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Some network-type properties of the space of G-permutation degree

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Abstract

In this paper the network-type properties (network, $cs-$network, cs^*-network, $cn-$network and $ck-$network) of the space $\mathsf{SP}^n_G X$ of G-permutation degree of X are studied. It is proved that:
(1) If X is a T_1-space that has a network of cardinality $\leq \kappa$, then $\mathsf{SP}^n_G X$ has a network of cardinality $\leq \kappa$;
(2) If X is a T_1-space that has a cs-network (resp. cs^*-network) of cardinality $\leq \kappa$, then $\mathsf{SP}^n_G X$ has a cs-network (resp. cs^*-network) of cardinality $\leq \kappa$;
(3) If X is a T_1-space that has a cn-network (resp. ck-network) of cardinality $\leq \kappa$, then $\mathsf{SP}^n_G X$ has a cn-network (resp. $ck-$network) of cardinality $\leq \kappa$.

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1. Introduction

The study of the influence of normal, weakly normal and seminormal functors to topological and geometric properties of topological spaces, in particular to the cardinal properties (density, weak density, local density, tightness, set tightness, T-tightness, functional tightness, mini-tightness), has been developed in recent investigations (see [1, 2, 3, 4, 6, 7, 9, 10, 11, 12, 14, 15, 16, 17]). For example, in [2] it was proved that the exponential functor of finite degree preserves the functional tightness and minimal tightness of compact sets, while in [11] a similar investigation (related to the $T-$tightness, set tightness, functional tightness, mini-tightness) was done for the functor SP^n_G of G-permutation degree. We mention here that tightness-type properties of function spaces with the compact-open topology have been studied previously in [9].

In this paper, we study the behavior of the network-type properties of topological spaces under the influence of the functor of G-permutation degree. We prove that this functor preserves the network, $cs-$network, cs^*-network, $cn-$network and $ck-$network of topological spaces.

Recall that the concept of functor of $G-$permutation degree was first introduced by V. V. Fedorchuk and V. V. Filippov in [6, 7].

Throughout this paper, all spaces are topological spaces and κ is an infinite cardinal number.

2. Preliminaries

The following are definitions that we need in the rest of this paper.

The set of all non-empty closed subsets of a topological space X is denoted by $\mathsf{exp}X$. The family of all sets of the form

$$O\langle U_1, U_2, \dots, U_n\rangle = \{F:\ F \in \mathsf{exp}X,\ F \subset \bigcup_{i=1}^{n} U_i,\ F \cap U_i \neq \varnothing,\ i = 1, \dots, n\},$$

where $U_1, U_2, \dots, U_n$ are open subsets of X, is a base of the topology on the set $\mathsf{exp}X$, called the *Vietoris topology.* The set $\mathsf{exp}X$ with the Vietoris topology is called the *exponential space* or *hyperspace* of a space X. Also, we set [7]

$$\mathsf{exp_n}X = \{F \in \mathsf{exp}X :\ |F| \leq n\}.$$

Let G be a subgroup of the symmetric group S_n of all permutations of the set $\{1, 2, \dots, n\}$. The group G acts on the n-th power X^n of a space X as permutation of coordinates, i.e. the points $\mathbf{x} = (x_1, x_2, \dots, x_n), \mathbf{y} = (y_1, y_2, \dots, y_n) \in X^n$ are *G-equivalent* if there exists a permutation $\sigma \in G$ such that $y_i = x_{\sigma(i)}$, $i \leq n$. This relation is called the *symmetric G-equivalence relation* on X. The G-equivalence class of an element $\mathbf{x} \in X^n$ is denoted by $[\mathbf{x}]_G$ or $[(x_1, x_2, \dots, x_n)]_G$, and the set of all orbits of actions of the group G is denoted by $\mathsf{SP}^n_G X$.

The quotient mapping $\pi^s_{n,G} : X^n \to \mathsf{SP}^n_G X$ defined by

$$\pi^s_{n,G}(\mathbf{x}) = [\mathbf{x}]_G$$

generates the quotient topology on the set $\mathsf{SP}^n_G X$, and this space is called the *space of n–G-permutation degree*, or simply the *space of G-permutation degree* of X.

Let $f : X \to Y$ be a continuous mapping. Define the mapping $\mathsf{SP}^n_G f : \mathsf{SP}^n_G X \to \mathsf{SP}^n_G Y$ by

$$\mathsf{SP}^n_G f([\mathbf{x}]_G) = [(f(x_1), f(x_2), \dots, f(x_n))]_G.$$

In this way, one obtains the functor $\mathsf{SP^n_G}$ which is a normal functor in the category of compacta and called the *functor of G-permutation degree* [7].

Recall also that the functor $\mathsf{exp_n}$ is the factor functor of the functor SP^n_G [6, 7].

We are going now to define notions that we use in the sequel.

Definition 2.1 ([13]). A sequence $\{x_n\}$ in a space X is said to be *eventually* in $P \subset X$ if $\{x_n\}$ converges to x, and there exists $m \in \mathbb{N}$ such that $\{x\} \bigcup \{x_n : \ n \geq m\} \subset P$.

Definition 2.2. A family ν of subsets of a topological space X is called:

(1) ([5]) a *network* for X if for every $x \in X$ and any neighbourhood U of x there exists $V \in \nu$ such that $x \in V \subset U$;
(2) ([13]) a *cs-network* at a point $x \in X$ if for any sequence $\{x_n\}$ converging to x and any neighborhood U of x, there exists $P \in \nu$ such that $P \subset U$ and $\{x_n\}$ is eventually in P;
(3) ([13]) a *cs*-network* at a point $x \in X$ if whenever $\{x_n\}$ is a sequence converging to a point $x \in U$ with U open in X, then $\{x_{n_i} : \ i \in \mathbb{N}\} \subset P \subset U$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and some $P \in \nu$;
(4) ([8]) a *cn-network* at a point $x \in X$ if for each neighbourhood O_x of x the set $\bigcup\{M \in \nu : x \in M \subseteq O_x\}$ is a neighbourhood of x; ν is a *cn*-network in X if it is a *cn*-network at each point $x \in X$;
(5) ([8]) a *ck-network* at a point $x \in X$ if for any neighbourhood O_x of x there is a neighbourhood $U_x \subset O_x$ of x such that for each compact subset $K \subset U_x$ there exists a finite subfamily $\mu \subset \nu$ satisfying $x \in \bigcap \mu$ and $K \subset \bigcup \mu \subset O_x$; ν is a *ck*-network in X if it a *ck*-network at each point $x \in X$.

Evidently, we have
base (at x)$\Rightarrow$ *ck*-network (at x)$\Rightarrow$ *cn*-network (at x)$\Rightarrow$ network (at x).

It can be easily seen from the definitions that for each network ν in a topological space X the family $\nu \vee \nu := \{A \cup B : A, B \in \nu\}$ is a *cn*-network in X. Also, if ν is a *ck*-network (at a point $x \in X$), then ν is a *cn*-network and a *cs**-network (at x).

The tightness $t(x, X)$ at x in X is the minimal infinite cardinal κ such that for any set $A \subset X$ with $x \in \overline{A}$, there is a subset B of A with $|B| \leq \kappa$ and $x \in \overline{B}$.

3. Main results

In this section we study how the functor of $G-$permutation degree acts to defined kinds of networks. If a network (cs-network, cs^*-network, cn-network, ck-network) of a space has cardinality $\leq \kappa$, then we say that it is a κ network (κ cs-network, κ cs^*-network, κ cn-network, κ ck-network).

Theorem 3.1. *If an infinite T_1-space X has a network of cardinality $\leq \kappa$, then the space $\mathsf{SP}^n_G X$ also has a network of cardinality $\leq \kappa$.*

Proof. First we show that if the infinite T_1-spaces $X_1, X_2, ..., X_n$ have a κ network at every point, then the product $Z = \prod_{i=1}^n X_i$ also has a κ network at every point. Let $\mathcal{P}(x_i) = \{P^\alpha_{x_i} : x_i \in X_i, i = \overline{1,n}, \alpha < \kappa\}$ be a network at the point $x_i \in X_i$, where $i = \overline{1,n}$. Now we will show that $\mathcal{P}(x_1)\times\mathcal{P}(x_2)\times...\times\mathcal{P}(x_n)$ is a network at the point $\mathbf{x} = (x_1, x_2, ..., x_n) \in Z$. Assume that W is a neighbourhood of the point $\mathbf{x} \in Z$. Let $Ox_1, Ox_2, ...Ox_n$ be neighbourhoods of the points $x_1, x_2, ..., x_n$, such that $\mathbf{x} \in Ox_1 \times Ox_2 \times ... \times Ox_n \subset W$. Since $\mathcal{P}(x_i)$, $i \leq n$, are networks for the spaces $X_1, X_2, ..., X_n$ at the points $x_1, x_2, ..., x_n$, there exist $P^{\alpha_1}_{x_1}, P^{\alpha_2}_{x_2}, ...P^{\alpha_n}_{x_n}$ such that $x_1 \in P^{\alpha_1}_{x_1} \subset Ox_1$, $x_2 \in P^{\alpha_2}_{x_2} \subset Ox_2$, ..., $x_n \in P^{\alpha_n}_{x_n} \subset Ox_n$. It follows that $(x_1, x_2, ..., x_n) \in P^{\alpha_1}_{x_1} \times P^{\alpha_2}_{x_2} \times ... \times P^{\alpha_n}_{x_n} \subset Ox_1 \times Ox_2 \times ... \times Ox_n \subset W$. It means that

$$\mathcal{P}(x_1)\times\mathcal{P}(x_2)\times...\times\mathcal{P}(x_n) = \{P^{\alpha_1}_{x_1}\times P^{\alpha_2}_{x_2}\times...\times P^{\alpha_n}_{x_n} : x_i \in X_i, \alpha_i < \kappa, i = \overline{1,n}\}$$

is a κ network at the point $(x_1, x_2, ..., x_n) \in X_1 \times X_2 \times ... \times X_n$. In particular, if X has a κ network (at every point), then X^n also has a κ network (at every point).

Now, we prove that if the space X^n has a κ network at every point $\mathbf{x} = (x_1, x_2, ..., x_n)$, then the space $\mathsf{SP}^n_G X$ also has a κ network at a point $[\mathbf{x}]_G = [(x_1, x_2, ..., x_n)]_G$. Let $[\mathbf{x}]_G \in \mathsf{SP}^n_G X$ be an arbitrary point and $O[\mathbf{x}]_G$ be an arbitrary neighbourhood of the point $[\mathbf{x}]_G$ in $\mathsf{SP}^n_G X$. We consider the sets $(\pi^s_{n,G})^{\leftarrow}([\mathbf{x}]_G)$ and $(\pi^s_{n,G})^{\leftarrow}(O[x]_G)$. Since $\pi^s_{n,G}$ is continuous, the set $(\pi^s_{n,G})^{\leftarrow}(O[\mathbf{x}]_G)$ is open and contains the finite set

$$(\pi^s_{n,G})^{\leftarrow}([\mathbf{x}]_G) = (\pi^s_{n,G})^{\leftarrow}([x_1, x_2, ..., x_n]_G) \subset (\pi^s_{n,G})^{\leftarrow}(O[\mathbf{x}]_G).$$

Suppose that $\mathcal{P}(x_1), \mathcal{P}(x_2), ..., \mathcal{P}(x_n)$ are the κ networks at the points $x_1, x_2, ...$, x_n, respectively. It means that there exist the sets $P_1 \in \mathcal{P}(x_1), P_2 \in \mathcal{P}(x_2), ...$, $P_n \in \mathcal{P}(x_n)$ such that $x_1 \in P_1$, $x_2 \in P_2$, ..., $x_n \in P_n$ and $[\mathbf{x}]_G \in (\pi^s_{n,G})(\prod_{i=1}^n P_i)$ $\subset O[\mathbf{x}]_G$, where $(\pi^s_{n,G})(\prod_{i=1}^n P_i) \in (\pi^s_{n,G})(\prod_{i=1}^n \mathcal{P}(x_i))$. It shows that the space $\mathsf{SP}^n_G X$ has a κ network at the point $[\mathbf{x}]_G$. □

The functor of G-permutation degree SP^n_G preserves the cardinality of cs-networks of a topological space, i.e., the following holds.

Theorem 3.2. *If the families $\mathcal{P}_1, \mathcal{P}_2, ..., \mathcal{P}_n$ are the cs-networks at the points $x_1, x_2, ..., x_n \in X$ respectively, then $\mathsf{SP}^n_G\mathcal{P} = \pi^s_{n,G}(\prod_{i=1}^n \mathcal{P}_i)$ is a cs-network at the point $[\mathbf{x}]_G = [x_1, x_2, \ldots, x_n]_G \in \mathsf{SP}^n_G X$.*

Proof. First we show that if the families $\{\mathcal{P}_i\}_{i=1}^n$ are cs-networks for $X_1, X_2, ..., X_n$ at the points $x_1, x_2, ..., x_n$ respectively, then the product $\prod_{i=1}^n \mathcal{P}_i$ is also a cs-network at the point $(x_1, x_2, ..., x_n) \in \prod_{i=1}^n X_i$. Let G be a neighbourhood of the point $(x_1, x_2, ..., x_n)$ and $\{(x_1^m, x_2^m, ..., x_n^m)\}$ be a sequence converging to $(x_1, x_2, ..., x_n)$. There there exist neighbourhoods $U_1, U_2, ..., U_n$ of the points $x_1, x_2, ..., x_n$, respectively, such that $U_1 \times U_2 \times ... \times U_n \subset G$. Observe that the sequences $\{x_1^m\}$, $\{x_2^m\}$, ..., $\{x_n^m\}$ converge to x_1, x_2, ..., x_n, respectively. For every $i = \overline{1, n}$ there exist $P_i \in \mathcal{P}_i$ and $m_0^i \in \mathbb{N}$ such that $\{x_i^m\} \subset P_i \subset U_i$ for each $m^i > m_0^i$. Let $m^0 = \max_{i=\overline{1,n}}(m_0^i)$. Then we have that $\{(x_1^m, x_2^m, ..., x_n^m)\} \subset \prod_{i=1}^n P_i \subset G$ for each $m > m^0$. It means that $\prod_{i=1}^n \mathcal{P}_i$ is cs-network at the point $(x_1, x_2, ..., x_n)$.

We are now going to prove that if a family $\prod_{i=1}^n \mathcal{P}_i$ is cs-network at the point $(x_1, x_2, ..., x_n) \in X^n$, then the family $\mathsf{SP}_G^n \mathcal{P} = \pi_{n,G}^s(\prod_{i=1}^n \mathcal{P}_i)$ is a cs-network at the point $[\mathbf{x}]_G = [(x_1, x_2, ..., x_n)]_G \in \mathsf{SP}_G^n X$. By the continuity of $\pi_{n,G}^s$ for every neighbourhood $\mathsf{SP}_G^n V$ of the point $[\mathbf{x}]_G$ there exists a neighbourhood U of the point $\mathbf{x} = (x_1, x_2, ..., x_n) \in X^n$ such that $\pi_{n,G}^s(U) \subset \mathsf{SP}_G^n V$. Since the family $\prod_{i=1}^n \mathcal{P}_i$ is the cs-network at the point $\mathbf{x} \in X^n$, there exists $P \in \prod_{i=1}^n \mathcal{P}_i$ such that $P \subset U$. Therefore, there exists $\pi_{n,G}^s(P) \in \pi_{n,G}^s(\prod_{i=1}^n \mathcal{P}_i) = \mathsf{SP}_G^n \mathcal{P}$ such that $\pi_{n,G}^s(P) \subset \mathsf{SP}_G^n V$. Now we will prove that for every sequence $\{[(x_1^m, x_2^m, ..., x_n^m)]_G\}$ converging to $[(x_1, x_2, ..., x_n)]_G$ there exists $m_0 \in \mathbb{N}$ such that $\{[(x_1^m, x_2^m, ..., x_n^m)]_G\} \subset \pi_{n,G}^s(P)$ for every $m > m_0$. On the other hand, $\{[(x_1^m, x_2^m, ..., x_n^m)]_G\}$ is the image of the sequence $(x_1^m, x_2^m, ..., x_n^m)$ of the space X^n converging to $(x_1, x_2, ..., x_n)$. Then there exists $m_1 \in \mathbb{N}$ such that $\{(x_1^m, x_2^m, ..., x_n^m)\} \subset P$ for every $m > m_1$ and so $\{\pi_{n,G}^s((x_1^m, x_2^m, ..., x_n^m))\} = \{[(x_1^m, x_2^m, ..., x_n^m)]_G\} \subset \pi_{n,G}^s(P)$ for every $m > m_1$. Hence, $\mathsf{SP}_G^n \mathcal{P}$ is the cs-network at the point $[(x_1, x_2, ..., x_n)]_G$. □

Corollary 3.3. *If the families $\mathcal{P}_1, \mathcal{P}_2, ..., \mathcal{P}_n$ are the cs-networks in X, then $\mathsf{SP}_G^n \mathcal{P}$, $\mathcal{P} = \prod_{i=1}^n \mathcal{P}_i$, is a cs-network in $\mathsf{SP}_G^n X$.*

The following theorem shows that the functor SP_G^n preserves the cardinality of cs^*-network of a space X.

Theorem 3.4. *If the families $\mathcal{P}_1, \mathcal{P}_2, ..., \mathcal{P}_n$ are the cs^*-networks at the point $x_1, x_2, ..., x_n \in X$, respectively, then $\mathsf{SP}_G^n \mathcal{P}$, $\mathcal{P} = \prod_{i=1}^n \mathcal{P}_i$, is a cs^*-network at the point $[(x_1, x_2, ..., x_n)]_G \in \mathsf{SP}_G^n X$.*

Proof. First we show that if the families $\{\mathcal{P}_i\}_{i=1}^n$ are cs^*-networks for X at the points $x_1, x_2, ..., x_n$, respectively, then the product $\prod_{i=1}^n \mathcal{P}_i$ is a cs^*-network at the point $(x_1, x_2, ..., x_n) \in X^n$. Let G be a neighbourhood of a point $(x_1, x_2, ..., x_n) \in X^n$ and $U_1, U_2, ..., U_n$ neighbourhoods of the points $x_1, x_2, ..., x_n$, respectively, such that $U_1 \times U_2 \times ... \times U_n \subset G$. Let $\{(x_1^m, x_2^m, ..., x_n^m)\}$ be a sequence converging to $(x_1, x_2, ..., x_n)$. Then the sequences $\{x_1^m\}, \{x_2^m\}, ..., \{x_n^m\}$ converge to the points $x_1, x_2, ..., x_n$, respectively. By the definition of cs^*-network there exist $P_i \in \mathcal{P}_i$, $i \leq n$, and subsequences $\{x_1^{m_k}\}, \{x_2^{m_k}\}, ..., \{x_n^{m_k}\}$ of the sequences $\{x_1^m\}, \{x_2^m\}, ..., \{x_n^m\}$, respectively, such that $\{x_i^{m_k}\} \subset P_i \subset$

U_i, $i = \overline{1,n}$. Consequently, we have $\{(x_1^{m_k}, x_2^{m_k}, ..., x_n^{m_k}) : k \in N\} \subset \prod_{i=1}^n P_i \subset G$, i.e. $\prod_{i=1}^n \mathcal{P}_i$ is a cs^*-network at the point $(x_1, x_2, ..., x_n) \in X^n$.

Ir remains to prove that if a family $\mathcal{P} = \prod_{i=1}^n \mathcal{P}_i$ is the cs^*-network at the point $(x_1, x_2, ..., x_n) \in X^n$, then the family $\mathsf{SP}_G^n \mathcal{P}$ is a cs^*-network at the point $[(x_1, x_2, ..., x_n)]_G \in \mathsf{SP}_G^n X$. It is sufficient to show that for every sequence $\{[(x_1^m, x_2^m, ..., x_n^m)]_G\}$ converging to the point $[(x_1, x_2, ..., x_n)]_G \in \mathsf{SP}_G^n V$ with an open set $SP_G^n V$ in $\mathsf{SP}_G^n X$, there exist subsequence $\{[(x_1^{m_k}, x_2^{m_k}, ..., x_n^{m_k})]_G : k \in N\}$ and $\mathsf{SP}_G^n U \in \pi_{n,G}^s(\prod_{i=1}^n \mathcal{P}_i)$ such that $\{[(x_1^{m_k}, x_2^{m_k}, ..., x_n^{m_k})]_G\} \subset \mathsf{SP}_G^n U \subset \mathsf{SP}_G^n V$. In addition, there exist $(y_1, y_2, ..., y_n) \in (\pi_{n,G}^s)^{\leftarrow}([(x_1, x_2, ..., x_n)]_G)$ and $(x_1^m, x_2^m, ..., x_n^m) \in (\pi_{n,G}^s)^{\leftarrow}([(x_1^m, x_2^m, ..., x_n^m)]_G)$ such that $(x_1^m, x_2^m, ..., x_n^m)$ converges to $(y_1, y_2, ..., y_n)$. Suppose that $(x_1, x_2, ..., x_n) \neq (y_1, y_2, ..., y_n)$. Then $\pi_{n,G}^s(x_1^m, x_2^m, ..., x_n^m) = [(x_1^{m_k}, x_2^{m_k}, ..., x_n^{m_k})]_G$ does not converge to the point $[(x_1, x_2, ..., x_n)]_G$ because of continuity of the mapping $\pi_{n,G}^s$, and therefore $(x_1, x_2, ..., x_n) = (y_1, y_2, ..., y_n)$. Since $\prod_{i=1}^n \mathcal{P}_i$ is the cs^*-network at the point $(x_1, x_2, ..., x_n) \in X^n$, there exists a subsequence $\{(x_1^{m_k}, x_2^{m_k}, ..., x_n^{m_k}) : k \in N\}$ of $(x_1^m, x_2^m, ..., x_n^m)$ and $P \in \prod_{i=1}^n \mathcal{P}_i$ such that $\{(x_1^{m_k}, x_2^{m_k}, ..., x_n^{m_k})\} \subset P$. It follows that

$$\{\pi_{n,G}^s(x_1^{m_k}, x_2^{m_k}, ..., x_n^{m_k})\} = \{[(x_1^{m_k}, x_2^{m_k}, ..., x_n^{m_k})]_G\}$$

$$\subset \{[(x_1^m, x_2^m, ..., x_n^m)]_G\} \subset \pi_{n,G}^s(P) = \mathsf{SP}_G^n U.$$

Hence, $\pi_{n,G}^s(\prod_{i=1}^n \mathcal{P}_i) = \{\pi_{n,G}^s(P) : P \in \prod_{i=1}^n \mathcal{P}_i\}$ is a cs^*-network at the point $[(x_1, x_2, ..., x_n)]_G$. □

The functor SP_G^n preserves also the cardinality of a *cn*-network, i.e. the following holds.

Theorem 3.5. *If the families $\mathcal{N}_1, \mathcal{N}_2, ..., \mathcal{N}_n$ are cn-networks at the points $x_1, x_2, ..., x_n$ in X, respectively, then the family $\mathsf{SP}_G^n \mathcal{N} = \pi_{n,G}^s(\prod_{i=1}^n \mathcal{N}_i)$ is a cn-network at the point $[(x_1, x_2, ..., x_n)]_G \in \mathsf{SP}_G^n X$.*

Proof. Let us prove that if $\mathcal{N}_i = \{N^i\}$ is a *cn*-network at a point x_i of a space X, $1 \leq i \leq n$, then $\mathcal{N} = \prod_{i=1}^n \mathcal{N}_i = \{N_{m_1}^1 \times N_{m_2}^2 \times ... \times N_{m_n}^n : x_i \in N_{m_i}^i \in \mathcal{N}_i\}$ is a *cn*-network at the point $\mathbf{x} = (x_1, x_2, ..., x_n)$ of the space X^n. Assume that $U = \prod_{i=1}^n U_i$ is a neighbourhood of the point $\mathbf{x}$, where U_i is a neighbourhood of x_i for all $1 \leq i \leq n$. Since $\mathcal{N}_i$ is the *cn*-network for all $1 \leq i \leq n$, the set $W_i = \bigcup\{N_k^i \in \mathcal{N}_i : x_i \in N_k^i \subseteq U_i\}$ is a neighbourhood of x_i. It is easy to see that $\prod_{i=1}^n W_i \subseteq \bigcup\{\prod_{i=1}^n N_k^i \in \mathcal{N} : x_i \in N_k^i \subseteq U_i, 1 \leq i \leq n\}$. It means that $\mathcal{N}$ is a *cn*-network at the point $\mathbf{x}$ in X^n.

Now we prove that if $\mathcal{N} = \prod_{i=1}^n \mathcal{N}_i$ is the *cn*-network at the point $\mathbf{x} = (x_1, x_2, ..., x_n)$, then $\mathsf{SP}_G^n \mathcal{N} = \pi_{n,G}^s(\prod_{i=1}^n \mathcal{N}_i)$ is a *cn*-network at the point $[\mathbf{x}]_G = [(x_1, x_2, ..., x_n)]_G$. Suppose that $\mathcal{N} = \prod_{i=1}^n \mathcal{N}_i$ is the *cn*-network at the point x. Let $[x]_G$ be any point of $\mathsf{SP}_G^n X$ and $O_{[\mathbf{x}]_G}$ be a neighbourhood of $[\mathbf{x}]_G$. By the continuity of the mapping $\pi_{n,G}^s$, we have that the set $(\pi_{n,G}^s)^{\leftarrow}(O_{[\mathbf{x}]_G}) = U_x \subset X^n$ is a neighbourhood of $\mathbf{x}$. Since $\mathcal{N} = \prod_{i=1}^n \mathcal{N}_i$ is a *cn*-network at the point $\mathbf{x}$, the set $\bigcup\{N \in \mathcal{N} : x \in N \subseteq U_x\}$ is a neighbourhood of $\mathbf{x}$. On the other hand, we have that

$$\pi^s_{n,G}(\bigcup\{N \in \mathcal{N} : x \in N \subseteq U_x) = \bigcup\{\pi^s_{n,G}(N) \in \pi^s_{n,G}(\mathcal{N}) : x \in N \subseteq U_x\} =$$

$$\bigcup\{\mathsf{SP}^n_G N \in \mathsf{SP}^n_G\mathcal{N} : [\mathbf{x}]_G \in \mathsf{SP}^n_G N \subseteq O_{[\mathbf{x}]_G}\}$$

and this set is a neighbourhood of the point $[\mathbf{x}]_G$ (by the openness of the mapping $\pi^s_{n,G}$). It means that $\mathsf{SP}^n_G\mathcal{N}$ is a *cn*-network at the point $[\mathbf{x}]_G$. □

Corollary 3.6. *If the families $\mathcal{N}_1, \mathcal{N}_2, ..., \mathcal{N}_n$ are the cn-networks in X, then the family $\mathsf{SP}^n_G\mathcal{N} = \pi^s_{n,G}(\prod_{i=1}^n \mathcal{N}_i)$ is a cn-network in $\mathsf{SP}^n_G X$.*

Proposition 3.7. *If $\mathcal{N} = \{N_\alpha : \alpha < \kappa\}$ is the cn-network at a point $x \in X$, then $t(x, X) \leq \kappa$.*

Proof. Let $\mathcal{N} = \{N_\alpha : \alpha < \kappa\}$ be a *cn*-network at a point x and A be a subset of X such that $x \in \overline{A} \setminus A$. Put $\lambda = \{\alpha < \kappa : N_\alpha \cap A \neq \varnothing\}$. For every $\alpha < \lambda$ we take $a_\alpha \in N_\alpha \cap A$ arbitrarily and set $B := \{a_\alpha : \alpha < \lambda\}$. Now we prove that $x \in \overline{B}$. For any neighbourhood U of x we can take a set $\mathcal{I}(U) := \{\alpha < \kappa : x \in N_\alpha \subseteq U\}$. By definition, the set $\bigcup_{\alpha \in \mathcal{I}(U)} N_\alpha$ contains another neighbourhood V of x. Since $A \cap V \neq \varnothing$, we can find $\alpha \in \mathcal{I}(U) \cap \lambda$. Then it implies that $a_\alpha \in B \cap U$. It means that the tightness of X at the point x does not exceed κ. □

Corollary 3.8. *If a topological space X has a countable cn-network at the points $x_1, x_2, ..., x_n$, then the space $\mathsf{SP}^n_G X$ also has a countable cn-network and a countable tightness at the point $[(x_1, x_2, ..., x_n)]_G$.*

Finally, we prove that the functor SP^n_G preserves the cardinality of *ck*-networks of a space X.

Theorem 3.9. *If the families $\mathcal{N}_1, \mathcal{N}_2, ..., \mathcal{N}_n$ are ck-networks at the points $x_1, x_2, ..., x_n \in X$, respectively, then the family $\mathsf{SP}^n_G\mathcal{N} = \pi^s_{n,G}(\prod_{i=1}^n \mathcal{N}_i)$ is a ck-network at the point $[(x_1, x_2, ..., x_n)]_G \in \mathsf{SP}^n_G X$.*

Proof. First we show that if $\mathcal{N}_i$ is a *ck*-network at the point x_i of a space X, $i \leq n$, then $\mathcal{N} = \prod_{i=1}^n \mathcal{N}_i$ is a *ck*-network at the point $\mathbf{x} = (x_1, x_2, ..., x_n)$ of the space X^n. Let $O_x = \prod_{i=1}^n O_{x_i}$ be a basic neighbourhood of the point $\mathbf{x}$, where O_{x_i} is a neighbourhood of x_i for all $1 \leq i \leq n$ and $U_x = \prod_{i=1}^n U_{x_i} \subset O_x$ be another neighbourhood of $\mathbf{x}$. For every compact subset K of U_x we have that $pr_i(K) = K_i$ is also compact and $K_i \subset U_{x_i}$ for all $1 \leq i \leq n$. Since $\mathcal{N}_i$ is the *ck*-network at the point x_i there exists a finite subfamily $\mathcal{F}_i$ of $\mathcal{N}_i$ such that $x_i \in \bigcap \mathcal{F}_i$ and $K_i \subset \bigcup \mathcal{F}_i \subset O_{x_i}$ for all $1 \leq i \leq n$. Consequently, it can be constructed finite subfamily $\mathcal{F} = \prod_{i=1}^n \mathcal{F}_i$ satisfying $x \in \bigcap \mathcal{F}$ and $K \subset \bigcup \mathcal{F} \subset O_x$. It means that $\mathcal{N}$ is a *ck*-network at the point $x \in X^n$.

Now we prove that if $\mathcal{N} = \prod_{i=1}^n \mathcal{N}_i$ is a *ck*-network at a point $\mathbf{x} = (x_1, x_2, ..., x_n) \in X^n$, then $\mathsf{SP}^n_G\mathcal{N} = \pi^s_{n,G}(\prod_{i=1}^n \mathcal{N}_i)$ is a *ck*-network at the point $[\mathbf{x}]_G$. Let $O_{[\mathbf{x}]_G}$ be an arbitrary neighbourhood of the point $[\mathbf{x}]_G$. Clearly, the set $K = (\pi^s_{n,G})^{\leftarrow}(C)$ is compact for every compact subset C of $O_{[\mathbf{x}]_G}$. In addition, by the continuity of the mapping $\pi^s_{n,G}$, we have that $(\pi^s_{n,G})^{\leftarrow 1}(O_{[\mathbf{x}]_G}) = V_{\mathbf{x}}$ is a neighbourhood of the point $\mathbf{x} \in X^n$. Since $\mathcal{N}$ is the *ck*-network at a point $\mathbf{x}$ there is $W_{\mathbf{x}} \subset V_{\mathbf{x}}$ such that for every compact $K \subset W_{\mathbf{x}}$ there exists finite subfamily $\mathcal{F}$ of the family $\mathcal{N}$ satisfying $\mathbf{x} \in \bigcap \mathcal{F}$ and $K \subset \bigcup \mathcal{F} \subset V_{\mathbf{x}}$. Since the mapping $\pi^s_{n,G}$ is open and closed onto mapping, we have that there is $\pi^s_{n,G}(W_{\mathbf{x}}) = U_{[\mathbf{x}]_G} \subset O_{[\mathbf{x}]_G}$ such that for any compact $C \subset U_{[\mathbf{x}]_G}$ there exists a finite subfamily $\mathsf{SP}^n_G\mathcal{F} = \pi^s_{n,G}(\mathcal{F})$ of the family $\mathsf{SP}^n_G\mathcal{N}$ satisfying $[\mathbf{x}]_G \in \bigcap \mathsf{SP}^n_G\mathcal{F}$ and $C \subset \bigcup \mathsf{SP}^n_G\mathcal{F} \subset O_{[\mathbf{x}]_G}$. It means that $\mathsf{SP}^n_G\mathcal{N}$ is a *ck*-network at the point $[\mathbf{x}]_G$. □

Corollary 3.10. *If the families $\mathcal{N}_1, \mathcal{N}_2, ..., \mathcal{N}_n$ are ck-networks in X, then the family $\mathsf{SP}^n_G\mathcal{N} = \pi^s_{n,G}(\prod_{i=1}^n \mathcal{N}_i)$ is a ck-network in $\mathsf{SP}^n_G X$.*

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