Applied General Topology

*Appl. Gen. Topol. 24, no. 1 (2023), 205-227*
*doi:10.4995/agt.2023.18783*


# ⊤-quasi-Cauchy spaces — a non-symmetric theory of completeness and completion

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*Communicated by J. Rodríguez-López*

Abstract

*Based on the concept of Cauchy pair ⊤-filters, we develop an axiomatic theory of completeness for non-symmetric spaces, such as ⊤-quasi-uniform (limit) spaces or* L*-metric spaces. We show that the category of ⊤-quasi-Cauchy spaces is topological and Cartesian closed and we construct a finest completion for a non-complete ⊤-quasi-Cauchy space. In the special case of symmetry, ⊤-quasi-Cauchy spaces can be identified with ⊤-Cauchy spaces.*

2020 MSC: *54A40; 54A05; 54A20; 54E15.*

Keywords: *fuzzy topology; pair ⊤-filter; Cauchy pair ⊤-filter; ⊤-quasi-Cauchy space; ⊤-quasi-uniform space; ⊤-quasi-uniform limit space;* L*-metric space.*

## 1. Introduction

For studying completeness and completion, apart from uniform (limit) spaces, Cauchy spaces have been considered an appropriate framework, see for example Reed's fundamental paper [20]. A Cauchy space is a set, together with a set of filters satisfying axioms that are derived from the properties of Cauchy filters of a uniform limit space [16]. As uniform limit spaces, uniform spaces and metric spaces, are symmetric, Cauchy spaces are from their very nature designed to describe completeness in symmetric spaces.

*Received 16 November 2022 – Accepted 20 February 2023*

For many years it has been observed that non-symmetric spaces are of interest. Non-symmetric metric spaces, as so-called $\mathsf{L}$-*categories*, are at the basis of monoidal topology, and allow a "categorical" treatment of completeness and completion [18, 12]. Non-symmetric uniform spaces, bearing the name *quasi-uniform spaces*, have been studied since the middle of the last century, see e.g. [17]. Completeness and completion in these spaces were defined and studied, among others, by Lindgren and Fletcher [6] with the help of Cauchy pair filters. Recently, this approach has been used – in the realm of many-valued topology – in $\top$-quasi-uniform spaces [25] using $\top$-Cauchy pair filters. In [15], we have stated the properties of the $\top$-Cauchy pair filters in a $\top$-quasi-uniform space (with a small error that we will correct in this paper). These properties can be used as an axiomatic definition of $\top$-quasi-Cauchy spaces, thus allowing a non-symmetric theory of completeness and completion that encompasses the corresponding approaches in $\top$-quasi-uniform spaces or in $\mathsf{L}$-metric spaces. We use commutative and integral quantales $\mathsf{L}$ as basis for treating "many-valuedness" and would like to point out that for the two-point chain $\mathsf{L} = \{0, 1\}$, $\top$-filters can be identified with ordinary filters and the theory developed in this paper can be used also in the classical case of quasi-uniform (limit) spaces.

The paper is organized as follows. In the next section, we collect the necessary facts and concepts about quantales, $L$-sets and $\top$-filters. In the third section we define our category of $\top$-quasi-Cauchy spaces and give important examples, and the fourth section is devoted to the categorical properties. Completeness and completion are studied in section 5 and section 6 shows that if a symmetry axiom is satisfied, then our spaces can be identified with the $\top$-Cauchy spaces of [21]. Finally we draw some conclusions.

## 2. Preliminaries

The triple $\mathsf{L} = (L, \leq, *)$, where $(L, \leq)$ is a complete lattice with order relation $\leq$ and with distinct top and bottom elements $\top \neq \bot$, and $(L, *)$ is a commutative semigroup for which the top element of $L$ acts as the unit, i.e. $\alpha * \top = \alpha$ for all $\alpha \in L$, and $*$ is distributive over arbitrary joins, i.e. $(\bigvee_{i\in J} \alpha_i) * \beta = \bigvee_{i\in J}(\alpha_i * \beta)$, is called a *commutative and integral quantale*, see e.g. [12]. Important examples of such quantales are e.g. the unit interval $[0, 1]$ with a left-continuous *t-norm* [22], or *Lawvere's quantale*, the interval $[0, \infty]$ with the opposite order and addition $\alpha * \beta = \alpha + \beta$, extended by $\alpha + \infty = \infty + a = \infty$, see e.g. [5]. A further important example is the quantale $(\Delta^+, \leq, *)$ of distance distribution functions [5]. This quantale finds applications, for example, in the theory of probabilistic metric spaces [22].

In a quantale, we can define an *implication* by $\alpha \to \beta = \bigvee\{\delta \in L \,:\, \alpha * \delta \leq \beta\}$. The implication is characterized by $\delta \leq \alpha \to \beta$ if, and only if, $\delta * \alpha \leq \beta$.

We list some of the properties of the implication, that we will use later on. We omit the straightforward proofs.

**Lemma 2.1.** *Let $\alpha, \beta, \gamma, \delta, \alpha_j, \beta_j \in L$ for $j \in J$. The following assertions hold.*

(1) *If $\alpha \leq \beta$, then $\alpha \to \gamma \geq \beta \to \gamma$ and $\gamma \to \alpha \leq \gamma \to \beta$.*

(2) $(\alpha \to \beta) * (\gamma \to \delta) \leq (\alpha * \gamma) \to (\beta * \delta)$.
(3) $\bigwedge_{j\in J}(\alpha_j \to \beta) = (\bigvee_{j\in J} \alpha_j) \to \beta$.
(4) $\bigwedge_{j\in J}(\alpha_j \to \beta_j) \leq (\bigvee_{j\in J} \alpha_j) \to (\bigvee_{j\in J} \beta_j)$.

We denote the set of *L-sets in* $X$, or, more precisely, *L-subsets of* $X$, $a, b, c, \ldots$ by $L^X = \{a : X \longrightarrow L\}$. In particular, we denote for $A \subseteq X$, the *characteristic function* $\top_A \in L^X$ by $\top_A(x) = \top$ if $x \in A$ and $\top_A(x) = \bot$ otherwise. The lattice operations are extended pointwisely from $L$ to $L^X$. If $a \in L^X$, $b \in L^Y$ and $\varphi : X \longrightarrow Y$ is a mapping, then we define $\varphi(a) \in L^Y$ by $\varphi(a)(y) = \bigvee_{x:\varphi(x)=y} a(x)$ for $y \in Y$ and $\varphi^{\leftarrow}(b) = b \circ \varphi$.

For $a \in L^X$ and $b \in L^Y$ we define the *monoidal product* $a \otimes b \in L^{X\times Y}$ by $(a \otimes b)(x, y) = a(x) * b(y)$ for all $(x, y) \in X \times Y$.

For $b, d \in L^X$ we denote the *fuzzy inclusion order* [2] by $[b, d] = \bigwedge_{x\in X}(b(x) \to d(x))$. We collect some of the properties that we will need later.

**Lemma 2.2.** *Let* $a, a', b, b', c \in L^X$, $d \in L^Y$ *and let* $\varphi : X \longrightarrow Y$ *be a mapping. Then*

(i) $a \leq b$ *if and only if* $[a, b] = \top$;
(ii) $a \leq a'$ *implies* $[a', b] \leq [a, b]$ *and* $b \leq b'$ *implies* $[a, b] \leq [a, b']$;
(iii) $[a, c] \wedge [b, c] = [a \vee b, c]$;
(iv) $[\varphi(a), d] = [a, \varphi^{\leftarrow}(d)]$.

For $L$-sets $b, c \in L^{X\times X}$ we define the *composition*, $b \circ c \in L^{X\times X}$, by $b \circ c(x, y) = \bigvee_{z\in X} c(x, z) * b(z, y)$ for all $x, y \in X$ and the *inverse*, $b^{-1} \in L^{X\times X}$, by $b^{-1}(x, y) = b(y, x)$ for all $x, y \in X$.

**Definition 2.3** ([24, 9, 7])**.** A subset $\mathbb{F} \subseteq L^X$ is called a ⊤-*filter* on $X$ if

(⊤-F1) $\bigvee_{x\in X} b(x) = \top$ for all $b \in \mathbb{F}$;
(⊤-F2) $a, b \in \mathbb{F}$ implies $a \wedge b \in \mathbb{F}$;
(⊤-F3) $\bigvee_{b\in\mathbb{F}}[b, d] = \top$ implies $d \in \mathbb{F}$.

We denote the set of all ⊤-filters on $X$ by $\mathsf{F}^{\top}_{\mathsf{L}}(X)$.

**Example 2.4.** For $x \in X$, $[x] = \{a \in L^X : a(x) = \top\}$ is a ⊤-filter. More general, if $a(x) = \top$ for some $x \in X$, then $[a] = \{b \in L^X : a \leq b\}$ is a ⊤-filter.

**Definition 2.5** ([24, 9, 7])**.** A subset $\mathbb{B} \subseteq L^X$ is called a ⊤-*filter basis* if

(⊤-B1) $\bigvee_{x\in X} b(x) = \top$ for all $b \in \mathbb{B}$;
(⊤-B2) $a, b \in \mathbb{B}$ implies $\bigvee_{c\in\mathbb{B}}[c, a \wedge b] = \top$.

For a ⊤-filter base $\mathbb{B}$, $[\mathbb{B}] = \{a \in L^X : \bigvee_{b\in\mathbb{B}}[b, a] = \top\}$ is a ⊤-filter, the ⊤-filter generated by $\mathbb{B}$. For an ordinary filter $\mathcal{F}$ on $X$, the set $\{\top_F : F \in \mathcal{F}\}$ is a ⊤-filter basis and we denote the generated ⊤-filter by $\top_{\mathcal{F}}$.

For the following definitions and properties, we refer to [9, 24, 14]. The set $\mathsf{F}^{\top}_{\mathsf{L}}(X)$ is ordered by $\mathbb{F} \leq \mathbb{G}$ if $\mathbb{F} \subseteq \mathbb{G}$. The meet of a non-empty family $(\mathbb{F}_j)_{j\in J}$ of ⊤-filters on $X$ is given by $\bigwedge_{j\in J} \mathbb{F}_j = \bigcap_{j\in J} \mathbb{F}_j$ and a ⊤-filter base for $\bigwedge_{j\in J} \mathbb{F}_j$ is given by $\{\bigvee_{j\in J} f_j : f_j \in \mathbb{F}_j \, \forall j \in J\}$.

For a $\top$-filter $\mathbb{F} \in \mathsf{F}_{\mathsf{L}}^{\top}(X)$ and a mapping $\varphi : X \longrightarrow Y$, the set $\{\varphi(f) \ : \ f \in \mathbb{F}\}$ is a $\top$-filter basis and we call the generated $\top$-filter, $\varphi(\mathbb{F}) \in \mathsf{F}_{\mathsf{L}}^{\top}(Y)$, the *image of* $\mathbb{F}$ *under* $\varphi$. We then have $\varphi([x]) = [\varphi(x)]$ and for two mappings $\varphi : X \longrightarrow Y$ and $\psi : Y \longrightarrow Z$ we have $\psi(\varphi(\mathbb{F})) = (\psi \circ \varphi)(\mathbb{F})$. For a $\top$-filter $\mathbb{G} \in \mathsf{L}_{\mathsf{L}}^{\top}(Y)$, the set $\{\varphi^{\leftarrow}(g) \ : \ g \in \mathbb{G}\}$ is a $\top$-filter basis if and only if $\bigvee_{y \in \varphi(X)} g(y) = \top$ for all $g \in \mathbb{G}$. In this case, we denote the generated $\top$-filter by $\varphi^{\leftarrow}(\mathbb{G}) \in \mathsf{F}_{\mathsf{L}}^{\top}(X)$ and call it the *inverse image of* $\mathbb{G}$ *under* $\varphi$. We also say that $\varphi^{\leftarrow}(\mathbb{G})$ *exists*. In the special case of a subset $A \subseteq X$ and the embedding $i_A : A \longrightarrow X$, we denote for $\mathbb{G} \in \mathsf{F}_{\mathsf{L}}^{\top}(X)$, in case of existence, $i_A^{\leftarrow}(\mathbb{G}) = \mathbb{G}_A$ and call it the *trace of* $\mathbb{G}$ *on* $A$.

For $\mathbb{F} \in \mathsf{F}_{\mathsf{L}}^{\top}(X), \mathbb{G} \in \mathsf{F}_{\mathsf{L}}^{\top}(U)$ we define $\mathbb{F} \otimes \mathbb{G}$ as the $\top$-filter on $X \times U$ generated by the $\top$-filter basis $\{f \otimes g \ : \ f \in \mathbb{F}, g \in \mathbb{G}\}$. For mappings $\varphi : X \longrightarrow Y, \psi : U \longrightarrow V$ we have, with the product mapping $(\varphi \times \psi)(x, y) = (\varphi(x), \psi(y))$, $(\varphi \times \psi)(\mathbb{F} \otimes \mathbb{G}) = \varphi(\mathbb{F}) \otimes \psi(\mathbb{G})$.

Finally, for $\top$-filters $\Phi, \Psi$ on $X \times X$, we define the *inverse* $\Phi^{-1} = \{a^{-1} \ : \ a \in \Phi\}$ and the *composition* $\Phi \circ \Psi$ as the $\top$-filter generated by the $\top$-filter basis $\{b \circ c \ : \ b \in \Phi, c \in \Psi\}$.

## 3. $\top$-quasi-Cauchy spaces

Following [6] and [25], we call, for $\mathbb{F}, \mathbb{G} \in \mathsf{F}_{\mathsf{L}}^{\top}(X)$, $(\mathbb{F}, \mathbb{G})$ a *pair* $\top$*-filter* if for all $f \in \mathbb{F}, g \in \mathbb{G}$ we have $\bigvee_{x \in X} f(x) * g(x) = \top$. If this condition is satisfied, we also say that $\mathbb{F}$ and $\mathbb{G}$ are *linked*. The set of all pair $\top$-filters on $X$ is denoted by $\mathsf{PF}_{\mathsf{L}}^{\top}(X)$. For $(\mathbb{F}, \mathbb{G}), (\mathbb{F}', \mathbb{G}') \in \mathsf{PF}_{\mathsf{L}}^{\top}(X)$ we write $(\mathbb{F}, \mathbb{G}) \leq (\mathbb{F}', \mathbb{G}')$ if $\mathbb{F} \leq \mathbb{F}'$ and $\mathbb{G} \leq \mathbb{G}'$.

**Lemma 3.1.** *Let $\varphi : X \longrightarrow X'$ be a mapping and let $(\mathbb{F}, \mathbb{G}) \in \mathsf{PF}_{\mathsf{L}}^{\top}(X)$. Then $(\varphi(\mathbb{F}), \varphi(\mathbb{G})) \in \mathsf{PF}_{\mathsf{L}}^{\top}(X')$.*

*Proof.* This follows as for $f \in \mathbb{F}$ and $g \in \mathbb{G}$ we have

$$\bigvee_{x' \in X'} \varphi(f)(x') * \varphi(g)(x') \geq \bigvee_{x' \in X'} \bigvee_{\varphi(x) = x'} f(x) * g(x) = \bigvee_{x \in X} f(x) * g(x) = \top.$$

□

**Definition 3.2.** A set $\mathcal{CP} \subseteq \mathsf{PF}_{\mathsf{L}}^{\top}(X)$ is called a $\top$*-quasi-Cauchy structure* if

(TQC1) $([x], [x]) \in \mathcal{CP}$ for all $x \in X$;
(TQC2) $(\mathbb{F}', \mathbb{G}') \in \mathcal{CP}$ whenever $(\mathbb{F}, \mathbb{G}) \in \mathcal{CP}$ and $(\mathbb{F}, \mathbb{G}) \leq (\mathbb{F}', \mathbb{G}') \in \mathsf{PF}_{\mathsf{L}}^{\top}(X)$;
(TQC3) $(\mathbb{F} \wedge \mathbb{F}', \mathbb{G} \wedge \mathbb{G}') \in \mathcal{CP}$ whenever $(\mathbb{F}, \mathbb{G}), (\mathbb{F}', \mathbb{G}') \in \mathcal{CP}$ and $\bigvee_{x \in X} f(x) * g'(x) = \top = \bigvee_{x \in X} f'(x) * g(x)$ for all $f \in \mathbb{F}, f' \in \mathbb{F}', g \in \mathbb{G}, g' \in \mathbb{G}'$, that is, if $\mathbb{F}$ and $\mathbb{G}'$ as well as $\mathbb{F}'$ and $\mathbb{G}$ are linked.

The pair $(X, \mathcal{CP})$ is called a $\top$*-quasi-Cauchy space*. A mapping $\varphi : (X, \mathcal{CP}) \longrightarrow (X', \mathcal{CP}')$ is called *Cauchy continuous* if $(\varphi(\mathbb{F}), \varphi(\mathbb{G})) \in \mathcal{CP}'$ whenever $(\mathbb{F}, \mathbb{G}) \in \mathcal{CP}$.

We note that a $\top$-quasi-Cauchy structure is not just a pair of $\top$-Cauchy structures as defined in [21], see Section 7 below, one for each component of a

pair ⊤-filter, but that the condition (TQC3) links different components of the pair ⊤-filters. The "linking condition" in this form stems from the following example.

**Example 3.3** (⊤-QUnif and L-Met)**.** A ⊤-*quasi-uniform space* (*fuzzy* L-*quasi-uniform space* [9], *probabilistic quasi-uniform space* [25]) is a pair $(X,\mathcal{U})$ of a set $X$ and a ⊤-filter $\mathcal{U} \in \mathsf{F}^{\top}_{\mathsf{L}}(X \times X)$ with the properties (TU1) $\mathcal{U} \leq [(x,x)]$ for all $x \in X$, (TU2) $\mathcal{U} \leq \mathcal{U} \circ \mathcal{U}$. A mapping $\varphi : (X,\mathcal{U}) \longrightarrow (X',\mathcal{U}')$ between two ⊤-quasi-uniform spaces $(X,\mathcal{U}),(X',\mathcal{U}')$ is called *uniformly continuous* if $\mathcal{U}' \leq (\varphi \times \varphi)(\mathcal{U})$.

A pair ⊤-filter $(\mathbb{F},\mathbb{G})$ is called a *Cauchy pair* ⊤*-filter* in $(X,\mathcal{U})$ if $\mathcal{U} \leq \mathbb{G} \otimes \mathbb{F}$, [25]. We denote the set of Cauchy pair ⊤-filters in $(X,\mathcal{U})$ by $\mathcal{CP}^{\mathcal{U}}$. Then $(X,\mathcal{CP}^{\mathcal{U}})$ is a ⊤-quasi-Cauchy space. The axiom (TQC1) follows with $[x]\otimes[x] = [(x,x)]$. The axiom (TQC2) is obvious. We show (TQC3). Let $(\mathbb{F},\mathbb{G}),(\mathbb{F}',\mathbb{G}') \in \mathcal{CP}^{\mathcal{U}}$ and let $\bigvee_{x\in X} f(x) * g'(x) = \top$ and $\bigvee_{x\in X} f'(x) * g(x) = \top$ for all $f \in \mathbb{F}, f' \in \mathbb{F}', g \in \mathbb{G}, g' \in \mathbb{G}'$. Then $\mathcal{U} \leq \mathbb{G} \otimes \mathbb{F}$ and $\mathcal{U} \leq \mathbb{G}' \otimes \mathbb{F}'$ and hence, by (TU2), $\mathcal{U} \leq \mathcal{U} \circ \mathcal{U} \leq (\mathbb{G} \otimes \mathbb{F}) \circ (\mathbb{G}' \otimes \mathbb{F}')$. For $g \in \mathbb{G}, f \in \mathbb{F}, g' \in \mathbb{G}', f' \in \mathbb{F}'$ we have

$$\begin{aligned}(g \otimes f) \circ (g' \otimes f')(s,t) &= \bigvee_{x\in X} g'(s) * f'(x) * g(x) * f(t)\\ &= g'(s) * f(t) * \bigvee_{x\in X} f'(x) * g(x) \;=\; g' \otimes f(s,t).\end{aligned}$$

Therefore, we have $\mathcal{U} \leq (\mathbb{G} \otimes \mathbb{F}) \circ (\mathbb{G}' \otimes \mathbb{F}') = \mathbb{G}' \otimes \mathbb{F}$. Similarly, we can show that $\mathcal{U} \leq \mathbb{G} \otimes \mathbb{F}'$ and we conclude

$$\mathcal{U} \leq (\mathbb{G} \otimes \mathbb{F}) \wedge (\mathbb{G} \otimes \mathbb{F}') \wedge (\mathbb{G}' \otimes \mathbb{F}) \wedge (\mathbb{G}' \otimes \mathbb{F}') = (\mathbb{G} \wedge \mathbb{G}') \otimes (\mathbb{F} \wedge \mathbb{F}').$$

Hence $(\mathbb{F} \wedge \mathbb{F}', \mathbb{G} \wedge \mathbb{G}') \in \mathcal{CP}^{\mathcal{U}}$.

A uniformly continuous mapping $\varphi : (X,\mathcal{U}) \longrightarrow (X',\mathcal{U}')$ is Cauchy continuous as a mapping $\varphi : (X,\mathcal{CP}^{\mathcal{U}}) \longrightarrow (X',\mathcal{CP}^{\mathcal{U}'})$: For $(\mathbb{F},\mathbb{G}) \in \mathcal{CP}^{\mathcal{U}}$ we have $\mathcal{U} \leq \mathbb{G} \otimes \mathbb{F}$ and hence $\mathcal{U}' \leq (\varphi \times \varphi)(\mathcal{U}) \leq (\varphi \times \varphi)(\mathbb{G} \otimes \mathbb{F}) = \varphi(\mathbb{G}) \otimes \varphi(\mathbb{F})$, that is, we have $(\varphi(\mathbb{F}),\varphi(\mathbb{G})) \in \mathcal{CP}^{\mathcal{U}'}$.

This example encompasses L-*metric spaces* (also called *continuity spaces* [5], L-*categories* [18, 12] or L-*preordered sets* [26]) $(X,d)$ with (LM1) $d(x,x) = \top$ for all $x \in X$ and (LM2) $d(x,y) * d(y,z) \leq d(x,z)$ for all $x,y,z \in X$. Then $[d] = \{u \in L^{X\times X} \;:\; d \leq u\}$ is a ⊤-quasi-uniformity [13] and we call a pair ⊤-filter $(\mathbb{F},\mathbb{G})$ a Cauchy pair ⊤-filter if $[d] \leq \mathbb{G} \otimes \mathbb{F}$, i.e. if $\top = \bigvee_{g\in\mathbb{G}, f\in\mathbb{F}} [g \otimes f, d]$. If $\mathbb{F} = \top_{\mathcal{F}}$ and $\mathbb{G} = \top_{\mathcal{G}}$ with filters $\mathcal{F},\mathcal{G}$ on $X$, then $(\top_{\mathcal{F}},\top_{\mathcal{G}})$ is a Cauchy pair ⊤-filter if and only if $\mathcal{F} \vee \mathcal{G}$ exists and $\top = \bigvee_{G\in\mathcal{G}, F\in\mathcal{F}} \bigwedge_{y\in G, x\in F} d(y,x)$.

For Lawvere's quantale $\mathsf{L} = ([0,\infty),\geq,+)$, this establishes a connection with the definition of Cauchy sequences in quasi-metric spaces given by Doitchinov [4]. A sequence $(x_n)$ in a quasi-metric space $(X,d)$ is called a *Cauchy sequence* if there is a co-sequence $(y_n)$ such that for all $\epsilon > 0$ there is $N \in \mathbb{N}$ such that for all $m,n \geq N$ we have $d(y_m,x_n) \leq \epsilon$. If we denote the generated filters by $\mathcal{F} = \langle (x_n) \rangle$ and $\mathcal{G} = \langle (y_n) \rangle$, then $(\top_{\mathcal{F}},\top_{\mathcal{G}})$ being a Cauchy pair implies that

$(x_n)$ is a Cauchy sequence with $(y_n)$ as co-sequence. We note that Doitchinov does not demand that $\top_{\mathcal{F}}$ and $\top_{\mathcal{G}}$ are linked.

*Remark* 3.4. In [25], completeness for $\top$-quasi-uniform spaces was studied using Cauchy pair $\top$-filters and also using the concept of adjoint promodules. It was shown that both approaches are equivalent. The keypoint is, that adjoint promodules can be identified with (minimal) Cauchy pair $\top$-filters, see also [15]. The condition $\bigvee_{x\in X} f(x) * g(x) = \top$ for all $f \in \mathbb{F}, g \in \mathbb{G}$ for a Cauchy pair $\top$-filter $(\mathbb{F}, \mathbb{G})$ is the one half of the adjointness. We would like to mention in this respect also the work of Preuß [19] who studied completeness in so-called preuniform convergence spaces by means of pre-Cauchy filters $\mathcal{F}$ on $X$. Also here the existence of a "co-filter" $\mathcal{G}$ on $X$ is required such that $\mathcal{G} \times \mathcal{F}$ belongs to the preuniform convergence structure, without demanding the existence of the join of $\mathcal{F}$ and $\mathcal{G}$. It seems that both Preuß' and Doitchinov's theory do not fully fit into a lax algebraic setting as it was developed e.g. in [3].

**Example 3.5** ($\top$-QULim)**.** Relaxing one axiom of a definition given in [21], we call a pair $(X, \Lambda)$ a $\top$-*quasi-uniform limit space* if $\Lambda \subseteq \mathsf{F}_{\mathsf{L}}^{\top}(X \times X)$ satisfies

(TULS1) $[(x, x)] \in \Lambda$ for all $x \in X$;
(TULS2) If $\Phi \leq \Psi$ and $\Phi \in \Lambda$, then $\Psi \in \Lambda$;
(TULS3) $\Phi \wedge \Psi \in \Lambda$ whenever $\Phi, \Psi \in \Lambda$;
(TULS4) $\Phi \circ \Psi \in \Lambda$ whenever $\Phi, \Psi \in \Lambda$ and $\Phi \circ \Psi$ exists.

A mapping $\varphi : (X, \Lambda) \longrightarrow (X', \Lambda')$ between the $\top$-quasi-uniform limit spaces $(X, \Lambda)$ and $(X', \Lambda')$ is called *uniformly continuous* if $(\varphi \times \varphi)(\Phi) \in \Lambda'$ whenever $\Phi \in \Lambda$.

We call a pair $\top$-filter $(\mathbb{F}, \mathbb{G}) \in \mathsf{PF}_{\mathsf{L}}^{\top}(X)$ a *Cauchy pair* $\top$*-filter* if $\mathbb{G} \otimes \mathbb{F} \in \Lambda$ and we denote the set of all Cauchy pair $\top$-filters on $(X, \Lambda)$ by $\mathcal{CP}^{\Lambda}$. It is then not difficult to see that $(X, \mathcal{CP}^{\Lambda})$ is a $\top$-quasi-Cauchy space and that a uniformly continuous mapping $\varphi : (X, \Lambda) \longrightarrow (X', \Lambda')$ is Cauchy continuous as a mapping $\varphi : (X, \mathcal{CP}^{\Lambda}) \longrightarrow (X', \mathcal{CP}^{\Lambda'})$.

**Example 3.6** ($\top$-PUConv)**.** Generalizing a definition of Preuß [19] to the lattice-valued case, we define a $\top$-*preuniform convergence space* $(X, \Lambda)$ with $\Lambda \subseteq \mathsf{F}_{\mathsf{L}}^{\top}(X \times X)$ satisfying (TULS1) and (TULS2). We further call $\mathbb{F} \in \mathsf{F}_{\mathsf{L}}^{\top}(X)$ a *Cauchy* $\top$*-filter* if $\mathbb{F} \otimes \mathbb{F} \in \Lambda$ and we call a pair $\top$-filter $(\mathbb{F}, \mathbb{G})$ a *Cauchy pair* $\top$*-filter* if $\mathbb{G} \otimes \mathbb{F} \in \Lambda$. We denote the set of Cauchy filters by $\mathcal{C}^{\Lambda}$ and the set of Cauchy pair $\top$-filters by $\mathcal{CP}^{\Lambda}$. It is then not difficult to show that with $\widetilde{\Lambda}$ defined by

$$\widetilde{\Lambda} = \{\Phi \in \mathsf{F}_{\mathsf{L}}^{\top}(X \times X) \ : \ \exists\, \mathbb{F} \in \mathcal{C}^{\Lambda} \text{ s.t. } \mathbb{F} \otimes \mathbb{F} \leq \Phi\},$$

$(X, \widetilde{\Lambda})$ is the finest $\top$-preuniform convergence space such that $\mathcal{C}^{\widetilde{\Lambda}} = \mathcal{C}^{\Lambda}$, i.e. for any $\top$-preuniform convergence space $(X, \Lambda^*)$ with $\mathcal{C}^{\Lambda^*} = \mathcal{C}^{\Lambda}$ we have $\widetilde{\Lambda} \subseteq \Lambda^*$. We note that $\widetilde{\Lambda}$ is *symmetric* in the sense that $\Phi \in \widetilde{\Lambda}$ implies $\Phi^{-1} \in \widetilde{\Lambda}$. If we are looking for the finest non-symmetric $\top$-preuniform convergence space

$(X, \overline{\Lambda})$ such that $\mathcal{CP}^{\Lambda} = \mathcal{CP}^{\overline{\Lambda}}$, then analogously we get

$$\overline{\Lambda} = \{\Phi \in \mathsf{F}_{\mathsf{L}}^{\top}(X \times X) \ : \ \exists\, (\mathbb{F}, \mathbb{G}) \in \mathcal{CP}^{\Lambda} \text{ s.t. } \mathbb{G} \otimes \mathbb{F} \leq \Phi\}.$$

As $\mathbb{F} \in \mathcal{C}^{\Lambda}$ implies $(\mathbb{F}, \mathbb{F}) \in \mathcal{CP}^{\Lambda}$ we see that $\widetilde{\Lambda} \subseteq \overline{\Lambda}$.

## 4. Categorical properties

It is clear that for ⊤-quasi-Cauchy spaces $(X, \mathcal{CP}), (X', \mathcal{CP}')$ and $(X'', \mathcal{CP}'')$, the identity mapping $id_X : (X, \mathcal{CP}) \longrightarrow (X, \mathcal{CP})$ is Cauchy continuous and that for two Cauchy continuous mappings $\varphi : (X, \mathcal{CP}) \longrightarrow (X', \mathcal{CP}')$, $\psi : (X', \mathcal{CP}') \longrightarrow (X'', \mathcal{CP}'')$ the composition $\psi \circ \varphi : (X, \mathcal{CP}) \longrightarrow (X'', \mathcal{CP}'')$ is Cauchy continuous. Therefore, we can form a category which has as objects the ⊤-quasi-Cauchy spaces and as morphisms the Cauchy continuous mappings. We denote this category by ⊤-$\mathsf{QChy}$.

**Proposition 4.1.** ⊤-$\mathsf{QChy}$ *is a well-fibred and topological category.*

*Proof.* The class of all ⊤-quasi-Cauchy structures on a fixed set $X$ is a subset of $\{0,1\}^{\mathsf{PF}_{\mathsf{L}}^{\top}(X)}$, that is, it is a set and hence ⊤-$\mathsf{QChy}$ is fibre-small. Furthermore, on a one-point set $X = \{x\}$, there is exactly one ⊤-quasi-Cauchy structure, namely $\mathcal{CP} = \{([x], [x])\}$. Hence ⊤-$\mathsf{QChy}$ is well-fibred.

We show the existence of initial constructions. Consider a source $(\varphi_j : X \longrightarrow (X_j, \mathcal{CP}_j))_{j \in J}$. We define the initial ⊤-quasi-Cauchy structure on $X$ as follows. For $(\mathbb{F}, \mathbb{G}) \in \mathsf{PF}_{\mathsf{L}}^{\top}(X)$, we define $(\mathbb{F}, \mathbb{G}) \in \mathcal{CP}$ if for all $j \in J$ we have $(\varphi_j(\mathbb{F}), \varphi_j(\mathbb{G})) \in \mathcal{CP}_j$.

Then $(X, \mathcal{CP})$ is a ⊤-quasi-Cauchy space. As for each $j \in J$ and each $x \in X$ we have $(\varphi_j([x]), \varphi_j([x])) = ([\varphi_j(x)], [\varphi_j(x)]) \in \mathcal{CP}_j$, we see that $([x], [x]) \in \mathcal{CP}$ for all $x \in X$ and (TQC1) is valid. For (TQC2) consider $(\mathbb{F}', \mathbb{G}') \geq (\mathbb{F}, \mathbb{G}) \in \mathcal{CP}$. Then for all $j \in J$ we have $(\varphi_j(\mathbb{F}), \varphi_j(\mathbb{G})) \in \mathcal{CP}_j$ and $\varphi_j(\mathbb{F}') \geq \varphi_j(\mathbb{F})$ and $\varphi_j(\mathbb{G}') \geq \varphi_j(\mathbb{G})$. Hence $(\varphi_j(\mathbb{F}'), \varphi_j(\mathbb{G}')) \in \mathcal{CP}_j$ for all $j \in J$ which implies $(\mathbb{F}', \mathbb{G}') \in \mathcal{CP}$. To show (TQC3), let $(\mathbb{F}, \mathbb{G}), (\mathbb{F}', \mathbb{G}') \in \mathcal{CP}$ and $\bigvee_{x \in X} f(x) * g'(x) = \top = \bigvee_{x \in X} f'(x) * g(x)$ for all $f \in \mathbb{F}, f' \in \mathbb{F}', g \in \mathbb{G}, g' \in \mathbb{G}'$. Then $\bigvee_{x_j \in X_j} \varphi_j(f)(x_j) * \varphi_j(g')(x_j) = \top = \bigvee_{x_j \in X_j} \varphi_j(f')(x_j) * \varphi_j(g)(x_j)$ for all $j \in J$. As $(\varphi_j(\mathbb{F}), \varphi_j(\mathbb{G})), (\varphi_j(\mathbb{F}'), \varphi_j(\mathbb{G}')) \in \mathcal{CP}_j$ we conclude $(\varphi_j(\mathbb{F} \wedge \mathbb{F}'), \varphi_j(\mathbb{G} \wedge \mathbb{G}')) = (\varphi_j(\mathbb{F}) \wedge \varphi_j(\mathbb{F}'), \varphi_j(\mathbb{G}) \wedge \varphi_j(\mathbb{G}')) \in \mathcal{CP}_j$ for all $j \in J$ and therefore $(\mathbb{F} \wedge \mathbb{F}', \mathbb{G} \wedge \mathbb{G}') \in \mathcal{CP}$.

It is furthermore clear that all $\varphi_j : (X, \mathcal{CP}) \longrightarrow (X_j, \mathcal{CP}_j)$ are Cauchy continuous. Consider finally a mapping $\psi : (X'', \mathcal{CP}'') \longrightarrow (X, \mathcal{CP})$ such that $\varphi_j \circ \psi$ are Cauchy continuous for all $j \in J$. For $(\mathbb{F}'', \mathbb{G}'') \in \mathcal{CP}''$ then $(\varphi_j(\psi(\mathbb{F}'')), \varphi_j(\psi(\mathbb{G}''))) = (\varphi_j \circ \psi(\mathbb{F}''), \varphi_j \circ \psi(\mathbb{G}'')) \in \mathcal{CP}_j$ for all $j \in J$. This implies $(\psi(\mathbb{F}''), \psi(\mathbb{G}'')) \in \mathcal{CP}$ and $\psi$ is Cauchy continuous. □

**Example 4.2** (Subspace)**.** Let $(X, \mathcal{CP})$ be a ⊤-quasi-Cauchy space and let $A \subseteq X$. The initial ⊤-quasi-Cauchy structure on $A$ for the embedding $i_A : A \longrightarrow X, a \longmapsto a$, $\mathcal{CP}_A$, is defined, for $(\mathbb{F}, \mathbb{G}) \in \mathsf{PF}_{\mathsf{L}}^{\top}(A)$, by

$$(\mathbb{F}, \mathbb{G}) \in \mathcal{CP}_A \iff ([\mathbb{F}], [\mathbb{G}]) = (i_A(\mathbb{F}), i_A(\mathbb{G})) \in \mathcal{CP}.$$

The following result shows that a subspace of a $\top$-quasi uniform space induces the subspace of the $\top$-quasi-Cauchy space.

**Proposition 4.3.** *Let $(X,\mathcal{U})$ be a $\top$-quasi-uniform space and let $A \subseteq X$. Then $(\mathbb{F},\mathbb{G}) \in (\mathcal{CP}^{\mathcal{U}})_A$ if and only if $\mathbb{G} \otimes \mathbb{F} \geq \mathcal{U}_{A\times A}$.*

*Proof.* This follows from $[\mathbb{G}\otimes\mathbb{F}] = (i_A \times i_A)(\mathbb{G}\otimes\mathbb{F}) = i_A(\mathbb{G})\otimes i_A(\mathbb{F}) = [\mathbb{G}]\otimes[\mathbb{F}]$ and $[\mathbb{G}\otimes\mathbb{F}]_{A\times A} = \mathbb{G}\otimes\mathbb{F}$ and $[\mathcal{U}_{A\times A}] \geq \mathcal{U}$, see e.g. [15]. □

In a similar way, we can show the next result.

**Proposition 4.4.** *Let $(X,\Lambda)$ be a $\top$-quasi-uniform limit space and let $A \subseteq X$. Then $(\mathbb{F},\mathbb{G}) \in (\mathcal{CP}^{\Lambda})_A$ if and only if there is $\Phi \in \Lambda$ such that $\Phi_{A\times A}$ exists and $\mathbb{G}\otimes\mathbb{F} \geq \Phi_{A\times A}$.*

**Example 4.5** (Product space)**.** Let $(X_j,\mathcal{CP}_j)$ be $\top$-quasi-Cauchy spaces for all $j \in J$. The initial $\top$-quasi-Cauchy structure on the Cartesian product $\prod_{j\in J} X_j$ with respect to the projects $pr_i : \prod_{j\in J} X_j \longrightarrow X_i$, $\pi - \mathcal{CP}$, is defined by

$$(\mathbb{F},\mathbb{G}) \in \pi - \mathcal{CP} \iff (pr_j(\mathbb{F}), pr_j(\mathbb{G})) \in \mathcal{CP}_j \ \forall j \in J.$$

For our next result we will assume that $\bigvee A = \top$ for $A \subseteq L$ implies $\bigvee_{\alpha\in A} \alpha * \alpha = \top$. It was shown in [13] that this can e.g. be ensured if the quantale $\mathsf{L}$ is *divisible* [10], i.e. if for all $\alpha,\beta \in L$ with $\alpha \leq \beta$ there is $\gamma \in L$ such that $\alpha = \beta * \gamma$. Another sufficient condition for this is the existence of a $\top$-*approximating sequence* $(\alpha_1,\alpha_2,...)$ in $L$ with the properties $\bot \neq \alpha_1 \leq \alpha_2 \leq ... \lhd \top$ and $\bigvee_{k\in\mathbb{N}} \alpha_k = \top$, [15]. Here, the *well-below relation* (sometimes also called the *totally-below relation*) is defined by $\alpha \lhd \beta$ if for all subsets $D \subseteq L$ such that $\beta \leq \bigvee D$ there is $\delta \in D$ such that $\alpha \leq \delta$.

We note that $(\Delta^+,\leq)$ satisfies this property [15], however $(\Delta^+,\leq,*)$ is in general not divisible, see [8]. Also, $\mathsf{L}$ being a *value quantale* [5] ensures the property, see [13].

**Theorem 4.6.** *Let the quantale $\mathsf{L}$ satisfy that $\bigvee_{\alpha\in A} \alpha * \alpha = \top$ whenever $\bigvee A = \top$ for $A \subseteq L$. Then the category $\top$-$\mathsf{QChy}$ is Cartesian closed.*

*Proof.* We show that $\top$-$\mathsf{QChy}$ has function spaces in the sense of [1]. As a well-fibred topological category, it is then Cartesian closed.

For $(X,\mathcal{CP}),(X',\mathcal{CP}') \in |\top\text{-}\mathsf{QChy}|$ we denote

$$H = H((X,\mathcal{CP}),(X',\mathcal{CP}')) = \{\varphi : (X,\mathcal{CP}) \longrightarrow (X',\mathcal{CP}') \text{ Cauchy continuous}\}.$$

We define $\mathcal{CP}_c \subseteq \mathsf{PF}^{\top}_{\mathsf{L}}(H)$ by

$$(\mathbb{H},\mathbb{K}) \in \mathcal{CP}_c \iff \text{for all } (\mathbb{F},\mathbb{G}) \in \mathcal{CP} \text{ we have } (\mathrm{ev}(\mathbb{H}\otimes\mathbb{F}), \mathrm{ev}(\mathbb{K}\otimes\mathbb{G})) \in \mathcal{CP}',$$

with the *evaluation mapping* $\mathrm{ev} : H \times X \longrightarrow X', (\varphi,x) \longmapsto \varphi(x)$.

We first show that $(H,\mathcal{CP}_c)$ is a $\top$-quasi-Cauchy space.

(TQC1) We note that for $\varphi \in H$ and $a \in L^X$ we have $\mathrm{ev}(\top_{\{\varphi\}} \otimes a)(y) = \bigvee_{\mathrm{ev}(\psi,x)=y} \top_{\{\varphi\}}(\psi) * a(x) = \bigvee_{\varphi(x)=y} a(x) = \varphi(a)(y)$. Hence, for $\mathbb{F} \in \mathsf{F}^{\top}_{L}(X)$ we have $\varphi(\mathbb{F}) = \mathrm{ev}([\varphi]\otimes\mathbb{F})$ and we obtain for $(\mathbb{F},\mathbb{G}) \in \mathcal{CP}$ that $(\mathrm{ev}([\varphi]\otimes$

$\mathbb{F}), \mathrm{ev}([\varphi] \otimes \mathbb{G})) = (\varphi(\mathbb{F}), \varphi(\mathbb{G})) \in \mathcal{CP}'$ by the Cauchy continuity of $\varphi$. This shows $([\varphi], [\varphi]) \in \mathcal{CP}_c$.

(TQC2) follows from $(\mathrm{ev}(\mathbb{H}' \otimes \mathbb{F}), \mathrm{ev}(\mathbb{K}' \otimes \mathbb{G})) \geq (\mathrm{ev}(\mathbb{H} \otimes \mathbb{F}), \mathrm{ev}(\mathbb{K} \otimes \mathbb{G}))$ for a pair $\top$-filter $(\mathbb{H}', \mathbb{K}') \geq (\mathbb{H}, \mathbb{K}) \in \mathcal{CP}_c$.

(TQC3) Let $(\mathbb{H}, \mathbb{K}), (\mathbb{H}', \mathbb{K}') \in \mathcal{CP}_c$ such that $\bigvee_{\varphi \in H} h(\varphi) * k'(\varphi) = \top = \bigvee_{\varphi \in H} h'(\varphi) * k(\varphi)$ for all $h \in \mathbb{H}, h' \in \mathbb{H}', k \in \mathbb{K}, k' \in \mathbb{K}'$. For $(\mathbb{F}, \mathbb{G}) \in \mathcal{CP}$ we have $(\mathrm{ev}(\mathbb{H} \otimes \mathbb{F}), \mathrm{ev}(\mathbb{K} \otimes \mathbb{F})), (\mathrm{ev}(\mathbb{H}' \otimes \mathbb{F}), \mathrm{ev}(\mathbb{K}' \otimes \mathbb{F})) \in \mathcal{CP}'$ and

$$(\mathrm{ev}((\mathbb{H} \wedge \mathbb{H}') \otimes \mathbb{F}), \mathrm{ev}((\mathbb{K} \wedge \mathbb{K}') \otimes \mathbb{F})) = (\mathrm{ev}(\mathbb{H} \otimes \mathbb{F}) \wedge \mathrm{ev}(\mathbb{H}' \otimes \mathbb{F}), \mathrm{ev}(\mathbb{K} \otimes \mathbb{F}) \wedge \mathrm{ev}(\mathbb{K}' \otimes \mathbb{F})).$$

The assumption on the quantale yields

$$\begin{aligned}
& \bigvee_{y \in X'} \mathrm{ev}(h \otimes f)(y) * \mathrm{ev}(k' \otimes f)(y) \\
= & \bigvee_{y \in X'} \bigvee_{\varphi(x) = y} h(\varphi) * f(x) * \bigvee_{\psi(z) = y} k'(\psi) * f(z) \\
\geq & \bigvee_{y \in X'} \bigvee_{\varphi(x) = y} h(\varphi) * f(x) * k'(\varphi) * f(x) \\
= & \bigvee_{\varphi \in H} h(\varphi) * k'(\varphi) * \bigvee_{x \in X} f(x) * f(x) \\
= & \top * \top = \top.
\end{aligned}$$

Similarly, $\bigvee_{y \in X'} \mathrm{ev}(h' \otimes f)(y) * \mathrm{ev}(k \otimes f)(y) = \top$. This is true for all $h \in \mathbb{H}, h' \in \mathbb{H}', k \in \mathbb{K}, k' \in \mathbb{K}'$ and $f \in \mathbb{F}$ and we conclude that $(\mathrm{ev}((\mathbb{H} \wedge \mathbb{H}') \otimes \mathbb{F}), \mathrm{ev}((\mathbb{K} \wedge \mathbb{K}') \otimes \mathbb{F})) \in \mathcal{CP}'$. Hence $(\mathbb{H} \wedge \mathbb{H}', \mathbb{K} \wedge \mathbb{K}') \in \mathcal{CP}_c$.

Next we show that $\mathrm{ev} : (H, \mathcal{CP}_c) \times (X, \mathcal{CP}) \longrightarrow (X', \mathcal{CP}')$ is Cauchy continuous. To this end, let $(\mathbb{H}, \mathbb{K}) \in \mathcal{CP}_c \times \mathcal{CP}$. Then $(pr_H(\mathbb{H}), pr_H(\mathbb{K})) \in \mathcal{CP}_c$ and $(pr_X(\mathbb{H}), pr_X(\mathbb{K})) \in \mathcal{CP}$ with the corresponding projection mappings. By the definition of $\mathcal{CP}_c$ we obtain $(\mathrm{ev}(pr_H(\mathbb{H}) \otimes pr_X(\mathbb{H})), \mathrm{ev}(pr_H(\mathbb{K}) \otimes pr_X(\mathbb{K}))) \in \mathcal{CP}'$ and as $(\mathrm{ev}(\mathbb{H}), \mathrm{ev}(\mathbb{K})) \geq (\mathrm{ev}(pr_H(\mathbb{H}) \otimes pr_X(\mathbb{H})), \mathrm{ev}(pr_H(\mathbb{K}) \otimes pr_X(\mathbb{K})))$ we deduce $(\mathrm{ev}(\mathbb{H}), \mathrm{ev}(\mathbb{K})) \in \mathcal{CP}'$.

Finally, let $\varphi : (X, \mathcal{CP}) \times (X', \mathcal{CP}') \longrightarrow (X'', \mathcal{CP}'')$ be a Cauchy continuous mapping. For $x \in X$, we define $\varphi_x : X' \longrightarrow X''$ by $\varphi_x(x') = \varphi(x, x')$. Let $(\mathbb{F}', \mathbb{G}') \in \mathcal{CP}'$. As $([x], [x]) \in \mathcal{CP}$ we see that $([x] \otimes \mathbb{F}', [x] \otimes \mathbb{G}') \in \mathcal{CP} \times \mathcal{CP}'$ and hence, by continuity of $\varphi$, $(\varphi([x] \otimes \mathbb{F}'), \varphi([x] \otimes \mathbb{G}')) \in \mathcal{CP}''$. It is not difficult to show that $(\varphi_x(\mathbb{F}'), \varphi_x(\mathbb{G}')) \geq (\varphi([x] \otimes \mathbb{F}'), \varphi([x] \otimes \mathbb{G}'))$ and therefore $\varphi_x$ is Cauchy continuous. We can hence define $\varphi^* : X \longrightarrow H((X', \mathcal{CP}'), (X'', \mathcal{CP}''))$ by $\varphi^*(x) = \varphi_x$ and we need to show that $\varphi^*$ is Cauchy continuous. Let $(\mathbb{F}, \mathbb{G}) \in \mathcal{CP}$. Then, for all $(\mathbb{H}, \mathbb{K}) \in \mathcal{CP}'$ we have $(\mathbb{F} \otimes \mathbb{H}, \mathbb{G} \otimes \mathbb{K}) \in \mathcal{CP} \times \mathcal{CP}'$. From $\mathrm{ev}(\varphi^* \times id_{X'}) = \varphi$ we obtain with the Cauchy continuity of $\varphi$,

$$(\mathrm{ev}(\varphi^*(\mathbb{F}) \otimes \mathbb{H}), \mathrm{ev}(\varphi^*(\mathbb{G}) \otimes \mathbb{K})) = (\varphi(\mathbb{F} \otimes \mathbb{H}), \varphi(\mathbb{G} \otimes \mathbb{K})) \in \mathcal{CP}''.$$

This shows that $(\varphi^*(\mathbb{F}), \varphi^*(\mathbb{G})) \in \mathcal{CP}_c$ and the proof is complete. □

## 5. Convergence

For a $\top$-quasi-Cauchy space $(X, \mathcal{CP})$ we say that a pair $\top$-filter $(\mathbb{F}, \mathbb{G}) \in \mathsf{PF}_{\mathsf{L}}^{\top}(X)$ *converges to* $x \in X$, and we write $x \in q^{\mathcal{CP}}(\mathbb{F}, \mathbb{G})$, if $(\mathbb{F} \wedge [x], \mathbb{G} \wedge [x]) \in \mathcal{CP}$. We note that convergent pair $\top$-filters are Cauchy pair $\top$-filters by (TQCS2).

**Example 5.1.** Let $(X, \mathcal{U})$ be a $\top$-quasi-uniform space. In [25], see also [15], it is defined that a pair $\top$-filter $(\mathbb{F}, \mathbb{G})$ converges to $x$ if and only if $[x] \otimes \mathbb{F} \geq \mathcal{U}$ and $\mathbb{G} \otimes [x] \geq \mathcal{U}$. In [15] it was shown that this requirement is equvalent to $(\mathbb{F} \wedge [x], \mathbb{G} \wedge [x])$ being a Cauchy pair $\top$-filter.

**Example 5.2.** Let $(X, \Lambda)$ be a $\top$-quasi-uniform limit space. We say that a pair $\top$-filter $(\mathbb{F}, \mathbb{G})$ converges to $x$ if $[x] \otimes \mathbb{F} \in \Lambda$ and $\mathbb{G} \otimes [x] \in \Lambda$. It is not difficult to show that this is equivalent to $(\mathbb{F} \wedge [x], \mathbb{G} \wedge [x])$ being a Cauchy pair $\top$-filter.

**Proposition 5.3.** *Let $(X, \mathcal{CP})$ be a $\top$-quasi-Cauchy space. Convergence of pair $\top$-filters has the following properties. For all $(\mathbb{F}, \mathbb{G}), (\mathbb{F}', \mathbb{G}') \in \mathsf{PF}_{\mathsf{L}}^{\top}(X)$ and all $x \in X$ we have:*

(TQL1) $x \in q^{\mathcal{CP}}([x], [x])$;
(TQL2) $(\mathbb{F}, \mathbb{G}) \leq (\mathbb{F}', \mathbb{G}')$ *implies* $q^{\mathcal{CP}}(\mathbb{F}, \mathbb{G}) \subseteq q^{\mathcal{CP}}(\mathbb{F}', \mathbb{G}')$;
(TQL3) $q^{\mathcal{CP}}(\mathbb{F}, \mathbb{G}) \cap q^{\mathcal{CP}}(\mathbb{F}', \mathbb{G}') \subseteq q^{\mathcal{CP}}(\mathbb{F} \wedge \mathbb{F}', \mathbb{G} \wedge \mathbb{G}')$.

*Proof.* We only show (TQL3). We have $(\mathbb{F} \wedge [x], \mathbb{G} \wedge [x]), (\mathbb{F}' \wedge [x], \mathbb{G}' \wedge [x]) \in \mathcal{CP}$. The axiom (TQC3) then yields $(\mathbb{F} \wedge \mathbb{F}' \wedge [x], \mathbb{G} \wedge \mathbb{G}' \wedge [x]) \in \mathcal{CP}$. □

**Proposition 5.4.** *Let $(X, \mathcal{CP})$ and $(X', \mathcal{CP}')$ be $\top$-quasi-Cauchy spaces and let $\varphi : (X, \mathcal{CP}) \longrightarrow (X', \mathcal{CP}')$ be Cauchy continuous. Then $x \in q^{\mathcal{CP}}(\mathbb{F}, \mathbb{G})$ implies $\varphi(x) \in q^{\mathcal{CP}'}(\varphi(\mathbb{F}), \varphi(\mathbb{G}))$.*

*Proof.* If $(\mathbb{F} \wedge [x], \mathbb{G} \wedge [x]) \in \mathcal{CP}$, then by Cauchy continuity, $(\varphi(\mathbb{F}) \wedge [\varphi(x)], \varphi(\mathbb{G}) \wedge [\varphi(x)]) = (\varphi(\mathbb{F} \wedge [x]), \varphi(\mathbb{G} \wedge [x])) \in \mathcal{CP}'$. □

**Definition 5.5.** Let $(X, \mathcal{P})$ be a $\top$-quasi-Cauchy space and let $A \subseteq X$. We define the *closure of* $A$, $\overline{A} = \overline{A}^{\mathcal{CP}}$, by $x \in \overline{A}$ if there is a pair $\top$-filter $(\mathbb{F}, \mathbb{G}) \in \mathsf{PF}_{\mathsf{L}}^{\top}(A)$ such that $x \in q^{\mathcal{CP}}([\mathbb{F}], [\mathbb{G}])$.

The following result shows that this notion of closure coincides with the definition of closure in $\top$-quasi-uniform spaces given by Wang and Yue [23].

**Proposition 5.6.** *Let $(X, \mathcal{U})$ be a $\top$-quasi-uniform space and let $A \subseteq X$. Then $x \in \overline{A}^{\mathcal{CP}^{\mathcal{U}}}$ if, and only if, for all $u \in \mathcal{U}$ we have $\bigvee_{a \in A} u(x, a) * u(a, x) = \top$.*

*Proof.* Let $x \in \overline{A}^{\mathcal{CP}^{\mathcal{U}}}$. By the definition of $\mathcal{CP}^{\mathcal{U}}$ there is a pair $\top$-filter $(\mathbb{F}, \mathbb{G})$ on $A$ such that $[x] \otimes [\mathbb{F}] \geq \mathcal{U}$ and $[\mathbb{G}] \otimes [x] \geq \mathcal{U}$. Hence, for $u \in \mathcal{U}$, we have

$\top = \bigvee_{f\in\mathbb{F}}[\top_{\{x\}} \otimes f, u] = \bigvee_{g\in\mathbb{G}}[g \otimes \top_{\{x\}}, u]$. We conclude

$$\begin{aligned}
\top &= \bigvee_{f\in\mathbb{F},g\in\mathbb{G}} \bigwedge_{a\in A} (f(a) \to u(x,a)) * (g(a) \to u(a,x)) \\
&\leq \bigvee_{f\in\mathbb{F},g\in\mathbb{G}} \bigwedge_{a\in A} ((f(a) * g(a)) \to (u(x,a) * u(a,x))) \\
&\leq \bigvee_{f\in\mathbb{F},g\in\mathbb{G}} \left( (\bigvee_{a\in A} f(a) * g(a)) \to (\bigvee_{a\in A} u(x,a) * u(a,x)) \right) \\
&= \bigvee_{a\in A} u(x,a) * u(a,x)
\end{aligned}$$

because $(\mathbb{F},\mathbb{G})$ is a pair $\top$-filter on $A$.

Conversely, let $\bigvee_{a\in A} u(x,a) * u(a,x) = \top$ for all $u \in \mathcal{U}$. We define $\mathbb{F}$ as the $\top$-filter on $A$ with $\top$-filter basis on $A$, $\{u(x,\cdot) \ : \ u \in \mathcal{U}_{A\times A}\}$. Here, $u(x,\cdot)(a) = u(x,a)$ for $a \in A$, that is, $u(x,\cdot) \in L^A$. Likewise, $\mathbb{G}$ is the $\top$-filter on $A$ with $\top$-filter basis $\{u(\cdot,x) \ : \ u \in \mathcal{U}_{A\times A}\}$. The given condition guarantees that $(\mathbb{F},\mathbb{G})$ is a pair $\top$-filter on $A$. Moreover, we have $[x]\otimes[\mathbb{F}] \geq \mathcal{U}$, as for $u \in \mathcal{U}$ we have $\top_{\{x\}} \otimes i_A(u(x,\cdot))(s,t) \leq u(s,t)$ for $s,t \in X$. Similarly, $[\mathbb{G}] \otimes [x] \geq \mathcal{U}$. This implies $x \in q^{\mathcal{CP}^{\mathcal{U}}}([\mathbb{F}],[\mathbb{G}])$, that is, $x \in \overline{A}^{\mathcal{CP}^{\mathcal{U}}}$. □

We note that for an L-metric space $(X,d)$, the closure of $A$ in $(X,[d])$ is characterized by $x \in \overline{A}^{\mathcal{CP}^{[d]}}$ if and only if $\bigvee_{a\in A} d(x,a) * d(a,x) = \top$. This is a characterization of closure in $(X,d)$ used in [11].

Using the concept of convergence, we can introduce the following separation axiom. We call a $\top$-quasi-Cauchy space $(X,\mathcal{CP})$ *separated* if $x,y \in q^{\mathcal{CP}}(\mathbb{F},\mathbb{G})$ implies $x = y$.

Separation for $\top$-quasi-uniform spaces was defined in [23]. The following result shows that our definition applies also there.

**Proposition 5.7.** *Let $(X,\mathcal{U})$ be a $\top$-quasi-uniform space. Then $(X,\mathcal{CP}^{\mathcal{U}})$ is separated if and only if $x = y$ whenever $u(x,y) = \top = u(y,x)$ for all $u \in \mathcal{U}$.*

*Proof.* Let $(X,\mathcal{CP}^{\mathcal{U}})$ be separated and let $u(x,y) = \top = u(y,x)$ for all $u \in \mathcal{U}$. Then $[x]\otimes[y] \geq \mathcal{U}$ and $[y]\otimes[x] \geq \mathcal{U}$. This implies $x \in q^{\mathcal{U}}([y],[y])$ and because $y \in q^{\mathcal{U}}([y],[y])$ we obtain $x = y$.

For the converse, let $x,y \in q^{\mathcal{U}}(\mathbb{F},\mathbb{G})$. Then $[x]\otimes\mathbb{F} \geq \mathcal{U}$ and $\mathbb{G}\otimes[y] \geq \mathcal{U}$ and with (TU2), we conclude $\mathcal{U} \leq \mathcal{U}\circ\mathcal{U} \leq (\mathbb{G}\otimes[y]) \circ ([x]\otimes\mathbb{F}) = [x]\otimes[y]$, as $(\mathbb{F},\mathbb{G})$ is a pair $\top$-filter. Similarly, we see $[y]\otimes[x] \geq \mathcal{U}$. Hence, for all $u \in \mathcal{U}$ we have $u(x,y) = \top$ and $u(y,x) = \top$ and therefore $x = y$. □

For the special case of an L-metric space $(X,d)$, we have that $(X,\mathcal{CP}^{[d]})$ is separated if and only if $x = y$ whenever $d(x,y) = \top = d(y,x)$. Again, this is a characterization of separation of an L-metric space in [11].

## 6. Completeness and completion

We say that a pair $\top$-filter $(\mathbb{F},\mathbb{G})$ is *convergent* in $(X,\mathcal{CP})$ if there is $x\in X$ such that $x\in q^{\mathcal{CP}}(\mathbb{F},\mathbb{G})$, that is, if $(\mathbb{F}\wedge[x],\mathbb{G}\wedge[x])\in\mathcal{CP}$. Otherwise, we call $(\mathbb{F},\mathbb{G})$ *non-convergent*. A $\top$-quasi-Cauchy space $(X,\mathcal{CP})$ is called *complete* if every $(\mathbb{F},\mathbb{G})\in\mathcal{CP}$ is convergent.

**Proposition 6.1.** *Let $(X_j,\mathcal{CP}_j)$ be complete $\top$-quasi-Cauchy spaces for all $j\in J$. Then also the product space $(\prod_{j\in J}X_j,\pi-\mathcal{CP})$ is complete.*

*Proof.* Let $(\mathbb{F},\mathbb{G})\in\pi-\mathcal{CP}$. Then for all $j\in J$, $(pr_j(\mathbb{F}),pr_j(\mathbb{G}))\in\mathcal{CP}_j$ and hence there is $x_j\in X_j$ such that $(pr_j(\mathbb{F})\wedge[x_j],pr_j(\mathbb{G})\wedge[x_j])\in\mathcal{CP}_j$. We define $x=(x_j)_{j\in J}$. Then $(pr_j(\mathbb{F}\wedge[x]),pr_j(\mathbb{G}\wedge[x]))=(pr_j(\mathbb{F})\wedge[x_j],pr_j(\mathbb{G})\wedge[x_j])\in\mathcal{CP}_j$ for all $j\in J$ and hence $(\mathbb{F}\wedge[x],\mathbb{G}\wedge[x])\in\pi-\mathcal{CP}$. □

A completion $((X^+,\mathcal{CP}^+),\phi)$ of a $\top$-quasi-Cauchy space $(X,\mathcal{CP})$ is a complete $\top$-quasi-Cauchy space $(X^+,\mathcal{CP}^+)$ and a dense Cauchy embedding $\phi:(X,\mathcal{CP})\longrightarrow(X^+,\mathcal{CP}^+)$. This means that $\phi$ is injective and that we have $(\mathbb{F},\mathbb{G})\in\mathcal{CP}$ if, and only if, $(\phi(\mathbb{F}),\phi(\mathbb{G}))\in\mathcal{CP}^+$ and that $\overline{\phi(X)}^{\mathcal{CP}^+}=X^+$.

For two completions $((X^+,\mathcal{CP}^+),\phi),((X^\sim,\mathcal{CP}^\sim),\psi)$ of $(X,\mathcal{CP})$ we call $((X^+,\mathcal{CP}^+),\phi)$ *finer than* $((X^\sim,\mathcal{CP}^\sim),\psi)$, and we write $((X^+,\mathcal{CP}^+),\phi)\geq((X^\sim,\mathcal{CP}^\sim),\psi)$, if there is a Cauchy continuous mapping $h:(X^+,\mathcal{CP}^+)\longrightarrow(X^\sim,\mathcal{CP}^\sim)$ such that $h\circ\phi=\psi$.

We are now going to construct a completion of a non-complete $\top$-quasi-Cauchy space $(X,\mathcal{CP})$. To this end, the following relation on $\mathcal{CP}$ is useful. Let $(\mathbb{F},\mathbb{G}),(\mathbb{F}',\mathbb{G}')\in\mathcal{CP}$. We define

$$(\mathbb{F},\mathbb{G})\sim(\mathbb{F}',\mathbb{G}')\iff(\mathbb{F}\wedge\mathbb{F}',\mathbb{G}\wedge\mathbb{G}')\in\mathcal{CP}.$$

It is clear that for $(\mathbb{F},\mathbb{G})\in\mathcal{CP}$ we have $(\mathbb{F},\mathbb{G})\sim([x],[x])$ if and only if $x\in q^{\mathcal{CP}}(\mathbb{F},\mathbb{G})$.

**Proposition 6.2.** *Let $(X,\mathcal{CP})$ be a $\top$-quasi-Cauchy space. The relation $\sim$ is an equivalence relation.*

*Proof.* Reflexivity and symmetry of the relation are clear. We check the transitivity. Let $(\mathbb{F},\mathbb{G})\sim(\mathbb{F}',\mathbb{G}')$ and $(\mathbb{F}',\mathbb{G}')\sim(\mathbb{F}'',\mathbb{G}'')$. As $\mathbb{F}\wedge\mathbb{F}'\leq\mathbb{F}'$ and $\mathbb{G}'\wedge\mathbb{G}''\leq\mathbb{G}'$ we see that for $f\in\mathbb{F}\wedge\mathbb{F}'$ and $g\in\mathbb{G}'\wedge\mathbb{G}''$ we have $\bigvee_{x\in X}f(x)*g(x)=\top$, because $(\mathbb{F}',\mathbb{G}')$ is a pair $\top$-filter. Similarly, we see that for $g\in\mathbb{G}\wedge\mathbb{G}'$ and $f\in\mathbb{F}'\wedge\mathbb{F}''$ we have $\bigvee_{x\in X}f(x)*g(x)=\top$. Hence, from (TQC3) we obtain $(\mathbb{F}\wedge\mathbb{F}'\wedge\mathbb{F}'',\mathbb{G}\wedge\mathbb{G}'\wedge\mathbb{G}'')\in\mathcal{CP}$ and, using (TQC2), we conclude $(\mathbb{F}\wedge\mathbb{F}'',\mathbb{G}\wedge\mathbb{G}'')\in\mathcal{CP}$, that is, $(\mathbb{F},\mathbb{G})\sim(\mathbb{F}'',\mathbb{G}'')$. □

We denote the equivalence class of $(\mathbb{F},\mathbb{G})\in\mathcal{CP}$ by $\langle(\mathbb{F},\mathbb{G})\rangle$.

The equivalence relation allows simple proofs of the following results.

**Lemma 6.3.** *Let $(X,\mathcal{CP})$ be a $\top$-quasi-Cauchy space and let $(\mathbb{F},\mathbb{G}),(\mathbb{F}',\mathbb{G}')\in\mathcal{CP}$. If $(\mathbb{F}',\mathbb{G}')\leq(\mathbb{F},\mathbb{G})$ and if $x\in q^{\mathcal{CP}}(\mathbb{F},\mathbb{G})$, then also $x\in q^{\mathcal{CP}}(\mathbb{F}',\mathbb{G}')$.*

*Proof.* If $(\mathbb{F}',\mathbb{G}')\leq(\mathbb{F},\mathbb{G})$, then $(\mathbb{F}',\mathbb{G}')\sim(\mathbb{F},\mathbb{G})\sim([x],[x])$ and, by transitivity, $(\mathbb{F}',\mathbb{G}')\sim([x],[x])$. □

**Lemma 6.4.** *Let $(X,\mathcal{CP})$ be a $\top$-quasi-Cauchy space. If there is a pair $\top$-filter $(\mathbb{F},\mathbb{G})$ converging to both $x$ and $y$, then $\{(\mathbb{H},\mathbb{K})\in \mathsf{PF}^{\top}_{\mathsf{L}}(X)\ :\ x\in q^{\mathcal{CP}}(\mathbb{H},\mathbb{K})\} = \{(\mathbb{H},\mathbb{K})\in \mathsf{PF}^{\top}_{\mathsf{L}}(X)\ :\ y\in q^{\mathcal{CP}}(\mathbb{H},\mathbb{K})\}$.*

*Proof.* Clearly, $(\mathbb{F},\mathbb{G})\in\mathcal{CP}$ and we have $([y],[y])\sim(\mathbb{F},\mathbb{G})\sim([x],[x])$. So if $(\mathbb{H},\mathbb{K})\sim([x],[x])$, then by transitivity $(\mathbb{H},\mathbb{K})\sim([y],[y])$ and vice versa. □

**Lemma 6.5.** *Let $(X,\mathcal{CP})$ be a $\top$-quasi-Cauchy space. If $x\in q^{\mathcal{CP}}(\mathbb{F},\mathbb{G})$ and $(\mathbb{F}',\mathbb{G}')\in\mathcal{CP}$ satisfies that for all $f\in\mathbb{F}, g\in\mathbb{G}, f'\in\mathbb{F}', g'\in\mathbb{G}'$ we have $\bigvee_{x\in X} f'(x)*g(x)=\top=\bigvee_{x\in X} f(x)*g'(x)$, then $x\in q^{\mathcal{CP}}(\mathbb{F}',\mathbb{G}')$.*

*Proof.* We have $(\mathbb{F}\wedge\mathbb{F}',\mathbb{G}\wedge\mathbb{G}')\in\mathcal{CP}$ and $\leq(\mathbb{F},\mathbb{G})$. According to Lemma 6.3, $x\in q^{\mathcal{CP}}((\mathbb{F}\wedge\mathbb{F}',\mathbb{G}\wedge\mathbb{G}'))\subseteq q^{\mathcal{CP}}((\mathbb{F}',\mathbb{G}'))$. □

We define now $X^*=X\cup\{\langle(\mathbb{F},\mathbb{G})\rangle\ :\ (\mathbb{F},\mathbb{G})\in\mathcal{CP}\text{ non-convergent}\}$ and we denote $j:X\longrightarrow X^*, x\longmapsto x$ the embedding injection of $X$ into $X^*$. We define $\mathcal{CP}^*\subseteq\mathsf{PF}^{\top}_{\mathsf{L}}(X^*)$ as follows. $(\mathbb{H},\mathbb{K})\in\mathcal{CP}^*$ if there is $(\mathbb{F},\mathbb{G})\in\mathcal{CP}$ convergent such that $\mathbb{H}\geq j(\mathbb{F})$ and $\mathbb{K}\geq j(\mathbb{G})$ or if there is $(\mathbb{F},\mathbb{G})\in\mathcal{CP}$ non-convergent such that $\mathbb{H}\geq j(\mathbb{F})\wedge[\langle(\mathbb{F},\mathbb{G})\rangle]$ and $\mathbb{K}\geq j(\mathbb{G})\wedge[\langle(\mathbb{F},\mathbb{G})\rangle]$.

**Theorem 6.6.** *Let $(X,\mathcal{CP})$ be a $\top$-quasi-Cauchy space. Then $((X^*,\mathcal{CP}^*),j)$ is a completion of $(X,\mathcal{CP})$.*

*Proof.* We first show that $(X^*,\mathcal{CP}^*)$ is a $\top$-quasi-Cauchy space. The axiom (TQC1) follows, as for $x\in X$ we have $j([x])=[j(x)]$. For $\langle(\mathbb{F},\mathbb{G})\rangle$ with $(\mathbb{F},\mathbb{G})\in\mathcal{CP}$ non-convergent, we have $[\langle(\mathbb{F},\mathbb{G})\rangle]\geq j(\mathbb{F})\wedge[\langle(\mathbb{F},\mathbb{G})\rangle]$ and $[\langle(\mathbb{F},\mathbb{G})\rangle]\geq j(\mathbb{G})\wedge[\langle(\mathbb{F},\mathbb{G})\rangle]$ and hence also $([\langle(\mathbb{F},\mathbb{G})\rangle],[\langle(\mathbb{F},\mathbb{G})\rangle])\in\mathcal{CP}^*$.

The axiom (TQC2) is obvious.

For the axiom (TQC3), let $(\mathbb{H},\mathbb{K}),(\mathbb{H}',\mathbb{K}')\in\mathcal{CP}^*$ such that $\bigvee_{z^*\in X^*} h(z^*)*k'(z^*)=\top=\bigvee_{z^*\in X^*} h'(z^*)*k(z^*)$ for all $h\in\mathbb{H},h'\in\mathbb{H}',k\in\mathbb{K},k'\in\mathbb{K}'$. We distinguish four cases.

*Case 1:* There are $(\mathbb{F},\mathbb{G}),(\mathbb{F}',\mathbb{G}')\in\mathcal{CP}$ convergent such that $\mathbb{H}\geq j(\mathbb{F}),\mathbb{K}\geq j(\mathbb{G})$ and $\mathbb{H}'\geq j(\mathbb{F}'),\mathbb{K}'\geq j(\mathbb{G}')$. We then have

$$\begin{aligned}\top &= \bigvee_{z^*\in X^*} j(f)(z^*)*j(g')(z^*)\\ &= \bigvee_{z\in X} j(f)(z)*j(g')(z)\\ &\quad\vee \bigvee_{(\mathbb{F},\mathbb{G})\in\mathcal{CP}\text{ non-conv.}} j(f)(\langle(\mathbb{F},\mathbb{G})\rangle)*j(g')(\langle(\mathbb{F},\mathbb{G})\rangle)\\ &= \bigvee_{z\in X} f(z)*g'(z)\end{aligned}$$

Similarly, $\bigvee_{z\in X} f'(z)*g(z)=\top$ and hence $(\mathbb{F}\wedge\mathbb{F}',\mathbb{G}\wedge\mathbb{G}')\in\mathcal{CP}$ convergent, from Lemma 6.3. Therefore $(j(\mathbb{F}\wedge\mathbb{F}'),j(\mathbb{G}\wedge\mathbb{G}'))\leq(\mathbb{H}\wedge\mathbb{H}',\mathbb{K}\wedge\mathbb{K}')$ and we obtain $(\mathbb{H}\wedge\mathbb{H}',\mathbb{K}\wedge\mathbb{K}')\in\mathcal{CP}^*$.

*Case 2:* There is $(\mathbb{F},\mathbb{G})\in\mathcal{CP}$ convergent such that $\mathbb{H}\geq j(\mathbb{F})$ and $\mathbb{K}\geq j(\mathbb{G})$ and there is $(\mathbb{F}',\mathbb{G}')\in\mathcal{CP}$ non-convergent such that $\mathbb{H}'\geq j(\mathbb{F}')\wedge[\langle(\mathbb{F}',\mathbb{G}')\rangle]$ and

$\mathbb{K}' \geq j(\mathbb{G}') \wedge [\langle(\mathbb{F}', \mathbb{G}')\rangle]$. If $\bigvee_{x\in X} f'(x) * g(x) = \top = \bigvee_{x\in X} f(x) * g'(x)$ for all $f \in \mathbb{F}, f' \in \mathbb{F}', g \in \mathbb{G}, g' \in \mathbb{G}'$, then $(\mathbb{F}', \mathbb{G}')$ would be convergent by Lemma 6.5. Hence we may assume without loss of generality that there are $f' \in \mathbb{F}', g \in \mathbb{G}$ such that $\bigvee_{x\in X} f'(x) * g(x) \neq \top$. As $j(f') \vee \top_{\langle(\mathbb{F}',\mathbb{G}')\rangle} \in j(\mathbb{F}') \wedge [\langle(\mathbb{F}', \mathbb{G}')\rangle] \leq \mathbb{H}'$ we conclude, with $j(g) \in j(\mathbb{G}) \leq \mathbb{K}$ that $\bigvee_{x^*\in X^*}(j(f') \vee \top_{\langle(\mathbb{F}',\mathbb{G}')\rangle})(x^*) * j(g)(x^*) = \bigvee_{x\in X} f'(x) * g(x) \neq \top$, a contradiction. Hence this case does not occur.

*Case 3:* There is $(\mathbb{F}', \mathbb{G}') \in \mathcal{CP}$ convergent such that $\mathbb{H}' \geq j(\mathbb{F}')$ and $\mathbb{K}' \geq j(\mathbb{G}')$ and there is $(\mathbb{F}, \mathbb{G}) \in \mathcal{CP}$ non-convergent such that $\mathbb{H} \geq j(\mathbb{F}) \wedge [\langle(\mathbb{F}, \mathbb{G})\rangle]$ and $\mathbb{K} \geq j(\mathbb{G}) \wedge [\langle(\mathbb{F}, \mathbb{G})\rangle]$. The arguments of case 2 can be used to show that also this case does not occur.

*Case 4:* There are $(\mathbb{F}, \mathbb{G}), (\mathbb{F}', \mathbb{G}') \in \mathcal{CP}$ non-convergent such that $\mathbb{H} \geq j(\mathbb{F}) \wedge [\langle(\mathbb{F}, \mathbb{G})\rangle], \mathbb{K} \geq j(\mathbb{G}) \wedge [\langle(\mathbb{F}, \mathbb{G})\rangle]$ and $\mathbb{H}' \geq j(\mathbb{F}') \wedge [\langle(\mathbb{F}', \mathbb{G}')\rangle], \mathbb{K}' \geq j(\mathbb{G}') \wedge [\langle(\mathbb{F}', \mathbb{G}')\rangle]$. Then we have, for $f \in \mathbb{F}, f' \in \mathbb{F}', g \in \mathbb{G}, g' \in \mathbb{G}'$,

$$\begin{aligned} \top &= \bigvee_{x^*\in X^*} (j(f) \vee \top_{\langle(\mathbb{F},\mathbb{G})\rangle})(x^*) * (j(g') \vee \top_{\langle(\mathbb{F}',\mathbb{G}')\rangle})(x^*) \\ &= \bigvee_{x\in X} f(x) * g'(x) \vee \bigvee_{x^*\in X^*\setminus X} \top_{\langle(\mathbb{F},\mathbb{G})\rangle}(x^*) * \top_{\langle(\mathbb{F}',\mathbb{G}')\rangle}(x^*), \end{aligned}$$

and, similarly,

$$\top = \bigvee_{x\in X} f'(x) * g(x) \vee \bigvee_{x^*\in X^*\setminus X} \top_{\langle(\mathbb{F}',\mathbb{G}')\rangle}(x^*) * \top_{\langle(\mathbb{F},\mathbb{G})\rangle}(x^*).$$

If $\langle(\mathbb{F}, \mathbb{G})\rangle \neq \langle(\mathbb{F}', \mathbb{G}')\rangle$, then $\bigvee_{x\in X} f(x) * g'(x) = \top = \bigvee_{x\in X} f'(x) * g(x)$ and $(\mathbb{F} \wedge \mathbb{F}', \mathbb{G} \wedge \mathbb{G}') \in \mathcal{CP}$. As this pair $\top$-filter is $\leq (\mathbb{F}, \mathbb{G}), (\mathbb{F}', \mathbb{G}')$ we conclude $\langle(\mathbb{F}, \mathbb{G})\rangle = \langle(\mathbb{F} \wedge \mathbb{F}', \mathbb{G} \wedge \mathbb{G}')\rangle = \langle(\mathbb{F}', \mathbb{G}')\rangle$, a contradiction. Hence $\langle(\mathbb{F}, \mathbb{G})\rangle = \langle(\mathbb{F}', \mathbb{G}')\rangle$. Then $(\mathbb{F}, \mathbb{G}) \sim (\mathbb{F}', \mathbb{G}')$ and hence $(\mathbb{F} \wedge \mathbb{F}', \mathbb{G} \wedge \mathbb{G}') \in \mathcal{CP}$ is non-convergent, too. Therefore $\mathbb{H} \wedge \mathbb{H}' \geq j(\mathbb{F} \wedge \mathbb{F}') \wedge [\langle(\mathbb{F} \wedge \mathbb{F}', \mathbb{G} \wedge \mathbb{G}')\rangle]$ and $\mathbb{K} \wedge \mathbb{K}' \geq j(\mathbb{G} \wedge \mathbb{G}') \wedge [\langle(\mathbb{F} \wedge \mathbb{F}', \mathbb{G} \wedge \mathbb{G}')\rangle]$ and we have $(\mathbb{H} \wedge \mathbb{H}', \mathbb{K} \wedge \mathbb{K}') \in \mathcal{CP}'$.

The mapping $j : (X, \mathcal{CP}) \longrightarrow (X^*, \mathcal{CP}^*)$ is Cauchy continuous, as for $(\mathbb{F}, \mathbb{G}) \in \mathcal{CP}$ either $(\mathbb{F}, \mathbb{G})$ is convergent and then $(j(\mathbb{F}), j(\mathbb{G})) \in \mathcal{CP}^*$. Or $(\mathbb{F}, \mathbb{G})$ is non-convergent and then, because $j(\mathbb{F}) \geq j(\mathbb{F}) \wedge [\langle(\mathbb{F}, \mathbb{G})\rangle]$ and $j(\mathbb{G}) \geq j(\mathbb{G}) \wedge [\langle(\mathbb{F}, \mathbb{G})\rangle]$, again $(j(\mathbb{F}), j(\mathbb{G})) \in \mathcal{CP}^*$.

Conversely, if $(j(\mathbb{F}), j(\mathbb{G})) \in \mathcal{CP}^*$, then either there is $(\mathbb{F}', \mathbb{G}') \in \mathcal{CP}$ convergent such that $\mathbb{F} \geq \mathbb{F}'$ and $\mathbb{G} \geq \mathbb{G}'$ and hence $(\mathbb{F}, \mathbb{G}) \in \mathcal{CP}$ by (TQC2). Or there is $(\mathbb{F}', \mathbb{G}') \in \mathcal{CP}$ non-convergent such that $j(\mathbb{F}) \geq j(\mathbb{F}') \wedge [\langle(\mathbb{F}', \mathbb{G}')\rangle]$ and $j(\mathbb{G}) \geq j(\mathbb{G}') \wedge [\langle(\mathbb{F}', \mathbb{G}')\rangle]$. This implies $\mathbb{F}' \leq \mathbb{F}$ and $\mathbb{G}' \leq \mathbb{G}$ and again with (TQC2), $(\mathbb{F}, \mathbb{G}) \in \mathcal{CP}$.

Furthermore, $(X^*, \mathcal{CP}^*)$ is complete. If $(\mathbb{G}, \mathbb{K}) \in \mathcal{CP}^*$ such that $\mathbb{H} \geq j(\mathbb{F})$ and $\mathbb{K} \geq j(\mathbb{G})$ with $(\mathbb{F}, \mathbb{G}) \in \mathcal{CP}$ convergent to $x \in X$, then $(\mathbb{H}, \mathbb{K})$ is convergent to $x$ in $(X^*, \mathcal{CP}^*)$. If $(\mathbb{G}, \mathbb{K}) \in \mathcal{CP}^*$ such that $\mathbb{H} \geq j(\mathbb{F}) \wedge [\langle(\mathbb{F}, \mathbb{G})\rangle]$ and $\mathbb{K} \geq j(\mathbb{G}) \wedge [\langle(\mathbb{F}, \mathbb{G})\rangle]$ with $(\mathbb{F}, \mathbb{G}) \in \mathcal{CP}$ non-convergent, then $(\mathbb{H} \wedge [\langle(\mathbb{F}, \mathbb{G})\rangle], \mathbb{K} \wedge [\langle(\mathbb{F}, \mathbb{G})\rangle]) \in \mathcal{CP}^*$, that is $(\mathbb{H}, \mathbb{K})$ converges to $\langle(\mathbb{F}, \mathbb{G})\rangle$ in $(X^*, \mathcal{CP}^*)$.

Finally, we show that $\overline{j(X)}^{\mathcal{CP}^*} = X^*$. Let $x^* \in X^*$. If $x \in X$, then $(j([x]), j([x]))$ converges to $x$ in $(X^*, \mathcal{CP}^*)$. If $x^* = \langle(\mathbb{F}, \mathbb{G})\rangle$ with $(\mathbb{F}, \mathbb{G}) \in \mathcal{CP}$ non-convergent, then $(j(\mathbb{F}) \wedge [\langle(\mathbb{F}, \mathbb{G})\rangle], j(\mathbb{G}) \wedge [\langle(\mathbb{F}, \mathbb{G})\rangle]) \in \mathcal{CP}^*$ and hence $(j(\mathbb{F}), j(\mathbb{G}))$ converges to $x^*$ in $(X^*, \mathcal{CP}^*)$. Hence $x^* \in \overline{j(X)}^{\mathcal{CP}^*}$ and the proof is complete. □

**Theorem 6.7.** *Let $(X, \mathcal{CP}), (X', \mathcal{CP}')$ be ⊤-quasi-Cauchy spaces, let $(X', \mathcal{CP}')$ be complete and let $\varphi : (X, \mathcal{CP}) \longrightarrow (X', \mathcal{CP}')$ be Cauchy continuous. Then there exists a Cauchy continuous mapping $\varphi^* : (X^*, \mathcal{CP}^*) \longrightarrow (X', \mathcal{CP}')$ such that $\varphi = \varphi^* \circ j$.*

*Proof.* We define $\varphi^*$ as follows. For $x^* = j(x)$ with $x \in X$, we define $\varphi^*(x^*) = \varphi(x)$ and for $x^* = \langle(\mathbb{F}, \mathbb{G})\rangle$ with $(\mathbb{F}, \mathbb{G}) \in \mathcal{CP}$ non-convergent, we define $\varphi^*(x^*) = y$, where $y$ is one of the limits of $(\varphi(\mathbb{F}), \varphi(\mathbb{G}))$ in $(X', \mathcal{CP}')$. We note that we consider $y$ as a fixed choice and that it does not depend on the representative of $\langle(\mathbb{F}, \mathbb{G})\rangle$. For if $(\mathbb{F}, \mathbb{G}) \sim (\mathbb{F}', \mathbb{G}')$ then $(\mathbb{F} \wedge \mathbb{F}', \mathbb{G} \wedge \mathbb{G}') \in \mathcal{CP}$ and is $\leq (\mathbb{F}, \mathbb{G})$. Hence, if $y \in q^{\mathcal{CP}}(\varphi(\mathbb{F}), \varphi(\mathbb{G}))$, with Lemma 6.3 also $y \in q^{\mathcal{CP}}(\varphi(\mathbb{F} \wedge \mathbb{F}'), \varphi(\mathbb{G} \wedge \mathbb{G}'))$ and using (TQC2) then also $y \in q^{\mathcal{CP}}(\varphi(\mathbb{F}'), \varphi(\mathbb{G}'))$.

Clearly, with this definition, we have $\varphi^* \circ j = \varphi$ and we have to show that $\varphi^*$ is Cauchy continuous. To this end, let $(\mathbb{H}, \mathbb{K}) \in \mathcal{CP}^*$. We distinguish two cases.

*Case 1:* There is $(\mathbb{F}, \mathbb{G}) \in \mathcal{CP}$ convergent such that $\mathbb{H} \geq j(\mathbb{F})$ and $\mathbb{K} \geq j(\mathbb{G})$. Let $x \in q^{\mathcal{CP}}(\mathbb{F}, \mathbb{G})$. Then $\varphi^*(\mathbb{H}) \geq \varphi(\mathbb{F})$, $\varphi^*(\mathbb{K}) \geq \varphi(\mathbb{G})$ and $\varphi(x) \in q^{\mathcal{CP}'}(\varphi(\mathbb{F}), \varphi(\mathbb{G}))$. This means $(\varphi(\mathbb{F}) \wedge [\varphi(x)], \varphi(\mathbb{G}) \wedge [\varphi(x)]) \in \mathcal{CP}'$ and (TQC2) yields $(\varphi^*(\mathbb{H}), \varphi^*(\mathbb{K})) \in \mathcal{CP}'$.

*Case 2:* There is $(\mathbb{F}, \mathbb{G}) \in \mathcal{CP}$ non-convergent such that $\mathbb{H} \geq j(\mathbb{F}) \wedge [\langle(\mathbb{F}, \mathbb{G})\rangle]$ and $\mathbb{K} \geq j(\mathbb{G}) \wedge [\langle(\mathbb{F}, \mathbb{G})\rangle]$. We conclude $\varphi^*(\mathbb{H}) \geq \varphi^*(j(\mathbb{F})) \wedge [\varphi^*(\langle(\mathbb{F}, \mathbb{G})\rangle)] = \varphi(\mathbb{F}) \wedge [y]$ and, similarly, $\varphi^*(\mathbb{K}) \geq \varphi(\mathbb{G}) \wedge [y]$, where $y \in q^{\mathcal{CP}'}(\varphi(\mathbb{F}), \varphi(\mathbb{G}))$. Hence $(\varphi(\mathbb{F}) \wedge [y], \varphi(\mathbb{G}) \wedge [y]) \in \mathcal{CP}'$ and, again with (TQC2), $(\varphi^*(\mathbb{H}), \varphi^*(\mathbb{K})) \in \mathcal{CP}'$. □

**Corollary 6.8.** *Let $(X, \mathcal{CP})$ be a ⊤-quasi-Cauchy space. Then $((X^*, \mathcal{CP}^*), j)$ is the finest completion of $(X, \mathcal{CP})$.*

*Proof.* If $((X^\sim, \mathcal{CP}^\sim), \psi)$ is a further completion, then $\psi : (X, C) \longrightarrow (X^\sim, \mathcal{CP}^\sim)$ is Cauchy continuous and $(X^\sim, \mathcal{CP}^\sim)$ is complete. Theorem 6.7 ensures that there exists a Cauchy continuous mapping $\psi^* : (X^*, \mathcal{CP}^*) \longrightarrow (X^\sim, \mathcal{CP}^\sim)$ such that $\psi^* \circ j = \psi$. Hence, $((X^*, \mathcal{CP}^*), j) \geq ((X^\sim, \mathcal{CP}^\sim), \psi)$. □

**Corollary 6.9.** *Let $(X, \mathcal{CP}_X), (Y, \mathcal{CP}_Y)$ be ⊤-quasi-Cauchy spaces and let $\varphi : (X, \mathcal{CP}_X) \longrightarrow (Y, \mathcal{CP}_Y)$ be Cauchy continuous. We denote the finest completions of $(X, \mathcal{CP}_X)$ and $(Y, \mathcal{CP}_Y)$ by $((X^*, \mathcal{CP}_X^*), j_X)$ and $((Y^*, \mathcal{CP}_Y^*), j_Y)$, respectively. Then there exists a Cauchy continuous mapping $\varphi^* : (X^*, \mathcal{CP}_X^*) \longrightarrow (Y^*, \mathcal{CP}_Y^*)$ such that $\varphi^* \circ j_X = j_Y \circ \varphi$.*

*Proof.* We consider the Cauchy continuous mapping $g = j_Y \circ \varphi : (X, \mathcal{CP}) \longrightarrow (Y^*, \mathcal{CP}_Y^*)$. Then there exists a Cauchy continuous mapping $\varphi^* : (X^*, \mathcal{CP}^*) \longrightarrow (Y^*, \mathcal{CP}_Y^*)$ such that $\varphi^* \circ j_X = g$. □

Finally, we are showing that separation carries over from $(X,\mathcal{CP})$ to $(X^*,\mathcal{CP}^*)$.

**Proposition 6.10.** *Let $(X,\mathcal{CP})$ be a separated $\top$-quasi-Cauchy space. Then also $(X^*,\mathcal{CP}^*)$ is separated.*

*Proof.* Let $x^*,y^*\in q^{\mathcal{CP}^*}(\mathbb{H},\mathbb{K})$. We distinguish three cases.

*Case 1:* $x^*=j(x),y^*=j(y)$ with $x,y\in X$. Then, by definition of $q^{\mathcal{CP}^*}$, we have $(\mathbb{H}\wedge[j(x)],\mathbb{K}\wedge[j(x)])\in\mathcal{CP}^*$ and $(\mathbb{H}\wedge[j(y)],\mathbb{K}\wedge[j(y)])\in\mathcal{CP}^*$.

If $\mathbb{H}\wedge[j(x)]\geq j(\mathbb{F})\wedge[\langle(\mathbb{F},\mathbb{G})\rangle]$ and $\mathbb{K}\wedge[j(x)]\geq j(\mathbb{G})\wedge[\langle(\mathbb{F},\mathbb{G})\rangle]$ with $(\mathbb{F},\mathbb{G})\in\mathcal{CP}$ non-convergent, then for $f\in\mathbb{F}$ we have that $j(f)\vee\top_{\langle(\mathbb{F},\mathbb{G})\rangle}\in\mathbb{H}\wedge[j(x)]$. Hence

$$\top=\bigvee_{h\in\mathbb{H}}[h\vee\top_{j(x)},j(f)\vee\top_{\langle(\mathbb{F},\mathbb{G})\rangle}]\leq j(f)(j(x))=f(x),$$

and we have $\mathbb{F}\leq[x]$. Similarly, we obtain $\mathbb{G}\leq[x]$ and hence $(\mathbb{F},\mathbb{G})$ converges to $x$ in $(X,\mathcal{CP})$, a contradiction. Hence we must have $\mathbb{H}\wedge[j(x)]\geq j(\mathbb{F})$ and $\mathbb{K}\wedge[j(x)]\geq j(\mathbb{G})$ with $(\mathbb{F},\mathbb{G})\in\mathcal{CP}$ convergent. If we assume that $(\mathbb{F},\mathbb{G})$ converges to $\overline{x}$, then $\mathbb{H}\wedge[j(x)]\geq j(\mathbb{F})\wedge[j(\overline{x})]$ and for $f\in\mathbb{F}$ we have $j(f)\vee\top_{j(\overline{x})}\in\mathbb{H}\wedge[j(x)]$. This implies

$$\top=\bigvee_{h\in\mathbb{H}}[h\vee\top_{j(x)},j(f)\vee\top_{j(\overline{x})}]\leq j(f)(j(x))\vee\top_{j(\overline{x})}(j(x)).$$

If $j(\overline{x})\neq j(x)$, then $f(x)=\top$ and we see that $\mathbb{F}\leq[x]$. Similarly, $\mathbb{G}\leq[x]$ and hence $(\mathbb{F},\mathbb{G})$ converges also to $x$. As $(X,\mathcal{CP})$ is separated, we obtain $x=\overline{x}$.

As a result, we have that the convergence of $(\mathbb{H},\mathbb{K})$ to $j(x)$ in $(X^*,\mathcal{CP}^*)$ implies the existence of $(\mathbb{F},\mathbb{G})\in\mathcal{CP}$ converging to $x$, such that $\mathbb{H}\geq j(\mathbb{F})$ and $\mathbb{K}\geq j(\mathbb{G})$. In the same way, $(\mathbb{F}',\mathbb{G}')\in\mathcal{CP}$ exists, converging to $y$ and $\mathbb{H}\geq j(\mathbb{F}')$ and $\mathbb{K}\geq j(\mathbb{G}')$. We conclude that $j^{\leftarrow}(\mathbb{H})$ and $j^{\leftarrow}(\mathbb{K})$ exist. We show this for $j^{\leftarrow}(\mathbb{H})$. For $f\in\mathbb{F}$ and $h\in\mathbb{H}$ we have $h\wedge j(f)\in\mathbb{H}$ and as $\mathbb{H}$ is a $\top$-filter, we conclude

$$\top=\bigvee_{x^*\in X^*}(h\wedge j(f))(x^*)\leq\bigvee_{x\in X}h(j(x))=\bigvee_{x\in X}j^{\leftarrow}(h)(x).$$

We conclude $j^{\leftarrow}(\mathbb{H})\geq\mathbb{F}$, $j^{\leftarrow}(\mathbb{K})\geq\mathbb{G}$ and also $j^{\leftarrow}(\mathbb{H})\geq\mathbb{F}'$ and $j^{\leftarrow}(\mathbb{K})\geq\mathbb{G}'$. We note that $(\mathbb{H},\mathbb{K})$ is a pair $\top$-filter and $\mathbb{H}\geq j(\mathbb{F}),\mathbb{K}\geq j(\mathbb{G})$ and we fix $f\in\mathbb{F},g\in\mathbb{G}$. Then for $h\in\mathbb{H},k\in\mathbb{K}$ we have $h\wedge j(f)\in\mathbb{H},k\wedge j(g)\in\mathbb{K}$ and hence,

$$\begin{aligned}\top &= \bigvee_{x^*\in X^*}(h\wedge j(f))(x^*)*(k\wedge j(g))(x^*)\\ &\leq \bigvee_{x\in X}h(j(x))*k(j(x)) = \bigvee_{x\in X}j^{\leftarrow}(h)(x)*j^{\leftarrow}(k)(x)\end{aligned}$$

and hence $(j^{\leftarrow}(\mathbb{H}),j^{\leftarrow}(\mathbb{K}))$ is a pair $\top$-filter converging to both $x$ and $y$ in $(X,\mathcal{CP})$. Separation yields $x=y$, i.e. $x^*=y^*$.

*Case 2:* $x^*=\langle(\mathbb{F},\mathbb{G})\rangle$, $y^*=\langle(\mathbb{F}',\mathbb{G}')\rangle$ with $(\mathbb{F},\mathbb{G}),(\mathbb{F}',\mathbb{G}')\in\mathcal{CP}$ non-convergent. Then $(\mathbb{H}\wedge[\langle(\mathbb{F},\mathbb{G})\rangle],\mathbb{K}\wedge[\langle(\mathbb{F},\mathbb{G})\rangle])\in\mathcal{CP}^*$. If we assume $\mathbb{H}\wedge$

$[\langle(\mathbb{F},\mathbb{G})\rangle] \geq j(\overline{\mathbb{F}})$ and $[\langle(\mathbb{F},\mathbb{G})\rangle] \geq j(\overline{\mathbb{G}})$ with $(\overline{\mathbb{F}},\overline{\mathbb{G}}) \in \mathcal{CP}$ convergent, then for $\overline{f} \in \overline{\mathbb{F}}$ we had $\top = \bigvee_{h\in\mathbb{H}}[h \vee \top_{\langle(\mathbb{F},\mathbb{G})\rangle}, j(\overline{f})] \leq j(\overline{f})(\langle(\mathbb{F},\mathbb{G})\rangle) = \bot$, a contradiction. Hence we must have $\mathbb{H} \wedge [\langle(\mathbb{F},\mathbb{G})\rangle] \geq j(\overline{\mathbb{F}}) \wedge [\langle(\overline{\mathbb{F}},\overline{\mathbb{G}})\rangle]$ with $(\overline{\mathbb{F}},\overline{\mathbb{G}}) \in \mathcal{CP}$ non-convergent. For $\overline{f} \in \overline{\mathbb{F}}$ then

$$\top = \bigvee_{h\in\mathbb{H}} [h \vee \top_{\langle(\mathbb{F},\mathbb{G})\rangle}, j(\overline{f}) \vee \top_{\langle(\overline{\mathbb{F}},\overline{\mathbb{G}})\rangle}] \leq \top_{\langle(\overline{\mathbb{F}},\overline{\mathbb{G}})\rangle}(\langle(\mathbb{F},\mathbb{G})\rangle).$$

Hence $\langle(\mathbb{F},\mathbb{G})\rangle = \langle(\overline{\mathbb{F}},\overline{\mathbb{G}})\rangle$ and we conclude $\mathbb{H} \geq j(\mathbb{F}) \wedge [\langle(\mathbb{F},\mathbb{G})\rangle], \mathbb{K} \geq j(\mathbb{G}) \wedge [\langle(\mathbb{F},\mathbb{G})\rangle]$. In the same way we see that also $\mathbb{H} \geq j(\mathbb{F}') \wedge [\langle(\mathbb{F}',\mathbb{G}')\rangle], \mathbb{K} \geq j(\mathbb{G}') \wedge [\langle(\mathbb{F}',\mathbb{G}')\rangle]$. As $(\mathbb{H},\mathbb{K})$ is a pair $\top$-filter, we conclude, for $f \in \mathbb{F}$ and $g' \in \mathbb{G}'$ that

$$\begin{aligned}\top &= \bigvee_{x^*\in X^*} (j(f) \vee \top_{\langle(\mathbb{F},\mathbb{G})\rangle})(x^*) * (j(g') \vee \top_{\langle(\mathbb{F}',\mathbb{G}')\rangle})(x^*)\\ &= \bigvee_{x\in X} f(x) * g'(x)\\ &\quad \vee \bigvee_{(\overline{\mathbb{F}},\overline{\mathbb{G}})\in\mathcal{CP} \text{ non-conv.}} \top_{\langle(\mathbb{F},\mathbb{G})\rangle}(\langle(\overline{\mathbb{F}},\overline{\mathbb{G}})\rangle) * \top_{\langle(\mathbb{F}',\mathbb{G}')\rangle}(\langle(\overline{\mathbb{F}},\overline{\mathbb{G}})\rangle).\end{aligned}$$

If $\langle(\mathbb{F},\mathbb{G})\rangle \neq \langle(\mathbb{F}',\mathbb{G}')\rangle$, then $\bigvee_{x\in X} f(x) * g'(x) = \top$ and, with analogous arguments, $\bigvee_{x\in X} f'(x) * g(x) = \top$ for $f' \in \mathbb{F}', g \in \mathbb{G}$. Therefore, $(\mathbb{F}\wedge\mathbb{F}', \mathbb{G}\wedge\mathbb{G}') \in \mathcal{CP}, \leq (\mathbb{F},\mathbb{G}), (\mathbb{F}',\mathbb{G}')$ and we conclude $(\mathbb{F},\mathbb{G}) \sim (\mathbb{F}\wedge\mathbb{F}', \mathbb{G}\wedge\mathbb{G}') \sim (\mathbb{F}',\mathbb{G}')$ and we have also in this case $x^* = \langle(\mathbb{F},\mathbb{G})\rangle = \langle(\mathbb{F}',\mathbb{G}')\rangle = y^*$.

*Case 3:* $x^* = j(x)$ with $x \in X$ and $y^* = \langle(\mathbb{F},\mathbb{G})\rangle$ with $(\mathbb{F},\mathbb{G}) \in \mathcal{CP}$ non-convergent. As we have seen before, then $\mathbb{H} \geq j(\mathbb{F}')$ and $\mathbb{K} \geq j(\mathbb{G}')$ with $(\mathbb{F}',\mathbb{G}') \in \mathcal{CP}$ convergent and $\mathbb{H} \geq j(\mathbb{F}) \wedge [\langle(\mathbb{F},\mathbb{G})\rangle]$ and $\mathbb{H} \geq j(\mathbb{G}) \wedge [\langle(\mathbb{F},\mathbb{G})\rangle]$. For $f' \in \mathbb{F}'$ and $g \in \mathbb{G}$ we have $j(f') \in \mathbb{H}$ and $j(g) \vee \top_{\langle(\mathbb{F},\mathbb{G})\rangle} \in \mathbb{K}$ and because $(\mathbb{H},\mathbb{H})$ is a $\top$-pair filter we obtain

$$\top = \bigvee_{x^*\in X^*} j(f')(x^*) * (j(g) \vee \top_{\langle(\mathbb{F},\mathbb{G})\rangle})(x^*) = \bigvee_{x\in X} f'(x) * g(x).$$

Hence, $\mathbb{F}', \mathbb{G}$ are linked. Similarly, we can show that also $\mathbb{F}, \mathbb{G}'$ are linked and hence $(\mathbb{F}\wedge\mathbb{F}', \mathbb{G}\wedge\mathbb{G}') \in \mathcal{CP}$. As this pair $\top$-filter is $\leq (\mathbb{F}',\mathbb{G}')$ it is convergent and therefore also $(\mathbb{F},\mathbb{G})$ is convergent, a contradiction. Hence this case cannot occur. □

## 7. Symmetry – ⊤-Cauchy spaces

Let $(X,\mathcal{CP})$ be a $\top$-quasi-Cauchy space. We call $(X,\mathcal{CP})$ *symmetric* if the axiom

(TQCS) $(\mathbb{F},\mathbb{G}) \in \mathcal{CP}$ implies $(\mathbb{G},\mathbb{F}) \in \mathcal{CP}$

is satisfied.

For $\top$-quasi-uniform spaces and L-metric spaces, we obtain the usual concepts.

**Proposition 7.1.** *Let $(X,\mathcal{U})$ be a $\top$-quasi-uniform space. Then $(X,\mathcal{CP}^{\mathcal{U}})$ is symmetric if, and only if, $\mathcal{U} \leq \mathcal{U}^{-1}$.*

*Proof.* Let first $(X,\mathcal{CP}^{\mathcal{U}})$ be symmetric. It was shown in [25] that, for each $x\in X$, $(\mathcal{U}(x,\cdot),\mathcal{U}(\cdot,x))\in\mathcal{CP}^{\mathcal{U}}$ where $\mathcal{U}(x,\cdot)$ is generated by the $\top$-filter basis $\{u(x,\cdot)\ :\ u\in\mathcal{U}\}$ and $\mathcal{U}(\cdot,x)$ is generated by the $\top$-filter basis $\{u(\cdot,x)\ :\ u\in\mathcal{U}\}$. A $\top$-filter basis for $\bigwedge_{x\in X}\mathcal{U}(\cdot,x)\otimes\mathcal{U}(x,\cdot)$ is given by the $L$-sets $b=\bigvee_{x\in X}u(\cdot,x)\otimes u(x,\cdot)$ with $u\in\mathcal{U}$ and we have $b(s,t)\geq u(s,s)*u(s,t)=u(s,t)$. Hence we have $\bigwedge_{x\in X}\mathcal{U}(\cdot,x)\otimes\mathcal{U}(x,\cdot)\leq\mathcal{U}$ and we conclude

$$\mathcal{U}\leq\bigwedge_{(\mathbb{F},\mathbb{G})\in\mathcal{CP}^{\mathcal{U}}}\mathbb{G}\otimes\mathbb{F}\leq\bigwedge_{x\in X}\mathcal{U}(\cdot,x)\otimes\mathcal{U}(x,\cdot)\leq\mathcal{U}.$$

Therefore, using the symmetry of $(X,\mathcal{CP}^{\mathcal{U}})$, we conclude

$$\mathcal{U}^{-1}=\bigwedge_{(\mathbb{F},\mathbb{G})\in\mathcal{CP}^{\mathcal{U}}}(\mathbb{G}\otimes\mathbb{F})^{-1}\geq\bigwedge_{(\mathbb{G},\mathbb{F})\in\mathcal{CP}^{\mathcal{U}}}\mathbb{F}\otimes\mathbb{G}=\mathcal{U}.$$

For the converse, let $(\mathbb{F},\mathbb{G})\in\mathcal{CP}^{\mathcal{U}}$. Then $\mathbb{G}\otimes\mathbb{F}\geq\mathcal{U}$ and hence $\mathbb{F}\otimes\mathbb{G}=(\mathbb{G}\otimes\mathbb{F})^{-1}\geq\mathcal{U}^{-1}\geq\mathcal{U}$, that is, $(\mathbb{G},\mathbb{F})\in\mathcal{CP}^{\mathcal{U}}$. □

**Corollary 7.2.** *Let $(X,d)$ be an $\mathsf{L}$-metric space. Then $(X,\mathcal{CP}^{[d]})$ is symmetric if, and only if, $d(x,y)=d(y,x)$ for all $x,y\in X$.*

In the sequel we are going to show that symmetric $\top$-quasi-Cauchy spaces can be identified with $\top$-Cauchy spaces. Reid and Richardson [21] gave the following definition. A $\top$-*Cauchy space* $(X,\mathcal{C})$) is a set $X$ with a set of $\top$-filters $\mathcal{C}\subseteq\mathsf{F}_{\mathsf{L}}^{\top}(X)$ such that

(TC1) $[x]\in\mathcal{C}$ for all $x\in X$;
(TC2) If $\mathbb{G}\leq\mathbb{F}$ and $\mathbb{F}\in\mathcal{C}$, then $\mathbb{G}\in\mathcal{C}$;
(TC3) $\mathbb{F}\wedge\mathbb{G}\in\mathcal{C}$ whenever $\mathbb{F},\mathbb{G}\in\mathcal{C}$ and $\bigvee_{x\in X}f(x)*g(x)=\top$ for all $f\in\mathbb{F},g\in\mathbb{G}$.

A mapping $\varphi:(X,\mathcal{C})\longrightarrow(X',\mathcal{C}')$ is called Cauchy continuous if $\varphi(\mathbb{F})\in\mathcal{C}'$ whenever $\mathbb{F}\in\mathcal{C}$. We denote the category of $\top$-Cauchy spaces with Cauchy continuous mappings as morphisms by $\top$-$\mathsf{Chy}$.

Let now $(X,\mathcal{CP})$ be a $\top$-quasi-Cauchy space. We define

$$\mathcal{C}_{\mathcal{CP}}=\{\mathbb{H}\in\mathsf{F}_{\mathsf{L}}^{\top}(X)\ :\ (\mathbb{H},\mathbb{H})\in\mathcal{CP}\}.$$

The proofs of the following propositions are straightforward and not shown.

**Proposition 7.3.** *Let $(X,\mathcal{CP})$ be a $\top$-quasi-Cauchy space. Then $(X,\mathcal{C}_{\mathcal{CP}})$ is a $\top$-Cauchy space.*

**Proposition 7.4.** *If $\varphi:(X,\mathcal{CP})\longrightarrow(X',\mathcal{CP}')$ is Cauchy continuous, then $\varphi:(X,\mathcal{C}_{\mathcal{CP}})\longrightarrow(X',\mathcal{C}_{\mathcal{CP}'})$ is Cauchy continuous.*

Hence we have a functor $\mathsf{F}:\top$-$\mathsf{QChy}\longrightarrow\top$-$\mathsf{Chy}$ which maps a $\top$-quasi-Cauchy space $(X,\mathcal{CP})$ to the $\top$-Cauchy space $(X,\mathcal{C}_{\mathcal{CP}})$ and leaves morphisms unchanged.

Let now $(X,\mathcal{C})$ be a $\top$-Cauchy space. We define

$$\mathcal{CP}_{\mathcal{C}}=\{(\mathbb{F},\mathbb{G})\in\mathsf{PF}_{\mathsf{L}}^{\top}(X)\ :\ \exists\,\mathbb{H}\in\mathcal{C}\text{ s.t. }(\mathbb{H},\mathbb{H})\leq(\mathbb{F},\mathbb{G})\}.$$

Again, we omit the straightforward proofs of the following results.

**Proposition 7.5.** *Let $(X,\mathcal{C})$ be a $\top$-Cauchy space. Then $(X,\mathcal{CP}_{\mathcal{C}})$ is a symmetric $\top$-quasi-Cauchy space.*

**Proposition 7.6.** *If $\varphi : (X,\mathcal{C}) \longrightarrow (X',\mathcal{C}')$ is Cauchy continuous, then $\varphi : (X,\mathcal{CP}_{\mathcal{C}}) \longrightarrow (X',\mathcal{CP}_{\mathcal{C}'})$ is Cauchy continuous.*

Hence we have another functor $\mathsf{G} : \top\text{-}\mathsf{Chy} \longrightarrow \top\text{-}\mathsf{QChy}$ which maps a $\top$-Cauchy space $(X,\mathcal{C})$ to the $\top$-quasi-Cauchy space $(X,\mathcal{CP}_{\mathcal{C}})$ and leaves morphisms unchanged.

**Proposition 7.7.** *We have $\mathsf{F}\circ\mathsf{G} = id_{\top-\mathsf{Chy}}$ and $\mathsf{G}\circ\mathsf{F} \geq id_{\top-\mathsf{QChy}}$.*

*Proof.* For a $\top$-Cauchy space $(X,\mathcal{C})$, we have $\mathbb{F} \in \mathcal{C}_{(\mathcal{CP}_{\mathcal{C}})}$ if, and only if, $(\mathbb{F},\mathbb{F}) \in \mathcal{CP}_{\mathcal{C}}$, if, and only if, there is $\mathbb{H} \in \mathcal{C}$ such that $(\mathbb{H},\mathbb{H}) \leq (\mathbb{F},\mathbb{F})$ if, and only if, there is $\mathbb{H} \in \mathcal{C}$ such that $\mathbb{H} \leq \mathbb{F}$. This is equivalent to $\mathbb{F} \in \mathcal{C}$.

For a $\top$-quasi-Cauchy space $(X,\mathcal{CP})$, we have $(\mathbb{F},\mathbb{G}) \in \mathcal{CP}_{(\mathcal{C}_{\mathcal{CP}})}$ if, and only if, there is $\mathbb{H} \in \mathcal{C}_{\mathcal{CP}}$ such that $(\mathbb{H},\mathbb{H}) \leq (\mathbb{F},\mathbb{G})$, if, and only if, there is $\mathbb{H} \in \mathsf{F}_L^{\top}(X)$ such that $(\mathbb{H},\mathbb{H}) \in \mathcal{CP}$ and $(\mathbb{H},\mathbb{H}) \leq (\mathbb{F},\mathbb{G})$. This implies $(\mathbb{F},\mathbb{G}) \in \mathcal{CP}$. Hence we have shown $\mathcal{CP}_{(\mathcal{C}_{\mathcal{CP}})} \subseteq \mathcal{CP}$. We note that if $(X,\mathcal{CP})$ is symmetric, we even have equality, as $(\mathbb{F},\mathbb{G}) \in \mathcal{CP}$ implies $(\mathbb{G},\mathbb{F}) \in \mathcal{CP}$ and hence $(\mathbb{F}\wedge\mathbb{G},\mathbb{F}\wedge\mathbb{G}) \in \mathcal{CP}$ and we can choose $\mathbb{H} = \mathbb{F}\wedge\mathbb{G}$. □

**Corollary 7.8.** *The category $\top$-$\mathsf{Chy}$ is isomorphic to a reflective subcategory of the category $\top$-$\mathsf{QChy}$*

We denote the subcategory of symmetric $\top$-quasi-Cauchy spaces by $\top$-$\mathsf{sQChy}$. We restrict the domain of the functor $\mathsf{F}$ to this subcategory, and note that the codomain of $\mathsf{G}$ is automatically in this subcategory. Denoting these resulting functors again by $\mathsf{F},\mathsf{G}$, we even have $\mathsf{F}\circ\mathsf{G} = id_{\top-\mathsf{Chy}}$ and $\mathsf{G}\circ\mathsf{F} = id_{\top-\mathsf{sQChy}}$.

**Corollary 7.9.** *The categories $\top$-$\mathsf{Chy}$ and $\top$-$\mathsf{sQChy}$ are isomorphic.*

In this way we can identify symmetric $\top$-quasi-Cauchy spaces and $\top$-Cauchy spaces.

**Proposition 7.10.** *Let $(X,\mathcal{CP})$ be a $\top$-quasi-Cauchy space and $((X^*,\mathcal{CP}^*),j)$ be the finest completion. If $(X,\mathcal{CP})$ is symmetric, then so is $(X^*,\mathcal{CP}^*)$.*

*Proof.* This follows directly from the definition of $\mathcal{CP}^*$. □

We will finally outline, that for a symmetric $\top$-quasi-Cauchy space $(X,\mathcal{CP})$, we can construct $(X^*,\mathcal{CP}^*)$ in a different way. We first need some definitions [21].

Let $(X,\mathcal{C})$ be a $\top$-Cauchy space. A $\top$-filter $\mathbb{F} \in \mathsf{F}_{\mathsf{L}}^{\top}(X)$ converges to $x \in X$ if $\mathbb{F}\wedge[x] \in \mathcal{C}$. The space $(X,\mathcal{C})$ is called *complete* if each $\mathbb{F} \in \mathcal{C}$ converges to some $x \in X$. For a subset $A \subseteq X$ we define the $\mathcal{C}$-closure of $A$ by $x \in \overline{A}^{\mathcal{C}}$ if there is $\mathbb{F} \in \mathsf{F}_{\mathsf{L}}^{\top}(A)$ such that $[\mathbb{F}]\wedge[x] \in \mathcal{C}$. A completion $((X^+,\mathcal{C}^+),\phi)$ of a $\top$-Cauchy space $(X,\mathcal{C})$ is a complete $\top$-Cauchy space $(X^+,\mathcal{C}^+)$ with a dense embedding $\phi : X \longrightarrow X^+$, that is, we have $\phi(\mathbb{F}) \in \mathcal{C}^+$ if, and only if, $\mathbb{F} \in \mathcal{C}$ and $\overline{\phi(X)}^{\mathcal{C}^+} = X^+$.

**Lemma 7.11.** *A $\top$- Cauchy space $(X,\mathcal{C})$ is complete if, and only if, $(X,\mathcal{CP}_{\mathcal{C}})$ is complete.*

*Proof.* If $(X,\mathcal{C})$ is complete and $(\mathbb{F},\mathbb{G}) \in \mathcal{CP}_{\mathcal{C}}$ then there is $\mathbb{H} \in \mathcal{C}$ such that $(\mathbb{H},\mathbb{H}) \leq (\mathbb{F},\mathbb{G})$. Hence there is $x \in X$ such that $\mathbb{H} \wedge [x] \in \mathcal{C}$ and $(\mathbb{H} \wedge [x], \mathbb{H} \wedge [x]) \leq (\mathbb{F} \wedge [x], \mathbb{G} \wedge [x])$. This means that $(\mathbb{F},\mathbb{G})$ converges to $x$ in $(X,\mathcal{CP}_{\mathcal{C}})$. Hence, $(X,\mathcal{CP}_{\mathcal{C}})$ is complete.

Conversely, if $(X,\mathcal{CP}_{\mathcal{C}})$ is complete and $\mathbb{H} \in \mathcal{C}$, then $(\mathbb{H},\mathbb{H}) \in \mathcal{CP}_{\mathcal{C}}$ and there is $x \in X$ such that $(\mathbb{H} \wedge [x], \mathbb{H} \wedge [x]) \in \mathcal{CP}_{\mathcal{C}}$. By definition of $\mathcal{CP}_{\mathcal{C}}$ there is $\mathbb{F} \in \mathcal{C}$ auch that $(\mathbb{F},\mathbb{F}) \leq (\mathbb{H} \wedge [x], \mathbb{H} \wedge [x])$ which shows $\mathbb{H} \wedge [x] \in \mathcal{C}$ and $(X,\mathcal{C})$ is complete. □

**Lemma 7.12.** *If the $\top$-quasi-Cauchy space $(X,\mathcal{CP})$ is complete, then also $(X,\mathcal{C}_{\mathcal{CP}})$ is complete. If $(X,\mathcal{CP})$ is symmetric, then we have equivalence.*

*Proof.* Let $(X,\mathcal{CP})$ be complete and let $\mathbb{F} \in \mathcal{C}_{\mathcal{CP}}$. Then $(\mathbb{F},\mathbb{F}) \in \mathcal{CP}$ and hence there is $x \in X$ such that $(\mathbb{F} \wedge [x], \mathbb{F} \wedge [x]) \in \mathcal{CP}$. This means that $\mathbb{F} \wedge [x] \in \mathcal{C}_{\mathcal{CP}}$ and $(X,\mathcal{C}_{\mathcal{CP}})$ is complete.

If $(X,\mathcal{CP})$ is symmetric and $(X,\mathcal{C}_{\mathcal{CP}})$ is complete, then for $(\mathbb{F},\mathbb{G}) \in \mathcal{CP}$ we have $(\mathbb{F} \wedge \mathbb{G}, \mathbb{F} \wedge \mathbb{G}) \in \mathcal{CP}$ and hence, $\mathbb{F} \wedge \mathbb{G} \in \mathcal{C}_{\mathcal{CP}}$. Therefore, there is $x \in X$ such that $\mathbb{F} \wedge \mathbb{G} \wedge [x] \in \mathcal{C}_{\mathcal{CP}}$ and we conclude $(\mathbb{F} \wedge [x], \mathbb{G} \wedge [x]) \in \mathcal{CP}$. Hence, $(X,\mathcal{CP})$ is complete. □

Reid and Richardson constructed a finest completion as follows [21]. With the equivalence relation on $\mathcal{C}$, given by $\mathbb{F} \sim \mathbb{G}$ if $\mathbb{F} \wedge \mathbb{G} \in \mathcal{C}$, we denote the equivalence class of $\mathbb{F} \in \mathcal{C}$ by $\langle \mathbb{F} \rangle$ and we define $X^+ = X \cup \{\langle \mathbb{F} \rangle \ : \ \mathbb{F} \in \mathcal{C} \text{ non-convergent}\}$. Then $((X^+,\mathcal{C}^+),j^+)$ is defined by $j^+(x) = x$ for $x \in X$ and $\mathbb{H} \in \mathcal{C}^+$ if either $\mathbb{H} \geq j^+(\mathbb{F})$ for some convergent $\mathbb{F} \in \mathcal{C}$ or $\mathbb{H} \geq j^+(\mathbb{F}) \wedge [\langle \mathbb{F} \rangle]$ with some $\mathbb{F} \in \mathcal{C}$ non-convergent.

**Proposition 7.13.** *Let $(X,\mathcal{CP})$ be a symmetric $\top$-quasi-Cauchy space. For $(\mathbb{F},\mathbb{G}),(\mathbb{F}',\mathbb{G}') \in \mathcal{CP}$ we have $(\mathbb{F},\mathbb{G}) \sim (\mathbb{F}',\mathbb{G}')$ in $(X,\mathcal{CP})$ if, and only if, $\mathbb{F} \sim \mathbb{F}'$ in $(X,\mathcal{C}_{\mathcal{CP}})$.*

*Proof.* Let first $(\mathbb{F},\mathbb{G}) \sim (\mathbb{F}',\mathbb{G}')$. The symmetry of $(X,\mathcal{CP})$ ensures $\mathbb{F},\mathbb{F}' \in \mathcal{C}_{\mathcal{CP}}$. From $(\mathbb{F} \wedge \mathbb{F}', \mathbb{G} \wedge \mathbb{G}') \in \mathcal{CP}$ we conclude, again by symmetry, that $(\mathbb{F} \wedge \mathbb{F}' \wedge \mathbb{G} \wedge \mathbb{G}', \mathbb{F} \wedge \mathbb{F}' \wedge \mathbb{G} \wedge \mathbb{G}') \in \mathcal{CP}$ and, using (TQC2), $(\mathbb{F} \wedge \mathbb{F}', \mathbb{F} \wedge \mathbb{F}') \in \mathcal{CP}$, that is, $\mathbb{F} \wedge \mathbb{F}' \in \mathcal{C}_{\mathcal{CP}}$. Therefore $\mathbb{F} \sim \mathbb{F}'$.

If $\mathbb{F} \wedge \mathbb{F}' \in \mathcal{C}_{\mathcal{CP}}$, then $(\mathbb{F} \wedge \mathbb{F}', \mathbb{F} \wedge \mathbb{F}') \in \mathcal{CP}$. As $(\mathbb{F},\mathbb{G}),(\mathbb{F}',\mathbb{G}') \in \mathcal{CP}$, we conclude $(\mathbb{F} \wedge \mathbb{G}, \mathbb{F} \wedge \mathbb{G}), (\mathbb{F}' \wedge \mathbb{G}', \mathbb{F}' \wedge \mathbb{G}') \in \mathcal{CP}$. Applying (TQC3) twice, we conclude $(\mathbb{F} \wedge \mathbb{F}' \wedge \mathbb{G} \wedge \mathbb{G}', \mathbb{F} \wedge \mathbb{F}' \wedge \mathbb{G} \wedge \mathbb{G}') \in \mathcal{CP}$. (TQC2) yields $(\mathbb{F} \wedge \mathbb{F}', \mathbb{G} \wedge \mathbb{G}') \in \mathcal{CP}$, that is $(\mathbb{F},\mathbb{G}) \sim (\mathbb{F}',\mathbb{G}')$. □

We note that by symmetry we also have $\mathbb{G} \sim \mathbb{G}'$ in $(X,\mathcal{C}_{\mathcal{CP}})$ if, and only if, $(\mathbb{F},\mathbb{G}) \sim (\mathbb{F}',\mathbb{G}') \in \mathcal{CP}$.

**Corollary 7.14.** *The pair $\top$-filter $(\mathbb{F},\mathbb{G})$ converges to $x$ in $(X,\mathcal{CP})$ if, and only if, $\mathbb{F}$ converges to $x$ in $(X,\mathcal{C}_{\mathcal{CP}})$.*

As a consequence, the mapping $\eta : \{\langle(\mathbb{F},\mathbb{G})\rangle \ : \ (\mathbb{F},\mathbb{G}) \in \mathcal{CP} \text{ non-convergent}\} \longrightarrow \{\langle\mathbb{F}\rangle \ : \ \mathbb{F} \in \mathcal{C}_{\mathcal{CP}} \text{ non-convergent}\}$ defined by $\eta(\langle(\mathbb{F},\mathbb{G})\rangle) = \langle\mathbb{F}\rangle$ is a bijection and we can identify in this way $X^*$ and $j$ (from the finest completion of $(X,\mathcal{CP})$) and $X^+$ and $j^+$ (from the finest completion of $(X,\mathcal{C}_{\mathcal{CP}})$ for a symmetric ⊤-quasi-Cauchy space.

Moreover, we can deduce the following result.

**Corollary 7.15.** *Let $(X,\mathcal{CP})$ be a symmetric ⊤-quasi-Cauchy space and let $A \subseteq X$. Then $\overline{A}^{\mathcal{CP}} = \overline{A}^{\mathcal{C}_{\mathcal{CP}}}$.*

If we have a symmetric ⊤-quasi-Cauchy space $(X,\mathcal{CP})$, then we can construct the finest completion $(X^*,\mathcal{CP}^*)$. Alternatively, we can move to the ⊤-Cauchy space $(X,\mathcal{C}_{\mathcal{CP}})$ and construct the finest completion $(X^+,(\mathcal{C}_{\mathcal{CP}})^+)$ and from there move to $(X^+,\mathcal{CP}_{(\mathcal{C}_{\mathcal{CP}})^+})$. Identifying $X^+$ with $X^*$, we will show that $(X^+,\mathcal{CP}_{(\mathcal{C}_{\mathcal{CP}})^+}) = (X^*,\mathcal{CP}^*)$. First of all, we notice that for $(\mathbb{F},\mathbb{G}) \in \mathcal{CP}$ we have $\mathbb{F},\mathbb{G} \in \mathcal{C}_{\mathcal{CP}}$ and hence $j(\mathbb{F}), j(\mathbb{G}) \in (\mathcal{C}_{\mathcal{CP}})^+$ which implies $(j(\mathbb{F}), j(\mathbb{G})) \in \mathcal{CP}_{(\mathcal{C}_{\mathcal{CP}})^+}$. Conversely, if $(j(\mathbb{F}), j(\mathbb{G})) \in \mathcal{CP}_{(\mathcal{C}_{\mathcal{CP}})^+}$, then there is $\mathbb{H} \in (\mathcal{C}_{\mathcal{CP}})^+$ such that $\mathbb{H} \leq j(\mathbb{F}), j(\mathbb{G})$. Hence $j(\mathbb{F}), j(\mathbb{G}) \in (\mathcal{C}_{\mathcal{CP}})^+$, which implies $\mathbb{F},\mathbb{G} \in \mathcal{C}_{\mathcal{CP}}$. This means that $(\mathbb{F},\mathbb{F}),(\mathbb{G},\mathbb{G}) \in \mathcal{CP}$, and as $(\mathbb{F},\mathbb{G})$ is a pair ⊤-filter, we conclude with (TQC3) $(\mathbb{F}\wedge\mathbb{G},\mathbb{F}\wedge\mathbb{G}) \in \mathcal{CP}$. Hence, also $(\mathbb{F},\mathbb{G}) \in \mathcal{CP}$. Corollary 7.15 and Lemma 7.11 thus show that $((X^*,\mathcal{CP}_{(\mathcal{C}_{\mathcal{CP}})^+}), j)$ is a completion of $(X,\mathcal{CP})$ and therefore, $((X^*,\mathcal{CP}^*), j)$ being the finest completion, we see that $\mathcal{CP}^* \subseteq \mathcal{CP}_{(\mathcal{C}_{\mathcal{CP}})^+}$. For the converse subsethood relation, we take $(\mathbb{H},\mathbb{K}) \in \mathcal{CP}_{(\mathcal{C}_{\mathcal{CP}})^+}$. Then there is $\mathbb{H}' \in (\mathcal{C}_{\mathcal{CP}})^+$ such that $(\mathbb{H}',\mathbb{H}') \leq (\mathbb{H},\mathbb{K})$. Then either there is a convergent $\mathbb{F} \in \mathcal{C}_{\mathcal{CP}}$, that is $(\mathbb{F},\mathbb{F}) \in \mathcal{CP}$ such that $j(\mathbb{F}) \leq \mathbb{H}' \leq \mathbb{H},\mathbb{K}$, which shows $(\mathbb{H},\mathbb{K}) \in \mathcal{CP}^*$. Or there is $\mathbb{F} \in \mathcal{C}_{\mathcal{CP}}$ non-convergent such that $j(\mathbb{F}) \wedge [\langle\mathbb{F}\rangle] \leq \mathbb{H}'$. Then $(\mathbb{F},\mathbb{F}) \in \mathcal{CP}$ is non-convergent and, upon identification $\langle\mathbb{F}\rangle = \langle(\mathbb{F},\mathbb{F})\rangle$, we conclude $j(\mathbb{F}) \wedge [\langle(\mathbb{F},\mathbb{F})\rangle] \leq \mathbb{H}' \leq \mathbb{H},\mathbb{K}$ which again implies $(\mathbb{H},\mathbb{K}) \in \mathcal{CP}^*$. Hence we have proven the following result.

**Proposition 7.16.** *For a symmetric ⊤-quasi-Cauchy space $(X,\mathcal{CP})$ we have $(X^*,\mathcal{CP}^*) = (X^*,\mathcal{CP}_{(\mathcal{C}_{\mathcal{CP}})^+})$.*

We note that in the non-symmetric case, $(X^+,\mathcal{CP}_{(\mathcal{C}_{\mathcal{CP}})^+})$ is in general not a completion of $(X,\mathcal{CP})$. To see this, we consider a complete ⊤-quasi-Cauchy space $(X,\mathcal{CP})$. Then $(X,\mathcal{C}_{\mathcal{CP}})$ is a complete ⊤-Cauchy space and we get $(X^*,\mathcal{CP}^*) = (X,\mathcal{CP})$ and $(X^+,(\mathcal{C}_{\mathcal{CP}})^+) = (X,\mathcal{C}_{\mathcal{CP}})$ and from there, the complete ⊤-quasi-Cauchy space $(X,\mathcal{CP}_{(\mathcal{C}_{\mathcal{CP}})})$. As $(X,\mathcal{CP})$ is non-symmetric, there is $(\mathbb{F},\mathbb{G}) \in \mathcal{CP}$ such that $(\mathbb{G},\mathbb{F}) \notin \mathcal{CP}$. Since $\mathcal{CP}_{(\mathcal{C}_{\mathcal{CP}})} \subseteq \mathcal{CP}$, this $(\mathbb{F},\mathbb{G})$ cannot be in $\mathcal{CP}_{(\mathcal{C}_{\mathcal{CP}})}$. Hence, $j = id_X : (X,\mathcal{CP}) \longrightarrow (X,\mathcal{CP}_{(\mathcal{C}_{\mathcal{CP}})})$ is not Cauchy continuous.

## 8. Conclusions

We defined a non-symmetric framework for studying completeness and completion, generalizing the ⊤-Cauchy pair filters in a ⊤-uniform space that were used in [25] and [15]. Our category of ⊤-quasi-Cauchy spaces has nice categorical properties and covers the important examples of ⊤-quasi-uniform spaces

and L-metric spaces. It allows a theory of completeness and completion which is patterned after the corresponding theory in the symmetric case [21]. In the future, completions with special properties, e.g. diagonal completions or regular completions, can be studied using similar techniques as in [21] and the connection to completions of $\top$-quasi-uniform (limit) spaces are of interest. It would also be interesting to see if such a non-symmetric theory of completeness and completion finds a place in the field of monoidal topology [12].

## References


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