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GDMP-inverses of a matrix and their duals

M.V. Hernández*, M.B. Lattanzi* , N. Thome†

Abstract

This paper introduces and investigates a new class of generalized inverses, called GDMP-inverses (and their duals), as a generalization of DMP-inverses. GDMP-inverses are defined from G-Drazin inverses and the Moore-Penrose inverse of a complex square matrix. In contrast to most other generalized inverses, GDMP-inverses are not only outer inverses but also inner inverses. Characterizations and representations of GDMP-inverses are obtained by means of the core-nilpotent and the Hartwig-Spindelböck decompositions.

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1 Introduction and background

Let $\mathbb{C}^{m\times n}$ be the set of $m \times n$ complex matrices. For $A \in \mathbb{C}^{m\times n}$, let A^*, A^{-1}, $rk(A)$, and $\mathcal{R}(A)$ denote the conjugate transpose, the inverse ($m = n$), the rank, and the range space of A, respectively. As usual, I_n stands for the $n \times n$ identity matrix and 0 denotes the zero matrix of adequate size.

Let $A \in \mathbb{C}^{m\times n}$. We consider the following list of matrix equations:

(1) $AXA = A$, i.e., X is a $\{1\}$-inverse (or inner inverse) of A,

(2) $XAX = X$, i.e., X is a $\{2\}$-inverse (or outer inverse) of A,

*Universidad Nacional de La Pampa. Facultad de Ciencias Exactas y Naturales. Santa Rosa, Argentina. E-mail: mvaleriahernandez@exactas.unlpam.edu.ar, mblatt@exactas.unlpam.edu.ar. Partially supported by Universidad Nacional de La Pampa, Facultad de Ingeniería (grant Resol. Nro. 135/19).

†Instituto Universitario de Matemática Multidisciplinar, Universitat Politècnica de València, 46022, Valencia, Spain. E-mail: njthome@mat.upv.es. Partially supported by Ministerio de Economía, Industria y Competitividad of Spain (Grant Red de Excelencia MTM2017-90682-REDT).

(3) $(AX)^* = AX$, i.e., X is a $\{3\}$-inverse of A,

(4) $(XA)^* = XA$, i.e., X is a $\{4\}$-inverse of A,

(5) $AX = XA$,

(6) $A^{k+1}X = A^k$, for some nonnegative integer k,

(7) $XA^{k+1} = A^k$, for some nonnegative integer k.

A matrix $X \in \mathbb{C}^{n\times m}$ that satisfies equations (1)-(4) is called the *Moore-Penrose inverse* of $A \in \mathbb{C}^{m\times n}$, it always exists and is unique. It is denoted by $A^\dagger$. The set of matrices $X \in \mathbb{C}^{n\times m}$ satisfying the equation (1) is denoted by $\mathcal{A}\{1\}$ and one element of $\mathcal{A}\{1\}$ is denoted by A^-. The set of matrices $X \in \mathbb{C}^{n\times m}$ satisfying the equations (1)-(2) is called a $\{1,2\}$-inverse of A and denoted by $\mathcal{A}\{1,2\}$. Some applications can be found in [1].

The following results are used later.

Theorem 1.1. *[2] Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$, and $C \in \mathbb{C}^{m\times q}$. The matrix equation $AXB = C$ has a solution if and only if there exist $\{1\}$-inverses A^- and B^- of A and B, respectively, such that $AA^-C = C$ and $CB^-B = C$. In this case, the set of all solutions of $AXB = C$ is given by*

$$X = A^-CB^- + Y - A^-AYBB^-,$$

for arbitrary $Y \in \mathbb{C}^{n\times p}$.

For $A \in \mathbb{C}^{n\times n}$, the *index* of A is the smallest nonnegative integer k such that $\mathcal{R}(A^k) = \mathcal{R}(A^{k+1})$, and is denoted by $k = ind(A)$. Let $A \in \mathbb{C}^{n\times n}$ with $k = ind(A)$. The *Drazin inverse* of A is the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying equations (2), (5), and (6), and is denoted by A^D. When $ind(A) \leq 1$, the Drazin inverse of A is called the *group inverse* of A (that is, it is satisfied (1), (2), and (5)) and is denoted by $A^\#$. A detailed analysis of all these generalized inverses can be found, for example, in [2–4] and some applications in [5].

Campbell and Meyer considered in [6] some modifications to the classic Drazin inverse by introducing weak Drazin inverses. A particular case of weak Drazin inverses was defined by Wang and Liu in [7] by means of equations (1), (6), and (7). More precisely, for a given $A \in \mathbb{C}^{n\times n}$ of index k, a matrix $X \in \mathbb{C}^{n\times n}$ is called a *G-Drazin inverse* of A if satisfies

$$AXA = A, \quad XA^{k+1} = A^k, \quad \text{and} \quad A^{k+1}X = A^k, \tag{1.1}$$

always exists and, in general, is not unique. The symbol $\mathcal{A}\{GD\}$ stands for the set of all G-Drazin inverses of A; an element of this set is denoted by A^{GD}. In [8], Coll et al. proved that the set of the equations (1.1) is equivalent to the more simplified system given by

$$AXA = A \quad \text{and} \quad A^k X = XA^k. \tag{1.2}$$

For more details about G-Drazin inverses see [7–9].

Several generalized inverses and their duals were defined from others previously studied. For a matrix $A \in \mathbb{C}^{n\times n}$, two classes of such (uniquely determined) hybrid generalized inverses, namely $A^{\#\!\!\!\bigcirc} := A^{\#}AA^{\dagger}$ and $A_{\#\!\!\!\bigcirc} := A^{\dagger}AA^{\#}$, were defined in [10, p.97]. Both classes were rediscovered by Baksalary and Trenkler in [11], who introduced the inverse $A^{\#\!\!\!\bigcirc}$ as the unique matrix $X \in \mathbb{C}^{n\times n}$ such that $AX = AA^{\dagger}$ and $\mathcal{R}(X) \subseteq \mathcal{R}(A)$. Clearly, the matrices $A^{\#\!\!\!\bigcirc}$ and $A_{\#\!\!\!\bigcirc}$ only exist when $ind(A) \leq 1$; such matrices are known as the *core* and the *dual core* inverse of A, respectively. In [12, Theorem 2.1], Wang and Liu proved that if $ind(A) \leq 1$ then the core inverse of A is the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying the following three equations:

$$AXA = A, \quad AX^2 = X, \quad \text{and} \quad (AX)^* = AX. \tag{1.3}$$

These inverses were generalized for matrices of arbitrary index by Malik and Thome in [13]. For a matrix $A \in \mathbb{C}^{n\times n}$ with index k, the authors introduced the *DMP-inverse* as the unique solution of the system of matrix equations

$$XAX = X, \quad XA = A^D A, \quad \text{and} \quad A^k X = A^k A^{\dagger}, \tag{1.4}$$

and denoted by $A^{D,\dagger}$. It was proved that $A^{D,\dagger} = A^D AA^{\dagger}$. Similarly, its dual can be defined by $A^{\dagger,D} = A^{\dagger}AA^D$. Some more properties and extensions of DMP-inverses and other inverses were given in [14–18].

On the other hand, for $A \in \mathbb{C}^{n\times n}$ of index k, it was recently introduced its Drazin-star matrix by Mosić in [19] as $A^{D,*} = A^D AA^*$, which is a $\{2\}$-inverse of $(A^{\dagger})^*$. Dually, $A^{*,D} = A^*AA^D$ is called the star-Drazin of A. Some related results and applications of these generalized inverses can be found in [13, 20–22].

The aforementioned inverses, except for G-Drazin inverses, exist and they are unique. Moreover, each of them can be represented as the (unique) solution of a system of suitable matrix equations. Most of them do not satisfy property (1) above.

The main aim of this paper is to introduce and investigate a new generalized hybrid inverse (and its dual): GDMP inverses (and dual GDMP inverses). This new class of matrices provides a generalization of DMP inverses to a more general class.

In [2, Lemma 3, p.45], it was proved that if B and C are $\{1\}$-inverses of A, then the product BAC is a $\{1,2\}$-inverse of A. We will exploit this general class of $\{1,2\}$-inverses by considering either B or C as the Moore-Penrose inverse of A.

The paper is organized as follows. In Section 2, we introduce new generalized inverses called GDMP-inverses, which provide a generalization for DMP-inverses (and their duals). In addition, we obtain a representation of GDMP-inverses by using the core-nilpotent decomposition. After giving a representation of G-Drazin inverses, in Section 3 we obtain a characterization for GDMP-inverses by using the HS-decomposition. Section 4 is devoted to present GDMP-inverses as the solution of a system of matrix equations. Finally, Section 5 shows that dual GDMP-inverses can be similarly introduced and analyzed.

2 Definition and properties of GDMP-inverses

This section introduces new generalized inverses, named GDMP-inverses, which can be considered as a generalization of DMP-inverse.

Definition 2.1. *Let $A \in \mathbb{C}^{n\times n}$ and $k = ind(A)$. For each $A^{GD} \in \mathcal{A}\{GD\}$, the GDMP-inverse of A, denoted by $A^{GD\dagger}$, is the $n \times n$ matrix*

$$A^{GD\dagger} = A^{GD}AA^{\dagger}.$$

The symbol $\mathcal{A}\{GD\dagger\}$ stands for the set of all GDMP-inverses of A; clearly $\mathcal{A}\{GD\dagger\} \neq \emptyset$ because $\mathcal{A}\{GD\}$ is a nonempty set. Hence,

$$\mathcal{A}\{GD\dagger\} = \{A^{GD}AA^{\dagger} : A^{GD} \in \mathcal{A}\{GD\}\}.$$

Therefore, GDMP-inverses of A always exist and, in general, they are not unique.

Proposition 2.2. *Let $A \in \mathbb{C}^{n\times n}$ and $k = ind(A)$. For each $A^{GD} \in \mathcal{A}\{GD\}$, the matrix $A^{GD\dagger}$ satisfies the following properties:*

(a) $A^{GD\dagger} \in \mathcal{A}\{1,2\}$.

(b) $AA^{GD\dagger} = AA^{\dagger}$ *and* $A^{GD\dagger}A = A^{GD}A$.

(c) $A^sA^{GD\dagger} = A^sA^\dagger$ *and* $A^{GD\dagger}A^s = A^{GD}A^s$ *for any positive integer* s.

(d) $A^{GD\dagger}A^{k+1} = A^k$.

(e) $A^{GD\dagger} = A^{GD}AA^{GD\dagger}$.

Proof. (a) Since $A^{GD}, A^\dagger \in A\{1\}$, we have that $A^{GD\dagger} = A^{GD}AA^\dagger \in \mathcal{A}\{1,2\}$.
(b) $AA^{GD\dagger} = (AA^{GD}A)A^\dagger = AA^\dagger$ and $A^{GD\dagger}A = A^{GD}(AA^\dagger A) = A^{GD}A$.
(c) Let s be a positive integer. Then, $A^sA^{GD\dagger} = A^sA^{GD}AA^\dagger = A^{s-1}(AA^{GD}A)A^\dagger = A^{s-1}AA^\dagger = A^sA^\dagger$. Similarly, $A^{GD\dagger}A^s = A^{GD}A^s$.
(d) $A^{GD\dagger}A^{k+1} = A^{GD\dagger}A^kA = A^{GD}A^kA = A^{GD}A^{k+1} = A^k$.
(e) $A^{GD\dagger} = (A^{GD\dagger}A)A^{GD\dagger} = (A^{GD}A)A^{GD\dagger}$. $\square$

Since G-Drazin inverses provide a generalization of Drazin inverses, if $A \in \mathbb{C}^{n\times n}$, for each $A^{GD} \in \mathcal{A}\{GD\}$, GDMP-inverses constitute a generalization of DMP-inverses. In [13, p.8] the authors considered the matrix

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

for which $ind(B) = 2$ and

$$B^D = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B^\dagger = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B^{D,\dagger} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that the matrix $B^{D,\dagger}$ is not a $\{1\}$-inverse of B since $rk(B) \not\leq rk(B^{D,\dagger})$. Then, by Proposition 2.2 (a), it is clear that $B^{D,\dagger} \notin \mathcal{B}\{GD\dagger\}$. By means of this example, the fact that both classes of matrices are different, in general, is stated.

Next, we give a decomposition for GDMP-inverses of a matrix A by using its core nilpotent decomposition [3,23]. The core nilpotent decomposition of a matrix $A \in \mathbb{C}^{n\times n}$, with $ind(A) = k$ and $rk(A) = a > 0$, is given by

$$A = P \begin{pmatrix} C & 0 \\ 0 & N \end{pmatrix} P^{-1}, \tag{2.1}$$

where $P \in \mathbb{C}^{n\times n}$ and $C \in \mathbb{C}^{a\times a}$ are nonsingular matrices, and $N \in \mathbb{C}^{(n-a)\times(n-a)}$ is nilpotent with nilpotence index k.

If A is written in its core nilpotent decomposition as in (2.1), then the G-Drazin inverses of A can be written as (see [7])

$$A^{GD} = P \begin{pmatrix} C^{-1} & 0 \\ 0 & N^{-} \end{pmatrix} P^{-1}, \tag{2.2}$$

where $N^{-} \in N\{1\}$.

The core nilpotent decomposition allows us to find necessary (but not sufficient) conditions for the characterization of the matrix $A^{GD\dagger}$.

Proposition 2.3. *Let $A \in \mathbb{C}^{n\times n}$ be written as in (2.1). Then, for each G-Drazin inverse A^{GD} of A, the GDMP-inverses of A can be represented as follows*

$$A^{GD\dagger} = P \begin{pmatrix} C^{-1} & A_2 \\ 0 & A_4 \end{pmatrix} P^{-1},$$

where A_2 and A_4 are matrices of suitable size such that $A_2N = 0$, $A_4N = N^{-}N$, and $A_4 = N^{-}NA_4$, for some $N^{-} \in N\{1\}$ (in consequence, $A_4 \in N\{1,2\}$) .

Proof. Suppose that A and A^{GD} are written as in (2.1) and (2.2), respectively.

Let $A^{GD\dagger} = P \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} P^{-1}$ be partitioned accordingly to the sizes of the blocks of A. It is easy to see that

$$A^{GD\dagger}A = P \begin{pmatrix} A_1C & A_2N \\ A_3C & A_4N \end{pmatrix} P^{-1} \quad \text{and} \quad A^{GD}A = P \begin{pmatrix} I_a & 0 \\ 0 & N^{-}N \end{pmatrix} P^{-1}.$$

From Proposition 2.2 (b), $A^{GD\dagger}A = A^{GD}A$. Thus, $A_1 = C^{-1}$, $A_2N = 0$, $A_3 = 0$, and $A_4N = N^{-}N$.

Moreover,

$$A^{GD\dagger} = P \begin{pmatrix} C^{-1} & A_2 \\ 0 & A_4 \end{pmatrix} P^{-1} \quad \text{and} \quad A^{GD\dagger}AA^{GD\dagger} = P \begin{pmatrix} C^{-1} & A_2 \\ 0 & N^{-}NA_4 \end{pmatrix} P^{-1}.$$

By Proposition 2.2 (a), $A^{GD\dagger}AA^{GD\dagger} = A^{GD\dagger}$. Hence, $N^{-}NA_4 = A_4$, which completes the proof. $\square$

Let A be written as in (2.1) in its core-nilpotent decomposition, where $C = \begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix}$, $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $P = \begin{pmatrix} -1 & -6 & 1 & 0 \\ 2 & 8 & 0 & 1 \\ 1 & -6 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{pmatrix}$. If we consider the matrices $A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $A_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, and $N^- = A_4$ then it is easy to check that the conditions required in Proposition 2.3 hold. Thus, $H = P \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & \frac{1}{2} & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} P^{-1} \in \mathcal{A}\{GD\dagger\}$. However, $HA \neq A^D A$. Hence, $H \neq A^{D\dagger}$. This fact means that $\mathcal{A}\{GD\dagger\} \nsubseteq \{A^{D\dagger}\}$, from where both classes of matrices are different each other.

Remark 2.4. The condition $\mathcal{A}\{GD\dagger\} = \{A^{D\dagger}\}$ holds if and only if $ind(A) \leq 1$, for every $A \in \mathbb{C}^{n\times n}$.

3 GDMP-inverses by using the HS-decomposition

In this section, we give a characterization for the GDMP-inverses of a square matrix A by using the Hartwig-Spindelböck decomposition [24, 25]. For a given matrix $A \in \mathbb{C}^{n\times n}$ of rank $a > 0$, the Hartwig-Spindelböck decomposition (HS-D, for short) is given by

$$A = U \begin{pmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{pmatrix} U^*, \tag{3.1}$$

where $U \in \mathbb{C}^{n\times n}$ is unitary, $\Sigma = \mathrm{diag}(\sigma_1 I_{a_1}, \sigma_2 I_{a_2}, \ldots, \sigma_t I_{a_t})$, the diagonal entries σ_i being singular values of A, $\sigma_1 > \sigma_2 > \ldots > \sigma_t > 0$, $a_1 + a_2 + \ldots + a_t = a$, and $K \in \mathbb{C}^{a\times a}$, $L \in \mathbb{C}^{a\times(n-a)}$ satisfy $KK^* + LL^* = I_a$.

In what follows, we give a decomposition for every $A^{GD} \in \mathcal{A}\{GD\}$ from the HS-D for A. Before that, we state some necessary properties.

Remark 3.1. Suppose that $A \in \mathbb{C}^{n\times n}$ is written in its HS-D as in (3.1), with $rk(A) = a$ and $ind(A) = k$. If $\Delta := (\Sigma K)^{GD}$ is a G-Drazin inverse of ΣK and $P := (\Sigma K)^k \Delta^k$ then the following properties are valid:

(p1) $ind(\Sigma K) = k - 1$, (see [13, Lemma 2.8]).

(p2) Δ^{k-1} is a G-Drazin inverse of $(\Sigma K)^{k-1}$, (see [26, Lemma 3.1]).

(p3) $\Delta(\Sigma K)^k = (\Sigma K)^k \Delta = (\Sigma K)^{k-1}$.

(p4) $(\Sigma K)^{k-1}\Delta = \Delta(\Sigma K)^{k-1}$.

(p5) $(\Sigma K)^k \Delta^k (\Sigma K)^k = (\Sigma K)^k$.

(p6) $P = (\Sigma K)^{k-1}\Delta^{k-1}$.

(p7) $P(\Sigma K)^{k-1} = (\Sigma K)^{k-1}P = (\Sigma K)^{k-1}$. In consequence, $P(\Sigma K)^k = (\Sigma K)^k P = (\Sigma K)^k$.

(p8) $P\Delta(\Sigma K)^k = (\Sigma K)^{k-1}$.

(p9) $\Sigma K P \Delta = P$.

Properties (p3) and (p4) result directly from the definition of G-Drazin inverse and (p1). Now, since $(\Sigma K)^{k-1}\Delta^{k-1}(\Sigma K)^{k-1} = (\Sigma K)^{k-1}$ by (p2), the statement (p5) follows from pre-multiplying by $(\Sigma K)\Delta$, post-multiplying by ΣK, and applying properties (p4) and (p3). Property (p6) is obtained by applying (p3), while (p7) follows from (p6), (p2), and (p4). Property (p8) is obtained from (p6), (p3), and (p2). Finally, (p9) follows from (p6).

Theorem 3.2. *Let $A \in \mathbb{C}^{n\times n}$ be written in its HS-D as in* (3.1)*, with $rk(A) = a$ and $ind(A) = k$. The following conditions are equivalent:*

(i) $X \in \mathcal{A}\{GD\}$,

(ii) *There exist matrices X_i, for $i = 1, 2, 3, 4$ of suitable sizes such that*

$$X = U \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} U^*, \tag{3.2}$$

and the following four conditions are satisfied:

(a) $\Sigma K X_1 + \Sigma L X_3 = I_a$,

(b) $X_1 (\Sigma K)^k = (\Sigma K)^{k-1}$,

(c) $X_3 (\Sigma K)^{k-1} = 0$,

(d) $(\Sigma K)^{k+1} X_2 + (\Sigma K)^k \Sigma L X_4 = (\Sigma K)^{k-1} \Sigma L$.

(iii) *There exists a G-Drazin inverse* $\Delta := (\Sigma K)^{GD}$ *of* ΣK *such that for arbitrary matrices* $S \in \mathbb{C}^{(n-a)\times a}$, $Z \in \mathbb{C}^{a\times a}$, $\Gamma_1 \in \mathbb{C}^{a\times(n-a)}$, *and* $\Gamma_2 \in \mathbb{C}^{(n-a)\times(n-a)}$, X *can be written as in* (3.2), *where* $P := (\Sigma K)^{k-1}\Delta^{k-1}$ *and*

$$
\begin{aligned}
&X_1 = P\Delta + Z(I_a - P),\\
&X_2 = K^*\Sigma^{-1}\Delta P\Sigma L + (I_a - K^*\Sigma^{-1}P\Sigma K)\Gamma_1 - K^*\Sigma^{-1}P\Sigma L\Gamma_2,\\
&X_3 = S(I_a - P),\\
&X_4 = L^*\Sigma^{-1}\Delta P\Sigma L + (I_{n-a} - L^*\Sigma^{-1}P\Sigma L)\Gamma_2 - L^*\Sigma^{-1}P\Sigma K\Gamma_1,\\
&(\Sigma K Z + \Sigma L S)(I_a - P) = (I_a - P).
\end{aligned}
$$

Proof. Let A be as in (3.1).

(i) $\Longrightarrow$ (ii) Let $X \in \mathcal{A}\{GD\}$ partitioned as in (3.2) accordingly to the size of the blocks of A. From $AXA = A$ and $KK^* + LL^* = I_a$ we obtain $\Sigma K X_1 + \Sigma L X_3 = I_a$. Also, $XA^{k+1} = A^k$ implies $X_1(\Sigma K)^{k+1} = (\Sigma K)^k$, $X_1(\Sigma K)^k\Sigma L = (\Sigma K)^{k-1}\Sigma L$, $X_3(\Sigma K)^{k+1} = 0$, and $X_3(\Sigma K)^k\Sigma L = 0$. Then, by making some computations, we obtain $X_1 (\Sigma K)^k = (\Sigma K)^{k-1}$ and $X_3 (\Sigma K)^{k-1} = 0$. Finally, $A^{k+1}X = A^k$ implies $(\Sigma K)^{k+1}X_2 + (\Sigma K)^k\Sigma L X_4 = (\Sigma K)^{k-1}\Sigma L$.

(ii) $\Longrightarrow$ (iii) First, we solve the matrix equation in (b) by using Theorem 1.1. Note that there exists a G-Drazin inverse $\Delta := (\Sigma K)^{GD}$ of ΣK such that Δ^k is a $\{1\}$-inverse of $(\Sigma K)^k$ (see Remark 3.1 (p5)). Clearly, the equation $X_1 (\Sigma K)^k = (\Sigma K)^{k-1}$ has (at least) a solution in X_1. The set of all solutions of this system is given by

$$X_1 = P\Delta + Z(I_a - P), \text{ for arbitrary } Z \in \mathbb{C}^{a\times a}.$$

Analogously, by Theorem 1.1 and using Remark 3.1 (p2) and (p6), we solve the matrix equation in (c) and obtain

$$X_3 = S(I_a - P), \text{ for arbitrary } S \in \mathbb{C}^{(n-a)\times a}.$$

By replacing X_1 and X_3 in the matrix equation in (a) and by making some computations, we have $(\Sigma K Z + \Sigma L S)(I_a - P) = I_a - P$. Now, we solve the matrix equation in (d) taking into account that it can be written as

$$(\Sigma K)^k \begin{pmatrix} \Sigma K & \Sigma L \end{pmatrix} \begin{pmatrix} X_2 \\ X_4 \end{pmatrix} = (\Sigma K)^{k-1}\Sigma L \tag{3.3}$$

and using Theorem 1.1. In fact, an easy computation shows that $\begin{pmatrix} K^* \\ L^* \end{pmatrix} \Sigma^{-1}\Delta^k$ is a $\{1\}$-inverse of $(\Sigma K)^k \begin{pmatrix} \Sigma K & \Sigma L \end{pmatrix}$. Thus, equation (3.3) has (at least) a solution in $\begin{pmatrix} X_2 \\ X_4 \end{pmatrix}$, and the set of all solutions of the system (3.3) is given by

$$
\begin{aligned}
X_2 &= K^*\Sigma^{-1}\Delta P\Sigma L + (I_a - K^*\Sigma^{-1}P\Sigma K)\Gamma_1 - K^*\Sigma^{-1}P\Sigma L\Gamma_2, \\
X_4 &= L^*\Sigma^{-1}\Delta P\Sigma L + (I_{n-a} - L^*\Sigma^{-1}P\Sigma L)\Gamma_2 - L^*\Sigma^{-1}P\Sigma K\Gamma_1,
\end{aligned}
$$

for arbitrary matrices $\Gamma_1 \in \mathbb{C}^{a\times(n-a)}$ and $\Gamma_2 \in \mathbb{C}^{(n-a)\times(n-a)}$.

(iii) $\Longrightarrow$ (i) It is easy to see that

$$
AXA = U \begin{pmatrix} (\Sigma K X_1 + \Sigma L X_3)\Sigma K & (\Sigma K X_1 + \Sigma L X_3)\Sigma L \\ 0 & 0 \end{pmatrix} U^*.
$$

By using the expressions for X_1 and X_3, (p9), and the last condition of hypothesis, some calculations yield $\Sigma K X_1 + \Sigma L X_3 = \Sigma K P\Delta + \Sigma K Z(I_a - P) + \Sigma L S(I_a - P) = P + (\Sigma K Z + \Sigma L S)(I_a - P) = P + (I_a - P) = I_a$. Then, $(\Sigma K X_1 + \Sigma L X_3)\Sigma K = \Sigma K$ and $(\Sigma K X_1 + \Sigma L X_3)\Sigma L = \Sigma L$. Hence, $AXA = A$.

On the other hand,

$$
XA^k = U \begin{pmatrix} X_1(\Sigma K)^k & X_1(\Sigma K)^{k-1}\Sigma L \\ X_3(\Sigma K)^k & X_3(\Sigma K)^{k-1}\Sigma L \end{pmatrix} U^*,
$$

and

$$
A^k X = U \begin{pmatrix} (\Sigma K)^k X_1 + (\Sigma K)^{k-1}\Sigma L X_3 & (\Sigma K)^k X_2 + (\Sigma K)^{k-1}\Sigma L X_4 \\ 0 & 0 \end{pmatrix} U^*.
$$

Now, we replace X_1, X_2, X_3, and X_4 in block matrices of XA^k and A^kX. From (p7), we get $X_3(\Sigma K)^{k-1} = 0$. Moreover, using (p7) and (p8), it is clear that $X_1(\Sigma K)^k = (P\Delta + Z(I_a - P))(\Sigma K)^k = P\Delta(\Sigma K)^k = (\Sigma K)^{k-1}$. From the equality $\Sigma K X_1 + \Sigma L X_3 = I_a$ we have $(\Sigma K)^k X_1 + (\Sigma K)^{k-1}\Sigma L X_3 = (\Sigma K)^{k-1}(\Sigma K X_1 + \Sigma L X_3) = (\Sigma K)^{k-1}$.

Now, from (p7), (p4), and (p2) we get $X_1(\Sigma K)^{k-1}\Sigma L = (\Sigma K)^{k-1}\Delta^k(\Sigma K)^{k-1}\Sigma L + Z(I_a - P)(\Sigma K)^{k-1}\Sigma L = (\Sigma K)^{k-1}\Delta\Sigma L$. Finally, $(\Sigma K)^k X_2 + (\Sigma K)^{k-1}\Sigma L X_4 = (\Sigma K)^{k-1}(\Delta P\Sigma L - P\Sigma K\Gamma_1 - P\Sigma L\Gamma_2 + \Sigma K\Gamma_1 + \Sigma L\Gamma_2) = (\Sigma K)^{k-1}(\Delta P\Sigma L + (I_a - P)\Sigma K\Gamma_1 + (I_a - P)\Sigma L\Gamma_2) = (\Sigma K)^{k-1}\Delta\Sigma L$, where we have used the distributive property and (p7). Therefore, $XA^k = A^kX$. Hence, $X \in \mathcal{A}\{GD\}$. $\square$

Next, we are able to state a characterization for GDMP-inverses. The following result gives the form of any $A^{GD\dagger} \in \mathcal{A}\{GD\dagger\}$ from the H-SD of A.

Theorem 3.3. *Let $A \in \mathbb{C}^{n\times n}$ be written in its HS-D as in* (3.1), *with $rk(A) = a$, and $ind(A) = k$. The following conditions are equivalent:*

(i) $Y \in \mathcal{A}\{GD\dagger\}$.

(ii) *There exist matrices $X_1 \in \mathbb{C}^{a\times a}$ and $X_3 \in \mathbb{C}^{(n-a)\times a}$ such that*

$$Y = U \begin{pmatrix} X_1 & 0 \\ X_3 & 0 \end{pmatrix} U^*, \tag{3.4}$$

and the following three conditions are satisfied:

(a) $\Sigma K X_1 + \Sigma L X_3 = I_a$,

(b) $X_1 (\Sigma K)^k = (\Sigma K)^{k-1}$,

(c) $X_3 (\Sigma K)^k = 0$.

(iii) *There exists a G-Drazin inverse $\Delta := (\Sigma K)^{GD}$ of ΣK such that for arbitrary matrices $S \in \mathbb{C}^{(n-a)\times a}$ and $Z \in \mathbb{C}^{a\times a}$, Y can be written as in* (3.4), *where $P := (\Sigma K)^{k-1}\Delta^{k-1}$ and*

$$\begin{aligned} X_1 &= P\Delta + Z(I_a - P), \\ X_3 &= S(I_a - P), \\ (\Sigma K Z &+ \Sigma L S - I_a)(I_a - P) = 0. \end{aligned}$$

Proof. Let A be as in (3.1).

(i) $\Longrightarrow$ (ii) Since $Y \in \mathcal{A}\{GD\dagger\}$, there exists $X \in \mathcal{A}\{GD\}$ such that $Y = XAA^\dagger$. By Theorem 3.2, the matrix X can be written as $X = U \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} U^*$, partitioned accordingly to the size of the blocks of A, and the conditions (a), (b), and (c) of item (ii) hold. Since $A^\dagger = U \begin{pmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{pmatrix} U^*$, it is easy to see that $Y = XAA^\dagger = U \begin{pmatrix} X_1 & 0 \\ X_3 & 0 \end{pmatrix} U^*$.

(ii) $\Longrightarrow$ (iii) It follows directly from Theorem 3.2.

(iii) $\Longrightarrow$ (i) Let $X := U \begin{pmatrix} X_1 & K^*\Sigma^{-1}\Delta P\Sigma L \\ X_3 & L^*\Sigma^{-1}\Delta P\Sigma L \end{pmatrix} U^*$. From Theorem 3.2, it is clear that $X \in \mathcal{A}\{GD\}$. Moreover, $XAA^\dagger = Y$. Hence, $Y \in \mathcal{A}\{GD\dagger\}$. □

Next result follows from Theorems 3.2 and 3.3.

Corollary 3.4. *Let $A \in \mathbb{C}^{n\times n}$ be written in its HS-D as in* (3.1) *with ind*$(A) = k$. *The following statements hold:*

(a) $\mathcal{A}\{GD\dagger\} \subseteq \mathcal{A}\{GD\}$ *if and only if* $(\Sigma K)^{k-1}\,\Sigma L = 0$.

(b) $\mathcal{A}\{GD\} \subseteq \mathcal{A}\{GD\dagger\}$ *if and only if* $(\Sigma K)^{k-1}\,\Sigma L = 0, X_2 = 0,$ *and* $X_4 = 0$.

(c) $\mathcal{A}\{GD\} = \mathcal{A}\{GD\dagger\}$ *if and only if* $\mathcal{A}\{GD\} \subseteq \mathcal{A}\{GD\dagger\}$.

Two subclasses of matrices are of particular interest are those given by EP matrices ($AA^\dagger = A^\dagger A$) and Partial Isometries ($A^\dagger = A^*$).

Corollary 3.5. *Let $A \in \mathbb{C}^{n\times n}$ be written in its HS-D as in* (3.1), *with rk*$(A) = a$, *ind*$(A) = k$. *The following statements hold.*

(1) *If A is an EP matrix, then*

$$Y \in \mathcal{A}\{GD\dagger\} \quad \textit{if and only if} \quad Y = U \begin{pmatrix} K^*\Sigma^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^*.$$

This is, the GDMP-inverse of an EP matrix is unique, and $A^{GD\dagger} = A^\dagger = A^\#$.

(2) *If A is a partial isometry, then the following conditions are equivalent:*

(i) $Y \in \mathcal{A}\{GD\dagger\}$.

(ii) *There exist matrices $X_1 \in \mathbb{C}_{a\times a}$ and $X_3 \in \mathbb{C}_{(n-a)\times a}$ such that*

$$Y = U \begin{pmatrix} X_1 & 0 \\ X_3 & 0 \end{pmatrix} U^*,$$

and the following conditions are satisfied:

(a) $KX_1 + LX_3 = I_a$,

(b) $X_1K^k = K^{k-1}$,

(c) $X_3K^k = 0$.

Proof. In [24, Corollary 6] is was proved that A is an EP matrix if and only if $L = 0$ (or equivalently, K is a unitary matrix); and A is a partial isometry if and only if $\Sigma = I_a$. Then, the proof is immediate from Theorem 3.3. $\square$

4 GDMP-inverses as solution of matrix equations

Let $A \in \mathbb{C}^{n\times n}$. From Proposition 2.2, if $X \in \mathcal{A}\{GD\dagger\}$, then there exists $Z \in \mathcal{A}\{GD\}$ such that $X = ZAA^\dagger$, and X satisfies the equations of the system

$$XAX = X, \quad AX = AA^\dagger, \quad \text{and} \quad XA^k = ZA^k. \tag{4.1}$$

Next, we will prove that the set $\mathcal{A}\{GD\dagger\}$ is the solution set of the system (4.1), for any $Z \in \mathcal{A}\{GD\}$.

Theorem 4.1. *Let $A \in \mathbb{C}^{n\times n}$. The following conditions are equivalent:*

(i) $X \in \mathcal{A}\{GD\dagger\}$.

(ii) *There exists $A^{GD} \in \mathcal{A}\{GD\}$ such that X is solution of the system*

$$XAX = X, \quad AX = AA^\dagger, \quad \textit{and} \quad XA^k = A^{GD}A^k.$$

(iii) *For each $Z \in \mathcal{A}\{GD\}$, X is solution of the system* (4.1).

Proof. (i) $\Longrightarrow$ (ii) It has been shown above.
(ii) $\Longrightarrow$ (i) Suppose that $A \in \mathbb{C}^{n\times n}$ be written in its HS-D as in (3.1), with $rk(A) = a$, $ind(A) = k$, and $X \in \mathbb{C}^{n\times n}$ being partitioned accordingly to the size of the blocks of A as $X = U \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} U^*$. From $AX = AA^\dagger$, some calculations yield

$$\Sigma K X_1 + \Sigma L X_3 = I_a. \tag{4.2}$$

Let $A^{GD} = U \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix} U^*$ be partitioned accordingly to the size of the blocks of A. By Theorem 3.2, the hypothesis $A^{GD} \in \mathcal{A}\{GD\}$ implies that the following conditions are satisfied:

(a) $\Sigma K Z_1 + \Sigma L Z_3 = I_a$,

(b) $Z_1 (\Sigma K)^k = (\Sigma K)^{k-1}$,

(c) $Z_3 (\Sigma K)^k = 0$,

(d) $(\Sigma K)^{k+1} Z_2 + (\Sigma K)^k \Sigma L Z_4 = (\Sigma K)^{k-1} \Sigma L$.

Comparing blocks in $XA^k = A^{GD}A^k$, we have

$$X_1 (\Sigma K)^k = (\Sigma K)^{k-1}, \tag{4.3}$$

$$X_1 (\Sigma K)^{k-1} \Sigma L = (\Sigma K)^k Z_2 + (\Sigma K)^{k-1} \Sigma L Z_4, \tag{4.4}$$

$$X_3 (\Sigma K)^k = 0, \tag{4.5}$$

$$X_3 (\Sigma K)^{k-1} \Sigma L = 0. \tag{4.6}$$

Let us consider the matrix

$$Z_0 := U \begin{pmatrix} X_1 & Z_2 \\ X_3 & Z_4 \end{pmatrix} U^*.$$

We will see that $Z_0 \in \mathcal{A}\{GD\}$. Indeed, pre-multiplying equality (4.4) by ΣK and using equalities (4.3), (4.2), and (4.6) we obtain

$$\begin{aligned}
(\Sigma K)^{k+1} Z_2 + (\Sigma K)^k \Sigma L Z_4 &= \Sigma K X_1 (\Sigma K)^{k-1} \Sigma L \\
&= \Sigma K X_1 X_1 (\Sigma K)^k \Sigma L \\
&= (I_a - \Sigma L X_3) X_1 (\Sigma K)^k \Sigma L \\
&= X_1 (\Sigma K)^k \Sigma L - \Sigma L X_3 (\Sigma K)^{k-1} \Sigma L \\
&= (\Sigma K)^{k-1} \Sigma L.
\end{aligned}$$

Hence, from this last equality, (4.2), (4.3), (4.5), and the Theorem 3.2 we obtain $Z_0 \in \mathcal{A}\{GD\}$. Therefore, there exists $Z_0 \in \mathcal{A}\{GD\}$ such that $X = Z_0 A A^\dagger$, i.e., $X \in \mathcal{A}\{GD\dagger\}$.

(ii) $\Longrightarrow$ (iii) From [26, Theorem 3.2], $Z \in \mathcal{A}\{GD\}$ if and only if for arbitrary T, W, $Z = A^{GD} + (I - P_{A^k})T(I - P_A) + (I - Q_A)W(I - P_{A^k})$, where $P_{A^k} = A^k(A^{GD})^k$, $P_A = AA^{GD}$ and $Q_A = A^{GD}A$. Now, it is easy to see that, for arbitrary T and W, X satisfies equations of system (4.1) because $P_A A^k = A^k$ and $P_{A^k} A^k = A^k$.

(iii) $\Longrightarrow$ (ii) It follows immediately since $\mathcal{A}\{GD\} \neq \emptyset$. $\square$

5 Dual GDMP-inverses

Similarly to GDMP-inverses, we can introduce a dual class of matrices called MPGD-inverses. Since its development is analogous to GDMP-inverses, we only provide the results without proofs.

Definition 5.1. *Let $A \in \mathbb{C}^{n\times n}$ and $k = ind(A)$. For each $A^{GD} \in \mathcal{A}\{GD\}$, the MPGD-inverse of A, denoted by $A^{\dagger GD}$, is the $n \times n$ matrix*

$$A^{\dagger GD} = A^{\dagger} A A^{GD}.$$

The symbol $\mathcal{A}\{\dagger GD\}$ stands for the set of all MPGD-inverses of A, that is

$$\mathcal{A}\{\dagger GD\} = \{A^{\dagger} A A^{GD} : \, A^{GD} \in \mathcal{A}\{GD\}\}.$$

Therefore, MPGD-inverses of A always exist and, in general, they are not unique.

The following result gives a characterization for any $A^{\dagger GD} \in \mathcal{A}\{\dagger GD\}$ from the H-SD of A.

Theorem 5.2. *Let $A \in \mathbb{C}^{n\times n}$ be written in its HS-D as in (3.1), with $rk(A) = a$, and $ind(A) = k$. The following conditions are equivalent:*

(i) $Y \in \mathcal{A}\{\dagger GD\}$.

(ii) *There exist matrices $X_2 \in \mathbb{C}^{a\times(n-a)}$ and $X_4 \in \mathbb{C}^{(n-a)\times(n-a)}$ such that*

$$Y = U \begin{pmatrix} K^*\Sigma^{-1} & K^*KX_2 + K^*LX_4 \\ L^*\Sigma^{-1} & L^*KX_2 + L^*LX_4 \end{pmatrix} U^*, \tag{5.1}$$

where $(\Sigma K)^{k+1}X_2 + (\Sigma K)^k \Sigma L X_4 = (\Sigma K)^{k-1}\Sigma L$.

Notice that X_2 and X_4 can be also determined as in Theorem 3.3.

Finally, dual GDMP-inverses can be characterized as follows.

Theorem 5.3. *Let $A \in \mathbb{C}^{n\times n}$ with $ind(A) = k$. The following conditions are equivalent:*

(i) $X \in \mathcal{A}\{\dagger GD\}$.

(ii) *There exists $A^{GD} \in \mathcal{A}\{GD\}$ such that X is solution of the system*

$$XAX = X, \quad XA = A^{\dagger}A, \quad \textit{and} \quad A^kX = A^kA^{GD}.$$

(iii) *For each $Z \in \mathcal{A}\{GD\}$, X is solution of the system*

$$XAX = X, \quad XA = A^{\dagger}A, \quad \textit{and} \quad A^kX = A^kZ.$$

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