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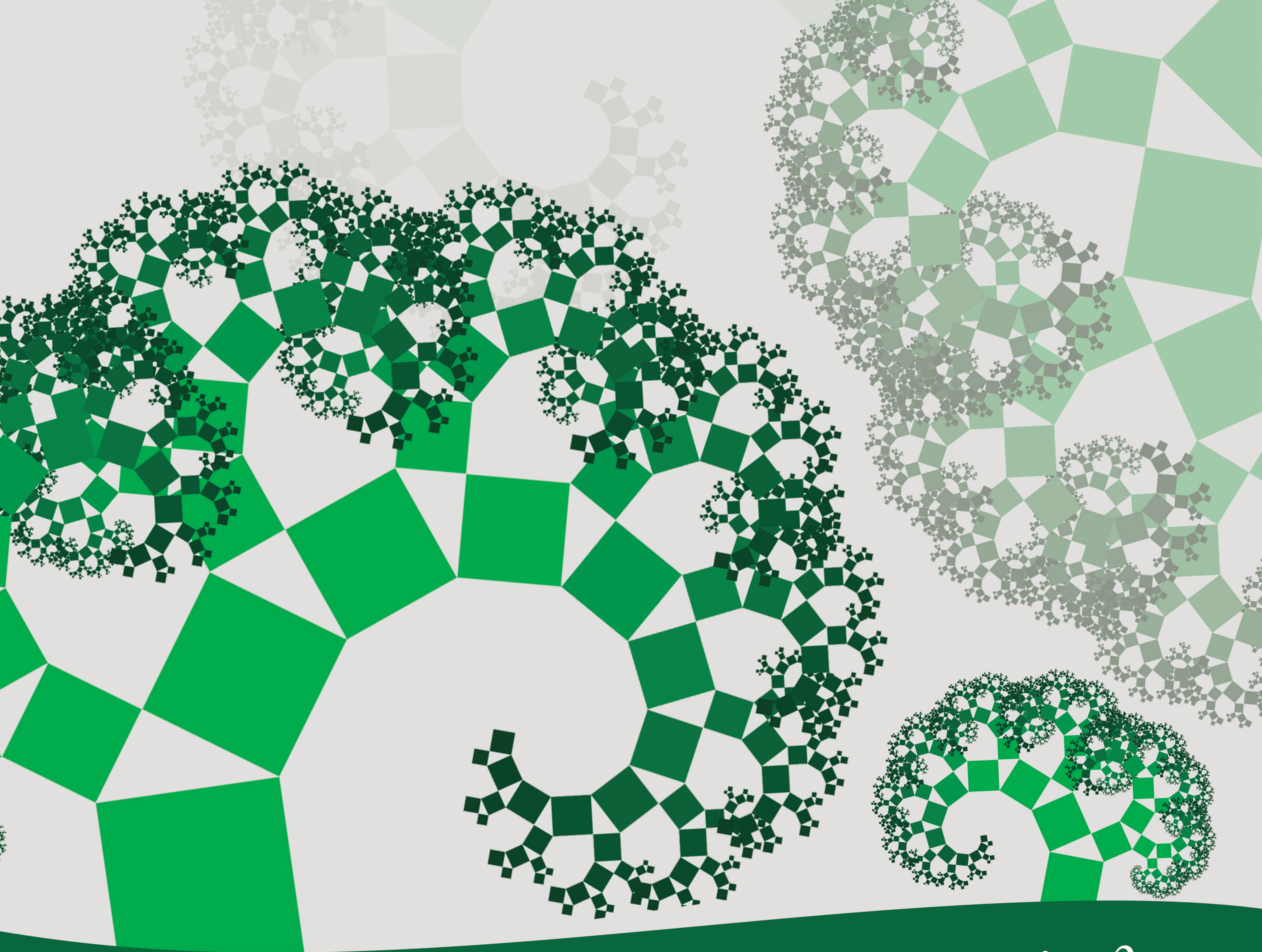

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Dynamical analysis of a new sixth-order parametric family for solving nonlinear systems of equations

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1 Introduction

A large number of problems in Computational Sciences and other disciplines can be stated in the form of a nonlinear equation or nonlinear system of equations using mathematical modelling. Finding the solution ξ of a nonlinear system of equations $F(x) = 0$ is a classical and difficult problem in Numerical Analysis, Applied Mathematics and Engineering, wherein $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is a sufficiently Frechet differentiable function in an open convex set D. We can find in [1, 2], and in the references therein, several overviews of the iterative methods for solving nonlinear systems published in the last years. The best known method for finding a solution $\xi \in D$ is Newton's scheme.

The dynamical behavior of the rational operator associated to iterative schemes for solving nonlinear systems, applied to low-degree polynomial systems, has shown to be an efficient tool for analyzing the stability and reliability of the methods, see for example [1, 3] and the references therein.

In this manuscript, we introduce a new sixth-order parametric family of multistep iterative schemes for solving nonlinear systems of equations as an extension of the family presented in [4] for solving nonlinear equations. This family is built from the Ostrowski's scheme, adding a Newton step with a "frozen" derivative and using a divided difference operator. We study its convergence, its real dynamics for stability and its numerical behavior. The dynamical planes are presented showing the complexity of the family. From the parameter spaces, presented in [4] for scalar functions, we have been able to determine different members of the family for vector functions that have bad convergence properties, since attracting periodic orbits and attracting strange fixed points appear in their dynamical planes. Moreover, this same study has allowed us to detect family members with specially stable behavior and suitable for solving practical problems. Several numerical tests are performed to illustrate the efficiency and stability of the presented family.

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2 New parametric family

The new triparametric family called MCTC(α,β,γ), object of study in this manuscript, has the following iterative expression:

$$\begin{cases} y^{(k)} = x^{(k)} - [F'(x^{(k)})]^{-1}F(x^{(k)}) \\ z^{(k)} = y^{(k)} - [2[x^{(k)},y^{(k)};F] - F'(x^{(k)})]^{-1}F(y^{(k)}) \\ x^{(k+1)} = z^{(k)} - (\alpha I + \beta u^{(k)} + \gamma v^{(k)})[F'(x^{(k)})]^{-1}F(z^{(k)}) \end{cases}, \quad (1)$$

where $u^{(k)} = I - [F'(x^{(k)})]^{-1}[x^{(k)},y^{(k)};F]$, $v^{(k)} = [x^{(k)},y^{(k)};F]^{-1}F'(x^{(k)})$, $k = 0,1,2,...$, and α, β and γ are arbitrary parameters. The divided difference operator $[x,y;F]$, defined in [5], is the map $[\cdot,\cdot;F] : D \times D \subset \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n)$, satisfying $[x,y;F](x-y) = F(x) - F(y), \forall x,y \in D$.

Theorem 13 (triparametric family). *Let $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a sufficiently differentiable function in an open convex set D and $\xi \in D$ a solution of the nonlinear system $F(x) = 0$. Let us suppose that $F'(x)$ is continuous and nonsingular at ξ, and $x^{(0)}$ is an initial estimate close enough to ξ. Then, sequence $\{x^{(k)}\}_{k\geq 0}$ obtained by using expression (1) converges to ξ with order four, being its error equation*

$$e^{(k+1)} = (1-\alpha-\gamma)\left(C_2^3 - C_3C_2\right)e^{(k)^4} + \mathcal{O}(e^{(k)^5}),$$

where $e^{(k)} = x^{(k)} - \xi$, $C_q = \frac{1}{q!}[F'(\xi)]^{-1}F^{(q)}(\xi)$ and $q = 2,3,...$

From Theorem 13, it follows that the new triparametric family has an order of convergence of four for any value of α, β and γ. However, convergence can be speed-up if only one parameter is held and the family is reduced to an uniparametric iterative scheme.

Theorem 14 (uniparametric family). *Let $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a sufficiently differentiable function in an open convex set D and $\xi \in D$ a solution of the nonlinear system $F(x) = 0$. Let us suppose that $F'(x)$ is continuous and nonsingular at ξ, and $x^{(0)}$ is an initial estimate close enough to ξ. Then, sequence $\{x^{(k)}\}_{k\geq 0}$ obtained by using expression (1) converges to ξ with order six, provided that $\beta = 1+\alpha$ and $\gamma = 1-\alpha$, being its error equation*

$$e^{(k+1)} = \left(C_3^2C_2 - C_3C_2^3 + 6C_2^5 - 6C_2^2C_3C_2\right)e^{(k)^6} + \mathcal{O}(e^{(k)^7}),$$

where $e^{(k)} = x^{(k)} - \xi$, $C_q = \frac{1}{q!}[F'(\xi)]^{-1}F^{(q)}(\xi)$ and $q = 2,3,...$

From Theorem 14, it follows that if we only hold α in (1), the triparametric family is reduced to an uniparametric family with an order of convergence of six, for any value of α, as long as $\beta = 1+\alpha$ and $\gamma = 1-\alpha$. So, the iterative expression of the new three-step uniparametric family, dependent only of α and which we will call MCTC(α) family, is defined as

$$\begin{cases} y^{(k)} = x^{(k)} - [F'(x^{(k)})]^{-1}F(x^{(k)}) \\ z^{(k)} = y^{(k)} - [2[x^{(k)},y^{(k)};F] - F'(x^{(k)})]^{-1}F(y^{(k)}) \\ x^{(k+1)} = z^{(k)} - (\alpha I + (1+\alpha)u^{(k)} + (1-\alpha)v^{(k)})[F'(x^{(k)})]^{-1}F(z^{(k)}) \end{cases}, \quad (2)$$

where $u^{(k)} = I - [F'(x^{(k)})]^{-1}[x^{(k)},y^{(k)};F]$, $v^{(k)} = [x^{(k)},y^{(k)};F]^{-1}F'(x^{(k)})$, $k = 0,1,2,...$, and α is an arbitrary parameter.

Because of the results obtained with the convergence analysis carried out, from now on we only work with MCTC(α) family of iterative methods and, to select the best members of this family, we use the real dynamics tools discussed in Section 3.

3 Real dynamics for stability

This section refers to the study of the dynamical behavior of the rational operator associated with iterative schemes of MCTC(α) family. This study give us important information about the stability and reliability of the family. We will construct dynamical planes in order to show the behavior of particular methods in terms of the basins of attraction of their fixed points, periodic points, etc.

3.1 Rational operator

The rational operator can be built on any nonlinear system; however, we construct this operator on the following low-degree nonlinear polynomial system:

$$F(x_1, x_2) = \left(x_1^2 - 1, x_2^2 - 1\right) = (0, 0)\,, \tag{3}$$

since the criterion of stability or instability of a method applied to this system can be generalized for other multidimensional cases.

Proposition 1 (rational operator R_F). Let the polynomial system $F(x_1, x_2)$ given in (3), with roots $(-1,-1)$, $(-1,1)$, $(1,-1)$, $(1,1) \in \mathbb{R}^2$. The rational operator associated with MCTC(α) family and applied on $F(x_1, x_2)$, with $\alpha \in \mathbb{R}$ an arbitrary parameter, is

$$R_F(x_1, x_2, \alpha) = \left(R_{F_{11}}, R_{F_{12}}\right), \tag{4}$$

where

$$\begin{aligned}
R_{F_{11}} &= \frac{1}{32}\left(\frac{\left(x_1^2-1\right)^4\left(\alpha+(\alpha-19)x_1^4-2(\alpha-1)x_1^2+1\right)}{4x_1^5\left(x_1^2+1\right)^2\left(3x_1^2+1\right)} + \frac{8\left(x_1^4+6x_1^2+1\right)}{x_1^3+x_1} - \frac{\alpha\left(x_2^2-1\right)^4}{x_2^3\left(x_2^2+1\right)^2}\right),\\
R_{F_{12}} &= \frac{1}{32}\left(\frac{\left(x_2^2-1\right)^4\left(\alpha+(\alpha-19)x_2^4-2(\alpha-1)x_2^2+1\right)}{4x_2^5\left(x_2^2+1\right)^2\left(3x_2^2+1\right)} + \frac{8\left(x_2^4+6x_2^2+1\right)}{x_2^3+x_2} - \frac{\alpha\left(x_1^2-1\right)^4}{x_1^3\left(x_1^2+1\right)^2}\right).
\end{aligned}$$

To simplify the rational operator R_F defined in Proposition 1, we can select a value of α that cancels terms of the expression and reduces it. It is easy to show that for $\alpha = 0$, the rational operator is simpler and there will be fewer fixed and critical points that can improve the stability of the associated method. Also, the components of this $R_F(x_1, x_2, 0)$ will be of separate variables.

3.2 Fixed points and their stability

We calculate the fixed points of the rational operator $R_F(x_1, x_2, \alpha)$ given in (4), and analyze their stability.

Proposition 2 (fixed points). The real fixed points of $R_F(x_1, x_2, \alpha)$ are the roots of the equation $R_F(x_1, x_2, \alpha) = (x_1, x_2)$. That is

$$fp_1 = (-1,-1), fp_2 = (-1,1), fp_3 = (1,-1), fp_4 = (1,1),$$

that correspond to the roots of the polynomial system $F(x_1, x_2)$ given in (3), and they are also superattracting. Other strange fixed points may appear but their components are roots of polynomials of very high degrees.

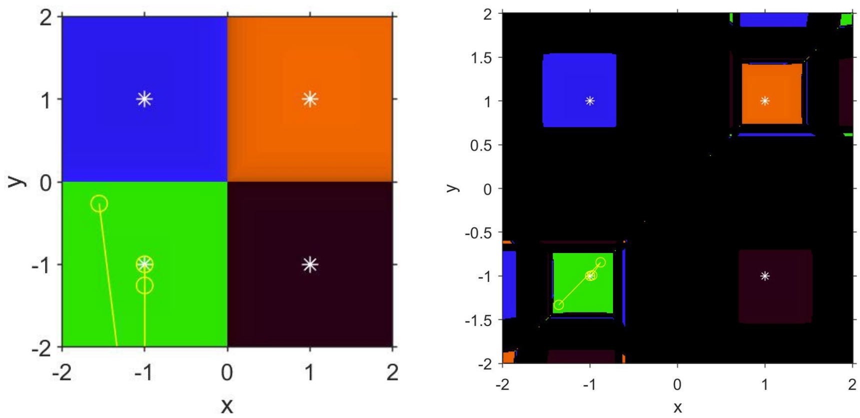
From Proposition 2, we establish there is a minimum of 4 fixed points. Of these, from fp_1 to fp_4 correspond to the roots of the original polynomial system $F(x_1, x_2)$ and are attractive and critical points.

3.3 Dynamical planes

Here, we study the stability of two MCTC(α) family methods as representatives. The first method is for $\alpha = 0$, whose value is inside the stability region of the parameter spaces shown in [4], that is, it is in the red area. The second method is for $\alpha = 200$, whose value is outside the stability region of the same parameter spaces, located in the black area.

Dynamical planes are built with a mesh from -2 to 2, with a step equal to 0.01. Every initial estimation is iterated 100 times (maximum) with a tolerance of 10^{-3}. The points in the mesh are represented based on the roots to which they converge: the color is brighter when lesser are the iterations. If all the iterations are completed and not convergence to any roots is reached, then the point is represented in black. Fixed points are illustrated with a white circle '$\bigcirc$', critical points with a white square '$\square$' and attractors with a white asterisk '$*$'. Also, the basins of attraction are depicted in different colors. The resulting graphic is made in Matlab R2020b with a resolution of 400x400 pixels.

Thus, the dynamical planes for $R_F(x_1, x_2, 0)$ and $R_F(x_1, x_2, 200)$, with some convergence orbits in yellow, are shown in Figure 1. On the one hand, the method for $\alpha = 0$ presents four basins of attraction associated with the roots. Also, there are no black areas of non-convergence to the solution. Consequently, this method shows good dynamical behavior: it is very stable. On the other hand, the method for $\alpha = 200$ presents the same four basins of attraction associated with the roots, but of reduced size, which minimizes the chances of convergence to the solution. Likewise, there are black areas of slow convergence of the method. Consequently, this method has poor dynamical behavior: it is unstable.



(a) $\alpha = 0$ and $pf = (-1, -1)$ (b) $\alpha = 200$ and $pf = (-1, -1)$

Figure 1: Dynamical planes for R_F.

4 Numerical results

In this section, we perform several numerical tests to illustrate the efficiency and stability of the presented family. We consider the same two members of MCTC(α) proposed in Section 3.3, for $\alpha = 0$ and $\alpha = 200$. These methods are applied on two nonlinear test systems, whose expressions and corresponding roots are shown in Table 1.

Table 1: Nonlinear test systems and corresponding roots.

Nonlinear test system	Roots
$F_1(x_1, x_2) = (x_1^2 - 1, x_2^2 - 1) = (0, 0)$	$\xi \approx (1, 1)^T$
$F_2(x_1, x_2) = \left(x_1^2 + x_2^2 - 1, x_1^2 - x_2^2 - \frac{1}{2}\right) = (0, 0)$	$\xi \approx \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)^T$

The calculations have been developed in Matlab R2020b programming package using variable precision arithmetics with 200 digits of mantissa. For each method, we analyze the number of iterations (iter) required to converge to the solution, so that the stopping criteria $||x^{(k+1)} - x^{(k)}|| < 10^{-100}$ or $||F(x^{(k+1)})|| < 10^{-100}$ are satisfied.

To check the theoretical order of convergence of the methods, we calculate the approximate computational order of convergence (ACOC) given in [12]. In the numerical results, if the ACOC vector inputs do not stabilize their values throughout the iterative process, it is marked as '-'; and, if any of the methods used does not reach convergence in a maximum of 50 iterations, it is marked as 'nc'.

Thus, in Table 2 we show the numerical performance of MCTC(0) for initial estimates near and far from the solution, that is, for $x^{(0)} \approx 2\xi$ and $x^{(0)} \approx 10\xi$. The results are encouraging because we can notice that MCTC(0) always converges to the solution in the two nonlinear test systems, regardless of the initial estimates used. Therefore, we verify this method is robust, according to the stability results shown in Section 3.

Table 2: Numerical performance of MCTC(0) on test problems.

System	$x^{(0)}$		$\|\|x^{(k+1)} - x^{(k)}\|\|$	$\|\|F(x^{(k+1)})\|\|$	iter	ACOC
F_1	$\approx 2\xi$	$(2, 2)^T$	6.4031e-71	8.0537e-173	5	3.1906
F_2	$\approx 2\xi$	$(1.7, 1)^T$	6.6576e-68	1.4261e-166	6	3.6518
F_1	$\approx 10\xi$	$(10, 10)^T$	6.7666e-41	3.4488e-113	10	6.4467
F_2	$\approx 10\xi$	$(9, 5)^T$	3.9e-70	5.5314e-172	12	3.4528

Now, in Table 3 we show the numerical performance of MCTC(200) for initial estimations very close to ($x^{(0)} \approx \xi$) and near to ($x^{(0)} \approx 2\xi$) the solution.

Table 3: Numerical performance of MCTC(200) on test problems.

System	$x^{(0)}$		$\|\|x^{(k+1)} - x^{(k)}\|\|$	$\|\|F(x^{(k+1)})\|\|$	iter	ACOC
F_1	$\approx \xi$	$(1.5, 1.5)^T$	8.1881e-94	1.5574e-207	4	2.5252
F_2	$\approx \xi$	$(1.3, 0.8)^T$	nc	nc	nc	nc
F_1	$\approx 2\xi$	$(2, 2)^T$	nc	nc	nc	nc
F_2	$\approx 2\xi$	$(1.7, 1)^T$	nc	nc	nc	nc

Note that the results shown in Table 3 also corroborate the stability analysis performed in Section 3. The MCTC(200) presents convergence problems even for estimates very close to the root ($x^{(0)} \approx \xi$), this method does not converge to the solution in one of two cases. Furthermore, for estimations near to the root ($x^{(0)} \approx 2\xi$), it does not converge to the solution in all cases, establishing a dependency on the initial estimates used. Therefore, the instability of this method is verified.

5 Conclusions

A highly efficient family of iterative methods MCTC(α) has been designed to solve nonlinear systems. This family proved to have an excellent numerical performance considering stable members as representatives. The method for $\alpha = 0$ proved to be robust (stable), according to the real dynamics analysis performed. The method for $\alpha = 200$ proved to be unstable, chaotic and cannot converge to the solution according to the initial estimate and the nonlinear system used. Also, the order of convergence is verified by the ACOC, which is close to 6. Numerical experiments confirm the theoretical results.

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