Applied General Topology

Appl. Gen. Topol. 24, no. 1 (2023), 1-8
doi:10.4995/agt.2023.12424

Selection principles and bitopological hyperspaces

Alexander V. Osipov

Krasovskii Institute of Mathematics and Mechanics, Ural Federal University, Ural State University of Economics, 620219, Yekaterinburg, Russia (OAB@list.ru)

Communicated by M. Sanchis

Abstract

In this paper we continue to research relationships between closure-type properties of hyperspaces over a space X and covering properties of X. For a Hausdorff space X we denote by 2^X the family of all closed subsets of X. We investigate selection properties of the bitopological space $(2^X, \Delta_1^+, \Delta_2^+)$ where Δ_i^+ is the upper Δ_i-topology for each $i = 1, 2$.

2020 MSC: *54B20; 54E55; 54D20.*

Keywords: *hyperspace; upper Fell topology; selection principles; bitopological space; $\mathbf{Z}^+$-topology; perfect space; k-perfect space.*

1. Introduction

For a Hausdorff space X we denote by 2^X the family of all closed subsets of X. If A is a subset of X and $\mathcal{A}$ a family of subsets of X, then

$A^c = X \setminus A$ and $\mathcal{A}^c = \{A^c : A \in \mathcal{A}\}$,

$A^- = \{F \in 2^X : F \cap A \neq \varnothing\}$,

$A^+ = \{F \in 2^X : F \subset A\}$.

Let Δ be a subset of 2^X closed for finite unions and containing all singletons. Then the *upper Δ-topology*, denoted by Δ^+, is the topology whose base is the collection $\{(D^c)^+ : D \in \Delta\} \cup \{2^X\}$.

We consider the next important cases:

- Δ is the collection $CL(X) = 2^X \setminus \{\varnothing\}$;
- Δ is the family $\mathbb{K}(X)$ of all non-empty compact subsets of X;

Received 7 October 2019 – Accepted 14 February 2023

• Δ is the family $\mathbb{F}(X)$ of all non-empty finite subsets of X.

When $\Delta = CL(X)$ we have the well-known *upper Vietoris topology* $\mathbf{V}^+$, when $\Delta = \mathbb{K}(X)$ we have the *upper Fell topology* (known also as the co-compact topology) $\mathbf{F}^+$, and when $\Delta = \mathbb{F}(X)$ we have the $\mathbf{Z}^+$-topology.

Many topological properties are defined or characterized in terms of the following classical selection principles [5, 19, 21]. Let $\mathcal{A}$ and $\mathcal{B}$ be sets consisting of families of subsets of an infinite set X. Then:

$S_1(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(b_n : n \in \mathbb{N})$ such that for each n, $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

$S_{fin}(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ of finite sets such that for each n, $B_n \subseteq A_n$, and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

In this paper, by a cover we mean a nontrivial one, that is, $\mathcal{U}$ is a cover of X if $X = \bigcup \mathcal{U}$ and $X \notin \mathcal{U}$.

An open cover $\mathcal{U}$ of a space X is called:

• an ω-cover (k-cover) if each finite (compact) subset C of X is contained in an element of $\mathcal{U}$;

• a γ-cover (γ_k-cover) if $\mathcal{U}$ is infinite and for each finite (compact) subset C of X the set $\{U \in \mathcal{U} : C \nsubseteq U\}$ is finite.

Because of these definitions all spaces are assumed to be *Hausdorff non-compact*, unless otherwise stated.

Definition 1.1. An open cover $\mathcal{U}$ of a space X is called Δ*-cover* if each element of Δ is contained in an element of $\mathcal{U}$.

In particular, a Δ-cover is a ω-cover (a k-cover) for $\Delta = \mathbb{F}(X)$ ($\Delta = \mathbb{K}(X)$).

Definition 1.2. An open cover $\mathcal{U}$ of a space X is called γ_Δ*-cover* if $\mathcal{U}$ is infinite and for each element C of Δ the set $\{U \in \mathcal{U} : C \nsubseteq U\}$ is finite.

In particular, a γ_Δ-cover is a γ-cover (a γ_k-cover) for $\Delta = \mathbb{F}(X)$ ($\Delta = \mathbb{K}(X)$).

Different Δ-covers (k-covers, ω-covers, k_F-covers, c_F-covers,...) exposed many dualities in hyperspace topologies such as co-compact topology $\mathbf{F}^+$, co-finite topology $\mathbf{Z}^+$, Pixley-Roy topology, Fell topology and Vietoris topology. They also play important roles in selection principles [2, 8, 10, 11, 13, 14, 16, 18].

In [15] we investigated selectors for sequence of subsets of the space 2^X with the $\mathbf{Z}^+$-topology and the upper Fell topology ($\mathbf{F}^+$-topology). Also we considered the selection properties of the bitopological space $(2^X, \mathbf{F}^+, \mathbf{Z}^+)$.

In this paper we continue to research relationships between closure-type properties of hyperspaces over a space X and covering properties of X. We investigate selection properties of the bitopological space $(2^X, \Delta_1^+, \Delta_2^+)$ where Δ_i^+ is the upper Δ_i-topology for each $i = 1, 2$.

2. Selective properties of bitopological hyperspaces

Definition 2.1 ([15])**.** Let X be a space and let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be an open cover of X. Then $\mathcal{U}^c = \{X \setminus U_\alpha : \alpha \in \Lambda\}$ *converges to* $\{\varnothing\}$ in $(2^X, \tau)$ where τ

is a topology on 2^X, if for every $F \in 2^X$ the $\mathcal{U}^c$ converges to F, i.e. for each neighborhood W of F in the space $(2^X, \tau)$, $|\{\alpha : (X \setminus U_\alpha) \nsubseteq W, \alpha \in \Lambda\}| < \aleph_0$.

Since every Γ_Δ-cover contains a countably Γ_Δ-cover, then each converging to $\{\varnothing\}$ subset of $(2^X, \Delta^+)$ contains a countable converging to $\{\varnothing\}$ subset of $(2^X, \Delta^+)$.

For a topological space X, consider Δ_i for $i = 1, 2$. Then $(2^X, \Delta_1^+, \Delta_2^+)$ is a bitopological space. We denote:

- $\mathcal{D}_\Omega^i$ — the family of dense subsets of $(2^X, \Delta_i^+)$;
- $\mathcal{D}_\Gamma^i$ — the family of converging to $\{\varnothing\}$ subsets of $(2^X, \Delta_i^+)$.
- Ω_{Δ_i} — the family of Δ_i-covers of X;
- Γ_{Δ_i} — the family of γ_{Δ_i}-covers of X.

Lemma 2.2. *Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be an open cover of X and $(2^X, \Delta^+)$ is a hyperspace. Then*

(a) $\mathcal{U}$ is an Δ-cover of $X \Leftrightarrow \mathcal{U}^c \in \mathcal{D}_\Omega$.

(b) $\mathcal{U}$ is an γ_Δ-cover of $X \Leftrightarrow \mathcal{U}^c \in \mathcal{D}_\Gamma$.

Proof. (a). Let $\mathcal{U}$ be an Δ-cover of X and let $(K^c)^+$ be a basic open subset of $(2^X, \Delta^+)$ where $K \in \Delta$. There is a member U_K of $\mathcal{U}$ containing K. Thus we have $U_K^c \in (K^c)^+$ and hence $\mathcal{U}^c \in \mathcal{D}_\Omega$.

Let $\mathcal{U}^c \in \mathcal{D}_\Omega$. Let $K \in \Delta$. Pick a set D in $(K^c)^+ \cap \mathcal{U}^c$. We have $D^c \in \mathcal{U}$ and $K \subset D^c$.

(b). Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be an γ_Δ-cover of X and $F \in 2^X$. For each neighborhood W of F in the space $(2^X, \tau)$, $|\{\alpha : (X \setminus U_\alpha) \nsubseteq W, \alpha \in \Lambda\}| < \aleph_0$. Hence $\mathcal{U}^c \in \mathcal{D}_\Gamma$.

Let $\mathcal{U}^c \in \mathcal{D}_\Gamma$ where $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$. Then for every $F \in 2^X$ and for each neighborhood W of F in the space $(2^X, \tau)$, $|\{\alpha : (X \setminus U_\alpha) \nsubseteq W, \alpha \in \Lambda\}| < \aleph_0$. Hence $\mathcal{U}$ is an γ_Δ-cover of X. □

Theorem 2.3. *Assume that $\Phi, \Psi \in \{\Omega, \Gamma\}$, $\star \in \{1, fin\}$. Then for a space X the following statements are equivalent:*

(1) *X satisfies $S_\star(\Phi_{\Delta_1}, \Psi_{\Delta_2})$;*

(2) *$(2^X, \Delta_1^+, \Delta_2^+)$ satisfies $S_\star(\mathcal{D}_\Phi^1, \mathcal{D}_\Psi^2)$.*

Proof. We prove the theorem for $\star = fin$, the other proofs being similar.

(1) $\Rightarrow$ (2). Let $(D_i : i \in \mathbb{N})$ be a sequence of dense subsets of $(2^X, \Delta_1^+)$ such that $D_i \in \mathcal{D}_\Phi^1$ for each $i \in \mathbb{N}$. Then $(D_i^c : i \in \mathbb{N})$ is a sequence of open covers of X such that $D_i^c \in \Phi_{\Delta_1}$ for each $i \in \mathbb{N}$. Since X satisfies $S_{fin}(\Phi_{\Delta_1}, \Psi_{\Delta_2})$, there is a sequence $(A_i : i \in \mathbb{N})$ of finite sets such that for each i, $A_i \subseteq D_i^c$, and $\bigcup_{i \in \mathbb{N}} A_i \in \Psi_{\Delta_2}$. It follows that $\bigcup_{i \in \mathbb{N}} A_i^c \in \mathcal{D}_\Psi^2$.

(2) $\Rightarrow$ (1). Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X such that $\mathcal{U}_n \in \Phi_{\Delta_1}$. For each n, $\mathcal{A}_n := \mathcal{U}_n^c$ is a dense subset of $(2^X, \Delta_1^+)$ such that $\mathcal{A}_n \in \mathcal{D}_\Phi^1$. Applying that $(2^X, \Delta_1^+, \Delta_2^+)$ satisfies $S_{fin}(\mathcal{D}_\Phi^1, \mathcal{D}_\Psi^2)$, there is a sequence $(A_n : n \in \mathbb{N})$ of finite sets such that for each n, $A_n \subseteq \mathcal{A}_n$, and $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{D}_\Psi^2$. Then $\bigcup_{n \in \mathbb{N}} U_n$ is an open cover of X where $U_n = A_n^c$ for each $n \in \mathbb{N}$ and $\bigcup_{n \in \mathbb{N}} U_n \in \Psi_{\Delta_2}$.

□

Corollary 2.4 (Theorem 6 in [9]). *For a space X the following are equivalent:*

(1) *X satisfies $S_1(\mathcal{K}, \Omega)$;*
(2) *$(2^X, \mathbf{F}^+, \mathbf{Z}^+)$ satisfies $S_1(\mathcal{D}_\Omega^{\mathbf{F}^+}, \mathcal{D}_\Omega^{\mathbf{Z}^+})$.*

Corollary 2.5 (Theorem 14 in [9]). *For a space X the following are equivalent:*

(1) *X satisfies $S_{fin}(\mathcal{K}, \Omega)$;*
(2) *$(2^X, \mathbf{F}^+, \mathbf{Z}^+)$ satisfies $S_{fin}(\mathcal{D}_\Omega^{\mathbf{F}^+}, \mathcal{D}_\Omega^{\mathbf{Z}^+})$.*

3. Local properties of hyperspaces

Let X be a topological space, and $x \in X$. A subset A of X *converges* to x, $x = \lim A$, if A is infinite, $x \notin A$, and for each neighborhood U of x, $A \setminus U$ is finite. Consider the following collection:

- $\Omega_x = \{A \subseteq X : x \in \overline{A} \setminus A\}$;
- $\Gamma_x = \{A \subseteq X : x = \lim A\}$.

Note that if $A \in \Gamma_x$, then there exists $\{a_n\} \subset A$ converging to x. So, simply Γ_x may be the set of non-trivial convergent sequences to x.

- A space X has *strictly Fréchet-Urysohn,* if X satisfies $S_1(\Omega_x, \Gamma_x)$.
- A space X has *strongly Fréchet-Urysohn,* (if $x \in \bigcap\limits_n A_n$ and $A_{n+1} \subset A_n$, then there exist $a_n \in A_n$ such that $a_n \mapsto x$), if X satisfies $S_{fin}(\Omega_x, \Gamma_x)$.

Theorem 3.1. *Assume that $\Phi, \Psi \in \{\Omega, \Gamma\}$, $\star \in \{1, fin\}$. Then for a space X the following statements are equivalent:*

(1) *Each open set $Y \subset X$ has the property $S_\star(\Phi_{\Delta_1}, \Psi_{\Delta_2})$;*
(2) *For each $E \in 2^X$, $(2^X, \Delta_1^+, \Delta_2^+)$ satisfies $S_\star(\Phi_E^{\Delta_1^+}, \Psi_E^{\Delta_2^+})$.*

Proof. (1) ⇒ (2). Let $E \in 2^X$ and let $(\mathcal{A}_n : n \in \mathbb{N})$ be a sequence such that $\mathcal{A}_n \in \Phi_E^{\Delta_1^+}$ for each $n \in \mathbb{N}$. Then $(\mathcal{A}_n^c : n \in \mathbb{N})$ is a sequence of open covers of E^c such that $\mathcal{A}_n^c \in \Phi_{\Delta_1}$ for each $n \in \mathbb{N}$. Since E^c has the property $S_\star(\Phi_{\Delta_1}, \Psi_{\Delta_2})$, there is a sequence $(A_n^c : n \in \mathbb{N})$ such that $A_n^c \in \mathcal{A}_n^c$ for each $n \in \mathbb{N}$ and $\{A_n^c : n \in \mathbb{N}\}$ is open cover of E^c such that $\{A_n^c : n \in \mathbb{N}\} \in \Psi_{\Delta_2}$. It follows that $\{A_n : n \in \mathbb{N}\} \in \Psi_E^{\Delta_2^+}$.

(2) ⇒ (1). Let Y be an open subset of X and let $(\mathcal{F}_n : n \in \mathbb{N})$ be a sequence of open covers of Y such that $\mathcal{F}_n \in \Phi_Y$ where Φ_Y is the Φ_{Δ_1} family of covers of Y. Let $E = X \setminus Y$. Put $\mathcal{A}_n = \mathcal{F}_n^c$ for each $n \in \mathbb{N}$. Then $\mathcal{A}_n \subset 2^X$ and $\mathcal{A}_n \in \Phi_E^{\Delta_1^+}$ for each $n \in \mathbb{N}$. Since, by (2), $(2^X, \Delta_1^+, \Delta_2^+)$ satisfies $S_\star(\Phi_E^{\Delta_1^+}, \Psi_E^{\Delta_2^+})$, there is a sequence $(A_n : n \in \mathbb{N})$ such that $A_n \in \mathcal{A}_n$ for each $n \in \mathbb{N}$ and $\{A_n : n \in \mathbb{N}\} \in \Psi_E^{\Delta_2^+}$. It follows that $\{F_n : F_n = A_n^c, n \in \mathbb{N}\} \in \Psi_{\Delta_2}$.

□

Corollary 3.2 (Theorem 3 in [6]). *For a space X the following statements are equivalent:*

(1) *Each open set $Y \subset X$ has the property $S_1(\Omega, \Gamma)$;*
(2) *$(2^X, \mathbf{Z}^+)$ is Fréchet-Urysohn;*
(3) *$(2^X, \mathbf{Z}^+)$ is strongly Fréchet-Urysohn.*

Corollary 3.3 (Theorem 9 in [9]). *For a space X the following are equivalent:*

(1) *Each open set $Y \subset X$ satisfies $S_{fin}(\Omega, \Omega)$;*
(2) *$(2^X, \mathbf{Z}^+)$ has countable fan tightness (For each $E \in 2^X$, $(2^X, \mathbf{Z}^+)$ satisfies $S_{fin}(\Omega_E, \Omega_E)$).*

Corollary 3.4 (Theorem 31 in [1]). *Assume that $\Phi \in \{\Gamma_k, \mathcal{K}\}$, $\Psi \in \{\Gamma, \Omega\}$, $\star \in \{1, fin\}$. Then for a space X the following statements are equivalent:*

(1) *Each open set $Y \subset X$ has the property $S_\star(\Phi, \Psi)$;*
(2) *For each $E \in 2^X$, $(2^X, \mathbf{F}^+, \mathbf{Z}^+)$ satisfies $S_\star(\Phi_E^{\mathbf{F}^+}, \Psi_E^{\mathbf{Z}^+})$.*

Recall that a space is *perfect* if every open subset is an F_σ-subset [7]. Clearly every semi-stratifiable space is perfect.

The well-known that all properties in the Scheepers Diagram [4, 20] are hereditary for F_σ subsets, i.e. if X satisfies $S_\star(\Phi_\Delta, \Psi_\Delta)$ for $\Delta = \mathbb{F}(X)$ then each F_σ-set $F \subset X$ satisfies $S_\star(\Phi_\Delta, \Psi_\Delta)$ (Corollary 2.4 in [12]).

Definition 3.5. A subset A of a space X is called an Δ-*F_σ-set* if A can be represented as $A = \bigcup\limits_{i=1}^{\infty} F_i$ where F_i is a closed set in X for each $i \in \mathbb{N}$ and for any set $B \in \Delta$ and $B \subseteq A$ there exists $i' \in \mathbb{N}$ such that $B \subseteq F_{i'}$.

In particular, Δ-F_σ-set is F_σ-set (k-F_σ-set) of X for $\Delta = \mathbb{F}(X)$ ($\Delta = \mathbb{K}(X)$) [15].

Definition 3.6. A space X is called Δ-*perfect* if every open subset is an Δ-F_σ-subset of X.

In particular, we get the definitions of perfect space for $\Delta = \mathbb{F}(X)$ and k-perfect space for $\Delta = \mathbb{K}(X)$ [15].

Note that every perfectly normal space is k-perfect (Proposition 4.10 in [15]) and, by definition, every k-perfect space is perfect. For the Sorgenfrey line $\mathbb{S}$, the space $\mathbb{S} \times \mathbb{S}$ is perfect [7], but not k-perfect [15].

In [15] we raised the question: Is there a k-perfect space which is not (perfectly) normal?

According to [17], a regular space with a σ-locally finite k-network is called an $\aleph$-space. Since every metric space has a σ-locally finite base, it is an $\aleph$-space. Recall that Foged in [3] constructed a non-normal space which is an $\aleph$-space.

Proposition 3.7. *Every $\aleph$-space is k-perfect.*

Proof. Let $\mathcal{B} = \bigcup \mathcal{B}_i$ be a σ-locally finite k-network of X where $\mathcal{B}_i$ is a locally finite family of closed subsets of X for each $i \in \mathbb{N}$. Fix an non-empty open set W of X. Let $F_i = \bigcup\{A \in \mathcal{B}_i : A \subset W\}$. Since every locally finite family of closed sets in X is closure-preserving, F_i is a closed subset of X for each $i \in \mathbb{N}$. Let $P_i = \bigcup\limits_{k<i+1} F_k$. Since $\mathcal{B}$ is a k-network of X, for any compact set $T \subset W$ there is a finite family $\{B_{i_1}, ..., B_{i_s}\}$ such that $T \subseteq \bigcup\{B_{i_m} : m = 1, ..., s\} \subseteq W$ and $B_{i_m} \in \mathcal{B}_{i_m}$ for each $m = 1, ..., s$. Then $T \subseteq P_t$ where $t = \max\{i_1, ..., i_s\}$.

□

Proposition 3.8. *There is a non-normal k-perfect space.*

Proof. Consider a non-normal $\aleph$-space (e.g. Foged's example in [3]).

□

Let X be a topological space. A family Δ of compact subsets of X is called *an ideal of compact sets* if $\bigcup \Delta = X$ and for any sets $A, B \in \Delta$ and compact subset $F \subset X$ we get $A \cup B \in \Delta$ and $A \cap F \in \Delta$, i.e. if Δ covers X and is closed under taking finite unions and closed subspaces. The most important cases are the ideal $\Delta = \mathbb{F}(X)$ and the ideal $\Delta = \mathbb{K}(X)$.

Theorem 3.9. *Assume that $\Phi, \Psi \in \{\Omega, \Gamma\}$, $\star \in \{1, fin\}$, X has the property $S_\star(\Phi_{\Delta_1}, \Psi_{\Delta_2})$, Δ_1 is an ideal of compact sets and A is an $\Delta_1 \cup \Delta_2$-F_σ-set. Then A has the property $S_\star(\Phi_{\Delta_1}, \Psi_{\Delta_2})$.*

Proof. We prove the theorem for $\star = fin$, the other proofs being similar.

Assume that X has the property $S_{fin}(\Phi_{\Delta_1}, \Psi_{\Delta_2})$ and A is an $\Delta_1 \cup \Delta_2$-F_σ-set. Consider a sequence $(\mathcal{U}_i : i \in \mathbb{N})$ of covers A such that $\mathcal{U}_i \in \Phi_A$ (where Φ_A is the Φ_{Δ_1} family of covers of A) for each $i \in \mathbb{N}$. Let $A = \bigcup\limits_{i=1}^{\infty} F_i$ where F_i is a closed set in X for each $i \in \mathbb{N}$ and for any compact set $B \subseteq A$ and $B \in \Delta_1 \cup \Delta_2$ there exists $i' \in \mathbb{N}$ such that $B \subseteq F_{i'}$. Consider $\mathcal{V}_i = \{(X \setminus F_i) \bigcup U : U \in \mathcal{U}_i\}$ for each $i \in \mathbb{N}$.

We claim that $\mathcal{V}_i \in \Phi_{\Delta_1}$ for each $i \in \mathbb{N}$. Let $S \in \Delta_1$. Then $S \bigcap F_i$ is a compact subset of A. Since Δ_1 is an ideal of compact sets, $S \bigcap F_i \in \Delta_1$. There is $U \in \mathcal{U}_i$ such that $S \bigcap F_i \subset U$. It follows that $S \subset (X \setminus F_i) \bigcup U$ for $(X \setminus F_i) \bigcup U \in \mathcal{V}_i$.

Since X has the property $S_{fin}(\Phi_{\Delta_1}, \Psi_{\Delta_2})$, there is a sequence $(B_i : i \in \mathbb{N})$ of finite sets such that for each i, $B_i \subset \mathcal{V}_i$, and $\bigcup_{i\in\mathbb{N}} B_i \in \Psi_{\Delta_2}$.

We claim that $\bigcup_{i\in\mathbb{N}}\{B_i \bigcap F_i : B_i \bigcap F_i \subset \mathcal{U}_i,\ i \in \mathbb{N}\} \in \Psi_A$ where Ψ_A is the Ψ_{Δ_2} family of covers of A. Let B be a compact subset of A such that $B \in \Delta_2$ then there exists $i' \in \mathbb{N}$ such that $B \subseteq F_{i'}$. Since $\bigcup_{i\in\mathbb{N}} B_i$ is a large cover of X there is $k \in \mathbb{N}$ and $V_k \in B_k \subset \mathcal{V}_k$ such that $k > i'$ and $B \subset V_k$. But $V_k = (X \setminus F_k) \bigcup U_k$ for $U_k \in \mathcal{U}_k$. Since $k > i'$, $B \subset U_k$. It follows that A has the property $S_{fin}(\Phi_{\Delta_1}, \Psi_{\Delta_2})$. □

Corollary 3.10. *Assume that X is a $\Delta_1 \cup \Delta_2$-perfect space and Δ_1 is an ideal of compact sets, $\Phi, \Psi \in \{\Omega, \Gamma\}$, $\star \in \{1, fin\}$. Then for a space X the following statements are equivalent:*

1. *X has the property $S_\star(\Phi_{\Delta_1}, \Psi_{\Delta_2})$;*
2. *For each $E \in 2^X$, $(2^X, \Delta_1^+, \Delta_2^+)$ satisfies $S_\star(\Phi_E^{\Delta_1^+}, \Psi_E^{\Delta_2^+})$.*

Corollary 3.11. *Assume that $\Phi, \Psi \in \{\Gamma, \Omega\}$, $\star \in \{1, fin\}$. Then for a perfectly normal space X the following statements are equivalent:*

(1) *X satisfies $S_\star(\Phi, \Psi)$;*
(2) *$(2^X, \mathbf{Z}^+)$ satisfies $S_\star(\mathcal{D}_\Phi, \mathcal{D}_\Psi)$;*
(3) *For each $E \in 2^X$, $(2^X, \mathbf{Z}^+)$ satisfies $S_\star(\Phi_E, \Psi_E)$.*

Corollary 3.12. *Assume that $\Phi, \Psi \in \{\Gamma_k, \mathcal{K}\}$, $\star \in \{1, fin\}$. Then for a $\aleph$-space X the following statements are equivalent:*

(1) *X satisfies $S_\star(\Phi, \Psi)$;*
(2) *$(2^X, \mathbf{F}^+)$ satisfies $S_\star(\mathcal{D}_\Phi, \mathcal{D}_\Psi)$;*
(3) *For each $E \in 2^X$, $(2^X, \mathbf{F}^+)$ satisfies $S_\star(\Phi_E, \Psi_E)$.*

We can summarize the relationships between considered selective properties in next theorem.

Theorem 3.13. *Assume that X is a $\Delta_1 \cup \Delta_2$-perfect space and Δ_1 is an ideal of compact sets, $\Phi, \Psi \in \{\Omega, \Gamma\}$, $\star \in \{1, fin\}$. Then for a space X the following statements are equivalent:*

(1) *X satisfies $S_\star(\Phi_{\Delta_1}, \Psi_{\Delta_2})$;*
(2) *$(2^X, \Delta_1^+, \Delta_2^+)$ satisfies $S_\star(\mathcal{D}^1_\Phi, \mathcal{D}^2_\Psi)$.*
(3) *Each open set $Y \subset X$ has the property $S_\star(\Phi_{\Delta_1}, \Psi_{\Delta_2})$;*
(4) *For each $E \in 2^X$, $(2^X, \Delta_1^+, \Delta_2^+)$ satisfies $S_\star(\Phi_E^{\Delta_1^+}, \Psi_E^{\Delta_2^+})$.*

In particular, this theorem is true for k-perfect spaces and, hence, for $\aleph$-spaces.

References

[1] A. Caserta, G. Di Maio, Lj. D. R. Kočinac and E. Meccariello, Applications of k-covers II, Topology and its Applications 153 (2006), 3277–3293.

[2] J. Fell, A Hausdorff topology for the closed subsets of a locally compact non-Hausdorff spaces, Proceedings of the American Mathematical Society 13 (1962), 472–476.

[3] L. Foged, Normality in k-and-$\aleph$ spaces, Topology and its Applications 22 (1986), 223–240.

[4] W. Just, A. W. Miller, M. Scheepers and P. J. Szeptycki, The combinatorics of open covers, II, Topology and its Applications 73 (1996), 241–266.

[5] Lj. D. R. Kočinac, Selected results on selection principles, in: Proceedings of the 3rd Seminar on Geometry and Topology (Sh. Rezapour, ed.), July 15–17, Tabriz, Iran, (2004), 71–104.

[6] Lj. D. R. Kočinac, γ-sets, γ_k-sets and hyperspaces, Mathematica Balkanica 19 (2005), 109–118.

[7] R. W. Heath and E. A. Michael, A property of the Sorgenfrey line, Compositio Mathematica 23, no. 2 (1971), 185–188.

[8] Z. Li, Selection principles of the Fell topology and the Vietoris topology, Topology and its Applications 212 (2016), 90–104.

[9] G. Di Maio, Lj. D. R. Kočinac and E. Meccariello, Selection principles and hyperspace topologies, Topology and its Applications 153 (2005), 912–923.

[10] G. Di Maio, Lj. D. R. Kočinac and T. Nogura, Convergence properties of hyperspaces, J. Korean Math. Soc. 44, no. 4 (2007), 845–854.
[11] M. Mršević and M. Jelić, Selection principles in hyperspaces with generalized Vietoris topologies, Topology and its Applications 156, no. 1 (2008), 124–129.
[12] T. Orenshtein and B. Tsaban, Linear σ-additivity and some applications, Transactions of the American Mathematical Society 363, no. 7 (2011), 3621–3637.
[13] A. V. Osipov, Application of selection principles in the study of the properties of function spaces, Acta Math. Hungar. 154, no. 2 (2018), 362–377.
[14] A. V. Osipov, Classification of selectors for sequences of dense sets of $C_p(X)$, Topology and its Applications 242 (2018), 20–32.
[15] A. V. Osipov, Selectors for sequences of subsets of hyperspaces, Topology and its Applications 275 (2020), 107007.
[16] A. V. Osipov, The functional characteristics of the Rothberger and Menger properties, Topology and its Applications 243 (2018), 146–152.
[17] P. O'Meara, On paracompactness in function spaces with the compact-open topology, Proc. Amer. Math. Soc. 29 (1971), 183–189.
[18] M. Sakai, Selective separability of Pixley-Roy hyperspaces, Topology and its Applications 159 (2012), 1591–1598.
[19] M. Sakai and M. Scheepers, The combinatorics of open covers, Recent Progress in General Topology III, (2013), 751–799.
[20] M. Scheepers, Combinatorics of open covers (I): Ramsey Theory, Topology and its Applications 69 (1996), 31–62.
[21] B. Tsaban, Some New Directions in Infinite-combinatorial Topology, in: Bagaria J., Todorcevic S. (eds) Set Theory. Trends in Mathematics. Birkhäuser Basel.(2006).