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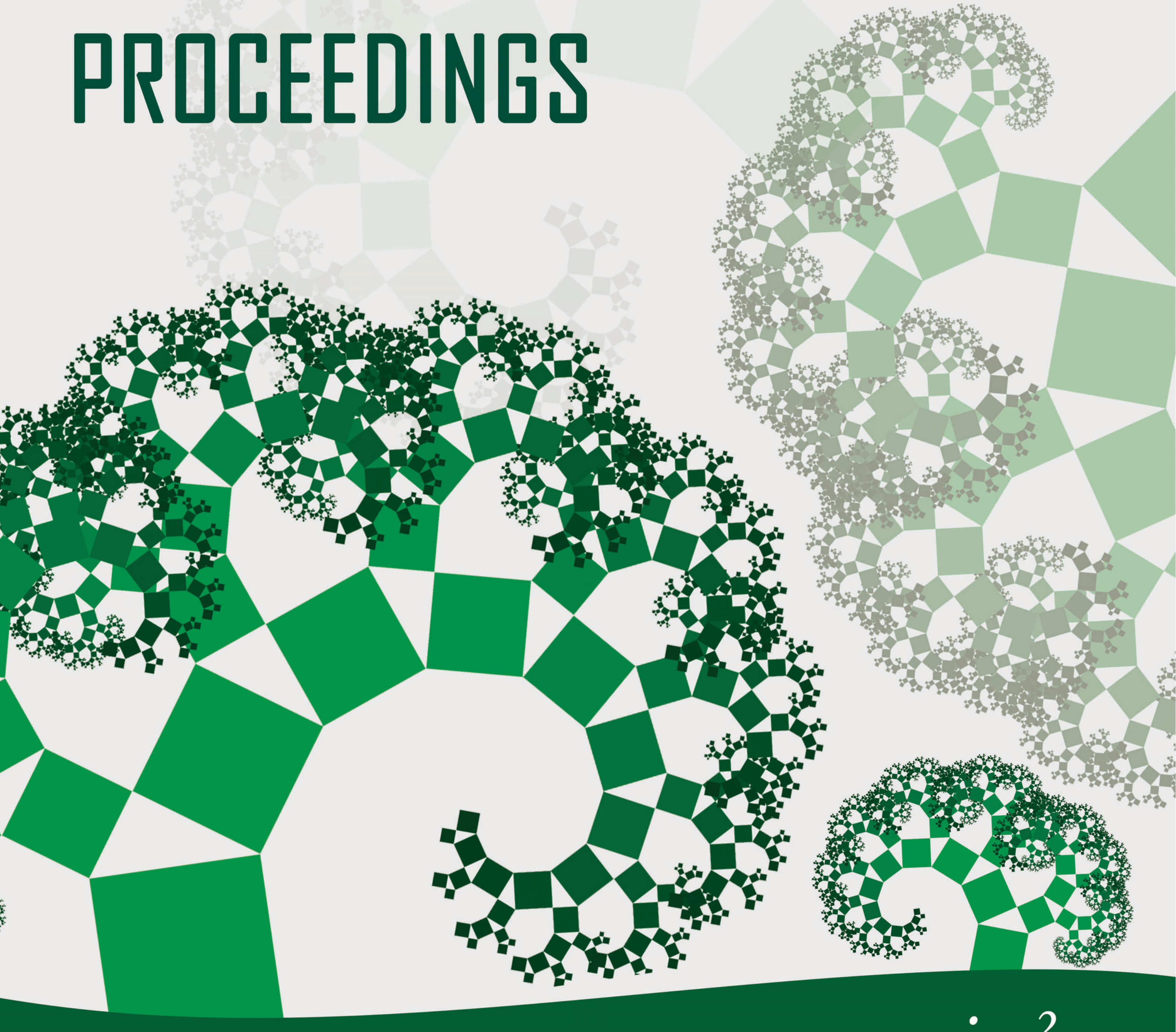

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Random Fractional Hermite Differential Equation: A full study un mean square sense

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Abstract

In this contribution a full probabilistic study for the Random Fractional Hermite differential equation is performed. Firstly, applying the random fractional Fröbenius method we will construct a solution convergent in mean square sense. Then, we will obtain reliable approximations for the mean and for the standard deviation taking into account that the solution described by a power series converges in mean square sense. After that, we will go a step further computing first probability density function of the solution. Finally, we show one numerical example to illustrate the theoretical findings.

1 Introduction

Hermite differential equation is applied in quantum physics to obtain the solution of problems which describes the dynamics of particles, as atoms protons and neutrons. An example of it is the quantum harmonic oscillator which is defined by the Schrödinger differential equation. Nevertheless, when we describe the dynamics of quantum particles, there exist non-local and memory effects which are not able to describe by classical derivatives. In the recent years, it has been observed that fractional operators can properly address this complex dynamics, [1]. In this work the classical derivative in the Hermite differential equation is reemplaced by the Caputo fractional operator.

It is also worthly mentioned that the parameters of the differential equations, forcing term, initial conditions, etc. include uncertainties which have to take into account to model accurately a real phenomena. In this contribution, besides including a fractional operator in the Hermite differential equation, we will consider that the parameters of the equation are random variables instead deterministic values.

In this work we will deal with the following Initial Value Problem (IVP)

$$({}^C D_0^{2\alpha} Y)(t) - 2t^{\alpha}({}^C D_0^{\alpha} Y) + \lambda Y(t) = 0, \quad Y(0) = Y_0, \quad Y'(0) = Y_1, \tag{1}$$

where λ, Y_0 and Y_1 are random variables and $({}^C D_0^{\alpha} Y)$ is the Caputo derivative of order $\alpha \in]0,1]$ of the stochastic process $Y(t)$ defined in mean square (m.s). The operator $({}^C D_0^{2\alpha} Y)$ is given by [2]

$$({}^C D_0^{2\alpha} Y) :={}^C D_0^{\alpha}({}^C D_0^{\alpha} Y(t))$$

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This contribution is organized as follows: Section 2 is devoted to construct a m.s. convergent solution of the IVP (1). In Section 3, we will obtain approximations for the mean and for the standard deviation of the solution. Then, in Section 4, we go an step further and applying the Random Variable Transformation Technique [3], we will obtain approximations for the first probability density function (1-PDF) of the solution Finally, the last Section, Section 5, is devoted to illustrate the theoretical findings with a numerical example.

2 Constructing mean square convergent solutions

This section is devoted to apply the random fractional Fröbenius method to compute the solution stochastic process of the RFIVP (1).

The fractional Fröbenius method allows us to obtain a solution in terms of a generalized power series,

$$Y(t) = \sum_{m=0}^{\infty} Y_m t^{\alpha m}. \tag{2}$$

Computing the coresponding Caputo fractional derivatives, $({}^C D_0^{\alpha})$ and $({}^C D_0^{2\alpha})$, of $Y(t)$, and substituting these expressions in the IVP (1), we can obtain the coefficients Y_m. Thus, the solution is given by

$$\begin{aligned} Y(t) =& Y_0 \left(1 + \sum_{m=1}^{\infty} \left[\frac{t^{2m\alpha}}{\Gamma(2\alpha m+1)} \left(\sum_{i=0}^{m-1} \lambda^{i+1}(-1)^{i+1} G_{m-1,i}\right)\right]\right) \\ &+ Y_1 \left(t^{\alpha} + \sum_{m=1}^{\infty} \left[\frac{\Gamma(\alpha+1) t^{(2m+1)\alpha}}{\Gamma((2m+1)\alpha+1)} \left(\sum_{i=0}^{m} \lambda^{i}(-1)^{i} \hat{G}_{m,i}\right)\right]\right), \end{aligned} \tag{3}$$

where

$$G_{m,i} = \begin{cases} \displaystyle\sum_{j_1<j_2<\ldots<j_{m-i}} 2\frac{\Gamma(2j_1\alpha+1)}{\Gamma((2j_1-1)\alpha+1)} \cdot 2\frac{\Gamma(2j_2\alpha+1)}{\Gamma((2j_2-1)\alpha+1)} \cdot \ldots \cdot 2\frac{\Gamma(2j_{m-i}\alpha+1)}{\Gamma((2j_{m-i}-1)\alpha+1)}, & \text{if } i<m, \\ 1, & \text{if } m=i, \\ 0, & \text{otherwise,} \end{cases} \tag{4}$$

and

$$\hat{G}_{m,i} = \begin{cases} \displaystyle\sum_{j_1<j_2<\ldots<j_{m-i}} 2\frac{\Gamma((2j_1-1)\alpha+1)}{\Gamma((2j_1-2)\alpha+1)} \cdot 2\frac{\Gamma((2j_2-1)\alpha+1)}{\Gamma((2j_2-2)\alpha+1)} \ldots 2\frac{\Gamma((2j_{m-i}-1)\alpha+1)}{\Gamma((2j_{m-i}-2)\alpha+1)}, & \text{if } i<m, \\ 1, & \text{if } m=i, \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

As it can be observed, the solution is described by a generalized random power series. Assuming the following hypotheses, we can guarantee that (3) is convergent in mean square sense.

- **H1:** The inputs parameters Y_0, Y_1 and λ are independent random variables.
- **H2:** The random variable λ is bounded random variable, i.e. exist b_1 and b_2 finite such that $b_1 < \lambda(\omega) < b_2$ for each $\omega \in \Omega$.

3 Approximating the mean and the standard deviation

Once a mean square convergent solution for the IVP (1) is obtained we compute reliable approximations for the mean and for the standard deviation of the solution.

To do it, it is important to take into account that mean square convergence of the random series (3)-(5) guarantees the convergence of the approximations of the mean and the standard deviation by truncating at a specific order, say $M > 0$ integer, the series solution, [3, Th. 4.3.1]. This leads to the following approximations for the mean and for the second order moment

$$\begin{aligned}\mathbb{E}[Y_M(t)] =&\mathbb{E}[Y_0]\left(1+\sum_{m=1}^{M}\left[\frac{t^{2m\alpha}}{\Gamma(2\alpha m+1)}\left(\sum_{i=0}^{m-1}\mathbb{E}[\lambda^{i+1}](-1)^{i+1}G_{m-1,i}\right)\right]\right)\\&+\mathbb{E}[Y_1]\left(t^{\alpha}+\sum_{m=1}^{M}\left[\frac{\Gamma(\alpha+1)t^{(2m+1)\alpha}}{\Gamma((2m+1)\alpha+1)}\left(\sum_{i=0}^{m}\mathbb{E}[\lambda^{i}](-1)^{i}\hat{G}_{m,i}\right)\right]\right)\end{aligned}\tag{6}$$

and

$$\begin{aligned}\mathbb{E}[Y_M^2(t)] =&\mathbb{E}[Y_0^2]\left(1+2\sum_{m=1}^{M}\left[\frac{t^{2\alpha m}}{\Gamma(2\alpha m+1)}\left(\sum_{i=0}^{m-1}\mathbb{E}[\lambda^{i+1}](-1)^{i+1}G_{m-1,i}\right)\right]\right.\\&+\left.\left(\sum_{m=1}^{M}\sum_{n=1}^{M}\left[\frac{t^{2\alpha(m+n)}}{\Gamma(2\alpha m+1)\Gamma(2\alpha n+1)}\left(\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\mathbb{E}[\lambda^{i+j+2}](-1)^{i+j+2}G_{m-1,i}G_{n-1,j}\right)\right]\right)\right)\\&+\mathbb{E}[Y_1^2]\left(t^{\alpha}+2t^{\alpha}\sum_{m=1}^{M}\left[\frac{\Gamma(\alpha+1)t^{(2m+1)\alpha}}{\Gamma((2m+1)\alpha+1)}\left(\sum_{i=0}^{m}\mathbb{E}[\lambda^{i}](-1)^{i}\hat{G}_{m,i}\right)\right]\right.\\&+\left.\sum_{m=1}^{M}\sum_{n=1}^{M}\left[\frac{\Gamma(\alpha+1)^2t^{(2n+2m+2)\alpha}}{\Gamma((2m+1)\alpha+1)\Gamma((2n+1)\alpha+1)}\left(\sum_{i=0}^{m}\sum_{j=0}^{n}\mathbb{E}[\lambda^{i+j}](-1)^{i+j}\hat{G}_{m,i}\hat{G}_{n,i}\right)\right]\right)\\&+2\mathbb{E}[Y_0]\mathbb{E}[Y_1]\left(t^{\alpha}+t^{\alpha}\sum_{m=1}^{M}\left[\frac{t^{2\alpha m}}{\Gamma(2\alpha m+1)}\left(\sum_{i=0}^{m-1}\mathbb{E}[\lambda^{i+1}](-1)^{i+1}G_{m-1,i}\right)\right]\right.\\&+\sum_{m=1}^{M}\left[\frac{\Gamma(\alpha+1)t^{(2m+1)\alpha}}{\Gamma((2m+1)\alpha+1)}\left(\sum_{i=0}^{m}\mathbb{E}[\lambda^{i}](-1)^{i}\hat{G}_{m,i}\right)\right]\\&+\left.\sum_{m=1}^{M}\sum_{n=1}^{M}\left[\frac{\Gamma(\alpha+1)t^{2\alpha m}t^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)\Gamma(2\alpha m+1)}\left(\sum_{i=0}^{m-1}\sum_{j=0}^{n}\mathbb{E}[\lambda^{i+j+1}](-1)^{i+j+1}G_{m-1,i}\hat{G}_{n,j}\right)\right]\right),\end{aligned}\tag{7}$$

respectively.

Taking into account (6) and, (7) and $\sigma[Y_M(t)] = \sqrt{E[Y_M(t)^2] - E[Y_M(t)]^2}$, we can compute the standard deviation.

4 Computing approximations for the 1-PDF of the solution SP

So far, approximations for the two first statistical moments have been obtained. Nevertheless, sometimes higher one-dimensional moments are also needed. The 1-PDF of the solution allows us to compute these higher moments. Applying the Random Variable Transformation technique [3, Ch. 2] to the truncated solution of order M we can construct approximations for the 1-PDF. After some involved computations, one gets

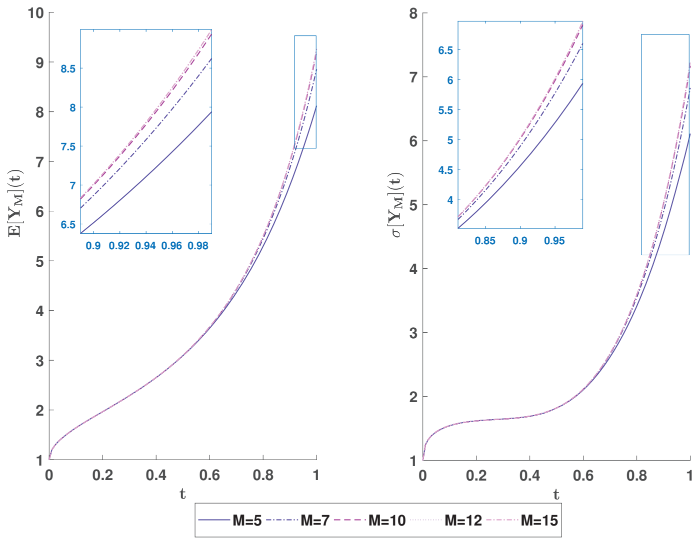

Figure 1: Mean and standard deviation of the solution for different orders of truncation M.

$$
\begin{aligned}
f_{Y_M(t)}(y) = \int_{\mathcal{D}(Y_1,\lambda)} f_{Y_0}\left(\frac{y - y_1\left(t^{\alpha} + \sum_{m=1}^{M}\left[\frac{\Gamma(\alpha+1)t^{(2m+1)\alpha}}{\Gamma((2m+1)\alpha+1)}\left(\sum_{i=0}^{m}\lambda^{i}(-1)^{i}\hat{G}_{m,i}\right)\right]\right)}{1+\sum_{m=1}^{M}\left[\frac{t^{2\alpha m}}{\Gamma(2\alpha m+1)}\left(\sum_{i=0}^{m}\lambda^{i+1}(-1)^{i+1}G_{m-1,i}\right)\right]}\right) f_{Y_1}(y_1) f_{\lambda}(\lambda) \\
\cdot \frac{1}{\left|1+\sum_{m=1}^{M}\left[\frac{t^{2\alpha m}}{\Gamma(1\alpha m+1)}\left(\sum_{i=0}^{m-1}\lambda^{i+1}(-1)^{i+1}G_{m-1,i}\right)\right]\right|} \mathrm{d}y_1 \mathrm{d}\lambda.
\end{aligned}
\tag{8}
$$

Assuming that the PDF of random variable Y_0, f_{Y_0}, is Lipschitz, it can be rigorously proved that (8) converges to the 1-PDF of the solution stochastic process, $Y(t)$, as $M \to \infty$.

5 Numerical Examples

This section is addressed to numerically illustrate the previous theoretical results. We will take $\alpha = 0.5$. We will assume that, $\lambda \sim Be(2,3)$, where $Be(2,3)$ denotes the beta distribution of parameters $(2,3)$; $Y_0 \sim Ga(1,1)$ and $Y_1 \sim N(2, \sqrt{2}^2)$, where $Ga(1,1)$ stands for the Gamma distribution of parameters $(1,1)$ and $N(2,\sqrt{2}^2)$ Gaussian distribution of parameters $(2, \sqrt{2}^2)$.

Figure 1 shows the mean and the standard deviation of the solution for different orders of truncation $M \in \{5, 7, 10, 12, 15\}$. We can observe in the zoomed area the convergence as M increases.

The 1-PDF has been plotted in Figure 2 considering different orders of truncation M. Each subplot corresponds to different $t \in \{0.25, 0.5, 0.75\}$. To verify the convergence as the order of truncation, M, increases, a zoom has been performed in each subplot.

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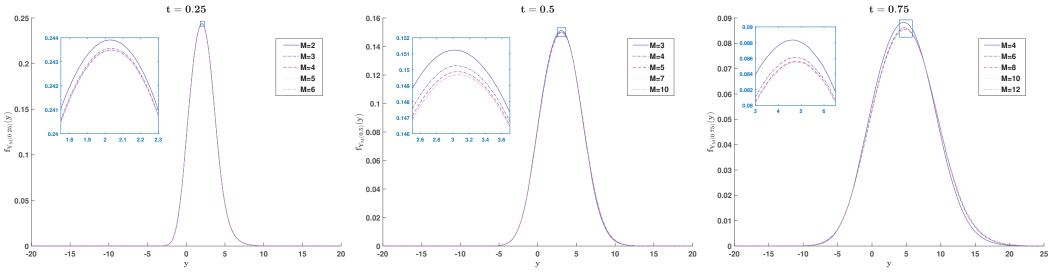

Figure 2: 1-PDF of the solution, (8), for different $t \in \{0.25, 0.5, 0.75\}$.

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