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Digital semicovering and digital quasicovering maps

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Abstract

In this paper we introduce notions of digital semicovering and digital quasicovering maps. We show that these are generalizations of digital covering maps and investigate their relations. We will also clarify the relationship between these generalizations and digital path lifting.

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1. Introduction and motivation

There are maps in digital topology that are not coverings, but have some properties of covering maps such as the path lifting property and unique path lifting property. As an example, a digital map obtained by restricting the domain of a digital covering map is not necessarily a digital covering map but has uniqueness of digital path liftings. This has made it important for us to generalize the notion digital covering map.

Han [9] has introduced a generalization of digital covering maps, named digital pseudocovering map, by weakening the local isomorphism condition in the definition of digital covering maps.

Pakdaman [13] has shown that any digital pseudocovering map (by Han's definition) is a digital covering map and changed the definition to achieve a correct generalization.

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Here, we will introduce two types of generalizations for digital covering maps: digital semicovering maps and digital quasicovering maps. We define these concepts by using local properties of digital maps, and then we stabilize them using the notions of digital path lifting property and digital unique path lifting property.

Then, we present the conditions when these generalizations are equivalent to each other and to digital covering maps. Although digital covering theory seemed to be a special case of graph covering theory, [14] has shown that it is not. We find that the digital quasicovering introduced here is a special case of graph covering.

2. Notations and preliminaries

For a positive integer u with $1 \leq u \leq n$, an adjacency relation of a digital image in $\mathbb{Z}^n$ is defined as follows:

Two distinct points $p = (p_1, p_2, ..., p_n)$ and $q = (q_1, q_2, ..., q_n)$ in $\mathbb{Z}^n$ are c_u-adjacent, denoted by $p \overset{c_u}{\longleftrightarrow} q$, [4] if there are at most u distinct indices i such that $|p_i - q_i| = 1$ and for all indices j, $p_j = q_j$ if $|p_j - q_j| \neq 1$. A c_u-adjacency relation on $\mathbb{Z}^n$ can be denoted by the number of points that are c_u-adjacent to a given point $p \in \mathbb{Z}^n$. For example,

- The c_1-adjacent points of $\mathbb{Z}$ are called 2-adjacent.
- The c_1-adjacent points of $\mathbb{Z}^2$ are called 4-adjacent and the c_2-adjacent points in $\mathbb{Z}_2$ are called 8-adjacent.
- The c_1-adjacent, c_2-adjacent and c_3-adjacent points of $\mathbb{Z}^3$ are called 6-adjacent, 18-adjacent, and 26-adjacent, respectively.

More general adjacency relations are studied in [10].

Let κ be an adjacency relation defined on $\mathbb{Z}^n$ and $X \subseteq \mathbb{Z}^n$. Then the pair (X, κ) is said to be a (binary) digital image. A digital image $X \subseteq \mathbb{Z}^n$ is **κ-connected** [16] if and only if for every pair of different points $x, y \in X$, there is a set $x_0, x_1, ..., x_r$ of points of a digital image X such that $x = x_0, y = x_r$ and x_i and x_{i+1} are κ-adjacent where $i = 0, 1, ..., r-1$.

Proposition 2.1 ([2, 15])**.** *Let (X, κ) in $\mathbb{Z}^n$ and (Y, λ) in $\mathbb{Z}^m$ be digital images. A function $f : X \longrightarrow Y$ is (κ, λ)-continuous if and only if for every κ-adjacent points $x_0, x_1 \in X$, either $f(x_0) = f(x_1)$ or $f(x_0)$ and $f(x_1)$ are λ-adjacent in Y.*

For $a, b \in \mathbb{Z}$ with $a < b$, a **digital interval** [1] is a set of the form

$$[a, b]_{\mathbb{Z}} = \{z \in \mathbb{Z} | a \leq z \leq b\}.$$

Definition 2.2. By a **digital κ-path** from x to y in a digital image (X, κ), we mean a $(2, \kappa)$-continuous function $f : [0, m]_{\mathbb{Z}} \longrightarrow X$ such that $f(0) = x$ and $f(m) = y$. If $f(0) = f(m)$ then the κ-path is said to be closed, and f is called a κ-loop.

Let $f : [0, m-1]_{\mathbb{Z}} \longrightarrow X \subseteq \mathbb{Z}^n$ be a $(2, \kappa)$-continuous function such that $f(i)$ and $f(j)$ are κ-adjacent if and only if $j = i \pm 1 \mod m$. Then the set $f([0, m-1]_{\mathbb{Z}})$ is a simple closed κ-curve containing m points, denoted by $SC_{\kappa}^{n,m}$.

If f is a constant function, it is called a trivial loop.

If $f : [0, m_1]_{\mathbb{Z}} \longrightarrow X$ and $g : [0, m_2]_{\mathbb{Z}} \longrightarrow X$ are digital κ-paths with $f(m_1) = g(0)$, then define the product [11] $(f * g) : [0, m_1 + m_2]_{\mathbb{Z}} \longrightarrow X$ by

$$(f * g)(t) = \begin{cases} f(t) & if \quad t \in [0, m_1]_{\mathbb{Z}}; \\ g(t - m_1) & if \quad t \in [m_1, m_1 + m_2]_{\mathbb{Z}}. \end{cases}$$

Let (E, κ) be a digital image and let $\varepsilon \in N$. The κ-neighborhood [8] of $e_0 \in E$ with radius ε is the set $N_\kappa(e_0, \varepsilon) = \{e \in E | \; l_\kappa(e_0, e) \leq \varepsilon\} \cup \{e_0\}$, where $l_\kappa(e_0, e)$ is the length of a shortest κ-path from e_0 to e in E.

By the notations above, the function $f : X \longrightarrow Y$ is a (κ, λ)**-isomorphism** [3], denoted by $X \overset{(\kappa,\lambda)}{\approx} Y$, if f is a (κ, λ)-continuous bijection and further $f^{-1} : Y \longrightarrow X$ is (λ, κ)-continuous. If $\kappa = \lambda$, then f is called a κ-isomorphism.

Definition 2.3 ([8])**.** For two digital spaces (X, κ) in $\mathbb{Z}^n$ and (Y, λ) in $\mathbb{Z}^m$, a (κ, λ)-continuous map $h : X \longrightarrow Y$ is called a **local** (κ, λ)**-isomorphism** if for every $x \in X$, $h|_{N_\kappa(x;1)}$ is a (κ, λ)-isomorphism onto $N_\lambda(h(x); 1)$. If $n = m$ and $\kappa = \lambda$, then the map h is called a local κ-isomorphism.

Definition 2.4 ([9])**.** For two digital spaces (X, κ) and (Y, λ), a map $h : X \longrightarrow Y$ is called a **weakly local** (κ, λ)**-isomorphism** if for every $x \in X$, h maps (κ, λ)-isomorphically $N_\kappa(x, 1)$ onto $h(N_\kappa(x, 1))$.

In the definition of local isomorphism we can remove the condition of the continuity of h, because continuity is a local notion and for every $x \in X$, $h|_{N_\kappa(x;1)}$ is a (κ, λ)-isomorphism and hence h is continuous. Also, it is notable that the difference between local isomorphisms and weakly local isomorphisms is surjectivity of $h|_{N_\kappa(x;1)}$.

For $n \in \mathbb{N}$, the map h is called an n-radius local isomorphism if for every $x \in X$, the restriction map $h|_{N_\kappa(x,n)} : N_\kappa(x, n) \longrightarrow N_\lambda(h(x), n)$ is a (κ, λ)-isomorphism.

Definition 2.5 ([6])**.** Let (E, κ) and (B, λ) be digital images and $p : E \longrightarrow B$ be a (κ, λ)-continuous surjection. The map p is called a (κ, λ)**-covering map** if there exists an index set M such that for each $b \in B$

(1) $p^{-1}(N_\lambda(b, 1)) = \bigsqcup_{i \in M} N_\kappa(e_i, 1)$ with $e_i \in p^{-1}(b)$;

(2) if $i, j \in M$, $i \neq j$, then $N_\kappa(e_i, 1) \cap N_\kappa(e_j, 1) = \varnothing$; and

(3) the restriction map $p|_{N_\kappa(e_i,1)} : N_\kappa(e_i, 1) \longrightarrow N_\lambda(b, 1)$ is a (κ, λ)-isomorphism for all $i \in M$.

Moreover, $(E; p; B)$ is said to be a (κ, λ)-covering and (E, κ) is called a digital (κ, λ)-covering space over (B, λ). Also, $N_\lambda(b, 1)$ is called an elementary λ-neighborhood of b or a coverable λ-neighborhood of b.

Definition 2.6 ([7])**.** Let (E, κ), (B, λ), and (X, μ) be digital images, let $p : E \longrightarrow B$ be a (κ, λ)-covering map, and let $f : X \longrightarrow B$ be (μ, λ)-continuous. A **lifting** of f with respect to p is a (μ, κ)-continuous function $\widetilde{f} : X \longrightarrow E$ such that $p \circ \widetilde{f} = f$.

Theorem 2.7 ([7]). *Let (E,κ) be a digital image and $e_0 \in E$. Let (B,λ) be a digital image and $b_0 \in B$. Let $p : E \longrightarrow B$ be a (κ,λ)-covering map such that $p(e_0) = b_0$. Then any λ-path $\alpha : [0,m]_{\mathbb{Z}} \longrightarrow B$ beginning at b_0 has a unique lifting to a path $\widetilde{\alpha}$ in E beginning at e_0.*

Definition 2.8 ([14]). Let $p : (E,\kappa) \to (B,\lambda)$ be a (κ,λ)-continuous surjection map. We say that
(i) p has **digital path lifting property** if for any digital path α in B and any $e \in p^{-1}(\alpha(0))$ there is a lifting $\widetilde{\alpha}$ of α in E such that $\widetilde{\alpha}(0) = e$.
(ii) p has the **uniqueness of digital path lifts property** if any two paths $\alpha, \beta : [0,m]_{\mathbb{Z}} \longrightarrow E$ are equal if $p \circ \alpha = p \circ \beta$ and $\alpha(0) = \beta(0)$.
(iii) p has the **unique path lifting property** (u.p.l, for abbreviation) if it has both the path lifting property and the uniqueness of path lifts property.

Although every digital covering map is a local isomorphism (by definition) and the converse is not true ([6]), the author and M.Zakki [14] showed the counterexample of [6] is incorrect and have proved the inverse as follows.

Theorem 2.9 ([14]). *Let $p : (E,\kappa) \to (B,\lambda)$ be a (κ,λ)-continuous surjection map. Then p is a digital (κ,λ)-covering map if and only if it is a local isomorphism.*

Definition 2.10 ([14]). Let $p : (E,\kappa) \to (B,\lambda)$ be a (κ,λ)-continuous map and $e \in E$. We say that e is a conciliator point for p if there exist $e', e'' \in N_\kappa(e,1)$ for which $e' \overset{\kappa}{\nleftrightarrow} e''$ and $p(e') \overset{\lambda}{\leftrightarrow} p(e'')$.

Theorem 2.11 ([14]). *A (κ,λ)-continuous surjection map $p : (E,\kappa) \to (B,\lambda)$ is a digital (κ,λ)-covering if it has u.p.l and has no conciliator point.*

In this paper, all the digital spaces assumed to be connected.

3. Digital pseudocovering maps

A digital pseudocovering map, introduced by Han [9], is a generalization of digital covering maps, obtained by weakening the local isomorphism condition in the definition of digital covering maps. Han [9] replaced the local isomorphism property by weak local isomorphism and proved some results for digital pseudocovering maps. The author [13] has shown that Han's examples of digital pseudocovering maps were either digital isomorphisms or did not satisfy the definition of digital pseudocovering map. Also, the author even has found a gap in the Han's definition of digital pseudocovering maps and has corrected it.

Definition 3.1 ([9]). Let $(E;\kappa_0)$ and $(B;\kappa_1)$ be digital spaces in $\mathbb{Z}^{n_0}$ and $\mathbb{Z}^{n_1}$, respectively. Let $p : (E;\kappa_0) \longrightarrow (B;\kappa_1)$ be a surjection. Suppose that for any $b \in B$ the map p has the following properties:

(1) for some index set M, $p^{-1}(N_{\kappa_1}(b;1)) = \bigsqcup_{i\in M} N_{\kappa_0}(e_i;1)$ with $e_i \in p^{-1}(b)$,

(2) if $i, j \in M$ and $i \neq j$, then $N_{\kappa_0}(e_i; 1) \cap N_{\kappa_0}(e_j; 1)$ is an empty set; and
(3) the restriction of p on $N_{\kappa_0}(e_i; 1)$ is a weakly local $(k_0; k_1)$-isomorphism for all $i \in M$.

Then the map p is called a $(k_0; k_1)$-pseudocovering map, $(E; p; B)$ is said to be a $(k_0; k_1)$-pseudocovering and $(E; k_0)$ is called a $(k_0; k_1)$-pseudocovering space over $(B; k_1)$.

This definition is like the definition of a digital covering map, but $p|_{N_{\kappa_0}(e_i;1)}$ is a weakly local $(k_0; k_1)$-isomorphism rather than a $(k_0; k_1)$-isomorphism.

In [9], Example 4.3(1) it is claimed that the map

$$f : \mathbb{Z}^+ \longrightarrow SC_\kappa^{n,l} := (s_i)_{i=0}^{l-1};\ l \geq 4,$$

defined by $f(i) = s_{i\ mod\ l}$, where $\mathbb{Z}^+ = \{k \in \mathbb{Z} | k \geq 0\}$ is a pseudocovering map. Let $l = 6$ and consider the point s_5. Since $s_0 \in N_\kappa(s_5; 1)$, we must have $0 \in p^{-1}(N_\kappa(s_5; 1))$. Also, $p^{-1}\{s_5\} = \{5k | k \in \mathbb{N}\}$ and by the condition (1), $p^{-1}(N_\kappa(s_5; 1)) = \bigsqcup_{k \in \mathbb{N}} N_2(5k; 1)$. But for every $k \in \mathbb{N}$, $0 \notin N_2(5k; 1)$. This contradiction shows that f can not be a pseudocovering map.

Theorem 3.2 ([13])**.** *Every map satisfying the conditions of Han's pseudocovering map is a covering map.*

According to the type of the gap in the definition, the author has corrected it as follows. Of course, the results that Han had achieved still hold true by this definition, since his (3) is a weaker condition than we give for (3) in the following.

Definition 3.3. Let $(E; \kappa_0)$ and $(B; \kappa_1)$ be digital spaces in $\mathbb{Z}^{n_0}$ and $\mathbb{Z}^{n_1}$, respectively. Let $p : (E; \kappa_0) \longrightarrow (B; \kappa_1)$ be a surjection. Suppose that the map p for each $b \in B$ has the following properties:

(1) there exist index set M such that $\bigsqcup_{i \in M} N_{\kappa_0}(e_i; 1) \subseteq p^{-1}(N_{\kappa_1}(b; 1))$ with $e_i \in p^{-1}(b)$,
(2) if $i, j \in M$ and $i \neq j$, then $N_{\kappa_0}(e_i; 1) \cap N_{\kappa_0}(e_j; 1)$ is an empty set; and
(3) the restriction map $p|_{N_{\kappa_0}(e_i;1)} : N_{\kappa_0}(e_i; 1) \longrightarrow p\big(N_{\kappa_0}(e_i; 1)\big)$ is a $(k_0; k_1)$-isomorphism for all $i \in M$.

Then the map p is called a $(k_0; k_1)$-pseudocovering map.

Remark 3.4. Note that the index set M is not necessarily same for each $b \in B$. For example, see Remark 4.7.

By this definition of digital pseudocovering map, the map

$$f : \mathbb{Z}^+ \longrightarrow SC_\kappa^{n,l} := (s_i)_{i=0}^{l-1};\ l \geq 4,$$

defined by $f(i) = s_{i\ mod\ l}$ is a pseudocovering map which is not a digital covering map and so we can consider digital pseudocovering maps as generalizations of digital covering maps.

4. Digital semicovering maps

S. E. Han [9] has used the weak version of local isomorphism instead of local isomorphism in the definition of digital covering map to define digital pseudocovering map, and investigated its various properties. For example, he has proved that a digital pseudocovering map with some hypothesis has the uniqueness of digital path lifts property. He named the uniqueness of digital path lifts property as the pseudolifting property, but we use the uniqueness of digital path lifts property to unify with common naming in general topology. We introduce a generalization of digital covering theory and study its properties.

Definition 4.1. A digital map $p : (E; \kappa) \longrightarrow (B; \lambda)$ is called **locally injective** if for every $e \in E$ the restricted map $p|_{N_\kappa(e,1)}$ is injective.

Definition 4.2. A digitally continuous surjection map $p : (E; \kappa) \longrightarrow (B; \lambda)$ is called a digital (κ, λ)**-semicovering** if it is locally injective.

When κ and λ are understood, we say p is a digital semicovering map.

Example 4.3. The map $f : \mathbb{Z}^+ \longrightarrow SC_\kappa^{n,l} := (s_i)_{i=0}^{l-1}$ defined by $f(i) = s_{i(mod\ l)}$ is a $(2; \kappa)$-semicovering, for $l \geq 4$.

The notion of a semicovering map generalizes that of a covering map. We see they are not equivalent, as follows.

Theorem 4.4. *Every digital covering is a digital semicovering, but the converse does not hold.*

Proof. The first assertion follows easily from the definition of digital covering map and digital semicovering map. For the converse, consider the map $f : \mathbb{Z}^+ \longrightarrow SC_\kappa^{n,l}$, discussed in Example 4.3 which is not a digital covering map, as it shown in Section 3. □

Note that for a digital semicovering $p : (E; \kappa) \longrightarrow (B; \lambda)$, the map $p|_{N_\kappa(e,1)} : N_\kappa(e, 1) \longrightarrow p(N_\kappa(e, 1))$ is not necessarily an isomorphism and therefore digital semicoverings do not satisfiy the third condition of digital pseudocovering maps (need not be weakly local isomorphisms). The following example shows this.

Example 4.5. Consider the map $p : \mathbb{Z}^+ \longrightarrow B$ by $p(i) = s_{i(mod\ 4)}$, where

$$B = \{s_0 = (0, 0), s_1 = (0, 1), s_2 = (-1, 1), s_3 = (-1, 0)\} \subset \mathbb{Z}^2$$

with 8-adjacency. Then for every $i \in \mathbb{Z}$, $N_2(i; 1) = \{i - 1, i, i + 1\}$ when indices are reduced mod 4 and obviously the inverse of $p|_{N_2(3;1)} : N_2(3; 1) \longrightarrow p(N_2(3; 1))$ is not continuous.

Theorem 4.6. *A digital semicovering map $p : (E; \kappa) \longrightarrow (B; \lambda)$ is a weakly local isomorphism if and only if it has no conciliator point.*

Proof. Let $p : (E; \kappa) \longrightarrow (B; \lambda)$ be a weakly local isomorphism. If for a given point $e \in E$, there exist $e', e'' \in N_\kappa(e, 1)$ for which $e' \overset{\kappa}{\nleftrightarrow} e''$ and $p(e') \overset{\lambda}{\leftrightarrow} p(e'')$,

then the inverse map of $p|_{N_\kappa(e,1)} : N_\kappa(e,1) \longrightarrow p(N_\kappa(e,1))$ is not continuous which is contradiction.
Conversely, assume that p is a digital semicovering map. Hence, for every $e \in E$, $p|_{N_\kappa(e,1)} : N_\kappa(e,1) \longrightarrow p(N_\kappa(e,1))$ is bijective and (κ,λ)-continuous. Also, p has no conciliator point which implies that the inverse of $p|_{N_\kappa(e,1)}$ is continuous. □

Remark 4.7. According to Definition 4.2, the fibers (inverse image of a singleton) of a digital semicovering need not have the same cardinal number and also, for $e, e' \in p^{-1}(b)$, $N_\kappa(e,1)$ need not be κ-isomorphic to $N_\kappa(e',1)$. For this, consider the map

$$h : [0,5]_{\mathbb{Z}} \longrightarrow SC_8^{2,4} := (s_j)_{j=0}^3$$

given by $h(i) = s_{i(mod\ 4)}$, which is a digital semicovering. Since h has no conciliator point, it is a digital pseudocovering map. Then $h^{-1}(s_0) = \{0,4\}$ and $h^{-1}(s_2) = \{2\}$ do not have the same cardinal number. Also, $N_2(0,1)$ is not 2-isomorphic to $N_2(4,1)$.

Proposition 4.8. *Let $p : (E;\kappa) \longrightarrow (B;\lambda)$ be locally injective and $b \in B$. Then*

(1) *if $e \neq e'$ and $p(e) = p(e')$ then $l_\kappa(e,e') > 2$.*
(2) *for every $e \neq e' \in p^{-1}(b)$, $N_\kappa(e;1) \cap N_\kappa(e';1) = \varnothing$.*

Proof. If $e \neq e'$ and $p(e) = p(e')$ then local injectivity implies $l_\kappa(e,e') > 2$. Part 2 comes easily from (1). □

Theorem 4.9. *Composition of any two digital semicovering map is a digital semicovering map.*

Proof. This comes easily from definitions and the fact that a composition of digitally continuous functions is digitally continuous [1, 2]. □

Theorem 4.10. *Let $p : (E;\kappa) \longrightarrow (B;\lambda)$ be a digitally continuous surjection map where E is κ-connected. Then*
(i) For digitally λ'-connected space $(X;\lambda')$ and two (λ',κ)-continuous maps $f_0, f_1 : X \longrightarrow E$ both coinciding at one point $x_0 \in X$ and satisfying $p \circ f_0 = p \circ f_1$, if p is a digital (κ,λ)-semicovering, then $f_0 = f_1$.
(ii) p is a (κ,λ)-semicovering if and only if it has the uniqueness of digital path lifts property.

Proof. (i) Assume that there exists a point $x \in X$ such that $f_o(x) \neq f_1(x)$ and $\alpha : [0,n]_{\mathbb{Z}} \longrightarrow E$ is a path from x_0 to x. If t_0 is the smallest $t \in [0,n]_{\mathbb{Z}}$ such that $f_0(\alpha(t)) \neq f_1(\alpha(t))$, then $f_0(\alpha(t_0-1)) = f_1(\alpha(t_0-1))$. Since $p \circ f_0 = p \circ f_1$, $p(f_0(\alpha(t_0))) = p(f_1(\alpha(t_0)))$ which is contradiction to injectivity of $p|_{N_\kappa(f_0(\alpha(t_0-1));1)}$.

(ii) By part (i), every digital semicovering map has the uniqueness of digital path lifts property. Conversely, for a given $e \in E$, let $e', e'' \in N_\kappa(e,1)$, $p(e') = p(e'') = b'$ and $p(e) = b$. Define $\alpha : [0,1]_{\mathbb{Z}} \longrightarrow B$ and $\beta, \gamma : [0,1]_{\mathbb{Z}} \longrightarrow E$ by $\alpha(0) = b$, $\alpha(1) = b'$, $\beta(0) = e$, $\beta(1) = e'$, $\gamma(0) = e$ and $\gamma(1) = e''$.

Then $p \circ \beta = p \circ \gamma = \alpha$ and $\beta(0) = \gamma(0)$. By the uniqueness of digital path lifts property we have $\beta = \gamma$ which implies that $e' = e''$. Hence $p|_{N_\kappa(e,1)}$ is injective. □

Although digital semicoverings have the uniqueness of digital path lifts property, they need not have digital path lifting property. For, if h is the map in Remark 4.7, then the map

$$\alpha : [0,1]_{\mathbb{Z}} \longrightarrow SC_8^{2,4}; \quad \alpha(0) = s_0,\ \alpha(1) = s_3,$$

is continuous (because s_0 and s_3 are 8-adjacent) and has no lifting started from 0.

Proposition 4.11. *Digital semicovering maps need not have digital path lifts property.*

Restriction of a digital covering map to a subspace of its domain is not necessarily a digital covering map. For example the map

$$p : \mathbb{Z} \longrightarrow SC_\kappa^{n,m} := (s_i)_{i=0}^{m-1}, \quad p(i) = s_{i \ (mod\ m)},$$

is a $(2,\kappa)$-covering map but $p|_{\mathbb{Z}^+}$ is not a digital covering map. This problem has been fixed in digital semicoverings.

Theorem 4.12. *Let $p : (E;\kappa) \longrightarrow (B;\lambda)$ be a digital semicovering and $(E';\kappa)$ be a subset of $(E;\kappa)$ such that $p|_{E'}$ is onto. Then $p|_{E'} : (E';\kappa) \longrightarrow (B;\lambda)$ is a (κ,λ)-semicovering.*

Proof. It is obvious because continuity and local injectivity are inherited from p. □

5. Digital quasicovering maps

In this section, another generalization of digital covering maps is provided. There are some examples of digital semicoverings that have stronger conditions than local injectivity and consequently, they also have the digital path lifts property.

Definition 5.1. A digitally continuous surjection map $p : (E;\kappa) \longrightarrow (B;\lambda)$ is called digital (κ,λ)**-quasicovering** if it is locally bijective, i.e. $p|_{N_\kappa(e,1)} : N_\kappa(e,1) \longrightarrow N_\lambda(p(e),1)$ is bijective, for every $e \in E$. When κ and λ are understood, we say p is a digital quasicovering map.

Obviously, every digital quasicovering map is a digital semicovering map, but the converse is not true. For example, the map p in Example 4.3 is a digital semicovering which is not a digital quasicovering because $p|_{N_2(0,1)}$ is not surjective. Also, we would like to know the difference between digital quasicoverings and digital covering maps.

Example 5.2. Consider the map $p : \mathbb{Z} \longrightarrow B$ by $p(i) = s_{i(mod\ 3)}$, where

$$B = \{s_0 = (0,0), s_1 = (0,1), s_2 = (-1,1)\} \subset \mathbb{Z}^2$$

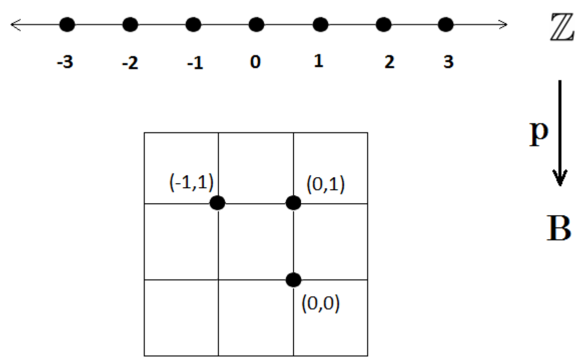

Figure 1. A (2,8)-quasicovering that is not (2,8)-covering

with 8-adjacency (see Figure 2). Then p is a $(2,8)$-continuous surjection that is also locally bijective and so a digital quasicovering. But p is not a digital $(2,8)$-covering because p does not satisfy condition (3) of Definition 2.5: For example, when $e = 0$ we have that $N_2(e,1) = \{-1,0,1\}$ and so the inverse of the restriction of p to $N_2(e,1)$ is not $(8,2)$-continuous, because it maps two 8-adjacent points s_1 and s_2 in $N_8(s_0,1) = B$ to $\{-1,1\}$ that are not 2-adjacent.

Theorem 5.3. *A digital semicovering map is a digital quasicovering map if and only if it has digital path lifting property.*

Proof. For a digital semicovering map $p : (E;\kappa) \longrightarrow (B;\lambda)$, let $e \in p^{-1}(b)$. Then $p|_{N_\kappa(e,1)}$ is injective. Suppose p has the path lifting property. If $b' \in N_\lambda(b,1)$, then the path $\alpha : [0,1]_{\mathbb{Z}} \longrightarrow B$ defined by $\alpha(0) = b$ and $\alpha(1) = b'$ has a lifting path $\widetilde{\alpha}$ started from e, by digital path lifting property of p. Since $p \circ \widetilde{\alpha} = \alpha$, $p(\widetilde{\alpha}(1)) = \alpha(1) = b'$ and hence $p : N_\kappa(e,1) \longrightarrow N_\lambda(b,1)$ is surjective. Therefore p is a digital quasicovering map.
Now, let p be a digital quasicovering map. We show that it has digital path lifting property. For a digital path $\alpha : [0,m]_{\mathbb{Z}} \longrightarrow B$, let $b_i = \alpha(i)$, where $i = 0,1,\ldots,m$. For every $e_0 \in p^{-1}(b_0)$, there exists $e_1 \in N_\kappa(e_0,1)$ such that $p(e_1) = b_1$ because $p|_{N_\kappa(e_0,1)}$ is bijective. Similarly, for every i, there exist $e_i \in N_\kappa(e_{i-1},1)$ such that $p(e_i) = b_i$ because $p|_{N_\kappa(e_{i-1},1)}$ is bijective. Define $\widetilde{\alpha} : [0,m]_{\mathbb{Z}} \longrightarrow E$ by $\widetilde{\alpha}(i) = e_i$. Since $e_i \in N_\kappa(e_{i-1},1)$, $\widetilde{\alpha}$ is continuous and $p \circ \widetilde{\alpha} = \alpha$ because $p(e_i) = b_i$. □

Corollary 5.4. *A digitally continuous surjection map $p : (E;\kappa) \longrightarrow (B;\lambda)$ is a (κ,λ)-quasicovering if and only if it has unique path lifiting property.*

Proof. Let the digitally continuous surjection map p has unique path lifting property. Since p has uniqueness of digital path lifts property, by Theorem 4.10, p is a digital semicovering map. Also, p has digital path lifting property

which implies that p is a digital quasicovering by Theorem 5.3.
Conversely, assume p is a digital quasicovering map. Since p is a digital semicovering map, p has uniqueness of digital path lifts property, by Theorem 4.10. By Theorem 5.3, p has the unique path lifiting property. □

Corollary 5.5. *Every digital quasicovering map is a digital covering map if it has no conciliator point.*

Proof. This follows from Theorem 2.11 and Corollary 5.4. □

Corollary 5.6. *Every digital semicovering is a digital covering map if it has digital path lifting property and has no conciliator point.*

Proof. By Part ii of Theorem 4.10, a digital semicovering $p : (E;\kappa) \longrightarrow (B;\lambda)$ has the uniqueness of digital path lifts property and hence by Corollary 5.4, it is a (κ,λ)-quasicovering. Since we are assuming B has no conciliator point, Corollary 5.5 implies p is a digital covering map. □

6. Corrections of some past proofs

In [14], Example 3.6, the authors claimed the map $h : \mathbb{Z}^+ \longrightarrow SC_8^{2,4} =: (s_i)_{i\in[0,3]_\mathbb{Z}}$ given by $h(i) = s_{i \ mod \ 4}$ has the uniqueness of digital path lifts property but does not have the digital path lifting property. The claim is true but some cases are missing from the proof. To prove that h has the uniqueness of digital path lifts property, let $\alpha, \beta : [0,m]_\mathbb{Z} \longrightarrow \mathbb{Z}^+$ be two paths in which $h\circ\alpha = h\circ\beta$ and $\alpha(0) = \beta(0) = d$. We show that $\alpha = \beta$. If not, then there is an $s \in [0,m]_\mathbb{Z}$ such that $\alpha(s) \neq \beta(s)$. We may assume that s is the smallest $t \in [0,m]_\mathbb{Z}$ such that $\alpha(t) \neq \beta(t)$. Thus we have the following:

$$\begin{cases} \alpha(s) \neq \beta(s), \\ \alpha(t) = \beta(t), \ \textit{for} \ \ \textit{all}\ t \in [0,s-1]_\mathbb{Z}, \\ h\circ\alpha(t) = h\circ\beta(t), \ \textit{for} \ \ \textit{all}\ t \in [0,m]_\mathbb{Z}. \end{cases}$$

If $k := \alpha(s-1) = \beta(s-1)$, then we have six cases

$$\begin{cases} \alpha(s) = k, \\ \beta(s) = k\pm 1. \end{cases} \ \textit{or} \ \begin{cases} \alpha(s) = k\pm 1, \\ \beta(s) = k. \end{cases} \ \textit{or} \ \begin{cases} \alpha(s) = k+1, \\ \beta(s) = k-1. \end{cases} \ \textit{or} \ \begin{cases} \alpha(s) = k-1, \\ \beta(s) = k+1. \end{cases}$$

Since $h\circ\alpha(s) = h\circ\beta(s)$, we must have either $h(k) = h(k\pm1)$ or $h(k-1) = h(k+1)$. These are contradictions, because $h(j) = h(k)$ if and only if $j = k \ mod \ 4$.

In [14, Theorem 4.5], also it is claimed that a (κ,λ)-covering map $p : (E,\kappa) \longrightarrow (B,\lambda)$ is a radius n covering map (a covering map that is a radius n local isomorphism) if and only if every lifting of any simple λ-loop with length at most $2n+1$ is a simple κ-loop. Although there are some problems in its proof, no counterexample has been found yet.

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