Multidimensional extension of conformable fractional iterative methods for solving nonlinear problems

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1 Introduction

Fractional calculus is an extension of classical calculus, and the theoretical aspects from this are held. Many problems can be described by using fractional calculus, because of the higher degree of freedom compared to classical calculus [1, 2].

In recent years, some Newton-type methods for solving nonlinear equations have been proposed by using the Riemann-Liouville, Caputo and conformable fractional derivatives [3–5].

First, let us introduce some preliminary concepts related to conformable derivative. The left conformable fractional derivative of a function $f : [a, \infty) \longrightarrow \mathbb{R}$, starting from a, of order $\alpha \in (0, 1]$, $\alpha, a, x \in \mathbb{R}$, $a < x$, is [6]

$$(T^a_\alpha f)(x) = \lim_{\varepsilon \longrightarrow 0} \frac{f(x + \varepsilon(x - a)^{1-\alpha}) - f(x)}{\varepsilon}. \tag{1}$$

If the limit exists, f is α-differentiable. If f is also differentiable, $(T^a_\alpha f)(x) = (x - a)^{1-\alpha} f'(x)$. If f is α-differentiable in (a, b), for some $b \in \mathbb{R}$, $(T^a_\alpha f)(a) = \lim_{x \to a^+} (T^a_\alpha f)(x)$. Note that $T^a_\alpha C = 0$, where C is a constant.

Newly, a Newton-type method by using conformable derivative was designed for solving nonlinear equations in [5] with the following iterative expression:

$$x_{k+1} = a + \left((x_k - a)^\alpha - \alpha \frac{f(x_k)}{(T^a_\alpha f)(x_k)} \right)^{1/\alpha}, \quad k = 0, 1, 2, \ldots \tag{2}$$

where $(T^a_\alpha f)(x_k)$ is the left conformable fractional derivative of order α, $\alpha \in (0, 1]$, starting from a, $a < x_k$, $\forall k$. When $\alpha = 1$, we obtain the classical Newton-Raphson method. The quadratic convergence of this method is stated in [5] by using an appropriate conformable Taylor series [7].

The method proposed in [5], as seen in equation (1), can be used only to solve scalar problems. To design a conformable vectorial Newton-type method in order to find the solution $\bar{x} \in \mathbb{R}^n$ of

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a nonlinear system $F(x) = \hat{0}$, with coordinate functions $f_1, \ldots, f_n$, where $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a sufficiently Fréchet-differentiable function in an open convex set D, we have to introduce the necessary existing concepts and results.

We can find in [8] a definition of conformable partial derivative:

Definition 1. *Let f be a function in n variables, $x_1, \ldots, x_n$, the conformable partial derivative of f of order $\alpha \in (0, 1]$ in $x_i > a = 0$ is defined as:*

$$\frac{\partial_0^\alpha}{\partial x_i^\alpha} f(x_1, \ldots, x_n) = \lim_{\epsilon \to 0} \frac{f(x_1, \ldots, x_i + \epsilon x_i^{1-\alpha}, \ldots, x_n) - f(x_1, \ldots, x_n)}{\epsilon}, \tag{3}$$

In [8] we can also find a definition of conformable Jacobian matrix:

Definition 2. *Let f, g be functions in 2 variables x and y, and the respective partial derivatives exist and are continuous. Being $x > a_1$ and $y > a_2$, where $a = (a_1, a_2) = (0, 0) = \hat{0}$, then the conformable Jacobian matrix is:*

$$F_{\hat{0}}^{\alpha(1)}(x) = \begin{pmatrix} \dfrac{\partial_0^\alpha f}{\partial x^\alpha} & \dfrac{\partial_0^\alpha f}{\partial y^\alpha} \\ \dfrac{\partial_0^\alpha g}{\partial x^\alpha} & \dfrac{\partial_0^\alpha g}{\partial y^\alpha} \end{pmatrix} = \begin{pmatrix} x^{1-\alpha} \dfrac{\partial f}{\partial x} & y^{1-\alpha} \dfrac{\partial f}{\partial y} \\ x^{1-\alpha} \dfrac{\partial g}{\partial x} & y^{1-\alpha} \dfrac{\partial g}{\partial y} \end{pmatrix}. \tag{4}$$

This concept can be directly extended to higher dimensions.

The new concepts and results required to design a conformable vectorial Newton-type method are stated in the next Section.

2 Methods

2.1 New concepts and results

Considering that in equation (3), $x_i \in (0, \infty)$, it can be defined the conformable partial derivative in $x_i \in (a, \infty)$:

Definition 3. *Let f be a function in n variables, $x_1, \ldots, x_n$, the conformable partial derivative of f of order $0 < \alpha \leq 1$ in $x_i \in (a, \infty)$ is*

$$\frac{\partial_a^\alpha}{\partial x_i^\alpha} f(x_1, \ldots, x_n) = \lim_{\epsilon \to 0} \frac{f(x_1, \ldots, x_i + \epsilon (x_i - a)^{1-\alpha}, \ldots, x_n) - f(x_1, \ldots, x_n)}{\epsilon}. \tag{5}$$

When $x_i = a$, $\dfrac{\partial_a^\alpha}{\partial x_i^\alpha} f(x_1, \ldots, a, \ldots, x_n) = \displaystyle\lim_{x_i \to a^+} \dfrac{\partial_a^\alpha}{\partial x_i^\alpha} f(x_1, \ldots, x_i, \ldots, x_n)$.

This conformable partial derivative is linear, and the product, quotient and chain rules are held, like conformable derivative in [6].

It can also be stated the definition of conformable Jacobian matrix for $x_1 \in (a_1, \infty)$ and $x_2 \in (a_2, \infty)$, being $x = (x_1, x_2)$ and $a = (a_1, a_2)$:

Definition 4. *Let f and g be the coordinate functions of a vector valued function $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with variables $x_1 > a_1$ and $x_2 > a_2$, being $x = (x_1, x_2)$ and $a = (a_1, a_2)$, such that the respective partial derivatives exist and are continuous. The conformable Jacobian matrix is*

$$F_a^{\alpha(1)}(x) = \begin{pmatrix} \dfrac{\partial_{a_1}^\alpha f}{\partial x_1^\alpha} & \dfrac{\partial_{a_2}^\alpha f}{\partial x_2^\alpha} \\ \dfrac{\partial_{a_1}^\alpha g}{\partial x_1^\alpha} & \dfrac{\partial_{a_2}^\alpha g}{\partial x_2^\alpha} \end{pmatrix} = \begin{pmatrix} (x_1 - a_1)^{1-\alpha} \dfrac{\partial f}{\partial x_1} & (x_2 - a_2)^{1-\alpha} \dfrac{\partial f}{\partial x_2} \\ (x_1 - a_1)^{1-\alpha} \dfrac{\partial g}{\partial x_1} & (x_2 - a_2)^{1-\alpha} \dfrac{\partial g}{\partial x_2} \end{pmatrix}. \tag{6}$$

This concept can also be directly extended to higher dimensions.

Likewise in [7] (Theorem 4.1), we can get a new Taylor series, where the conformable derivatives start from some point $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ distinct from another point $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$ where they are being evaluated:

Theorem 1. *Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be an infinitely α-differentiable vector valued function, for $\alpha \in (0,1]$, around some point $b_i \in (a_i, \infty)$, $\forall i = 1, \ldots, n$, being $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$. Then, F has the conformable Taylor power series*

$$F(t) = F(b) + \frac{F_a^{\alpha(1)}(b)}{\alpha}\Delta + \frac{F_a^{\alpha(2)}(b)}{2!\alpha^2}\Delta^2 + \cdots, \tag{7}$$

being $\Delta = H^{\odot\alpha} - L^{\odot\alpha}$; $H = t - a$, $L = b - a$, where $\odot$ is the Hadamard power.

In addition, in order to perform the convergence analysis, another concept has to be introduced.

Theorem 2. *Let $x, y \in \mathbb{R}^n$, $r \in \mathbb{R}$, and be $\odot$ the Hadamard product/power. The Newton's binomial theorem for fractional power and vector values is*

$$(x + y)^{\odot r} = \sum_{k=0}^{\infty} \binom{r}{k} x^{\odot(r-k)} \odot y^{\odot k}, \quad k \in \{0\} \cup \mathbb{N}, \tag{8}$$

where the generalized binomial coefficient (see [9]) is

$$\binom{r}{k} = \frac{\Gamma(r+1)}{k!\Gamma(r-k+1)}, \quad k \in \{0\} \cup \mathbb{N}. \tag{9}$$

Now, we set the design of conformable Newton-type method for solving nonlinear systems.

2.2 Design and convergence analysis

As we can see in [5], let us consider the approximation of a function F with the Taylor power series (7) up to order one, evaluated at the solution $\bar{x}$ of $F(x) = \hat{0}$:

$$F(x) \approx F(\bar{x}) + \frac{F_a^{\alpha(1)}(\bar{x})}{\alpha}\Delta. \tag{10}$$

Since $F(\bar{x}) = \hat{0}$, and $\Delta = H^{\odot\alpha} - L^{\odot\alpha}$; $H = x - a$, $L = \bar{x} - a$,

$$F(x) \approx \frac{F_a^{\alpha(1)}(\bar{x})}{\alpha}\left[(x-a)^{\odot\alpha} - (\bar{x}-a)^{\odot\alpha}\right]. \tag{11}$$

Multiplying both sides of (11), by $\alpha\left[F_a^{\alpha(1)}(\bar{x})\right]^{-1}$ from the left,

$$\alpha\left[F_a^{\alpha(1)}(\bar{x})\right]^{-1} F(x) \approx (x-a)^{\odot\alpha} - (\bar{x}-a)^{\odot\alpha}. \tag{12}$$

From $(\bar{x}-a)^{\odot\alpha}$, we can get $\bar{x}$, so

$$\bar{x} \approx a + \left((x-a)^{\odot\alpha} - \alpha\left[F_a^{\alpha(1)}(\bar{x})\right]^{-1} F(x)\right)^{\odot 1/\alpha}. \tag{13}$$

Considering iterates $x^{(k)}$ and $x^{(k+1)}$ as approximations of solution $\bar{x}$, we get the conformable Newton-type method for nonlinear systems:

$$x^{(k+1)} = a + \left[\left(x^{(k)} - a\right)^{\odot\alpha} - \alpha\left[F_a^{\alpha(1)}\left(x^{(k)}\right)\right]^{-1} F\left(x^{(k)}\right)\right]^{\odot 1/\alpha}, \; k = 0, 1, 2, \ldots \tag{14}$$

Next result shows that quadratic convergence of vectorial Newton-type method (14) by using the conformable Taylor series (7) is obtained.

Theorem 3. *Let $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a continuous function in an open convex set $D \subseteq \mathbb{R}^n$ holding a zero $\bar{x} \in \mathbb{R}^n$ of a vector valued function $F(x)$. Let $F_a^{\alpha(1)}(x)$ be the conformable Jacobian matrix of F starting at $a \in \mathbb{R}^n$, of order α, for any $\alpha \in (0,1]$. Let us suppose that $F_a^{\alpha(1)}(x)$ is continuous and non-singular at $\bar{x}$. If an starting point $x^{(0)} \in \mathbb{R}^n$ is quite close to $\bar{x}$, then the local order of convergence of conformable vectorial Newton's method*

$$x^{(k+1)} = a + \left[\left(x^{(k)} - a\right)^{\odot\alpha} - \alpha\left[F_a^{\alpha(1)}\left(x^{(k)}\right)\right]^{-1} F\left(x^{(k)}\right)\right]^{\odot 1/\alpha}, \quad k = 0,1,2,\dots$$

is at least 2, and the error equation is

$$e^{(k+1)} = \alpha C_2(\bar{x} - a)^{\odot(\alpha-1)} e^{(k)^2} + O\left(e^{(k)^3}\right), \tag{15}$$

where $C_j = \dfrac{1}{j!\alpha^{j-1}}\left[F_a^{\alpha(1)}(\bar{x})\right]^{-1} F_a^{\alpha(j)}(\bar{x})$, *$j = 2,3,4,\dots$, such that $a < x^{(k)}$, $\forall k$.*

Next, some numerical tests with some nonlinear systems of equations are made. We comment that, in all tests, a comparison with classical Newton-Raphson's method (when $\alpha = 1$) is being made. The dependence on initial estimates of both methods is also analyzed.

3 Results

The numerical tests are made by using Matlab R2020a with double precision arithmetic, $\|F(x^{(k+1)})\| < 10^{-8}$ or $\|x^{(k+1)} - x^{(k)}\| < 10^{-8}$ as stopping criterium, and a maximum of 500 iterations. We used $a = (a_1, \dots, a_n) = (-10, \dots, -10)$ for each test to be sure that $a_i < x_i$, $\forall i = 1, \dots, n$, as seen in Definitions 3 and 4, and $a < x^{(k)}$, $\forall k$, according to Theorem 3. We also use the Approximated Computational Order of Convergence (ACOC)

$$ACOC = \frac{\ln(\|x^{(k+1)} - x^{(k)}\|/\|x^{(k)} - x^{(k-1)}\|)}{\ln(\|x^{(k)} - x^{(k-1)}\|/\|x^{(k-1)} - x^{(k-2)}\|)}, \quad k = 0,1,2,\dots,$$

introduced in [10], to verify the theoretical order of convergence is got in practice, and $\alpha \in (0,1]$.

Our test function to vector values is $F(x,y) = (x^2 - 2x - y + 0.5, x^2 + 4y^2 - 4)^T$ with real and complex roots $\bar{x}_1 \approx (-0.2222, 0.9938)^T$, $\bar{x}_2 \approx (1.9007, 0.3112)^T$ and $\bar{x}_3 \approx (1.1608 - 0.6545i, -0.9025 - 0.2104i)^T$. The conformable Jacobian matrix of $F(x,y)$ is

$$F_a^{\alpha(1)}(x,y) = \begin{pmatrix} (x-a_1)^{1-\alpha}(2x-2) & (y-a_2)^{1-\alpha}(-1) \\ (x-a_1)^{1-\alpha}(2x) & (y-a_2)^{1-\alpha}(8y) \end{pmatrix},$$

where $a = (a_1, a_2) = (-10, -10)$.

α	$\bar{x}$	$\|F(x^{(k+1)})\|$	$\|x^{(k+1)} - x^{(k)}\|$	iter	ACOC
1	-	-	-	> 500	-
0.9	$\bar{x}_3$	5.40×10^{-11}	5.86×10^{-6}	54	2.00
0.8	$\bar{x}_3$	9.77×10^{-9}	7.87×10^{-5}	86	2.00
0.7	$\bar{x}_3$	2.27×10^{-14}	4.75×10^{-8}	36	1.98
0.6	$\bar{x}_2$	2.95×10^{-10}	1.16×10^{-5}	23	2.05
0.5	$\bar{x}_2$	4.89×10^{-10}	1.48×10^{-5}	122	2.05
0.4	$\bar{x}_2$	5.34×10^{-13}	5.01×10^{-7}	86	2.04
0.3	$\bar{x}_2$	4.94×10^{-10}	1.79×10^{-5}	35	2.03
0.2	$\bar{x}_2$	1.16×10^{-14}	6.76×10^{-8}	21	1.98
0.1	$\bar{x}_2$	2.16×10^{-10}	1.08×10^{-5}	39	2.06

Table 1: Results for $F(x,y) = \hat{0}$ with initial estimation $x^{(0)} = (-2, -1.5)^T$

α	$\bar{x}$	$\|F(x^{(k+1)})\|$	$\|x^{(k+1)} - x^{(k)}\|$	iter	ACOC
1	$\bar{x}_1$	8.31×10^{-11}	7.29×10^{-6}	5	2.00
0.9	$\bar{x}_1$	5.94×10^{-11}	6.15×10^{-6}	5	2.00
0.8	$\bar{x}_1$	4.21×10^{-11}	5.17×10^{-6}	5	2.00
0.7	$\bar{x}_1$	2.97×10^{-11}	4.33×10^{-6}	5	2.00
0.6	$\bar{x}_1$	2.09×10^{-11}	3.62×10^{-6}	5	2.00
0.5	$\bar{x}_1$	1.45×10^{-11}	3.01×10^{-6}	5	2.00
0.4	$\bar{x}_1$	1.01×10^{-11}	2.49×10^{-6}	5	2.00
0.3	$\bar{x}_1$	6.97×10^{-12}	2.06×10^{-6}	5	2.00
0.2	$\bar{x}_1$	4.80×10^{-12}	1.69×10^{-6}	5	2.00
0.1	$\bar{x}_1$	3.30×10^{-12}	1.39×10^{-6}	5	2.00

Table 2: Results for $F(x, y) = \hat{0}$ with initial estimation $x^{(0)} = (-2, 1.5)^T$

In Table 1, we can see for $F(x, y)$ that classical Newton's method (when $\alpha = 1$) does not get any solution in 500 iterations, while conformable vectorial Newton's procedure converges. We observe also that ACOC may be even slightly greater than 2 when $\alpha \neq 1$. Note also that complex root $\bar{x}_3$ is found with real initial estimate.

In Table 2, we can observe for $F(x, y)$, with a different initial estimation, that classical Newton's scheme and conformable Newton's method have a similar behaviour, regarding the number of iterations and the ACOC. Once again, the quadratic convergence of conformable Newton's method is held for any $\alpha \in (0, 1]$.

In order to study the stability of conformable vectorial Newton's method tested above, we analyze the dependence on initial estimates by observing convergence planes, which is defined in [11], and is also employed in [3–5].

For constructing convergence planes we consider from initial estimates (x_0, y_0), the points x_0 in the horizontal axis, and values of $\alpha \in (0, 1]$ in the vertical axis. Each one of 2 planes in the figure is performing a distinct value of y_0 from initial estimates (x_0, y_0). Each color in the planes represents different solutions, and when it is painted in black no solution was found in 500 iterations. Each plane is generated by a 400×400 grid, with a maximum of 500 iterations, and a tolerance of 0.001.

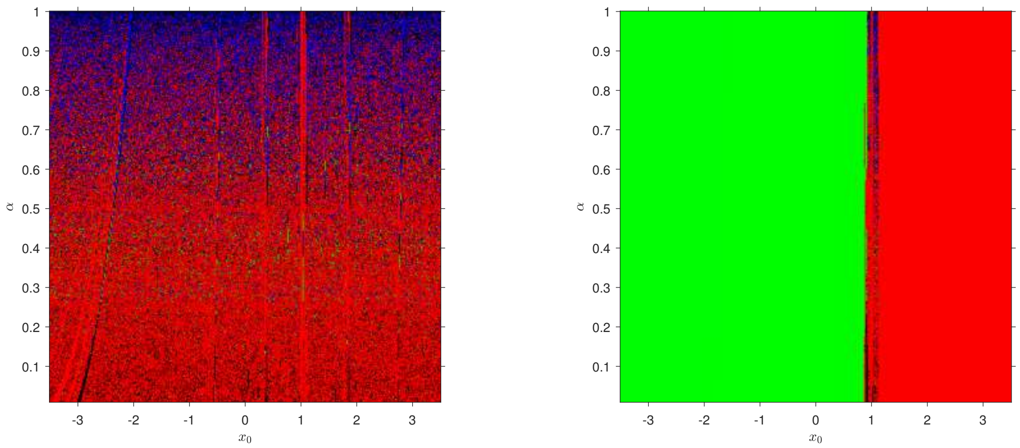
In Figure 1, it can be observed for $F(x, y)$ that in (b) is obtained almost 100% of convergence, while in (a) is obtained around 86% of convergence. This method converges to all roots for each case, even to complex root with real initial estimates.

We can also see, in general, it is possible to get several solutions with the same initial estimate by choosing different values of α.

4 Conclusions

The first conformable fractional Newton-type iterative method for solving nonlinear systems has been designed. We have introduced the analytical implements needed for the construction of this method. The convergence analysis has been made, and quadratic convergence of classical Newton's method is held. Numerical tests have been made, and the dependence on initial estimates was analyzed, sustaining the theory. We could see that conformable vectorial Newton-type method shows, in some cases, a better numerical behaviour than classical one in terms of number of iterations, ACOC, and wideness of basins of attractions of the roots. We could also see that complex roots may be found with real initial estimates, and several roots may be obtained with the same initial estimate by choosing distinct values for α.

Acknowledgement: This research was partially supported by PGC2018-095896-B-C22 (MCIU /AEI/10.13039/501100011033/FEDER Una manera de hacer Europa, UE) and by Dominican Re-



(a) $y_0 = -1.5$, 85.99% of convergence (b) $y_0 = 1.5$, 99.62% of convergence

Figure 1: Convergence planes of $F(x, y)$. $\bar{x}_1$: green, $\bar{x}_2$: red, $\bar{x}_3$: blue

public FONDOCYT 2018-2019-1D2-140.

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