Weak group inverses and partial isometries in proper $*$-rings

Mengmeng Zhou,* Jianlong Chen,† Yukun Zhou,‡ Néstor Thome§

Abstract: A weak group element is introduced in a proper $*$-ring. Several equivalent conditions of weak group elements are investigated. We prove that an element is pseudo core invertible if it is both partial isometry and weak group invertible. Reverse order law and additive property of the weak group inverse are presented. Finally, under certain assumption on a, equivalent conditions of $a^{ⓦ}a^* = a^*a^{ⓦ}$ are presented by using the normality of the group invertible part of an element in its group-EP decomposition.

Key words: Weak group inverse; weak group element; pseudo core inverse; partial isometry.
AMS subject classifications: 15A09, 16W10

1 Introduction

Let R be a unitary ring with an involution. An involution $*$: $R \to R$ is an anti-isomorphism of degree 2, i.e., for any $a,\ b \in R$, $(a^*)^* = a$, $(ab)^* = b^*a^*$, and $(a+b)^* = a^* + b^*$. Let $a \in R$. If there exists $x \in R$ satisfying the following equations:

$$(1)\ axa = a,\quad (2)\ xax = x,\quad (3)\ (ax)^* = ax,\quad (4)\ (xa)^* = xa,$$

x is called the Moore-Penrose inverse [12] of a. If such an x exists, it is unique and denoted by $a^\dagger$. If x satisfies equations (1) and (3), then x is a $\{1,3\}$-inverse of a and denoted by $a^{(1,3)}$. Let $a \in R$. If there exists $x \in R$ such that the following equations hold:

$$a^{k+1}x = a^k,\quad xax = x,\quad ax = xa,\ \text{for some positive integer } k,$$

x is called the Drazin inverse [2] of a. If such an x exists, it is unique and denoted by a^D. The integer k, denoted by $\mathrm{ind}(a)$, is called the Drazin index of a if k is the smallest positive integer such that above equations hold. The element x is called the group inverse of a and denoted by $a^\#$ when $k = 1$.

We denote by $\mathbb{Z}$ the set of all integers and by $\mathbb{Z}^+$ the set of all positive integers.

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In [8], Manjunatha Prasad et al. introduced the core-EP inverse of a complex matrix. Later, Gao et al. extended this generalized inverse to rings with involution in [4]. Let $a \in R$. If there exist $x \in R$ and $k \in \mathbb{Z}^+$ such that the following equations hold:

$$xa^{k+1} = a^k, \quad ax^2 = x, \quad (ax)^* = ax,$$

then x is called the pseudo core inverse of a, where $\mathbb{Z}^+$ is the set of all positive integers. If such an x exists, it is unique and denoted by $a^{\textcircled{\scriptsize D}}$. If k is the smallest positive integer satisfies above equations, then k is called the pseudo core index of a. Also, in [4] it was proved that an element a is pseudo core invertible if and only if a is Drazin invertible and a^k is $\{1,3\}$-invertible. Moreover, the authors proved that the pseudo core index of a is equal to its Drazin index. In particular, if $k = 1$, the pseudo core inverse of a is reduced to the core inverse of a [1, 13, 16], which is denoted by $a^{\textcircled{\scriptsize \#}}$.

In [14], Wang et al. introduced the weak group inverse of a complex matrix. In [17], Zhou et al. extended this generalized inverse to proper $*$-rings and characterized it by three equations. Let a be an element in a proper $*$-ring R. If there exist $x \in R$ and $k \in \mathbb{Z}^+$ satisfying the following equations:

$$xa^{k+1} = a^k, \quad ax^2 = x, \quad (a^k)^* a^2 x = (a^k)^* a,$$

then x is called the weak group inverse of a. If such an x exists, then it is unique and denoted by $a^{\textcircled{\scriptsize w}}$. When k is the smallest positive integer such that above equations hold, the integer k is called the weak group index of a. The authors showed that an element a is Drazin invertible if a is weak group invertible and proved that the weak group index of a is equal to the Drazin index of a. It is also known (see [17]) that an element a is weak group invertible if a is pseudo core invertible. The converse is not right (see [17, Example 4.5]). When $k = 1$, the weak group inverse is the group inverse.

Recently, Wang et al. [15] introduced the concept of the weak group matrix and presented some equivalent characterizations. Let A be a complex square matrix of order n with $\mathrm{ind}(A) = k$. If $AA^{\textcircled{\scriptsize w}} = A^{\textcircled{\scriptsize w}}A$, then A is called a weak group matrix. In [9], Mosić et al. investigated partial isometries and EP elements. Let $a \in R$. If a is Moore-Penrose invertible and $a^* = a^\dagger$, then a is called partial isometry. It is well known that $a^* = a^\dagger$ if and only if $aa^*a = a$. If a is both Moore-Penrose invertible and group invertible and $a^\dagger = a^\#$, then a is called an EP element. In [10], Mosić et al. further studied the relationship between partial isometry and EP element in rings with involution.

Motivated by above discussion, we define the weak group element in a proper $*$-ring. Some properties of the weak group element are obtained. In addition, we investigate relationships between weak group inverses and partial isometries.

The paper is organized as follows. In Section 2, the definition of a proper $*$-ring and some lemmas are presented. In Section 3, weak group elements are introduced. Some equivalent conditions of weak group elements, which are related to the Drazin inverse and $\{1,3\}$-inverses, are studied. In Section 4, relationships between partial isometries, pseudo core inverses and weak group inverses are given. In Section 5, reverse order law and additive property of the weak group inverse are obtained. In Section 6, under certain assumption on a, equivalent conditions of $a^{\textcircled{\scriptsize w}}a^* = a^*a^{\textcircled{\scriptsize w}}$ are presented by using group-EP decomposition and EP elements.

2 Preliminaries

In this section, some auxiliary definition and lemmas are given.

Definition 2.1. [6] *An element $a \in R$ is called left $*$-cancellable if $a^*ax = a^*ay$ implies $ax = ay$, it is called right $*$-cancellable if $xaa^* = yaa^*$ implies $xa = ya$. An element is called $*$-cancellable if it is both left and right $*$-cancellable. Moreover, R is called a proper $*$-ring if every element in R is $*$-cancellable.*

Throughout this paper, R is restricted to be a proper (unital) $*$-ring. The symbols $R^{(1,3)}$, $R^{\dagger}$, R^{D}, $R^{\#}$, R^{PI}, $R^{\textcircled{\#}}$, $R^{\textcircled{D}}$, and $R^{\textcircled{W}}$ denote the sets of all $\{1,3\}$-invertible elements, Moore-Penrose invertible elements, Drazin invertible elements, group invertible elements, partial isometries, core invertible elements, pseudo core invertible elements, and weak group invertible elements, respectively, in the ring R.

Lemma 2.2. [5] *(Core-EP decomposition) Let $a \in R$. Then $a \in R^{\textcircled{D}}$ if and only if $a = a_1 + a_2$, for unique elements $a_1,\ a_2 \in R$ such that following conditions hold:*

(i) $a_1 \in R^{\textcircled{\#}}$;

(ii) $a_2^k = 0$ *for some* $k \in \mathbb{Z}^+$;

(iii) $a_1^* a_2 = a_2 a_1 = 0$.

In this case, $a^{\textcircled{D}} = a_1^{\textcircled{\#}}$ and $(a^{\textcircled{D}})^2 a = a_1^{\#}$. This decomposition of a is called the core-EP decomposition of a.

Lemma 2.3. [17] *(Group-EP decomposition) Let $a \in R$. Then $a \in R^{\textcircled{W}}$ if and only if $a = a_1 + a_2$, for unique elements $a_1,\ a_2 \in R$ such that following conditions hold:*

(i) $a_1 \in R^{\#}$;

(ii) $a_2^k = 0$ *for some* $k \in \mathbb{Z}^+$;

(iii) $a_1^* a_2 = a_2 a_1 = 0$.

In this case, $a^{\textcircled{W}} = a_1^{\#}$. This decomposition of a is called the group-EP decomposition of a.

Corollary 2.4. *If $a \in R^{\textcircled{D}}$, then $a^{\textcircled{W}} = (a^{\textcircled{D}})^2 a$.*

Proof. By Lemma 2.2 and Lemma 2.3, we know that the core-EP decomposition is the group-EP decomposition with $a^{\textcircled{W}} = a_1^{\#} = (a^{\textcircled{D}})^2 a$. □

Lemma 2.5. [17] *Let $a \in R^{\textcircled{W}}$. Then the following conditions are equivalent:*

(i) $aa^{\textcircled{W}} = a^{\textcircled{W}} a$;

(ii) $a^{\textcircled{W}} = a^{D}$;

(iii) $a_1 a_2 = 0$, *where $a = a_1 + a_2$ is the group-EP decomposition of a.*

Lemma 2.6. [4, 18] *Let $a \in R$. If there exist $x \in R$ and $k \in \mathbb{Z}^+$ such that $xa^{k+1} = a^k$ and $ax^2 = x$, then*

(i) $ax = a^m x^m$ *for arbitrary* $m \in \mathbb{Z}^+$*;*

(ii) $xax = x$*;*

(iii) $axa^m = a^m$ *for any* $m \geq k$*;*

(iv) $a \in R^D$, $a^D = x^{k+1}a^k$ *with* $\mathrm{ind}(a) \leq k$*;*

(v) $a^D ax = x$*;*

(vi) $xaa^D = a^D$.

Lemma 2.7. [4] *Let $a \in R$ and let $k,\ m \in \mathbb{Z}^+$ with $k \geq m$. Then $a \in R^{\textcircled{D}}$ with the pseudo core index of a equals m if and only if $a \in R^D$ with the Drazin index of a equals m and $a^k \in R^{(1,3)}$. In this case, $a^{\textcircled{D}} = a^D a^k (a^k)^{(1,3)}$.*

Lemma 2.8. [7] *Let $a \in R$. Then $a \in R^D$ if and only if there exists $k \in \mathbb{Z}^+$ such that $a^k \in R^\#$. In this case, $(a^k)^\# = (a^D)^k$ for $k \geq \mathrm{ind}(a)$.*

Lemma 2.9. [3] *Let $a_1, a_2 \in R^D$ and $x \in R$. If $a_1 x = x a_2$, then $a_1^D x = x a_2^D$.*

Lemma 2.10. [18] *Let $a \in R$. Then x is a weak group inverse of a if and only if there exist $x \in R$ and $k \in \mathbb{Z}^+$ such that*

$$xa^{k+1} = a^k, \quad ax^2 = x, \quad (a^*a^2x)^* = a^*a^2x.$$

3 Weak group elements

We start this section by introducing the concept of weak group element. The weak group element is a natural extension of the weak group matrix defined in [15]. Secondly, some properties about the weak group element, which are related to the Drazin inverse, are studied. Finally, characterizations of weak group elements, which are related to $\{1,3\}$-inverses, are given.

Definition 3.1. *Let $a \in R^{\textcircled{W}}$. An element $a \in R$ is called a weak group element if $aa^{\textcircled{W}} = a^{\textcircled{W}}a$.*

Remark 3.2. *We notice that, in general, $aa^{\textcircled{W}} \neq a^{\textcircled{W}}a$ as we can see in References [14, 15] even in the matrix environment.*

Let $a \in R^{\textcircled{W}}$ and $m \in \mathbb{Z}^+$. By the definition of m-weak group inverse, the authors proved (see [18, Corollary 4.15]) that a is a weak group element if and only if $a^m a^{\textcircled{W}} = a^{\textcircled{W}} a^m$. For completeness, we prove this result by the group-EP decomposition in the following theorem.

Theorem 3.3. *Let $a \in R^{\textcircled{W}}$ and $m \in \mathbb{Z}^+$. Then the following conditions are equivalent:*

(i) *a is a weak group element;*

(ii) $a^m a^{ⓦ} = a^{ⓦ} a^m$;

(iii) $a^{ⓦ} a^m a^D = a^m a^D a^{ⓦ}$.

Proof. (i) ⇒ (ii) and (i) ⇒ (iii) are obvious by Lemma 2.5.

(ii) ⇒ (i) : Suppose that $a \in R^{ⓦ}$ with $\mathrm{ind}(a) = k$. Then a has a group-EP decomposition, i.e., $a = a_1 + a_2$, $a^{ⓦ} = a_1^{\#}$, $a_2^k = 0$, and $a_1^* a_2 = a_2 a_1 = 0$. When $k = 1$, we have $a^{ⓦ} = a^{\#}$. It is trivial. So, we take $k \geq 2$. Since

$$a^m a^{ⓦ} = (a_1 + a_2)^m a_1^{\#} = (a_1^m + \sum_{i=0}^{m-1} a_1^i a_2^{m-i}) a_1 (a_1^{\#})^2 = a_1^m a_1^{\#}$$

and

$$a^{ⓦ} a^m = a_1^{\#} (a_1 + a_2)^m = a_1^{\#} \sum_{i=0}^{m} a_1^i a_2^{m-i} = a_1^{\#} a_1^m + a_1^{\#} \sum_{i=0}^{m-1} a_1^i a_2^{m-i},$$

we have $a^m a^{ⓦ} = a^{ⓦ} a^m \Leftrightarrow a_1^{\#} \sum_{i=0}^{m-1} a_1^i a_2^{m-i} = 0 \Leftrightarrow \sum_{i=0}^{m-1} a_1^{i+1} a_2^{m-i} = 0 \Leftrightarrow \sum_{i=1}^{m} a_1^i a_2^{m+1-i} = 0$. From the equation

$$a_1 a_2^m + a_1^2 a_2^{m-1} + \cdots + a_1^{m-1} a_2^2 + a_1^m a_2 = 0, \tag{3.1}$$

we have the following cases:

(a): If $m = 1$, then $a_1 a_2 = 0$.

(b): Let $m = k-1$. We know that $a_2^{m+1} = 0$. Multiplying by a_2^{m-1} on the right of the equation (3.1), we obtain that $a_1^m a_2^m = 0$. Since a_1 is group invertible, we have $a_1 a_2^m = 0$. Multiplying by a_2^{m-2} on the right of the equation (3.1), we get that $a_1^m a_2^{m-1} = 0$. That is, $a_1 a_2^{m-1} = 0$. In the same way, we have $a_1 a_2^{m-2} = 0, \cdots, a_1 a_2 = 0$.

(c): If $1 < m < k-1$, multiplying by a_2^{k-2} on the right of the equation (3.1), we have $a_1^m a_2^{k-1} = 0$. Since a_1 is group invertible, we get that $a_1 a_2^{k-1} = 0$. In the same way, we obtain that $a_1 a_2^{k-2} = 0, \cdots, a_1 a_2 = 0$.

(d): If $m \geq k$, we have $a_2^m = 0$. Multiplying a_2^{m-2} on the right of the equation (3.1), we obtain that

$$a_1^2 a_2^{2m-3} + \cdots + a_1^{m-1} a_2^m + a_1^m a_2^{m-1} = 0.$$

Hence, $a_1^m a_2^{m-1} = 0$, i.e., $a_1 a_2^{m-1} = 0$. In the same way, we have $a_1 a_2^{m-2} = 0, \cdots, a_1 a_2 = 0$. By combining above four cases with Lemma 2.5, we obtain that $a a^{ⓦ} = a^{ⓦ} a$. By Definition 3.1, a is a weak group element.

(iii) ⇒ (i) : Assume that $a \in R^{ⓦ}$ with $\mathrm{ind}(a) = k$. Since

$$\begin{aligned} a^{ⓦ} a^D &= a^{ⓦ} a^D a^m (a^D)^m = a^{ⓦ} a^{k+1} (a^D)^{k+1} a^{m-1} (a^D)^m \\ &= a^k (a^D)^{k+1} a^{m-1} (a^D)^m = a^m (a^D)^{m+2} = (a^D)^2 \end{aligned}$$

and

$$\begin{aligned} a^D a^{ⓦ} &= (a^D)^m a^m a^D a^{ⓦ} = (a^D)^m a^{ⓦ} a^m a^D = (a^D)^m a^{ⓦ} a^{k+1} (a^D)^{k+1} a^{m-1} \\ &= (a^D)^m a^k (a^D)^{k+1} a^{m-1} = (a^D)^{m+1} a^{m-1} = (a^D)^2, \end{aligned}$$

we arrive at $a^D a^{\textcircled{W}} = a^{\textcircled{W}} a^D$. We know that $a^D \in R^{\#}$ and $(a^D)^{\#} = a^2 a^D$. Then, by Lemma 2.9, we have $a^{\textcircled{W}} a^D = a^D a^{\textcircled{W}} \Rightarrow a^{\textcircled{W}} (a^D)^{\#} = (a^D)^{\#} a^{\textcircled{W}} \Rightarrow a^{\textcircled{W}} a^2 a^D = a^2 a^D a^{\textcircled{W}}$. So,

$$\begin{aligned} a^{\textcircled{W}}(a^2a^D)^k &= a^2a^Da^{\textcircled{W}}(a^2a^D)^{k-1} = (a^2a^D)^2a^{\textcircled{W}}(a^2a^D)^{k-2} \\ &= \cdots = (a^2a^D)^{k-1}a^{\textcircled{W}}a^2a^D = (a^2a^D)^k a^{\textcircled{W}}. \end{aligned}$$

Since

$$a^{\textcircled{W}} a^k = a^{\textcircled{W}} a^{k+1} a^D = a^{\textcircled{W}} a^k a a^D = a^{\textcircled{W}} (a^2 a^D)^k$$

and

$$a^k a^{\textcircled{W}} = a^{k+1} a^D a^{\textcircled{W}} = a^k a a^D a^{\textcircled{W}} = (a^2 a^D)^k a^{\textcircled{W}},$$

we obtain $a^{\textcircled{W}} a^k = a^k a^{\textcircled{W}}$. By equivalence of conditions (i) and (ii), we have $a^{\textcircled{W}} a = a a^{\textcircled{W}}$. That is, a is a weak group element. □

Next lemma gives properties for the condition $xa^{k+1} = a^k$ in the definition of weak group inverse of a.

Lemma 3.4. *Let $a \in R^{\textcircled{W}}$ with $\mathrm{ind}(a) = k$. Then the following conditions hold:*

(i) $a^{\textcircled{W}} a^k = a^D a^k$;

(ii) $(a^D)^k = a^{\textcircled{W}} (a^D)^{k-1}$.

Proof. Since $a \in R^{\textcircled{W}}$ with $\mathrm{ind}(a) = k$, by the definition of the weak group inverse and Lemma 2.6, we obtain that $a \in R^D$ and $a^D = (a^{\textcircled{W}})^{k+1} a^k$.

(i) : By the definition of $a^{\textcircled{W}}$, we have

$$\begin{aligned} a^Da^k &= (a^{\textcircled{W}})^{k+1}a^ka^k = (a^{\textcircled{W}})^k a^{\textcircled{W}} a^{k+1}a^{k-1} = (a^{\textcircled{W}})^k a^k a^{k-1} \\ &= \cdots = (a^{\textcircled{W}})^2 a^k a = a^{\textcircled{W}} a^k. \end{aligned}$$

(ii) : By condition (i), we have

$$\begin{aligned} (a^{\textcircled{W}})^k a^k &= (a^{\textcircled{W}})^{k-1} a^k a^D = (a^{\textcircled{W}})^{k-2} a^k (a^D)^2 \\ &= \cdots = a^{\textcircled{W}} a^k (a^D)^{k-1} = a^k (a^D)^k. \end{aligned}$$

So,

$$(a^D)^k = (a^{\textcircled{W}})^{k+1} a^k (a^D)^{k-1} = (a^{\textcircled{W}}) a^k (a^D)^k (a^D)^{k-1} = a^{\textcircled{W}} (a^D)^{k-1}.$$

□

Combining with Lemma 3.4, we have the following characterization of weak group element in a proper $*$-ring.

Theorem 3.5. *Let $a \in R^{\textcircled{W}}$ with $\mathrm{ind}(a) = k$. Then the following conditions are equivalent:*

(i) *a is a weak group element;*

(ii) $a^D = a^k (a^{\textcircled{W}})^{k+1}$;

(iii) $a^k a^{\textcircled{W}} = a^k a^D$;

(iv) $a^k = a^{k+1}a^{\textcircled{w}}$;

(v) $a^{\textcircled{w}}a^D = a^Da^{\textcircled{w}}$;

(vi) $(a^D)^k = (a^D)^{k-1}a^{\textcircled{w}}$.

Proof. (i) $\Leftrightarrow$ (ii) : Suppose that $a^D = a^k(a^{\textcircled{w}})^{k+1}$. By Lemma 2.6, we obtain that $a^D = a^k(a^{\textcircled{w}})^{k+1} = aa^{\textcircled{w}}a^{\textcircled{w}} = a^{\textcircled{w}}$. Now, the result follows from Lemma 2.5.

(ii) $\Rightarrow$ (iii) : By Lemma 2.6, we have

$$a^ka^D = a^ka^k(a^{\textcircled{w}})^{k+1} = a^{k-1}a^{k+1}(a^{\textcircled{w}})^{k+1} = a^{k-1}aa^{\textcircled{w}} = a^ka^{\textcircled{w}}.$$

(iii) $\Rightarrow$ (iv) : It is clear that $a^k = aa^ka^D = a^{k+1}a^{\textcircled{w}}$.

(iv) $\Rightarrow$ (i) : Since $a^{\textcircled{w}}a^k = a^{\textcircled{w}}a^{k+1}a^{\textcircled{w}} = a^ka^{\textcircled{w}}$, we get by Theorem 3.3 that $aa^{\textcircled{w}} = a^{\textcircled{w}}a$. That is, a is a weak group element.

(i) $\Rightarrow$ (v) and (i) $\Rightarrow$ (vi) are evident by Lemma 2.5.

(v) $\Rightarrow$ (i) : By the proof of Theorem 3.3 (iii) $\Rightarrow$ (i), it is easy to see that $a^{\textcircled{w}}a^D = a^Da^{\textcircled{w}} \Rightarrow a^{\textcircled{w}}a^k = a^ka^{\textcircled{w}} \Rightarrow aa^{\textcircled{w}} = a^{\textcircled{w}}a$.

(vi) $\Rightarrow$ (i) : From Lemma 2.8, it is known that $(a^D)^k = (a^k)^\#$. Since $(a^D)^k = (a^D)^{k-1}a^{\textcircled{w}}$, we get by Lemma 3.4 (ii) that

$$(a^k)^\#a^{\textcircled{w}} = (a^D)^ka^{\textcircled{w}} = a^{\textcircled{w}}(a^D)^{k-1}a^{\textcircled{w}}$$

and

$$a^{\textcircled{w}}(a^k)^\# = a^{\textcircled{w}}(a^D)^k = a^{\textcircled{w}}(a^D)^{k-1}a^{\textcircled{w}}.$$

That is, $(a^k)^\#a^{\textcircled{w}} = a^{\textcircled{w}}(a^k)^\#$. By Lemma 2.9, it is easy to obtain $a^ka^{\textcircled{w}} = a^{\textcircled{w}}a^k$. By Theorem 3.3, we have $aa^{\textcircled{w}} = a^{\textcircled{w}}a$. $\square$

In the following, we characterize weak group elements by using $\{1,3\}$-inverses.

Theorem 3.6. *Let $a \in R^{\textcircled{w}}$ with $\mathrm{ind}(a) = k$ and let $a^k \in R^{(1,3)}$. The following conditions are equivalent:*

(i) *a is a weak group element;*

(ii) $a^{k+1} = a^{2k}(a^k)^{(1,3)}a$;

(iii) $a^*a^{k+1} = a^*a^{2k}(a^k)^{(1,3)}a$;

(iv) $a^{k+1+j} = a^{2k+j}(a^k)^{(1,3)}a$ *for arbitrary $j \in \mathbb{Z}^+$;*

(v) $a^{k+1+j} = a^{2k+j}(a^k)^{(1,3)}a$ *for some $j \in \mathbb{Z}^+$.*

Proof. Suppose that $a \in R^{\textcircled{w}}$ with $\mathrm{ind}(a) = k$ and $a^k \in R^{(1,3)}$. Then $a \in R^D$ and by Lemma 2.7, $a \in R^{\textcircled{D}}$. By Theorem 3.3, it is known that $aa^{\textcircled{w}} = a^{\textcircled{w}}a \Leftrightarrow a^ka^{\textcircled{w}} = a^{\textcircled{w}}a^k$.

(i) $\Leftrightarrow$ (ii) : From Corollary 2.4 and Lemma 2.7, we get

$$a^ka^{\textcircled{w}} = a^k(a^{\textcircled{D}})^2a = a^k(a^Da^k(a^k)^{(1,3)})^2a = (a^D)^2a^{2k}(a^k)^{(1,3)}a$$

and

$$a^{\textcircled{w}}a^k = (a^{\textcircled{D}})^2a^{k+1} = a^{\textcircled{D}}a^k = a^Da^k(a^k)^{(1,3)}a^k = a^Da^k.$$

Then,

$$\begin{aligned} aa^{\textcircled{w}} = a^{\textcircled{w}}a &\Leftrightarrow a^k a^{\textcircled{w}} = a^{\textcircled{w}} a^k \Leftrightarrow a^D a^k = (a^D)^2 a^{2k} (a^k)^{(1,3)} a \\ &\Leftrightarrow a^2 a^D a^k = a^2 (a^D)^2 a^{2k} (a^k)^{(1,3)} a \\ &\Leftrightarrow a^{k+1} = a^{2k} (a^k)^{(1,3)} a. \end{aligned}$$

(ii) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv) $\Rightarrow$ (v) are obvious.

(iii) $\Rightarrow$ (ii) : Since R is a proper $*$-ring and $a \in R$, a is left $*$-cancellable. So $a^* a^{k+1} = a^* a^{2k} (a^k)^{(1,3)} a \Rightarrow a^{k+1} = a^{2k} (a^k)^{(1,3)} a$.

(v) $\Rightarrow$ (ii) : Since $a^{k+1+j} = a^{2k+j} (a^k)^{(1,3)} a \Rightarrow (a^D)^j a^{k+1+j} = (a^D)^j a^{2k+j} (a^k)^{(1,3)} a$, $a^D a^{k+2} = a^D a a^{2k} (a^k)^{(1,3)} a \Rightarrow a^{k+1} = a^{2k} (a^k)^{(1,3)} a$. □

4 Partial isometry of the weak group inverse

In this section, we state relationships between weak group inverse, pseudo core inverse and partial isometry in a proper $*$-ring R.

Theorem 4.1. *Let $a \in R^{PI} \cap R^{\textcircled{w}}$. Then $a \in R^{\textcircled{D}}$.*

Proof. Suppose that $a = a_1 + a_2$ is a group-EP decomposition of a. Then $a_1 \in R^{\#}$ and $a_1^* a_2 = a_2 a_1 = 0$. Since $a \in R^{PI}$, $a = aa^*a$, i.e., $a^* = a^{\dagger}$. So

$$a_1 + a_2 = a = a(a_1 + a_2)^*(a_1 + a_2) = a(a_1^* a_1 + a_2^* a_2). \tag{4.1}$$

Multiplying by a_1 on the right of equality (4.1), we obtain

$$a_1^2 = a a_1^* a_1^2 = (a_1 + a_2) a_1^* a_1^2. \tag{4.2}$$

Multiplying by a_1^* on the left of equality (4.2), we have $a_1^* a_1^2 = a_1^* a_1 a_1^* a_1^2$. Since $a_1 \in R^{\#}$, $a_1^* a_1 = a_1^* a_1 a_1^* a_1$. Since R is a proper $*$-ring, it is known that a_1 is left $*$-cancellable. So, $a_1 = a_1 a_1^* a_1$. Therefore, a_1 is a partial isometry, i.e., $a_1 \in R^{\dagger}$ and $a_1^* = a_1^{\dagger}$. Since $a_1 \in R^{\#} \cap R^{\dagger}$, we have that $a_1 \in R^{\textcircled{\#}}$. By Lemma 2.2, $a \in R^{\textcircled{D}}$. □

Remark 4.2. *We claim that $R^{\textcircled{D}} \nsubseteq R^{PI} \cap R^{\textcircled{D}}$. For example, if we set $R = \mathbf{M}_3(\mathbb{C})$, we take $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ (with $\mathrm{ind}(A) = 2$), and the involution as the conjugate transpose, by a direct computation, we obtain $A^{\textcircled{D}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. But $AA^*A = \begin{pmatrix} 3 & 3 & 4 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \neq A$. That is, A is not a partial isometry.*

Remark 4.3. *We observe that $a \in R^{\textcircled{w}}$ may not imply $a^{\textcircled{w}} \in R^{\dagger}$. In fact, let $R = \mathbf{M}_2(\mathbb{Z})$ and take the involution as the transpose. Then, R is a proper $*$-ring. If $a = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \in R$ then $a^{\textcircled{w}} = a$ and, by [17, Example 4.5], a is not $\{1,3\}$-invertible, that is, $a^{\textcircled{w}}$ is not Moore-Penrose invertible.*

Next, we present a characterization of partial isometries by means of elements of $R^{\textcircled{w}}$.

Theorem 4.4. *Let $a \in R^{\textcircled{w}}$. The following conditions are equivalent:*

(i) *$a^{\textcircled{w}}$ is partial isometry;*

(ii) *$a^{\textcircled{w}} \in R^{\dagger}$ and $(a^{\textcircled{w}})^{\textcircled{\#}} = aa^{\textcircled{w}}(a^{\textcircled{w}})^*$;*

(iii) *$a^{\textcircled{w}} \in R^{\dagger}$ and $a^{\textcircled{w}}(a^{\textcircled{w}})^{\textcircled{\#}} = a^{\textcircled{w}}(a^{\textcircled{w}})^*$;*

(iv) *$a^{\textcircled{w}} \in R^{\dagger}$ and $(a^{\textcircled{w}})^{\textcircled{\#}} = (a^{\textcircled{w}})^{\textcircled{\#}}a^{\textcircled{w}}(a^{\textcircled{w}})^*$;*

(v) *$a^{\textcircled{w}} \in R^{\dagger}$ and $(a^{\textcircled{w}})^{\textcircled{\#}} = a^{\textcircled{w}}(a^{\textcircled{w}})^*(a^{\textcircled{w}})^{\textcircled{\#}}$.*

Proof. From [17, Theorem 3.8], we know that $(a^{\textcircled{w}})^{\textcircled{w}} = (a^{\textcircled{w}})^{\#} = a^2a^{\textcircled{w}}$.

(i) $\Rightarrow$ (ii) : In [13, Theorem 2.19], it was proved that $b^{\textcircled{\#}} = b^{\#}bb^{\dagger}$ if b is core invertible and $b \in R^{\dagger}$. From $a^{\textcircled{w}} = a^{\textcircled{w}}(a^{\textcircled{w}})^*a^{\textcircled{w}}$, $a^{\textcircled{w}} \in R^{\dagger}$ by the definition of the Moore-Penrose. Thus, $a^{\textcircled{w}} \in R^{\textcircled{\#}}$. $(a^{\textcircled{w}})^{\textcircled{\#}} = (a^{\textcircled{w}})^{\#}a^{\textcircled{w}}(a^{\textcircled{w}})^{\dagger} = a^2a^{\textcircled{w}}a^{\textcircled{w}}(a^{\textcircled{w}})^* = aa^{\textcircled{w}}(a^{\textcircled{w}})^*$.

(ii) $\Rightarrow$ (i) : Multiplying by $a^{\textcircled{w}}$ on the right and on the left of the equality $(a^{\textcircled{w}})^{\textcircled{\#}} = aa^{\textcircled{w}}(a^{\textcircled{w}})^*$, we get

$$a^{\textcircled{w}} = a^{\textcircled{w}}aa^{\textcircled{w}}(a^{\textcircled{w}})^*a^{\textcircled{w}} = a^{\textcircled{w}}(a^{\textcircled{w}})^*a^{\textcircled{w}}.$$

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) is evident.

(iii) $\Rightarrow$ (v) : According to $(a^{\textcircled{w}})^{\textcircled{\#}} = a^{\textcircled{w}}((a^{\textcircled{w}})^{\textcircled{\#}})^2$, we have

$$(a^{\textcircled{w}})^{\textcircled{\#}} = (a^{\textcircled{w}}(a^{\textcircled{w}})^{\textcircled{\#}})(a^{\textcircled{w}})^{\textcircled{\#}} = a^{\textcircled{w}}(a^{\textcircled{w}})^*(a^{\textcircled{w}})^{\textcircled{\#}}.$$

(iv) $\Rightarrow$ (ii) : Since $(a^{\textcircled{w}})^{\#} = a^2a^{\textcircled{w}}$, we obtain

$$(a^{\textcircled{w}})^{\textcircled{\#}} = (a^{\textcircled{w}})^{\#}a^{\textcircled{w}}(a^{\textcircled{w}})^* = a^2a^{\textcircled{w}}a^{\textcircled{w}}(a^{\textcircled{w}})^* = aa^{\textcircled{w}}(a^{\textcircled{w}})^*.$$

(v) $\Rightarrow$ (i) : Since $(a^{\textcircled{w}})^{\textcircled{\#}} = a^{\textcircled{w}}(a^{\textcircled{w}})^*(a^{\textcircled{w}})^{\textcircled{\#}}$, we get $(a^{\textcircled{w}})^{\#}a^{\textcircled{w}} = a^{\textcircled{w}}(a^{\textcircled{w}})^*(a^{\textcircled{w}})^{\#}a^{\textcircled{w}}$. So,

$$a^{\textcircled{w}} = a^{\textcircled{w}}(a^{\textcircled{w}})^*(a^{\textcircled{w}})^{\#}(a^{\textcircled{w}})^2 = a^{\textcircled{w}}(a^{\textcircled{w}})^*a^{\textcircled{w}}.$$

□

In general, the weak group inverse of an element is neither a partial isometry nor an EP element. In the following, we give an example to illustrate this fact.

Example 4.5. *Let* $A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbf{M}_4(\mathbb{C})$ *with* $\mathrm{ind}(A) = 2$ *and take the involution as the conjugate transpose. By a direct computation, we have* $A^{\textcircled{w}} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$,

$(A^{\textcircled{w}})^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, $(A^{\textcircled{w}})^{\#} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $(A^{\textcircled{w}})^{\dagger} = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \end{pmatrix}$. *In this case,* $(A^{\textcircled{w}})^{\#} \neq (A^{\textcircled{w}})^{\dagger}$ *and* $(A^{\textcircled{w}})^* \neq (A^{\textcircled{w}})^{\dagger}$. *Hence,* $A^{\textcircled{w}}$ *is neither partial isometry nor an EP element.*

5 Reverse order law and additive property of the weak group inverse

Let $a_1, a_2, x \in R$. By [19], we know that $a_1^{\textcircled{D}}x = xa_2^{\textcircled{D}}$ if $a_1, a_2 \in R^{\textcircled{D}}$ with $a_1x = xa_2$ and $a_1^*x = xa_2^*$. Furthermore, let $a,\ b \in R^{\textcircled{D}}$ with $ab = ba$ and $a^*b = ba^*$. The authors in [4] proved that $(ab)^{\textcircled{D}} = a^{\textcircled{D}}b^{\textcircled{D}} = b^{\textcircled{D}}a^{\textcircled{D}}$. Motivated by these results, we study reverse order law of the weak group inverse by using similar conditions.

Lemma 5.1. *Let $a_1, a_2, x \in R$ with $a_1x = xa_2$ and $a_1^*x = xa_2^*$. If $a_1, a_2 \in R^{\textcircled{W}}$, then $a_1^{\textcircled{W}}x = xa_2^{\textcircled{W}}$.*

Proof. By Lemma 2.9, we know that $a_1^Dx = xa_2^D$ if $a_1x = xa_2$. Since $a_1x = xa_2$ and $a_1^*x = xa_2^*$, by Lemma 2.6 and Lemma 2.10, we have

$$\begin{aligned} a_1^*a_1^2a_1^{\textcircled{W}}x &= (a_1^*a_1^2a_1^{\textcircled{W}})^*x = (a_1a_1^{\textcircled{W}})^*a_1^*a_1x = (a_1a_1^{\textcircled{W}})^*xa_2^*a_2 = (a_1^k(a_1^{\textcircled{W}})^k)^*xa_2^*a_2 \\ &= ((a_1^{\textcircled{W}})^k)^*x(a_2^k)^*a_2^*a_2 = ((a_1^{\textcircled{W}})^k)^*x(a_2a_2^{\textcircled{W}}a_2^k)^*a_2^*a_2 \\ &= ((a_1^{\textcircled{W}})^k)^*x(a_2^k)^*(a_2a_2^{\textcircled{W}})^*a_2^*a_2 = ((a_1^{\textcircled{W}})^k)^*x(a_2^k)^*(a_2^*a_2^2a_2^{\textcircled{W}})^* \\ &= ((a_1^{\textcircled{W}})^k)^*(a_1^k)^*x(a_2^*a_2^2a_2^{\textcircled{W}})^* = (a_1a_1^{\textcircled{W}})^*xa_2^*a_2^2a_2^{\textcircled{W}} = a_1^*a_1^2a_1^{\textcircled{W}}xa_2a_2^{\textcircled{W}}. \end{aligned}$$

Since R is proper $*$-ring, we obtain that $a_1^2a_1^{\textcircled{W}}x = a_1^2a_1^{\textcircled{W}}xa_2a_2^{\textcircled{W}}$. So, we have $(a_1^D)^2a_1^2a_1^{\textcircled{W}}x = (a_1^D)^2a_1^2a_1^{\textcircled{W}}xa_2a_2^{\textcircled{W}} \Rightarrow a_1^Da_1a_1^{\textcircled{W}}x = a_1^Da_1a_1^{\textcircled{W}}xa_2a_2^{\textcircled{W}}$. By Lemma 2.6 (v) and (vi), we get that

$$\begin{aligned} a_1^{\textcircled{W}}x &= a_1^{\textcircled{W}}xa_2a_2^{\textcircled{W}} = a_1^{\textcircled{W}}xa_2a_2^Da_2a_2^{\textcircled{W}} = a_1^{\textcircled{W}}a_1a_1^Dxa_2a_2^{\textcircled{W}} \\ &= a_1^Dxa_2a_2^{\textcircled{W}} = xa_2^Da_2a_2^{\textcircled{W}} = xa_2^{\textcircled{W}}. \end{aligned}$$

□

Theorem 5.2. *Let $a,\ b \in R^{\textcircled{W}}$ with $ab = ba$ and $a^*b = ba^*$. Then $ab \in R^{\textcircled{W}}$ with $(ab)^{\textcircled{W}} = a^{\textcircled{W}}b^{\textcircled{W}} = b^{\textcircled{W}}a^{\textcircled{W}}$.*

Proof. By Lemma 5.1, we have $a^{\textcircled{W}}b = ba^{\textcircled{W}}$ and $ab^{\textcircled{W}} = b^{\textcircled{W}}a$. Since $a^*b^* = b^*a^*$ and $a^*b = ba^*$, we get that $b^{\textcircled{W}}a^* = a^*b^{\textcircled{W}}$. Similarly, we obtain that $a^{\textcircled{W}}b^* = b^*a^{\textcircled{W}}$. Combining with $ab^{\textcircled{W}} = b^{\textcircled{W}}a$, we obtain that $a^{\textcircled{W}}b^{\textcircled{W}} = b^{\textcircled{W}}a^{\textcircled{W}}$. Suppose that $k = \max\{\mathrm{ind}(a),\ \mathrm{ind}(b)\}$. By a direct computation, we have

$$a^{\textcircled{W}}b^{\textcircled{W}}(ab)^{k+1} = a^{\textcircled{W}}a^{k+1}b^{\textcircled{W}}b^{k+1} = a^kb^k = (ab)^k,$$

$$ab(a^{\textcircled{W}}b^{\textcircled{W}})^2 = a(a^{\textcircled{W}})^2b(b^{\textcircled{W}})^2 = a^{\textcircled{W}}b^{\textcircled{W}}$$

and

$$\begin{aligned} ((ab)^k)^*(ab)^2a^{\textcircled{W}}b^{\textcircled{W}} &= (a^k)^*(b^k)^*a^2b^2a^{\textcircled{W}}b^{\textcircled{W}} \\ &= (a^k)^*a^2a^{\textcircled{W}}(b^k)^*b^2b^{\textcircled{W}} = (a^k)^*a(b^k)^*b = ((ab)^k)^*ab. \end{aligned}$$

Thus, $(ab)^{\textcircled{W}} = a^{\textcircled{W}}b^{\textcircled{W}} = b^{\textcircled{W}}a^{\textcircled{W}}$. □

Let $a,\ b \in R^{\textcircled{D}}$. In [4], it was proved that $a + b \in R^{\textcircled{D}}$ with $(a + b)^{\textcircled{D}} = a^{\textcircled{D}} + b^{\textcircled{D}}$ if $ab = ba = 0$ and $a^*b = 0$. Next, we give the additive property of the weak group inverse by using similar conditions.

Theorem 5.3. *Let $a,\ b \in R^{\textcircled{W}}$ with $ab = ba = 0$ and $a^*b = 0$. Then $a + b \in R^{\textcircled{W}}$ with $(a + b)^{\textcircled{W}} = a^{\textcircled{W}} + b^{\textcircled{W}}$.*

Proof. Suppose that $a,\ b \in R^{\textcircled{W}}$. Since $ab = ba = 0$, we get $ab^{\textcircled{W}} = ab(b^{\textcircled{W}})^2 = 0$. By Lemma 2.10, we have $a^*a^2a^{\textcircled{W}}b = (aa^{\textcircled{W}})^*a^*ab = 0$. Since R is a proper $*$-ring, we obtain that $a^2a^{\textcircled{W}}b = 0$. By Lemma 2.6 (v), we obtain $a^{\textcircled{W}}b = aa^Da^{\textcircled{W}}b = (a^D)^2a^2a^{\textcircled{W}}b = 0$. Similarly, $ba^{\textcircled{W}} = 0$ and $b^{\textcircled{W}}a = 0$. Hence, we have $a^{\textcircled{W}}b^{\textcircled{W}} = 0 = b^{\textcircled{W}}a^{\textcircled{W}}$. Assume that $k = \max\{\mathrm{ind}(a),\ \mathrm{ind}(b)\}$. By a direct computation,

$$(a^{\textcircled{W}} + b^{\textcircled{W}})(a + b)^{k+1} = (a^{\textcircled{W}} + b^{\textcircled{W}})(a^{k+1} + b^{k+1}) = a^k + b^k = (a + b)^k$$

and

$$(a + b)(a^{\textcircled{W}} + b^{\textcircled{W}})^2 = (a + b)((a^{\textcircled{W}})^2 + (b^{\textcircled{W}})^2) = a^{\textcircled{W}} + b^{\textcircled{W}}.$$

Since $a^*b = 0$, we finally obtain

$$\begin{aligned} ((a + b)^k)^*(a + b)^2(a^{\textcircled{W}} + b^{\textcircled{W}}) &= (a^k + b^k)^*(a^2a^{\textcircled{W}} + b^2b^{\textcircled{W}}) \\ &= (a^k)^*a^2a^{\textcircled{W}} + (a^k)^*b^2b^{\textcircled{W}} + (b^k)^*a^2a^{\textcircled{W}} + (b^k)^*b^2b^{\textcircled{W}} \\ &= (a^k)^*a^2a^{\textcircled{W}} + (b^k)^*b^2b^{\textcircled{W}} = (a^k)^*a + (b^k)^*b \\ &= ((a + b)^k)^*(a + b). \end{aligned}$$

Hence, $a + b \in R^{\textcircled{W}}$ and $(a + b)^{\textcircled{W}} = a^{\textcircled{W}} + b^{\textcircled{W}}$. □

Finally, we claim that the additive property for the pseudo core inverse (see [4]) and the weak group inverse (Theorem 5.3) may not be true in a proper $*$-ring if we remove the condition $ba = 0$. In the following, we present two examples.

Example 5.4. *Let $a,\ b \in R^{\textcircled{D}}$. Then $ab = a^*b = 0$ may not imply $a + b \in R^{\textcircled{D}}$. In fact, let $R = \mathbf{M}_2(\mathbb{Z})$ and take the involution as the transpose. Then R is a proper $*$-ring. By setting $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, a direct computation allows us to obtain that $a^{\textcircled{D}} = a$, $b^{\textcircled{D}} = 0$, $ab = a^*b = 0$ and $a + b = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. By [17, Example 4.5], we know that $a + b$ is not $\{1, 3\}$-invertible. Since $(a + b)^m = a + b$ for any $m \in \mathbb{Z}^+$, $(a + b)^m$ is not $\{1, 3\}$-invertible. So, $a + b$ is not pseudo core invertible.*

Example 5.5. *Let $a,\ b \in R^{\textcircled{W}}$. Then $ab = a^*b = 0$ may not imply $a + b \in R^{\textcircled{W}}$. We consider $R = \mathbf{M}_3(\mathbb{Z})$ and we take the involution as the transpose. Then, R is a proper $*$-ring. Setting $a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, we have $a^2 = a$, $a^{\textcircled{W}} = a$, $b^2 = 0$, $b^{\textcircled{W}} = 0$ and $ab = a^*b = 0$. Next, we prove that $a + b$ is not weak group invertible. Denoting $c := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = a + b$, it is easy to check that $c^2 = c^3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. If $c \in R^{\textcircled{W}}$, then there exist $x \in R$ and $k \in \mathbb{Z}^+$ such that $xc^{k+1} = c^k$, $cx^2 = x$ and $(c^k)^*c^2x = (c^k)^*c$. So, we have the following two cases:*

(i) : *If* $k \geq 2$*, then* $c^k = c^2$*. In this case we get*

$$(c^k)^* c^2 x = (c^2)^* c^2 x = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x, \ (c^k)^* c = (c^2)^* c = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So, we have $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$*, which produces a contradiction.*

(ii) : *If* $k = 1$*, then* $xc^2 = c$*. So, we obtain that* $x \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$*, which produces a contradiction.*

6 Characterizations of $a^{\circledW} a^* = a^* a^{\circledW}$

In Lemma 5.1 and Theorem 5.2 two conditions were required in order to get the reverse order law of the weak group inverse. Inspired by condition $a^*b = ba^*$, with $a \in R^{\circledW}$, $b \in R$, we provide equivalent conditions for $a^{\circledW} a^* = a^* a^{\circledW}$ under a certain additional condition.

Firstly, an auxiliary lemma is given.

Lemma 6.1. *Let* $a \in R$*. If* $a = a_1 + a_2$ *is group-EP decomposition and* a_1 *is EP, then* $a_1 a_2 = 0 = a_1 a_2^*$*.*

Proof. Suppose that $a = a_1 + a_2$ is the group-EP decomposition of a. Then $a_2 a_1 = a_1^* a_2 = 0$. Since a_1 is EP,

$$a_1 a_2 = a_1 (a_1^\dagger)^* a_1^* a_2 = 0$$

and $a_2 a_1^* = a_2 a_1^\dagger a_1 a_1^* = a_2 a_1 a_1^\# a_1^* = 0$, i.e., $a_1 a_2^* = 0$. □

Let $a \in R$. In [11], it was shown that a is an EP element if and only if $a \in R^\#$ and $aR = a^*R$. In [13], it was shown that a is an EP element if and only if $a \in R^{\text{\textcircled{\#}}}$ and $aa^{\text{\textcircled{\#}}} = a^{\text{\textcircled{\#}}} a$. By Lemma 2.3, we know that $a \in R^{\circledW}$ if and only if a has a group-EP decomposition. So, we have the following lemma.

Lemma 6.2. *Let* $a \in R$*. If* $a = a_1 + a_2$ *is group-EP decomposition, then the following conditions are equivalent:*

(i) $a_1 R = a_1^* R$*;*

(ii) a_1 *is an EP element;*

(iii) $a = a_1 + a_2$ *is core-EP decomposition and* $a^{\circledW} = a^{\textcircled{D}}$*.*

Proof. Since $a = a_1 + a_2$ is group-EP decomposition, $a_1 \in R^\#$.

(i) ⇔ (ii) : It is obvious by [11].

(ii) ⇒ (iii) : Since a_1 is an EP element, $a_1 \in R^\dagger \cap R^\#$. In this case, the group-EP decomposition of a is the core-EP decomposition. Thus, by Lemma 2.2 and Lemma 2.3, $a^{\circledW} = a_1^\# = a_1^{\text{\textcircled{\#}}} = a^{\textcircled{D}}$.

(iii) $\Rightarrow$ (ii) : Since a has a core-EP decomposition, we know that $a_1 \in R^{\textcircled{\#}}$. From $a^{\textcircled{W}} = a^{\textcircled{D}}$, we have $a_1^{\#} = a_1^{\textcircled{\#}}$. So,

$$a_1 a_1^{\textcircled{\#}} = a_1 a_1^{\#} = a_1^{\#} a_1 = a_1^{\textcircled{\#}} a_1.$$

By [13], the result is clear. □

We present equivalent conditions of $a^{\textcircled{W}} a^* = a^* a^{\textcircled{W}}$ in the following theorem. In addition, we characterize normality of a_1 on the class of EP elements.

Theorem 6.3. *Let $a \in R$. If $a = a_1 + a_2$ is group-EP decomposition and a_1 is an EP element, then the following conditions are equivalent:*

(i) $a^* a^{\textcircled{W}} = a^{\textcircled{W}} a^*$;

(ii) $a_1^* a_1 = a_1 a_1^*$ *(that is, a_1 is normal in R)*;

(iii) $a^* a^{\textcircled{D}} = a^{\textcircled{D}} a^*$;

(iv) $a^{\textcircled{W}} a a^* = a a^* a^{\textcircled{W}}$;

(v) $a a^{\textcircled{W}} a^* = a^{\textcircled{W}} a^* a$;

(vi) $a^* a a^{\textcircled{W}} = a^{\textcircled{W}} a^* a$;

(vii) $a a^* a^{\textcircled{W}} = a^* a^{\textcircled{W}} a$;

(viii) $a^{\textcircled{W}} a^* a = a^*$;

(ix) $a a^* a^{\textcircled{W}} = a^*$;

(x) $a^{\textcircled{W}} a^* a^{\textcircled{W}} = (a^{\textcircled{W}})^2 a^*$;

(xi) $(a^2)^* a^{\textcircled{W}} = a^* a^{\textcircled{W}} a^*$;

(xii) $a^* (a^{\textcircled{W}})^2 = a^{\textcircled{W}} a^* a^{\textcircled{W}}$;

(xiii) $a^* a^{\textcircled{W}} a^* = a^{\textcircled{W}} (a^2)^*$.

Proof. Suppose that $a = a_1 + a_2$ is the group-EP decomposition of a. Then, by Lemma 2.3, $a_1^{\#} = a^{\textcircled{W}}$ and $a_2 a_1 = a_1^* a_2 = 0$. Since a_1 is an EP, we have $a_1 a_2 = 0 = a_1 a_2^*$ by Lemma 6.1.

(i) $\Rightarrow$ (ii) : Suppose that $a^* a^{\textcircled{W}} = a^{\textcircled{W}} a^*$. Then we have $a_1^* a_1^{\#} = a_1^{\#} a_1^*$. By Lemma 2.9, we have $a_1^* a_1 = a_1 a_1^*$.

(ii) $\Rightarrow$ (i) : Since $a_1 \in R^{\#}$ and $a_1^* a_1 = a_1 a_1^*$, by Lemma 2.9, $a_1^* a_1^{\#} = a_1^{\#} a_1^*$. So,

$$a^* a^{\textcircled{W}} = (a_1^* + a_2^*) a_1^{\#} = a_1^* a_1^{\#} = a_1^{\#} a_1^* = a_1^{\#} (a_1^* + a_2^*) = a^{\textcircled{W}} a^*.$$

(i) $\Leftrightarrow$ (iii) : It is evident by Lemma 6.2.

Suppose that $a_1^* a_1 = a_1 a_1^*$, it is easy to check that conditions (iv) $-$ (xiii) hold.

(iv) $\Rightarrow$ (ii) : From $a^{\textcircled{w}}aa^* = aa^*a^{\textcircled{w}}$, we have

$$a_1^{\#}a_1a_1^* = a_1a_1^*a_1^{\#} \tag{6.1}$$

Multiplying by a_1 on the left and by a_1^2 on the right of equality (6.1), we obtain $a_1a_1^*a_1^2 = a_1^2a_1^*a_1$. Since a_1 is an EP element, $a_1R = a_1^*R$ and $Ra_1 = Ra_1^*$. There exist $p,\ q \in R$ such that $a_1 = a_1^*p = qa_1^*$. Since R is a proper $*$-ring, a_1 is $*$-cancellable. Hence,

$$a_1a_1^*qa_1^*a_1 = a_1a_1^*pa_1^*a_1 \Rightarrow a_1^*a_1 = a_1a_1^*.$$

(v) $\Rightarrow$ (ii) : Suppose that $aa^{\textcircled{w}}a^* = a^{\textcircled{w}}a^*a$. Then $a_1a_1^{\#}a_1^* = a_1^{\#}a_1^*a_1$, and so, $a_1^2a_1^* = a_1a_1^*a_1$. Now the reasoning follows as in (iv) $\Rightarrow$ (ii).

(vi) $\Rightarrow$ (ii) : From $a^*aa^{\textcircled{w}} = a^{\textcircled{w}}a^*a$, we have $a_1^*a_1a_1^{\#} = a_1^{\#}a_1^*a_1$. That is, $a_1^2a_1^*a_1 = a_1a_1^*a_1^2$. Now the reasoning follows as in (iv) $\Rightarrow$ (ii).

(vii) $\Rightarrow$ (ii) : Assume that $aa^*a^{\textcircled{w}} = a^*a^{\textcircled{w}}a$. Then $a_1a_1^*a_1^{\#} = a_1^*a_1^{\#}a_1$, and so, $a_1a_1^*a_1 = a_1^*a_1^2$. The reasoning follows as in (iv) $\Rightarrow$ (ii).

(viii) $\Rightarrow$ (ii) : From $a^{\textcircled{w}}a^*a = a^*$, we have

$$a_1^{\#}a_1^*a_1 = a_1^* + a_2^*. \tag{6.2}$$

Multiply a_1 on the right of equality (6.2), we obtain $a_1^{\#}a_1^*a_1^2 = a_1^*a_1$, and so, $a_1a_1^*a_1^2 = a_1^2a_1^*a_1$. Now the reasoning follows as in (iv) $\Rightarrow$ (ii).

(ix) $\Rightarrow$ (ii) : Suppose that $aa^*a^{\textcircled{w}} = a^*$. Then

$$a_1a_1^*a_1^{\#} = a_1^* + a_2^*. \tag{6.3}$$

Multiplying by a_1^2 on the right of equality (6.3), we obtain $a_1a_1^*a_1 = a_1^*a_1^2$. Now the reasoning follows as in (iv) $\Rightarrow$ (ii).

(x) $\Rightarrow$ (ii) : Assume that $a^{\textcircled{w}}a^*a^{\textcircled{w}} = (a^{\textcircled{w}})^2a^*$. Then $a_1^{\#}a_1^*a_1^{\#} = (a_1^{\#})^2a_1^*$, and so, $a_1^2a_1^*a_1 = a_1a_1^*a_1^2$. The reasoning follows as in (iv) $\Rightarrow$ (ii).

(xi) $\Rightarrow$ (ii) : From $(a^2)^*a^{\textcircled{w}} = a^*a^{\textcircled{w}}a^*$, we have $a_1^*a_1^*a_1^{\#} = a_1^*a_1^{\#}a_1^*$. Since a_1 is an EP, there exists r such that $a_1^* = a_1r$. So, we have $a_1^*a_1ra_1^{\#} = a_1^*a_1(a_1^{\#})^2a_1^* \Rightarrow a_1ra_1^{\#} = a_1(a_1^{\#})^2a_1^*$. That is, $a_1^*a_1^{\#} = a_1^{\#}a_1^*$. By Lemma 2.9, we have $a_1^*a_1 = a_1a_1^*$.

(xii) $\Rightarrow$ (ii) : Suppose that $a^*(a^{\textcircled{w}})^2 = a^{\textcircled{w}}a^*a^{\textcircled{w}}$. Then $a_1^*(a_1^{\#})^2 = a_1^{\#}a_1^*a_1^{\#}$, and so, $a_1^2a_1^*a_1 = a_1a_1^*a_1^2$. The reasoning follows as in (iv) $\Rightarrow$ (ii).

(xiii) $\Rightarrow$ (ii) : Assume that $a^*a^{\textcircled{w}}a^* = a^{\textcircled{w}}(a^2)^*$. We have $a_1^*a_1^{\#}a_1^* = a_1^{\#}a_1^*a_1^*$. Then, $a_1^2a_1^*a_1^{\#}a_1^* = a_1a_1^*a_1^*$. Since a_1 is EP, $Ra_1 = Ra_1^*$. There exists r_1 such that $a_1^* = r_1a_1$. Then, we have $a_1^2a_1^*(a_1^{\#})^2a_1a_1^* = a_1r_1a_1a_1^*$. Since a_1 is $*$-cancellable, $a_1^2a_1^*(a_1^{\#})^2a_1 = a_1r_1a_1$. So, $a_1^2a_1^*a_1^{\#} = a_1a_1^*$. Since $a_1R = a_1^*R$, $a_1 = a_1^*p$ for some $p \in R$. Thus, we have $a_1a_1^*pa_1^*a_1^{\#} = a_1a_1^* \Rightarrow a_1^*pa_1^*a_1^{\#} = a_1^*$, and so, $a_1a_1^*a_1 = a_1^*a_1^2$. Since $a_1 \in Ra_1^*$, there exists r_2 such that $a_1 = r_2a_1^*$. We have $a_1a_1^*a_1 = a_1^*r_2a_1^*a_1$. Since a_1 is $*$-cancellable, we get $a_1a_1^* = a_1^*r_2a_1^* = a_1^*a_1$. $\square$

In general, we can not obtain $a^*a^{\textcircled{w}} = a^{\textcircled{w}}a^*$ even if the assumed condition of Theorem 6.3 hold.

Example 6.4. *Let* $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbf{M}_4(\mathbb{C})$ *with* $\operatorname{ind}(A) = 2$. *We take the involution as the conjugate transpose. Let* $A_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ *and* $A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. *It is easy to check that* $A = A_1 + A_2$ *is group-EP decomposition. Obviously,* A_1 *is not a normal matrix and* $A = A_1 + A_2$ *is directly the core-EP decomposition of* A. *By a direct computation, we have* $A^{\textcircled{w}} = A_1^{\#} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $A_1^{\dagger} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $A^*A^{\textcircled{w}} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $A^{\textcircled{w}}A^* = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. *In this case,* A_1 *is EP. However,* $A^*A^{\textcircled{w}} \neq A^{\textcircled{w}}A^*$.

Acknowledgments

We would like to thank the Referees for their constructive suggestions which help us to improve considerably the presentation of the paper.

This research was supported by the National Natural Science Foundation of China (No. 11771076, 11871145), the Postgraduate Research and Practice Innovation Program of Jiangsu Province (No. KYCX18_0053), the China Scholarship Council (File No. 201906090122), the Qing Lan Project of Jiangsu Province. The fourth author was partially supported by Ministerio de Economía y Competitividad of Spain (grant Red de Excelencia MTM2017-90682-REDT) and by Universidad Nacional de Río Cuarto, Argentina, Res. Rectoral N 083/2020.

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