approaches to the problem of pseudo epochs. The following methods will be analyzed: the least squares estimation (LS), the Huber method (an example of robust M-estimation), and the Hodges-Lehmann weighted estimation (example of R-estimation; HLW), the squared M_{split}

estimation (SMS), and the absolute M_{split} estimation (AMS).

II. EMPIRICAL ANALYSES

Let us consider the following linear model (Eq. 1):

$$\mathbf{y} = \mathbf{AX} + \mathbf{v} \tag{1}$$

where $\mathbf{y}$ = observation vector of size $n \times 1$
$\mathbf{A}$ = coefficient matrix of size $n \times r$
$\mathbf{X}$ = parameter vector of size $r \times 1$
$\mathbf{v}$ = measurement error vector $n \times 1$

The LS estimator $\hat{\mathbf{X}}_{LS}$ of the parameter vector can be computed as (Eq. 2):

$$\hat{\mathbf{X}}_{LS} = \left(\mathbf{A}^T\mathbf{PA}\right)^{-1}\mathbf{A}^{\mathbf{T}}\mathbf{Py} \tag{2}$$

where $\mathbf{P}$ = weight matrix of size $n \times n$

Whereas the Huber estimator $\hat{\mathbf{X}}_H$ of the parameter vector is determined in the iterative process (Eqs. 3 and 4):

$$\hat{\mathbf{X}}_H = \left(\mathbf{A}^T\mathbf{WA}\right)^{-1}\mathbf{A}^T\mathbf{Wy} \tag{3}$$

$$\left[\mathbf{W}\right]_{ii} = \left[\mathbf{P}\right]_{ii} \cdot w(\hat{v}_i) \tag{4}$$

where $\mathbf{W}$ = diagonal matrix of weights of size $n \times n$
$w(\hat{v}_i)$ = weight function related to variant of M-estimation, $1 \le i \le n$
$\hat{v}_i$ = standardized error of ith observation
$\left[\circ\right]_{ii}$ = ith diagonal element of matrix

The new version of the matrix **W** is computed in each subsequent iterative step. The general formula of the Huber weight function can be written as (Yang, 1994; Ge *et al.*, 2013) (Eq. 5):

$$w(\hat{v}_i) = \begin{cases} 1 & \text{for } |\hat{v}_i| \le c \\ \dfrac{c}{|\hat{v}_i|} & \text{for } |\hat{v}_i| > c \end{cases} \tag{5}$$

where c = positive constant

Here, we assume the constant $c = 2$, which defines the interval of the acceptable, standardized measurement errors (Gui and Zhang, 1998).

Another robust estimator which can be applied in the paper context is the Hodges-Lehmann weighted estimator of the expected value $\hat{E}^{HLW}\left(X_j\right)$, $1 \le j \le r$. The general formula is as follows (Duchnowski, 2013) (Eq. 6):

$$\hat{E}^{HLW}\left(X_j\right) = \mathrm{medw}\left(\frac{z_k + z_l}{2}\right) \tag{6}$$

where medw = weighted median operator
z_k, z_l = elements of sample of size s ($1 \le k \le s$, $1 \le l \le s$)

The sample mentioned is created for each parameter X_j separately. In the paper context, the sample elements are created by computing the coordinates of a particular network point by applying the raw observations and the reference point coordinates in all possible independent ways (Duchnowski, 2013, 2021).

M_{split} estimation is the last method considered here. The general assumption of that estimation method is the split of the functional model (Equation 1) into two competitive ones (Wiśniewski, 2009; Wyszkowska and Duchnowski, 2019) (Eq. 7):

$$\mathbf{y} = \mathbf{AX} + \mathbf{v} \Rightarrow \begin{cases} \mathbf{y} = \mathbf{AX}_{(1)} + \mathbf{v}_{(1)} \\ \mathbf{y} = \mathbf{AX}_{(2)} + \mathbf{v}_{(2)} \end{cases} \tag{7}$$

where $\mathbf{X}_{(m)}$ = versions of parameter vector
$\mathbf{v}_{(m)}$ = versions of measurement error vector
$m = 1$ or 2

We examine two M_{split} estimation variants: the squared and absolute M_{split} estimation. These two methods differ in the objective function, hence also in the influence and weight functions. The weight functions of SMS estimation are as follows (Wiśniewski, 2009) (Eq. 8):

$$\begin{cases} w_{(1)}\left(v_{i(1)}, v_{i(2)}\right) = v_{i(2)}^2 \\ w_{(2)}\left(v_{i(1)}, v_{i(2)}\right) = v_{i(1)}^2 \end{cases} \tag{8}$$

while AMS estimation refers to the following weight functions (Wyszkowska and Duchnowski, 2019) (Eq. 9):

$$\begin{cases} w_{(1)}\left(v_{i(1)}, v_{i(2)}\right) = \begin{cases} -\dfrac{|v_{i(2)}|}{2v_{i(1)}} & \text{for } v_{i(1)} < 0 \\ \dfrac{|v_{i(2)}|}{2v_{i(1)}} & \text{for } v_{i(1)} > 0 \end{cases} \\ w_{(2)}\left(v_{i(1)}, v_{i(2)}\right) = \begin{cases} -\dfrac{|v_{i(1)}|}{2v_{i(2)}} & \text{for } v_{i(2)} < 0 \\ \dfrac{|v_{i(1)}|}{2v_{i(2)}} & \text{for } v_{i(2)} > 0 \end{cases} \end{cases} \tag{9}$$

The competitive M_{split} estimates, namely $\hat{X}_{(1)}$ and $\hat{X}_{(2)}$ of the parameter vector, are determined in the iterative process by applying the modified Newton method (Wiśniewski, 2009). Because of the difference in the weight functions, two variants of M_{split} estimation require different computing algorithms. SMS estimation uses the traditional iterative process, while AMS estimation requires a parallel iterative process (Wyszkowska and Duchnowski, 2020). The differences between the algorithms concern the starting point and computing the estimates in subsequent iterative steps. There is one starting point in the traditional iterative process (usually LS estimates of the parameters) and two different starting points in the parallel iterative process. The algorithms are described in detail in (Wiśniewski, 2009; Wyszkowska and Duchnowski, 2019).

Considering the problem of pseudo epochs, the competitive versions of the parameter vector should reflect the point displacements. One expects one version to correspond to the point coordinates before displacement and the second after the point movement.

III. Title

Let us consider a leveling network presented in Figure 1. The network consists of two reference points (fixed ones), *A* and *B,* and seven object points (points with unknown heights). The observations, namely twenty height differences, are assumed to be measured twice in one measurement epoch with the assumed standard deviation of 1 mm. Without loss of generality, we can assume that the theoretical heights of all network points are equal to 0 mm; hence, all theoretical height differences are equal to 0 mm.

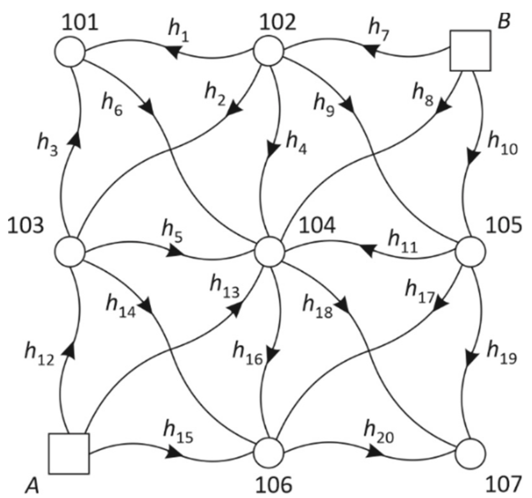

Figure 1. Simulated leveling network.

Let us assume that points 101 and 102 have moved vertically for $\Delta H_{101} = 10\,\text{mm}$, $\Delta H_{102} = 20\,\text{mm}$ during the measurements. Additionally, we consider two variants: Variant I, the second measurements of the height differences h_1, h_3, h_7 were measured after the point displacements; Variant II, the first measurements of the height differences h_3, h_7 and the second measurements of the height differences h_1, h_3, h_4, h_7, h_9 were measured after the point displacements. Two variants help us to understand how each estimation method reacts to a different number of observations after the point displacement. One should realize that in practice, we do not know which points are displaced and which observations are performed before and after the point displacement; thus, we cannot divide observations into subsets *a priori*.

Now let us examine how such an affected observation set influences the estimation results. Assuming that the observations are normally distributed, we simulated 5000 observation sets in each variant.

A. *Variant I*

The histograms of the heights of the chosen points estimated by applying the conventional methods are presented in Figure 2.

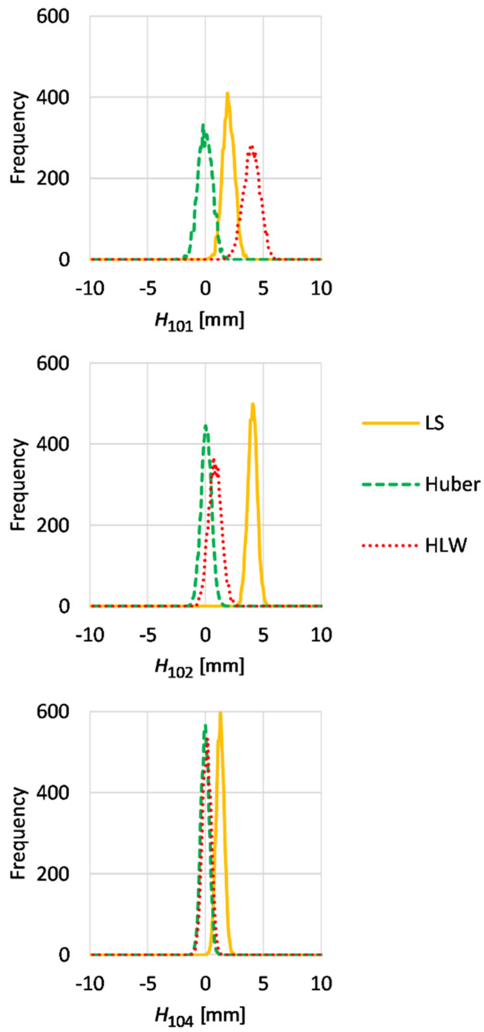

Figure 2. Histograms of estimated point heights (Variant I).

The histograms are determined for the heights of the object points that moved, namely 101 and 102, and one chosen stable point 104. The histograms obtained for the other stable points are very similar to the histogram of the point 104 height; hence they are omitted here.

The best results are obtained for the Huber method, where the height estimates are robust against outlying observations. The other robust method (HLW) does not provide correct results, especially for point 101. It results from the location of outliers and the low breakdown point of the method in the case of a small number of observations (Duchnowski, 2011). Note that we have 6 independent ways to compute the heights of point 101, 8 ways for point 102, and 12 ways for point 104, which determines the size of the samples in Equation 6.

The histograms of the estimated heights of the same variant and points obtained for the two variants of M_{split} estimation are shown in Figure 3. Following the split functional model of Equation 7 we have two competitive solutions (the competitive estimates of the point heights).

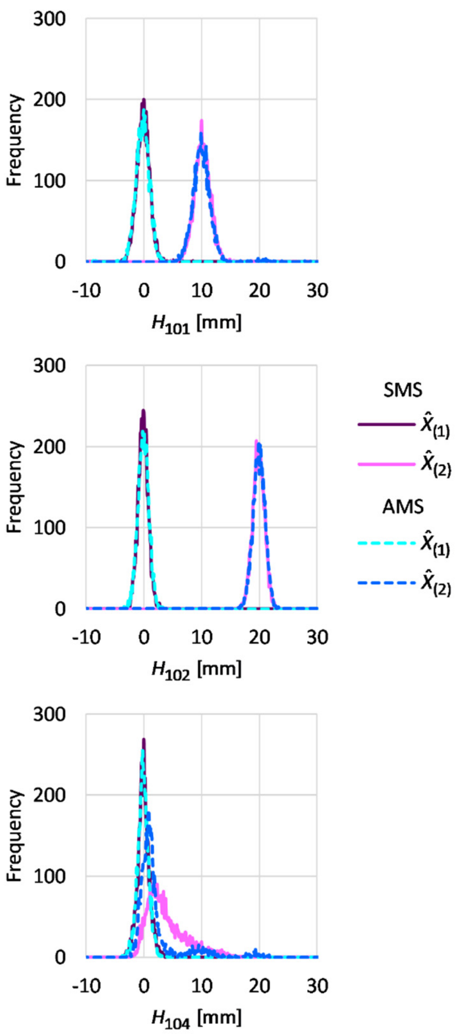

Figure 3. Histograms of estimated point heights (Variant I).

The histograms obtained for the estimated heights of points 101 and 102 show that the M_{split} estimation detects the point displacements correctly (the histograms coincide with the simulated point heights). The situation is different for the estimated height of point 104 (similar histograms are obtained for the rest of the object points). The first solutions of both variants seem correct; the histogram coincides with 0 mm. The histograms of the second solutions are generally proper; however, they seem a little bit skewed right, especially in the case of SMS estimation. Tables 1 and 2 present some descriptive statistics from the Monte Carlo simulations to complete the comparison. The medians here describe the central tendencies of the estimates. The mean values omitted here are usually equal to the respective medians or differ from them by no more than 0.2 mm. The accuracy of the estimates is described by the root-mean-square deviation (RMSD) computed as (Eq. 10):

$$\mathrm{RMSD} = \sqrt{\sum_{i=1}^{5000} \frac{\left(\hat{X}_i^{MC} - X\right)^2}{5000}} \tag{10}$$

where $\hat{X}_i^{MC}$ = estimated parameter in ith simulation
X = simulated parameter

For the conventional method, for each point $X = 0\,\mathrm{mm}$. In the case of M_{split} estimation, for the first solution, namely $X_{(1)}$, it is assumed that $X = 0\,\mathrm{mm}$, but for the second one, $X_{(2)}$, it holds that $X = \Delta H_{101} = 10\,\mathrm{mm}$, $X = \Delta H_{102} = 20\,\mathrm{mm}$ or $X = \Delta H_{104} = 0\,\mathrm{mm}$, respectively.

Table 1. Medians [mm] of the estimates of chosen object point heights from MC simulations (Variant I)

Parameter	Point 101	Point 102	Point 104
LS	2.0	4.1	1.3
Huber	0.0	0.1	0.0
HLW	4.1	0.9	0.2
SMS $\hat{X}_{(1)}$	-0.1	-0.1	0.0
SMS $\hat{X}_{(2)}$	10.1	19.9	3.0
AMS $\hat{X}_{(1)}$	-0.1	0.0	-0.1
AMS $\hat{X}_{(2)}$	10.0	20.2	1.0

Table 2. RMSDs [mm] of the estimates of chosen object point heights from MC simulations (Variant I)

Method	Point 101	Point 102	Point 104
LS	2.1	4.1	1.4
Huber	0.6	0.5	0.4
HLW	4.1	1.0	0.4
SMS $\hat{X}_{(1)}$	1.2	0.9	1.0
SMS $\hat{X}_{(2)}$	1.4	1.0	5.1
AMS $\hat{X}_{(1)}$	1.2	0.9	1.0
AMS $\hat{X}_{(2)}$	1.9	1.0	4.7

Tables confirm that the Huber method provided the correct results, and LS or HLW estimation did not. They also prove that M_{split} estimation might tell apart the first pseudo epoch from the second one. What is more, it can assess the point displacements correctly. However, M_{split} estimation provides worse assessments of $X_{(2)}$ for the points which do not displace during the measurements (see the results obtained in the case of point 104, especially RMSD). It might result from the small number of observations performed after the point displacements.

B. *Variant II*

In that variant, more measurements are carried out after the point displacements. The histograms of the heights of the chosen points estimated by using the conventional methods are shown in Figure 4. This time, any of the methods did not provide satisfactory results. Only histograms obtained for the height of point 104 are located close to 0 mm. It results from the fact that the most height differences between that point and the other network points were measured at the first pseudo epoch.

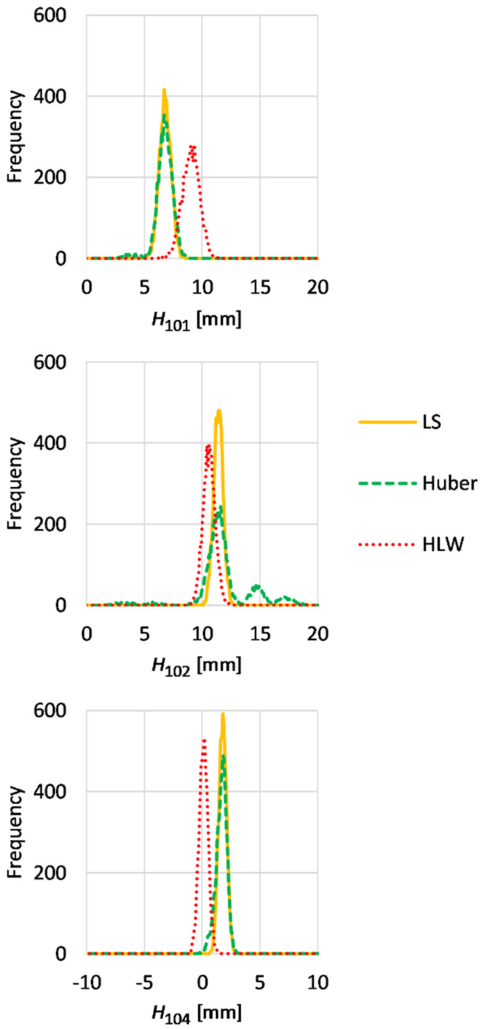

Figure 4. Histograms of estimated point heights (Variant II).

Figure 5 presents histograms of the point heights estimated by M_{split} estimation variants. The location of all the histograms is correct, and the histograms obtained for point 104 seem less skewed right than in the previous variant.

Like in the preceding subsection, we can also compare the descriptive statistics from the Monte Carlo method (Tables 3 and 4). The medians confirm the conclusions from the simple analysis of the histogram locations. The conventional methods cannot provide the correct results, and SMS as well as AMS estimation correctly identified the pseudo epochs and assessed the point displacements. The best results, the medians closest to the theoretical values, are obtained for AMS estimation. Empirical accuracies presented in Table 4 show that M_{split} estimation can deal with pseudo epochs better than the conventional methods when the number of observations carried out after the point displacements is greater. Even the Huber method, which succeeded in Variant I, cannot manage a higher number of outlying observations.

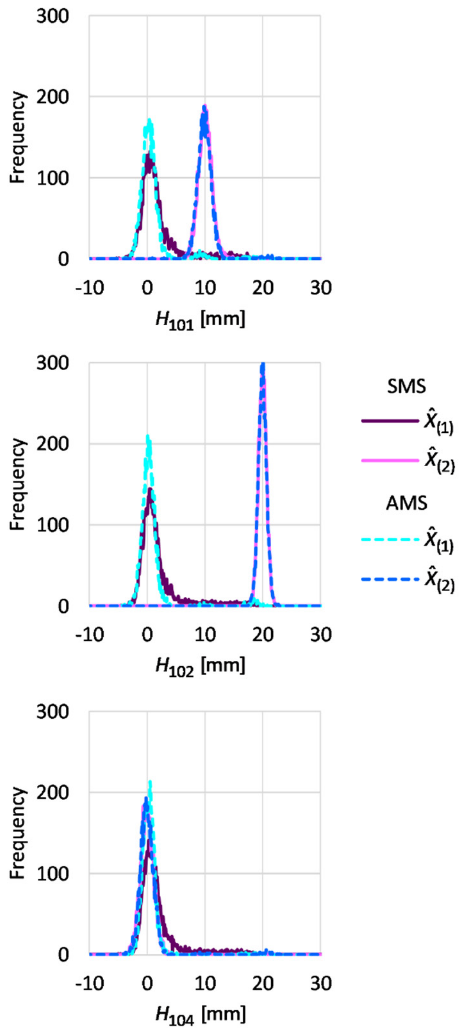

Figure 5. Histograms of estimated point heights (Variant II).

IV. Conclusions

The displacements of the object points during one measurement epoch, if they really happen, stay undetected at the stage of measurements. They mainly might concern large networks where the number of observations is high; hence the longer time of measurements is required. That situation might also happen in networks that were established to determine terrain surface deformations resulting, for example, from uplifts or mining damages, where the vertical displacements might be sudden and have a relatively high magnitude.

Table 3. Medians [mm] of the estimates of chosen object point heights from MC simulations (Variant II)

Method	Point 101	Point 102	Point 104
LS	6.8	11.4	1.8
Huber	6.8	11.6	1.8
HLW	9.1	10.6	0.2
SMS $\hat{X}_{(1)}$	0.9	0.9	0.9
SMS $\hat{X}_{(2)}$	10.1	20.0	0.0
AMS $\hat{X}_{(1)}$	0.4	0.3	0.3
AMS $\hat{X}_{(2)}$	10.1	20.0	0.0

Table 4. RMSDs [mm] of the estimates of chosen object point heights from MC simulations (Variant II)

Method	Point 101	Point 102	Point 104
LS	6.9	11.4	1.9
Huber	6.8	12.2	1.8
HLW	9.1	10.7	0.4
SMS $\hat{X}_{(1)}$	3.9	3.9	3.9
SMS $\hat{X}_{(2)}$	1.2	0.7	1.4
AMS $\hat{X}_{(1)}$	2.7	3.1	2.1
AMS $\hat{X}_{(2)}$	1.7	0.7	2.7

Loss or preservation information about point displacements is the essential difference between the conventional methods and the approach based on M_{split} estimation. The example presented in the paper proves that the conventional methods might not manage with point displacement during measurements. However, the robust methods can provide correct results only when the number of outlying observations is small enough. When applying robust M-estimation, it is assumed that observations from one pseudo epoch (before or after point displacements) are regarded as outliers. We would surely lose information about point displacements by “ignoring” such a group of observations. We would also not know which pseudo epoch is described finally. By applying M_{split} estimation, one can preserve the information about point displacements and keep track of point movements during the measurement epoch. In the given example, both variants of M_{split} estimation distinguish two groups of observations. What is more, they assess the point displacements between the pseudo epochs correctly. Comparing the results obtained for M_{split} estimation variants, one can suggest applying the AMS method rather than SMS estimation in the problem addressed in the paper.

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