

First Order Hamiltonian Systems

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1 Introduction

This paper presents a way to get a Hamiltonian function for a coupled first order differential equation system whose independent variable is time or, as it is called shortly in the literature about the subject, a dynamical system. In addition, the Hamiltonian is also first order in momenta, oppositely to physics dynamics, where the Hamiltonian is second order respect the momenta (note that the corresponding dynamical systems in physics are, by the Newton laws of Nature, coupled second order differential equation systems). Besides, while the temporal variables involved in physics are of spatial nature, in the dynamical systems here considered the temporal variables involved are abstract, i.e., they are of arbitrary nature, such as, for instance, biological populations, chemical components, or any other variables related with social or behavioural nature. The way used to get the Hamiltonian is that provided by Dirac by his Generalized Hamiltonian Dynamics method [1]. This method is applied for those systems for which the Hamiltonian cannot be provided from the Lagrangian function for two cases: either for those where the generalized velocities cannot be isolated as a function of the momenta, or for those with singular Hamiltonians, for which the momenta vanish. The second case is the corresponding to this paper. This method was developed by Dirac for fields in order to get the Hamiltonian of the electromagnetic field, which was also singular. Note that here, at difference of fields, which are infinite-dimensional systems, the system dimension is finite and equal to the number of the first order differential equations involved in the model studied. There are other approaches different to Dirac's method, such as Hava's approach [2] or Pontryagin's approach [3]. Hava's approach [2] focuses on the Lagrangian to discuss possible conservation laws, and Pontryagin's approach [3] uses a different Hamiltonian way steered to optimize dynamical functions, although Dirac's and Pontryagin's results are mathematically similar but with different objectives. In fact, the real Dirac's objective is to get the Schrödinger equation from the Hamiltonian. Different authors have followed Pontryagin's method [3] to get the same quantum approach objective [4, 5] in other contexts. However, this paper objective does not consider the quantum approach, which is so considered in [6, 7]. In addition, the equivalence between Hava's and Dirac's approach is addressed in [8] and also more recently in [6, 7]. Section 2 is devoted to get the Hamiltonian following Dirac's approach [1]. Section 3 is devoted to use the formalism presented to get a General System Thermodynamics (GST), while Section 4 points out how this formalism could reproduce the classical reversible Thermodynamics as a particular case of the GST. Section 5 is devoted to the paper discussion and the possible future applications of the formalism.

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2 Getting the Hamiltonian

Let $q_k(t)$, $k = 1, 2, \dots, n$, the abstract variables of a dynamical system, with $\mathbf{q} = (q_1, q_2, \dots, q_n)$:

$$\dot{q}_k(t) = f_k(t, \mathbf{q}); \quad k = 1, 2, \dots, n. \quad (1)$$

The problem of minimum action principle consists in finding an integral action $\Lambda(t)$:

$$\Lambda(t) = \int_{t_1}^{t_2} L(t, \mathbf{q}, \dot{\mathbf{q}}) dt \quad (2)$$

with arbitrary t_1 and t_2 times, being $L(t, \mathbf{q}, \dot{\mathbf{q}})$ the Lagrangian, such that $\Lambda(t)$ be a minimum, i.e., $\delta\Lambda(t) = 0$. This process of minimization provides the Euler-Lagrange equations:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0; \quad k = 1, 2, \dots, n. \quad (3)$$

Eq. (3) must reproduce the system of Eq. 1. This problem is known in the scientific literature as the Lagrange inverse problem [2].

As Dirac [1] and Havas [2] argue, the only possibility to solve the inverse Lagrange problem for Eq. 1 is as a linear combination for the Lagrangian such as the following:

$$L(t, \mathbf{q}, \dot{\mathbf{q}}) = \sum_{j=1}^n g_j(t, \mathbf{q}) \dot{q}_j - h(t, \mathbf{q}). \quad (4)$$

In Eq. (4) the $g_j(t, \mathbf{q})$ and $h(t, \mathbf{q})$ functions are unknown functions by the moment. Note that the momenta p_k can be defined from Eq. (4) as:

$$p_k = \frac{\partial L}{\partial \dot{q}_k} = g_k(t, \mathbf{q}) \quad (5)$$

As it is well known, the Hamiltonian ($H_0(t, \mathbf{q}, \mathbf{p})$) is defined from the Lagrangian as:

$$H_0(t, \mathbf{q}, \mathbf{p}) = \sum_{j=1}^n p_j \dot{q}_j - L(t, \mathbf{q}, \dot{\mathbf{q}}) = h(t, \mathbf{q}) \quad (6)$$

Note that the Hamiltonian $H_0(t, \mathbf{q}, \mathbf{p})$ of Eq. 6 is singular due to it does not depend on the momenta. Dirac's method [1] solves the problem by the steps described in the beginning. The first step is defining the primary constraints $\phi_j(t, \mathbf{q}, \mathbf{p})$ from Eq. 5:

$$\phi_j(t, \mathbf{q}, \mathbf{p}) = p_j - g_j(t, \mathbf{q}) = 0; \quad j = 1, 2, \dots, n \quad (7)$$

that are added to the Hamiltonian Eq. 6 by the $\lambda_j(t, \mathbf{q}, \mathbf{p})$ unknown multiplying functions as:

$$H(t, \mathbf{q}, \mathbf{p}) = H_0(t, \mathbf{q}, \mathbf{p}) + \sum_{j=1}^n \lambda_j(t, \mathbf{q}, \mathbf{p}) \phi_j(t, \mathbf{q}, \mathbf{p}) = h(t, \mathbf{q}) + \sum_{j=1}^n \lambda_j(t, \mathbf{q}, \mathbf{p}) \phi_j(t, \mathbf{q}, \mathbf{p}) \quad (8)$$

Note that the primary constraints are non-zero valued inside the Hamiltonian. By considering Eq. 7, the Hamilton equations for the new Hamiltonian $H(t, \mathbf{q}, \mathbf{p})$ are:

$$\left. \begin{aligned} \dot{q}_k &= \frac{\partial H(t, \mathbf{q}, \mathbf{p})}{\partial p_k} = \lambda_k(t, \mathbf{q}, \mathbf{p}) \\ \dot{p}_k &= -\frac{\partial H(t, \mathbf{q}, \mathbf{p})}{\partial q_k} = -\frac{\partial h(t, \mathbf{q})}{\partial q_k} + \sum_{j=1}^n \lambda_j(t, \mathbf{q}, \mathbf{p}) \frac{\partial g_j(t, \mathbf{q})}{\partial q_k} \end{aligned} \right\}; \quad k = 1, 2, \dots, n \quad (9)$$

The first result obtained by comparing Eqs. 1 and 9 is that:

$$\lambda_k(t, \mathbf{q}, \mathbf{p}) = f_k(t, \mathbf{q}); \quad k = 1, 2, \dots, n \quad (10)$$

Note that in Eq. 9 the zero value of the primary constraints outside the Hamiltonian has been used. In order to get the $g_j(t, \mathbf{q})$ and $h(t, \mathbf{q})$ functions of the formalism, the functions primary constants $\phi_j(t, \mathbf{q}, \mathbf{p})$ must hold the consistency conditions, i.e., $\dot{\phi}_l = 0$ ($l = 1, 2, \dots, n$), that is:

$$\dot{\phi}_l(t, \mathbf{q}, \mathbf{p}) = \frac{\partial \phi_l(t, \mathbf{q}, \mathbf{p})}{\partial t} + \sum_{k=1}^n \frac{\partial \phi_l(t, \mathbf{q}, \mathbf{p})}{\partial q_k} \dot{q}_k + \sum_{k=1}^n \frac{\partial \phi_l(t, \mathbf{q}, \mathbf{p})}{\partial p_k} \dot{p}_k = 0; \quad l = 1, 2, \dots, n \quad (11)$$

The substitution of Eqs. 9 and 10 in Eq. 11 provides, taking into account the zero value of the primary constraints, and after some calculations:

$$-\frac{\partial g_l(t, \mathbf{q})}{\partial t} - \frac{\partial h(t, \mathbf{q})}{\partial q_l} + \sum_{k=1}^n \left(-\frac{\partial g_l(t, \mathbf{q})}{\partial q_k} + \frac{\partial g_k(t, \mathbf{q})}{\partial q_l} \right) f_k(t, \mathbf{q}) = 0; \quad l = 1, 2, \dots, n \quad (12)$$

In Eq. 12 the following $F_{kl}(t, \mathbf{q})$ functions can be defined:

$$F_{kl}(t, \mathbf{q}) = -F_{lk}(t, \mathbf{q}) = -\frac{\partial g_l(t, \mathbf{q})}{\partial q_k} + \frac{\partial g_k(t, \mathbf{q})}{\partial q_l}; \quad k, l = 1, 2, \dots, n \quad (13)$$

Eq. 12 can be rewritten by using Eq. 13:

$$\sum_{k=1}^n F_{lk}(t, \mathbf{q}) f_k(t, \mathbf{q}) = -\frac{\partial g_l(t, \mathbf{q})}{\partial t} - \frac{\partial h(t, \mathbf{q})}{\partial q_l}; \quad l = 1, 2, \dots, n \quad (14)$$

In order to get an equation for the $F_{lk}(t, \mathbf{q})$ functions without the $h(t, \mathbf{q})$ function, such as in Eq. 14, the following steps are followed: 1. Take the derivative respect an arbitrary q_j in Eq. 14; 2. Rewrite Eq. 14 by replacing l by j ; 3. Take the derivative respect q_j in the rewritten equation; 4. Subtract both equations. Taking into account the equality of both $h(t, \mathbf{q})$ crossed derivatives, the result is:

$$\frac{\partial F_{jl}(t, \mathbf{q})}{\partial t} = \frac{\partial}{\partial q_j} \left(\sum_{k=1}^n F_{lk}(t, \mathbf{q}) f_k(t, \mathbf{q}) \right) - \frac{\partial}{\partial q_l} (F_{jk}(t, \mathbf{q}) f_k(t, \mathbf{q})); \quad j, l = 1, 2, \dots, n \quad (15)$$

Therefore, the process to get the $g_j(t, \mathbf{q})$ and $h(t, \mathbf{q})$ functions of the Hamiltonian given by Eq. 8 is: 1. Get the $F_{lk}(t, \mathbf{q})$ functions by Eq. 15; 2. Substitute these results in Eq. 13 to get the $g_j(t, \mathbf{q})$ functions; 3. Substitute these results in Eq. 14 to get the $h(t, \mathbf{q})$ function. Take into account in this process that $F_{ll}(t, \mathbf{q}) = 0$ and that $F_{kl}(t, \mathbf{q}) = -F_{lk}(t, \mathbf{q})$.

However, two different classes of solutions must be considered depending on the system dimension n . This is due to the antisymmetric definition of the $F_{lk}(t, \mathbf{q})$ functions. On the one hand, if n is even, then, in general $\det(F_{lk}(t, \mathbf{q})) \neq 0$. In this case Eqs. 14 or 15 are independent. On the other hand, if n is odd, then always $\det(F_{lk}(t, \mathbf{q})) = 0$, and being $\det(F_{lk}(t, \mathbf{q})) \neq 0$ for the $n-1$ even dimension, one of the Eq. 14 is dependent on the others, which makes that some $F_{lk}(t, \mathbf{q})$ functions become undetermined parameters from which the rest ones depend on.

Actually, this last case always happens due to Eq. 14 is a coupled set of n equations and $n+1$ unknown variables: $g_l(t, \mathbf{q})$ ($l = 1, 2, \dots, n$) and $h(t, \mathbf{q})$. For instance, let the special one-dimensional ($n = 1$) odd case be. The consistence conditions that provide Eq. 14 become:

$$\eta(t, q) := \frac{\partial g(t, q)}{\partial t} + \frac{\partial h(t, q)}{\partial q} = 0 \quad (16)$$

Note that Eq. 16 does not provide the $\lambda(t, q, p) = f(t, q)$ multiplying function. When this case happens, the Dirac's method [1] prescription is to consider the equations such as Eq. 16 as secondary constraints. In order to get the multiplying function in an equation, the time derivative is taken in Eq. 16:

$$\dot{\eta}(t, q) = \frac{\partial \eta(t, q, p)}{\partial t} + \frac{\partial \eta(t, q, p)}{\partial q} \dot{q} + \frac{\partial \eta(t, q, p)}{\partial p} \dot{p} = 0 \quad (17)$$

Taking into account the Hamilton equations Eq. 9, as for the primary constraints, Eq. 17 becomes, after some calculations:

$$\frac{\partial^2 g(t, q)}{\partial t^2} + \frac{\partial^2 h(t, q)}{\partial t \partial q} + f(t, q) \left(\frac{\partial^2 g(t, q)}{\partial q \partial t} + \frac{\partial^2 h(t, q)}{\partial q^2} \right) = 0 \quad (18)$$

Note in Eq. 18 that just an equation is provided for two unknown variables, $g(t, q)$ and $h(t, q)$, then one of the two variables become undetermined.

Now, with all the formalism background developed, the $g_j(t, \mathbf{q})$ ($j = 1, 2, \dots, n$) and $h(t, \mathbf{q})$ functions can be found, and the Hamiltonian can be written from Eqs. 7 and 8, as:

$$\begin{aligned} H(t, \mathbf{q}, \mathbf{p}) &= \sum_{j=1}^n f_j(t, \mathbf{q}) (p_j - g_j(t, \mathbf{q})) + h(t, \mathbf{q}) = \\ &= \sum_{j=1}^n f_j(t, \mathbf{q}) \cdot p_j - \sum_{j=1}^n f_j(t, \mathbf{q}) \cdot g_j(t, \mathbf{q}) + h(t, \mathbf{q}) \end{aligned} \quad (19)$$

Observe in addition that, if the dynamical model Eq. 1 is autonomous, thus $f_k(t, \mathbf{q}) = f_k(\mathbf{q})$ and both $g_j(t, \mathbf{q}) = g_j(\mathbf{q})$ ($j = 1, 2, \dots, n$) and $h(t, \mathbf{q}) = h(\mathbf{q})$ can be found as time independent functions, thus the Hamiltonian $H(t, \mathbf{q}, \mathbf{p}) = H(\mathbf{q}, \mathbf{p})$ and it is a constant of the dynamics, then it can be identified with the system energy E , that is:

$$E = \sum_{j=1}^n f_j(\mathbf{q}) (p_j - g_j(\mathbf{q})) + h(\mathbf{q}) = \sum_{j=1}^n f_j(\mathbf{q}) \cdot p_j - \sum_{j=1}^n f_j(\mathbf{q}) \cdot g_j(\mathbf{q}) + h(\mathbf{q}) \quad (20)$$

3 A proposal of a General System Thermodynamics (GST)

A first theoretical application of the Hamiltonian formalism developed is to state a general or abstract theory of systems, called here as a System General Thermodynamics (GST), by introducing some new postulates. The **First Postulate** is: **identify the Hamiltonian of Eq. 19 as the system Internal Energy**. The Hamilton equations will correspond to the state equations. Note that time is indispensable in this formalism, unlikely to the classical quasi-static evolution of Thermodynamics [9]. The **Second Postulate** is: **assume the two following equations**:

$$g_k(t, \mathbf{q}) = \frac{\partial \chi(t, \mathbf{q})}{\partial q_k}; \quad k = 1, 2, \dots, n \quad (21)$$

$$h(t, \mathbf{q}) = -\frac{\partial \chi(t, \mathbf{q})}{\partial t} \quad (22)$$

From Eqs. 21 and 22, Eq. 14 hold identically (even for the case $n = 1$ given by Eq. 16), being $\chi(t, \mathbf{q})$ an arbitrary function, and Eq. 15 and 18 are unnecessary to compute $g_k(t, \mathbf{q})$ and $h(t, \mathbf{q})$. Therefore, the primary constraints of Eq. 7 become:

$$\bar{\phi}_j(t, \mathbf{q}, \mathbf{p}) = p_j - \frac{\partial \chi(t, \mathbf{q})}{\partial q_j} = 0; \quad j = 1, 2, \dots, n \quad (23)$$

and the Hamiltonian (Eq. 19) becomes:

$$H(t, \mathbf{q}, \mathbf{p}) \sum_{j=1}^n f_j(t, \mathbf{q}) \cdot \bar{\phi}_j(t, \mathbf{q}, \mathbf{p}) - \frac{\partial \chi(t, q)}{\partial t} = \sum_{j=1}^n f_j(t, \mathbf{q}) \cdot p_j - \sum_{j=1}^n \frac{\partial \chi(t, \mathbf{q})}{\partial q_j} f_j(t, \mathbf{q}) - \frac{\partial \chi(t, q)}{\partial t} \quad (24)$$

The corresponding Hamilton equations to Eq. 24 are:

$$\left. \begin{aligned} \dot{q}_k &= \frac{\partial H(t, \mathbf{q}, \mathbf{p})}{\partial p_k} = f_k(t, q) \\ p_k &= -\frac{\partial H(t, \mathbf{q}, \mathbf{p})}{\partial q_k} = \sum_{j=1}^n f_j(t, \mathbf{q}) \frac{\partial^2 \chi(t, \mathbf{q})}{\partial q_k \partial q_j} + \frac{\partial^2 \chi(t, \mathbf{q})}{\partial q_k \partial t}; \end{aligned} \right\}; \quad k = 1, 2, \dots, n \quad (25)$$

and that the time derivative of the Hamiltonian is:

$$\frac{dH(t, \mathbf{q}, \mathbf{p})}{dt} = \frac{\partial H(t, \mathbf{q}, \mathbf{p})}{\partial t} = \frac{\partial^2 \chi(t, \mathbf{q})}{\partial t^2} - \sum_{k=1}^n f_k(t, \mathbf{q}) \frac{\partial^2 \chi(t, \mathbf{q})}{\partial q_k \partial t} \quad (26)$$

Note in Eq. 26, as it is well-known, that the fact that the Hamiltonian total time derivative be equal to its partial time derivative is a general property of the Hamiltonian systems. On the other hand, $\chi(t, \mathbf{q})$ can be fixed by the Third Postulate, which introduces in the formalism the generalized temperature $T(t, q)$ and the generalized Entropy and $S(t, q)$ in the Hamiltonian (Eq. 24) as:

$$\sum_{j=1}^n \frac{\partial \chi(t, \mathbf{q})}{\partial q_j} f_j(t, \mathbf{q}) + \frac{\partial \chi(t, \mathbf{q})}{\partial t} = -T(t, \mathbf{q}) \cdot S(t, q) \quad (27)$$

Then, this Hamiltonian of Eq. 24 can also be written as:

$$H(t, \mathbf{q}, \mathbf{p}) \sum_{j=1}^n f_j(t, \mathbf{q}) \cdot p_j + T(t, \mathbf{q}) \cdot S(t, \mathbf{q}) \quad (28)$$

that mathematically looks like much more to the Internal Energy of the classical Thermodynamics. The **Fourth Postulate** is a **generalized Gibbs-Duhem equation** [9], written as:

$$\sum_{k=1}^n f_k(t, \mathbf{q}) \cdot dp_k + S(t, \mathbf{q}) \cdot dT(t, \mathbf{q}) = 0 \quad (29)$$

Dividing Eq. 29 by dt, developing the total time derivative of $T(t, \mathbf{q})$ and making use subsequently of Eq. 25, the generalized Gibbs-Duhem equation becomes:

$$S(t, \mathbf{q}) \frac{\partial T(t, \mathbf{q})}{\partial t} + S(t, \mathbf{q}) \sum_{k=1}^n f_k(t, \mathbf{q}) \frac{\partial T(t, \mathbf{q})}{\partial q_k} = - \sum_{k=1}^n f_k(t, \mathbf{q}) \left(\sum_{j=1}^n f_j(t, \mathbf{q}) \frac{\partial^2 \chi(t, \mathbf{q})}{\partial q_k \partial q_j} + \frac{\partial^2 \chi(t, \mathbf{q})}{\partial q_k \partial t} \right) \quad (30)$$

Taking the differential of $H(t, \mathbf{q}, \mathbf{p})$ in Eq. 28 and considering the generalized Gibbs-Duhem equation Eq. 29:

$$dH(t, \mathbf{q}, \mathbf{p}) \sum_{k=1}^n dp_k \cdot df_k(t, \mathbf{q}) + T(t, \mathbf{q}) \cdot dS(t, \mathbf{q}) \quad (31)$$

Dividing Eq. 31 by dt, developing the total time derivative of $S(t, \mathbf{q})$ and making use subsequently of Eqs. 23, 25 and 26, Eq. 31 becomes:

$$T(t, \mathbf{q}) \frac{\partial S(t, \mathbf{q})}{\partial t} + T(t, \mathbf{q}) \sum_{k=1}^n f_k(t, \mathbf{q}) \frac{\partial S(t, \mathbf{q})}{\partial q_k} =$$

$$= -\frac{\partial^2 \chi(t, \mathbf{q})}{\partial t^2} - \sum_{k=1}^n f_k(t, \mathbf{q}) \frac{\partial^2 \chi(t, \mathbf{q})}{\partial q_k \partial t} - \sum_{k=1}^n \frac{\partial \chi(t, \mathbf{q})}{\partial q_k} \left(\frac{\partial f_k(t, \mathbf{q})}{\partial t} + \sum_{j=1}^n \frac{\partial f_k(t, \mathbf{q})}{\partial q_j} f_j(t, \mathbf{q}) \right) \quad (32)$$

Eqs. (27), (30) and (32) define a system of three partial differential equations for $\chi(t, \mathbf{q})$, $T(t, \mathbf{q})$ and $S(t, \mathbf{q})$. However, the $\chi(t, \mathbf{q})$ function can be uncoupled by taking the total time derivative of $T(t, \mathbf{q}) \cdot S(t, \mathbf{q})$ in Eq. 27, developing it by its partial derivatives, and comparing the result with Eqs. 30 and 32. This process provides the following equation for the $\chi(t, \mathbf{q})$ function:

$$\sum_{k=1}^n f_k(t, \mathbf{q}) \frac{\partial^2 \chi(t, \mathbf{q})}{\partial q_k \partial t} = 0 \quad (33)$$

Eq. 33 allows computing the $\chi(t, \mathbf{q})$ function, and at once simplifying Eqs. 30 and 32:

$$S(t, \mathbf{q}) \frac{\partial T(t, \mathbf{q})}{\partial t} + S(t, \mathbf{q}) \sum_{k=1}^n f_k(t, \mathbf{q}) \frac{\partial T(t, \mathbf{q})}{\partial q_k} = - \sum_{k=1}^n f_k(t, \mathbf{q}) \left(\sum_{j=1}^n f_j(t, \mathbf{q}) \frac{\partial^2 \chi(t, \mathbf{q})}{\partial q_k \partial q_j} \right) \quad (34)$$

$$\begin{aligned} T(t, \mathbf{q}) \frac{\partial S(t, \mathbf{q})}{\partial t} + T(t, \mathbf{q}) \sum_{k=1}^n f_k(t, \mathbf{q}) \frac{\partial S(t, \mathbf{q})}{\partial q_k} = \\ = -\frac{\partial^2 \chi(t, \mathbf{q})}{\partial t^2} - \sum_{k=1}^n \frac{\partial \chi(t, \mathbf{q})}{\partial q_k} \left(\frac{\partial f_k(t, \mathbf{q})}{\partial t} + \sum_{j=1}^n \frac{\partial f_k(t, \mathbf{q})}{\partial q_j} f_j(t, \mathbf{q}) \right) \end{aligned} \quad (35)$$

In conclusion, Eqs. (33), (34) and (35) are the base to compute the $\chi(t, \mathbf{q})$, $T(t, \mathbf{q})$ and $S(t, \mathbf{q})$ functions.

4 Reversible Thermodynamics is a particular case of the GST?

In order to answer the question of this section, the non-explicit time dependence of Hamiltonian Eq. 28 is assumed. Therefore: (a) the system Eq. 1 is autonomous ($f_k(t, \mathbf{q}) = f_k(\mathbf{q}); k = 1, 2, \dots, n$); (b) Temperature holds $T = T(\mathbf{q})$ and Entropy holds $S = S(\mathbf{q})$. In addition, it is assumed that there exists at least one steady state $\mathbf{q}e = (qe_1, qe_2, \dots, qe_n)$, such that $f_k(\mathbf{q}e) = 0; k = 1, 2, \dots, n$. Expanding at first order the $f_k(\mathbf{q})$ functions and substituting them in the Hamiltonian Eq. 28:

$$\begin{aligned} H(\mathbf{q}, \mathbf{p}) \sum_{j=1}^n \left(\sum_{l=1}^n \nu_{jl} (q_l - qe_l) \right) \cdot p_j + T(\mathbf{q}) \cdot S(\mathbf{q}) = \\ = \sum_{l=1}^n \left(\sum_{j=1}^n \nu_{jl} \cdot p_j \right) (q_l - qe_l) + T(\mathbf{q}) \cdot S(\mathbf{q}) \end{aligned} \quad (36)$$

In Eq. 36: $\nu_{jl} = \frac{\partial f_j}{\partial q_l}(\mathbf{q}e)$. If the following canonical transformation is done:

$$\left. \begin{aligned} Q_l &= q_l - qe_l \\ P_l &= \sum_{j=1}^n \nu_{jl} \cdot p_j \end{aligned} \right\}; \quad l = 1, 2, \dots, n \quad (37)$$

Then, the Hamiltonian Eq. 36 becomes:

$$H(\mathbf{Q}, \mathbf{P}) = \sum_{l=1}^n P_l \cdot Q_l + T(\mathbf{Q}) \cdot S(\mathbf{Q}) \quad (38)$$

Eq. 38 is then similar to that corresponding to a reversible thermodynamics.

5 Conclusions

Note that all the formalism presented is an attempt to develop a complete classical *analytical dynamics* or *mechanics* for dynamical systems, here represented as coupled first order differential equation systems.

On the one hand, the formalism is called classical versus the quantum possible development from the Hamiltonian. In fact, a first attempt to bring the formalism to the quantum context is done in [7]. On the other hand, it is a first attempt because more background can be developed, such as the Hamilton-Jacobi equation, or the corresponding canonical transformations.

It is important to emphasize that the formalism is completely open to solve many theoretical problems as well as its applications. For instance, about the relationship with the physical formalism, already faced by Havas in [2] from a Lagrangian perspective, in which the energy conservation should be also faced from a Hamiltonian perspective. This is important because a second order differential equation formalism can be reduced to a first order formalism by introducing the velocities as new variables. The same happens with the relationship of both formalisms in the quantum context [7]: the probability is conserved but the Hamilton-Jacobi equation does not present any stochastic further term.

Another problem faced from the presented formalism is, as pointed out already in [6], that the Hamiltonian can be reinterpreted as a nonlinear version of the Thermodynamics Internal Energy, expressed by the second Thermodynamics postulate, identifying the Internal Energy with the Hamiltonian. In fact this formalism goes beyond, because from the Hamiltonian function, an attempt to develop a General System Thermodynamics (GST) is presented. Observe that the GST presented is nonlinear and a comparing with the classical Thermodynamics referred to quasi-static or reversible systems. Thus, the irreversible systems [9] should be described by the GST. Note that, at this point, Section 4 shows how a Hamiltonian that does not depend explicitly on time in the linear context about a steady state reproduces the classical reversible Internal Energy. However, both linear and nonlinear general approaches should reproduce for Eqs. 34 and 35 the third Thermodynamics postulate, i.e., $T(t, \mathbf{q}) > 0$, while any sign of the Entropy time derivative ($\dot{S}(t, \mathbf{q}) > 0, \dot{S}(t, \mathbf{q}) = 0$ or $\dot{S}(t, \mathbf{q}) < 0$) should inform us whether the system is tending to order or to disorder. The most interesting application in this context is that related with the dynamics of the chemical reactions, due to it is modelled with coupled first order differential equation systems [9]. It could be a way to state the wished objective of physics to unify dynamics and thermodynamics.

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