UNIVERSITAT POLITÈCNICA DE VALÈNCIA

High Performance Multidimensional Iterative Processes for Solving Nonlinear Equations

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HACEN CONSTAR:

Que Doña Paula Triguero Navarro, Graduada en Matemáticas y Máster en Investigación Matemática, ha realizado, bajo nuestra dirección, el trabajo que se recoge en esta memoria para optar al título de Doctor en Matemáticas por la Universitat Politècnica de València.

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Fdo. Alicia Cordero	Fdo. Neus Garrido	Fdo. Juan R. Torregrosa

Abstract

In a large number of problems in applied mathematics, there is a need to solve nonlinear equations and systems, since many problems eventually are reduced to these. As the difficulty of the systems increases, obtaining the analytical solution becomes more complex. Furthermore, with the growth of computational tools, the dimensions of the problems to be solved have increased exponentially, making it more essential to obtain an approximation to the solution in a simple way that does not require significant time and computational cost. That is one of the reasons why iterative methods have increased their importance in recent years, as a multitude of schemes have been designed to converge rapidly to the solution and, in this way, to be able to solve problems that would be more arduous to solve using classical tools.

This Doctoral Thesis focuses on the study and design of numerous iterative methods that improve classical schemes in terms of their order of convergence, accessibility, number of solutions obtained or applicability to problems with special characteristics, such as non-differentiability or multiplicity of roots. The procedures studied in this report range from a family of optimal multi-step methods for solving equations, to a parametric derivative-free family of weight function schemes, to which memory is introduced for solving nonlinear systems. Additional procedures are described in this report such as iterative schemes that obtain roots with different multiplicities for equations and methods that approximate roots simultaneously for equations as well as for systems, and for simple as well as for multiples roots. In addition, part of this report focuses on how to perform the dynamical analysis for iterative schemes with memory that solve systems of nonlinear equations, as well as this study is carried out for different known iterative procedures. This dynamical analysis allows us to visualise and analyse the possible behaviours of the iterative methods depending on the initial approximations.

The results described above form part of this Doctoral Thesis to obtain the title of Doctor in Mathematics.

Resumen

En gran cantidad de problemas de la matemática aplicada, existe la necesidad de resolver ecuaciones y sistemas no lineales, dado que numerosos problemas, finalmente, se reducen a estos. Conforme aumenta la dificultad de los sistemas, la obtención de la solución analítica se vuelve más compleja. Además, con el aumento de las herramientas computacionales, las dimensiones de los problemas a resolver han crecido de manera exponencial, por lo que se vuelve más necesario obtener una aproximación a la solución de manera sencilla y que no requiera mucho tiempo y coste computacional. Esta es una de las razones por las que los métodos iterativos han aumentado su importancia en los últimos años, ya que se han diseñado multitud de procesos con el fin de que converjan rápidamente a la solución y, de esta forma, poder resolver problemas que con las herramientas clásicas resultaría más costoso.

La presente Tesis Doctoral, se centra en estudiar y diseñar numerosos métodos iterativos que mejoren a los esquemas clásicos en cuanto a su orden de convergencia, accesibilidad, cantidad de soluciones que obtienen o aplicabilidad a problemas con características especiales, como la no diferenciabilidad o la multiplicidad de las raíces. Entre los procesos que se estudian en esta memoria, se pueden encontrar desde una familia de métodos multipaso óptimos para la resolución de ecuaciones, hasta una familia paramétrica libre de derivadas de esquemas con función peso a la que se introduce memoria para la resolución de sistemas no lineales. Se destancan otros métodos en esta memoria como esquemas iterativos que obtienen raíces con diversas multiplicidades para ecuaciones y procesos que aproximan raíces de forma simultánea, tanto para ecuaciones como para sistemas, y, tanto para raíces simples como para múltiples. Además, parte de esta memoria se centra en cómo realizar el análisis dinámico para métodos iterativos con memoria que resuelven sistemas de ecuaciones no lineales, a la par que se realiza dicho estudio para diversos esquemas iterativos conocidos. Este análisis dinámico permite visualizar y analizar los posibles comportamientos de los procesos iterativos en función de las aproximaciones iniciales.

Los resultados anteriormente descritos forman parte de esta Tesis Doctoral para la obtención del título de Doctora en Matemáticas.

Resum

En gran quantitat de problemes de la matemàtica aplicada, existeix la necessitat de resoldre equacions i sistemes no lineals, atés que nombrosos problemes, finalment, es redueixen a aquests. Conforme augmenta la dificultat dels sistemes, l'obtenció de la solució analítica es torna més complexa. A més, amb l'augment de les eines computacionals, les dimensions dels problemes a resoldre han crescut de manera exponencial, per la qual cosa es torna més necessari obtindre una aproximació a la solució de manera senzilla i que no requerisca molt temps i cost computacional. Aquesta és una de les raons per les quals els mètodes iteratius han augmentat la seua importància en els últims anys, ja que s'han dissenyat multitud de processos amb la finalitat que convergisquen ràpidament a la solució i, d'aquesta manera, poder resoldre problemes que amb les eines clàssiques resultaria més costós.

La present Tesi Doctoral, es centra en estudiar i dissenyar nombrosos mètodes iteratius que milloren als esquemes clàssics en quant al seu ordre de convergència, accessibilitat, quantitat de solucions que obtenen o aplicabilitat a problemes amb característiques especials, com la no diferenciabilitat o la multiplicitat de les arrels. Entre els processos que s'estudien en aquesta memòria, es poden trobar des d'una família de mètodes multipas òptims per a la resolució d'equacions, fins a una família paramètrica lliure de derivades de esquemes amb funció pes a la que s'introdueix memòria per a la resolució de sistemes no lineals. Es destanquen altres mètodes en aquesta memòria com esquemes iteratius que obtenen arrels amb diverses multiplicitats per a equacions i processos que aproximen arrels de manera simultània, tant per a equacions com per a sistemes, i, tant per a arrels simples com per a múltiples. A més, part d'aquesta memòria es centra en com realitzar l'anàlisi dinàmic per a mètodes iteratius amb memòria que resolen sistemes d'equacions no lineals, al mateix temps que es realitza aquest estudi per a diversos esquemes iteratius coneguts. Aquest anàlisi dinàmic permet visualitzar i analitzar els possibles comportaments dels mètodes iteratius en funció de les aproximacions inicials.

Els resultats anteriorment descrits formen part d'aquesta Tesi Doctoral per a l'obtenció del títol de Doctora en Matemàtiques.

Contents

Chapter 1

Introduction

Many problems in physics, chemistry or applied mathematics are reduced to solve an equation or system of nonlinear equations, for example those problems modelled by differential equations. The appropriate tools are not always available to solve these nonlinear systems exactly, due, among other reasons, to the difficulty of the problem or its size, which has increased considerably with the advance of computer tools. For this reason, iterative methods have gained attention, since these schemes aim to find, through an iterative process using an initial approximation close to the solution, a sequence of approximations that, under certain conditions, converge to the solution.

An example of this is the resolution of polynomial equations. In the case of low-degree polynomials, there are analytical tools to obtain the roots, but as the degree of the polynomial increases, so does the complexity of the problem. This was one of the reasons why Newton, in 1669, developed a process to obtain the roots of polynomials. This method is still one of the most used and well-known and has the following iterative expression:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, \ldots,$$

where $f(x) = 0$ is the equation to be solved and $f'(x_k)$ is the derivative of this function evaluated at iterate x_k. As we will see later in this report, this method converges, under certain conditions, to the solution of nonlinear equation $f(x) = 0$.

Based on this method, many others have been designed in order to solve the cases in which the problems do not meet the convergence conditions of Newton's method. One of them is the applicability to systems of equations, which is why Ostrowski in [1] proposes the extension of Newton's method for the solution of nonlinear systems. Another problem is the condition of differentiability or the need to have simple roots in order to ensure convergence. In this report, we focus on several branches of iterative methods, many of them solving problems of differentiability, extension to systems or roots with different multiplicities.

In the following, we introduce each of the chapters that constitute this thesis.

In Chapter 2, the necessary preliminary concepts for the development of the subsequent chapters are introduced. This chapter is divided into two sections. In Section 2.1, we introduce iterative methods and the different categories into which they can be classified. Section 2.2 focuses on tools necessary for the dynamical analysis of iterative schemes, both from the point of view of complex unidimensional dynamical analysis and real multidimensional dynamical analysis, as well as establishing how we graphically represent the behaviour of these methods.

In Chapter 3, a family of optimal multi-step iterative procedures for solving nonlinear equations is designed. As we know, Newton's method is optimal, but if we compose n times Newton's method, let n be any natural number greater than or equal to 2, what we obtain is a method of order 2^n which is not optimal. In this chapter, we modify this method, whatever the value of n, in such a way that the order of convergence is maintained and optimal methods are obtained. As proven in this chapter, the family of multi-step schemes obtained is optimal for any number of steps. On certain elements of this family, complex dynamical analysis is performed and some dynamical planes are represented in order to make further comparisons between the proposed procedures and other known methods of similar order.

In Chapter 4, based on Traub's method [2], two parametric families of derivative-free iterative methods with weight function for nonlinear equations are designed, which, under certain conditions, have order 4 and 6, respectively, the family of order 4 is a class of optimal iterative schemes. Memory is introduced to both families in order to increase the order of convergence without performing more functional evaluations per iteration, increasing the order by up to two units for the family of order 4, and increasing it by up to three units for the family of order 6. A complex dynamical analysis is performed for the order 4 family, obtaining for which parameter values, the class of iterative methods holds more stable procedures, making parameter planes as a graphical representation. At the same time that this analysis is performed, a real multidimensional dynamical analysis is also performed for certain memory variants of this family, in order to make comparisons between the iterative class and its memory variants, beyond the order of convergence.

Chapter 5 focuses on the design of an iterative step for obtaining simple roots of a nonlinear equation simultaneously, given that sometimes we are interested in obtaining more than one solution to the problem as is illustrated in this chapter. It is obtained that the order of convergence of this step is 2, and it is also analysed that it can be added to any other method, thus generating a predictor-corrector method that approximates roots simultaneously with twice the order of convergence of the predictor method used for arbitrary equations and three times the order of convergence in the case of polynomial equations. How the behaviour of the methods is modified by adding this step of simultaneity is graphically represented in this chapter in order to motivate the reason of the introduction of simultaneity.

In Chapter 6, iterative schemes for obtaining roots of equations with multiplicity greater than 1 are presented, given that in many of the known iterative methods, it is required that this root is simple to ensure convergence or achieve a particular speed of convergence. This is not always the case given that many problems in applied mathematics have roots with different multiplicities. Many of the known procedures for multiple roots use the value of this multiplicity in their iterative expression, but to know this value, it is necessary to know the solutions of the problem, and if we want to obtain all the roots, we must change the value of the multiplicity depending on which root we want to converge to. For this reason, two iterative methods are designed from Kurchatov's scheme that obtain multiple roots without needing to know the multiplicity, one of them free of derivatives, and both of them maintain Kurchatov's quadratic convergence order. An exhaustive dynamical analysis of one of the proposed schemes is carried out on several polynomials, obtaining that although the multiplicities of the roots are different, the method converges to all of them. In addition, the iterative step defined in Chapter 5 is added, thus obtaining an iterative method that converges simultaneously to several roots without the need to take into account whether they are simple or multiple or whether they have different multiplicities. This method has convergence order 4 for arbitrary equations and order 6 in the case of polynomial equations.

In Chapter 7, based on two known iterative methods for nonlinear equations, a parametric class of iterative procedures for the approximation of nonlinear systems of equations is designed. This iterative class has convergence order 3, and increases to order 4 when the parameter has null value. We perform a unidimensional complex dynamical study for this family in order to find out for which parameter values the most stable methods are obtained.

In Chapter 8, the classes proposed in Chapter 4 are extended to the solution of nonlinear systems. In this case, the family of order 4 maintains the order, but the family of order 6 manages to increase the order of convergence by one unit, thus obtaining a parametric class of iterative schemes of order 7 for nonlinear systems. As in Chapter 4, memory is introduced to these families, increasing the order by two units and four units, respectively, that is, methods of up to order 6 are obtained for the case of the iterative class of order 4 and schemes of up to order 11 for the case of the parametric iterative class of order 7.

Chapter 9 focuses on the modification of the iterative step proposed in Chapter 5 in order to adapt it to the solution of nonlinear systems. The obtained step maintains the order of convergence that we had for nonlinear equations, and it is also proven that it can be added to any iterative method for systems obtaining a predictor-corrector method that duplicates the order of the predictor method.

In all the previous chapters, a section of numerical experiments is included to check the theoretical results obtained from the iterative schemes as well as to compare these iterative procedures with known methods of similar order. These numerical experiments have been carried out in Matlab as will be discussed in each corresponding section.

In Chapter 10, some theoretical results are obtained to carry out the dynamical study for iterative schemes with memory that solve systems of nonlinear equations, which until now in the literature it has been carried out only for nonlinear equations. Once these theoretical concepts have been defined, the dynamical analysis of two known methods, vectorial Steffensen's method and Kurchatov's method is carried out in order to illustrate different behaviours. On the one hand, we study what happens in the case where the system is uncoupled, that is, the components do not interact with each other, while on the other hand, we study what happens in the case of a coupled system, where the behaviour of each component of the fixed point operator on (x_1, x_2) involves both components.

To conclude, in Chapter 11 we present a summary of the results obtained and, finally, we end this report with a list of references that have been used during the development of this doctoral thesis.

Chapter 2

Preliminary concepts

In this chapter we are going to present the most used concepts of iterative methods in this dissertation.

Section 2.1 begins with the definition of nonlinear equations and nonlinear systems of equations and, since it is often not possible to obtain a solution, discusses how to approximate the solutions of these problems using iterative fixed-point methods. After that, the order of convergence of the iterative schemes is stated and results that will be used in the following chapters are presented. Next, we discuss some of the categories into which iterative procedures can be classified, which are if the method has memory or not, if the scheme consists of one or more steps, or if it is a scheme with derivatives or is derivative-free, defining then the divided difference operator for equations as well as for nonlinear systems.

In Section 2.2, the previous concepts for a dynamical analysis of iterative methods are introduced in order to illustrate the behaviour of them. The basic concepts of unidimensional complex dynamical and multidimensional real dynamical analysis required for the further development of the dynamical analysis of iterative schemes are introduced. We also discuss in each case how to perform graphical representations in order to compare between the different procedures.

2.1 Preliminary concepts of iterative methods

Some of the results developed in this thesis are focused on the resolution of nonlinear equations, that is, equations which have the form

$$f(x) = 0, \quad f : D \subset \mathbb{R} \to \mathbb{R}, \tag{2.1}$$

where f is a function defined on an open interval D and $\alpha \in D$ denotes a solution of the nonlinear equation (2.1). Specifically, we study the case in which α is a simple solution and in which it is a multiple solution, that is, there exists a $m \in \mathbb{N}\backslash\{1\}$ such that $f^{(i)}(\alpha) = 0$ for $i = 0, 1, \ldots, m-1$ and $f^{(m)}(\alpha) \neq 0$.

In addition to dealing with the unidimensional case, we also study the solution of multidimensional problems, in which the general expression is a system formed by n nonlinear equations with n unknowns as follows,

$$\begin{cases} f_1(x_1, x_2, \ldots, x_n) & = & 0, \\ f_2(x_1, x_2, \ldots, x_n) & = & 0, \\ & \vdots & \\ f_n(x_1, x_2, \ldots, x_n) & = & 0, \end{cases}$$

which can be simplified and denoted by

$$F(x) = 0, \tag{2.2}$$

where F is a vectorial function, $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$, defined on a non-empty open convex set D, which has as coordinate functions f_i, $i = 1, 2, \ldots, n$.

From this point onwards, we assume that we are solving a system of n nonlinear equations, and specify, if necessary, the unidimensional case $n = 1$.

Usually, and as it will be seen in ongoing chapters, the solution of nonlinear systems cannot be carried out analytically. For this reason, in recent years, it has been studied how to obtain an approximation to these solutions reliably through iterative methods.

These schemes generate a sequence $\{x^{(k)}\}$ by an iterative process. This sequence of approximations to the solution α is obtained from an initial approximation $x^{(0)}$ close to the solution and it is required that $\lim\limits_{k\to\infty} x^{(k)} = \alpha$ exists under certain error criteria. We denote the sequence by $\{x_k\}$ in the unidimensional case.

Many of the best-known iterative schemes focus on obtaining the approximation to a root as a fixed point of a certain function $\phi : \mathbb{R}^n \to \mathbb{R}^n$ by means of the following iterative scheme:

$$x^{(k+1)} = \phi(x^{(k)}), \quad k = 0, 1, \ldots$$

These schemes are distinguished from each other by the way the iteration function ϕ is defined. They are known as fixed-point methods.

One of the most well-known method is Newton's scheme [2], which is structured as follows

$$x^{(k+1)} = x^{(k)} - [F'(x^{(k)})]^{-1} F(x^{(k)}), \quad k = 0, 1, \ldots, \tag{2.3}$$

where $F'(x^{(k)})$ denotes the Jacobian matrix of F evaluated at $x^{(k)}$.

Based on their characteristics, iterative methods can be classified in several ways. One of the most relevant is the order of convergence, which provides a measure of the speed of convergence of the sequence to the solution. Some of the definitions relevant to these concepts are discussed in this section and can be found, for example, in [2].

Definition 1. *Let us consider a sequence $\{x^{(k)}\}_{k\geq 0}$ in $\mathbb{R}^n$ generated by an iterative method that converges to α. Then, the corresponding scheme has order of convergence p, $p \geq 1$, if there exists a strictly positive constant D_p such that*

$$\lim_{k\to\infty} \frac{\|x^{(k+1)} - \alpha\|}{\|x^{(k)} - \alpha\|^p} = D_p, \tag{2.4}$$

where D_p is called the asymptotic error constant. It must be satisfied that $D_p < 1$ if $p = 1$.

Ortega and Rheinboldt, realising that definition (2.4) of the order of convergence is quite restrictive, introduced in [3] the concepts of Q-order and R-order of convergence. They proved that these definitions coincide with the classical order when $0 < D_p < \infty$ exists for some $p \geq 1$.

Therefore, from now on, we assume that the definitions are analogous and work with the following definition of order of convergence.

Definition 2. *We denote $e_k = x^{(k)} - \alpha$ as the error made in the iteration $x^{(k)}$. Every method satisfies an equation of the type*

$$e_{k+1} = Le_k + O\left(e_k^{p+1}\right), \tag{2.5}$$

called the error equation, where L is a p-linear function $L \in \mathcal{L}\left(\mathbb{R}^n \times \cdots \times \mathbb{R}^n, \mathbb{R}^n\right)$, and p is the order of convergence of the method. Note that e_k^p denotes $(e_k, e_k, \ldots, e_k)$.

Considering $\{g_k\}_{k\geq 0}$ and $\{h_k\}_{k\geq 0}$ two non-zero scalar sequences, to carry out the analysis of the order of convergence of some methods, we use the notation given in [4]. We denote $g_k = O(h_k)$, or alternatively $g_k \sim h_k$, to denote that

$$\lim_{k\to\infty} \frac{g_k}{h_k} = C, \tag{2.6}$$

with C being a non-zero constant. As a consequence, the error equation (2.5) always satisfies a relation of the form

$$e_{k+1} \sim Le_k^p.$$

The theoretical definitions of the convergence order, however, cannot be used in numerical experiments. For this reason, besides obtaining the approximation to the solution, in numerical experiments we also obtain an approximation to the convergence order. One way to do this is by using the computational convergence order (COC), introduced by Weerakoon and Fernando in [5], which is defined as follows

$$p \approx COC = \frac{\ln\left(\|x^{(k+1)}-\alpha\|/\|x^{(k)}-\alpha\|\right)}{\ln\left(\|x^{(k)}-\alpha\|/\|x^{(k-1)}-\alpha\|\right)}.$$

where $x^{(k+1)}$, $x^{(k)}$ and $x^{(k-1)}$ are three successive approximations to α obtained in the iterative process.

Usually, the value of α is not known, therefore, to obtain the order of convergence it is more convenient to employ the approximate computational order of convergence (ACOC) defined by Cordero and Torregrosa in [6], which is given as follows

$$p \approx ACOC = \frac{\ln\left(\|x^{(k+1)}-x^{(k)}\|/\|x^{(k)}-x^{(k-1)}\|\right)}{\ln\left(\|x^{(k)}-x^{(k-1)}\|/\|x^{(k-1)}-x^{(k-2)}\|\right)}.$$

Another important category of iterative procedures is if they have memory or not, that is, how many previous iterations are used to obtain the next iteration. Thus, a method without memory can be described as follows

$$x^{(k+1)} = \phi(x^{(k)}), \quad k = 0, 1, 2, \ldots,$$

so we only use the immediately previous iteration to define the next iteration; while a method with memory has the following expression

$$x^{(k+1)} = \phi(x^{(k)}, x^{(k-1)}, x^{(k-2)}, \ldots), \quad k = 0, 1, 2, \ldots,$$

where several previous iterations are used to obtain the next one.

To prove the order of convergence of the methods with memory we use the following Ortega-Rheinboldt's Theorem, which can be found in [3].

Theorem 2.1.1. *Let ϕ be an iterative scheme with memory that generates a sequence $\{x^{(k)}\}_{k\geq 0}$ of approximations to the root α, and let this sequence converges to α. If there exist a nonzero constant η and positive numbers t_i, $i = 0, \ldots, m$ such that the inequality*

$$\|e_{k+1}\| \leq \eta \prod_{i=0}^{m} \|e_{k-i}\|^{t_i},$$

holds, then the R-order of convergence of the iterative method ϕ is at least p, where p is the unique positive root of the equation

$$p^{m+1} - \sum_{i=0}^{m} t_i p^{m-i} = 0.$$

On the other hand, Kung and Traub established in [7] the definition of optimal procedure, which is only applicable to methods without memory, and is defined below.

Conjecture 2.1.1.1. *The order of convergence of a method without memory that performs d functional evaluations per iteration fulfils*

$$p \leq 2^{d-1},$$

calling optimal scheme the one that satisfies $p = 2^{d-1}$.

When designing an iterative method, one of the main aims is to achieve the optimality, since this type of procedure uses the smallest possible number of functional evaluations to obtain the highest possible order.

Another category to classify iterative schemes is the number of steps involved. The reason for increasing the number of steps is because Traub in [2] showed that a one-step method, that only uses derivatives, must include derivatives of at least order $p-1$ to achieve order p. Consequently, increasing the number of steps is interesting to increase the order of convergence avoiding higher-order derivatives.

Multi-step methods that perform m steps, also referred to as predictor-corrector methods, can be described by

$$\begin{aligned}
y_1^{(k)} &= \Psi_1(x^{(k)}),\\
y_2^{(k)} &= \Psi_2(x^{(k)}, y_1^{(k)}),\\
&\dots\\
y_{m-1}^{(k)} &= \Psi_{m-1}(x^{(k)}, y_1^{(k)}, \dots, y_{m-2}^{(k)}),\\
x^{(k+1)} &= \Phi(x^{(k)}, y_1^{(k)}, \dots, y_{m-1}^{(k)}), \quad k = 0, 1, 2 \dots
\end{aligned}$$

In [3], it is shown that the order of convergence of the above multi-step method is p, being $p = p_1 p_2 p_{m-1} p_m$, where p_i denotes the order of convergence of scheme Ψ_i for $i = 1, \dots, m-1$ and p_m denotes the order of convergence of scheme Φ.

There are many known multi-step methods, among which we highlight Traub's scheme [2] and King's family [8].

The last category to classify iterative methods that we are going to establish in this chapter is the presence or absence of derivatives, giving rise to procedures with derivatives or derivative-free, which we call with Jacobians or Jacobian-free in the case of systems of nonlinear equations.

For the proofs of iterative methods with Jacobian matrices, it is necessary to recall the notation presented in [9], which is used in the following chapters.

Let $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$ be a sufficiently differentiable function on D. The q-th derivative of F in $u \in \mathbb{R}^n$, $q \geq 1$, is the q-linear function $F^{(q)}(u) : \mathbb{R}^n \times \dots \times \mathbb{R}^n \to \mathbb{R}^n$ such that $F^{(q)}(u)(v_1, \dots, v_q) \in \mathbb{R}^n$.

As it is stated in [9], the q-th derivative of F satisfies the following properties:

Proposition 2.1.1.1. *Consider $u \in \mathbb{R}^n$, $q \geq 1$, then the following properties are satisfied for the q-th derivative of F*

1. $F^{(q)}(u)(v_1, \dots, v_{q-1}, \cdot) \in \mathcal{L}(\mathbb{R}^n)$*, where $\mathcal{L}(\mathbb{R}^n)$ is the set of linear operators of $\mathbb{R}^n$.*
2. $F^{(q)}(u)(v_{\sigma(1)}, \dots, v_{\sigma(q)}) = F^{(q)}(u)(v_1, \dots, v_q)$ *for every permutation σ of $\{1, 2, \dots, q\}$.*
3. $F^{(q)}(u)(v_1, \dots, v_q) = F^{(q)}(u) \cdot v_1 \cdots v_q$.
4. $F^{(q)}(u)v^{q-1}F^{(p)}(u)v^p = F^{(q)}(u)F^{(p)}(u)v^{q+p-1}$.

On the other hand, for $\alpha + h \in \mathbb{R}^n$ in a neighbourhood of the solution α of $F(x) = 0$, one can apply Taylor developments on the derivative around α and, assuming that the Jacobian matrix $F'(\alpha)$ is non-singular, one obtains

$$F'(\alpha + h) = F'(\alpha)\left[h + \sum_{q=2}^{p-1} C_q h^q\right] + O(h^p), \tag{2.7}$$

where $C_q = \dfrac{1}{q!}[F'(\alpha)]^{-1}F^{(q)}(\alpha)$, $q \geq 2$. It is observed that $C_q h^q \in \mathbb{R}^n$ since $F^{(q)}(\alpha) \in \mathcal{L}(\mathbb{R}^n \times \cdots \times \mathbb{R}^n, \mathbb{R}^n)$ and $[F'(\alpha)]^{-1} \in \mathcal{L}(\mathbb{R}^n)$.

In some cases, the problems are non-differentiable or it is difficult to obtain the Jacobian matrix F' from the operator F, which is why Jacobian-free schemes arise, which allow us to solve both differentiable and non-differentiable problems.

One technique used to obtain derivative-free methods is to replace the derivatives in iterative methods by divided difference operators. The multidimensional first-order divided difference operator was defined by Ortega and Rheinboldt in [3] as the function

$$[\cdot, \cdot; F] : D \times D \subset \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n), \tag{2.8}$$

which satisfies

$$[x, y; F](x - y) = F(x) - F(y), \quad \forall x, y \in D. \tag{2.9}$$

In the case $n = 1$, the notation used to define the first-order divided difference between two points x and y is

$$f[x, y] = \frac{f(x) - f(y)}{x - y}. \tag{2.10}$$

The divided differences of order two for the scalar case, defined for the points x, y and z, are given by the expression

$$f[x, y, z] = \frac{f[x, y] - f[y, z]}{x - z}. \tag{2.11}$$

For the case of order greater than two, the process is equivalent using the divided differences of the immediately preceding order.

Since the convergence analysis of the proposed schemes is performed using Taylor series developments of the operator and its derivatives around the solution of the problem, it is necessary to obtain the development corresponding to the divided difference operator. With this aim, using the Genocchi-Hermite formula [3]

$$[x, x+h; F](x-y) = \int_0^1 F'(x+th)dt, \tag{2.12}$$

and developing $F'(x+th)$ in Taylor series around x, we obtain the expression of the operator

$$\int_0^1 F'(x+th)dt = F'(x) + \frac{1}{2}F''(x)h + \frac{1}{6}F'''(x)h^2 + O(h^3). \tag{2.13}$$

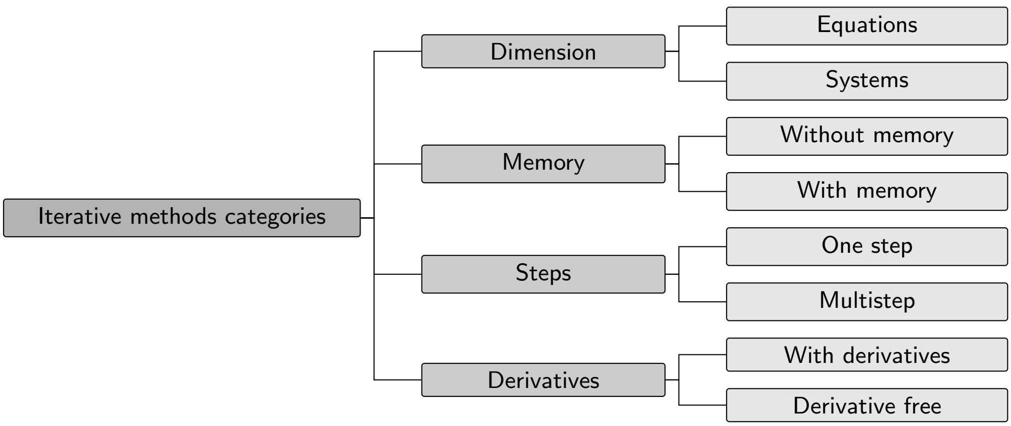
Also, for the implementation of the iterative methods in later chapters, the multidimensional divided difference operator is developed following the definition provided in [10], of the form

$$[x, y; F]_{ij} = \frac{F_i(y_1, y_2, \ldots, y_{j-1}, y_j, x_{j+1}, \ldots, x_n) - F_i(y_1, y_2, \ldots, y_{j-1}, x_j, x_{j+1}, \ldots, x_n)}{y_j - x_j}, \tag{2.14}$$

where $x = (x_1, x_2, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_{j-1}, y_j, y_{j+1}, \ldots, y_n)$ for all $1 \leq i, j \leq n$.

Among the derivative-free methods, Steffensen's scheme [11] and Kurchatov's scheme [12] stand out, both procedures with convergence order 2, but the second one is a method with memory.

The following is a summary of the categories discussed in this chapter that the iterative schemes studied in this report can possess:



2.2 Preliminary concepts of dynamical analysis

In the following subsections we focus on giving the necessary background concepts for the dynamical studies in the following chapters, both for unidimensional complex and multidimensional real dynamical analysis. These results can be consulted in more detail in [13, 14, 15].

Preliminary concepts of unidimensional complex dynamical study

In this section, we are going to introduce some of the essential concepts to carry out the complex dynamical study of iterative methods that solve nonlinear equations.

The first point to make is that, in general, a method applied to a polynomial $p(z) = 0$, provides a rational operator or function R, therefore, part of the definitions and results mentioned in this chapter, are about the study and conclusions about these rational operators.

Definition 3. *Let R be a rational function $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ denotes the Riemann sphere. The orbit of a point $z \in \hat{\mathbb{C}}$ is defined as*

$$\{z, R(z), R^2(z), \dots, R^m(z), \dots\},$$

that is, the set of its images by R.

- *A fixed point of R is a point $z \in \hat{\mathbb{C}}$ such that $R(z) = z$. A fixed point of R is said to be strange if $p(z) \neq 0$.*
- *It is said that a point $z \in \hat{\mathbb{C}}$ is a periodic point of R, of period K greater than 1, if it is satisfied that $R^K(z) = z$ and $R^k(z) \neq z$, for all $k < K$.*
- *A critical point of R is a point $z \in \hat{\mathbb{C}}$ such that $R'(z) = 0$. A critical point is said to be free if $p(z) \neq 0$.*

Next, we define the character of the fixed points of a rational operator.

Definition 4. *Let R be the rational operator R, we classify its fixed points as follows:*

- *If $|R'(z)| < 1$, then the fixed point z is an attractor. If it happens that the derivative at the point is exactly 0, that is, $R'(z) = 0$, then the fixed point z is a superattractor.*
- *If $|R'(z)| > 1$, then the fixed point z is a repelling point.*
- *If $|R'(z)| = 1$, then the fixed point z is a parabolic or neutral point.*

Definition 5. *The basin of attraction of an attracting fixed point (or periodic) $z \in \hat{\mathbb{C}}$ is constituted by the set of its pre-images of any order, that is,*

$$\mathcal{A} = \{z_1 \in \hat{\mathbb{C}} : R^m(z_1) \to z, m \to \infty\}.$$

Definition 6. *The Fatou set of R, denoted by $\mathcal{F}(R)$, consists of those points whose orbits tend to an attractor, and its complementary on the Riemann sphere is the Julia set, denoted by $\mathcal{J}(R)$.*

Therefore, the basin of attraction of any fixed point belongs to $\mathcal{F}(R)$, while $\mathcal{J}(R)$ contains all points that are repulsors and establishes the boundaries between the basins of attraction.

Another relevant result obtained by Julia and Fatou in [16] and [17], respectively, is the following:

Theorem 2.2.1. *Let R be a rational function. The basin of attraction of a periodic (or fixed) attractor point contains at least one critical point.*

In the following, we present results that allow us to reduce the dynamical study to simpler cases.

Definition 7. *Let f and g be functions of the Riemann sphere on itself. An analytic conjugation between f and g is a diffeomorphism $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $h \circ f = g \circ h$.*

Theorem 2.2.2. *(Scaling Theorem)*
Let $f(z)$ be an analytic function on $\hat{\mathbb{C}}$, $T(z) = \delta z + \gamma$ an affine application and R_f and R_g rational operators. If $g(z) = (f \circ T)(z)$, then fixed point operator R_f is analytically conjugate to R_g by T, that is, $(T \circ R_g \circ T^{-1})(z) = R_f(z)$.

Theorem 2.2.3. *Let $q(z) = a_1 z^2 + a_2 z + a_3$, with $a_1 \neq 0$, be a quadratic polynomial with simple roots and R_f and R_g rational operators. It can then be reduced to $p(z) = z^2 + c$ by an affine transformation, where $c = 4a_1 a_3 - a_2^2$. This affine application induces a conjugation between $R_q(z)$ and $R_p(z)$.*

Let $p(z) = (z - a)(z - b)$, with a and $b \in \mathbb{C}$. Since the operator associated to $p(z)$ depends on parameters a and b, we use the Möbius transformation and its inverse to simplify our rational function so that it does not depend on a and b. Then, we make the study of the dynamics easier, both in terms of obtaining fixed and critical points and also of its graphical representation.

To eliminate this dependence of a and b, we get the conjugate operator of R_p by h as follows:

$$O_p(z) = (h \circ R_p \circ h^{-1})(z), \tag{2.15}$$

where h is the application of Möbius and h^{-1} its inverse, defined as

$$\begin{aligned} h(z) &= \frac{z-a}{z-b}, \\ h^{-1}(z) &= \frac{zb-a}{z-1}, \end{aligned}$$

which satisfy the following properties

$$\begin{aligned} h(\infty) &= 1, \\ h(a) &= 0, \\ h(b) &= \infty. \end{aligned}$$

For the calculation of the dynamical planes, dynamical lines and parameter planes have been resorted to [14, 15], where the algorithms used for the design of these graphical representations of the iterative methods' dynamics are exposed and explained.

Preliminary concepts of multidimensional real dynamical study

In this section, we provide previous and necessary concepts for the dynamical study of iterative methods with memory to solve nonlinear equations, since these must be treated in a particular way as we will see below.

We recall that an iterative process with memory that uses m previous iterations in its iterative expression to compute the next iteration, can be expressed as

$$x_{k+1} = \phi(x_{k-m}, \dots, x_{k-1}, x_k), \quad k \geq m, \tag{2.16}$$

where $x_0, \dots, x_m$ are initial approximations.

Now, we explain the concepts for $m = 1$ to simplify the notation, but using more initial approximations the concepts are similar.

Therefore, the structure of the iterative methods that we are going to study is

$$x_{k+1} = \phi(x_{k-1}, x_k), \quad k = 1, 2, \dots \tag{2.17}$$

where x_0 and x_1 are initial approximations.

A function defined from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ cannot have fixed points since to be a fixed point of a function, the point and its image by the function must coincide. Therefore, an auxiliary function G is defined as follows:

$$G\left(x_{k-1}, x_k\right) = \left(x_k, x_{k+1}\right) = \left(x_k, \phi\left(x_{k-1}, x_k\right)\right), \ k = 1, 2, \dots$$

If (x_{k-1}, x_k) is a fixed point of G, then

$$G\left(x_{k-1}, x_k\right) = \left(x_{k-1}, x_k\right),$$

and by the definition of G, one has

$$\left(x_{k-1}, x_k\right) = \left(x_k, x_{k+1}\right).$$

To simplify the notation, we denote $z = x_{k-1}$ and $x = x_k$.

Thus, the discrete dynamical system $G\colon \mathbb{R}^2 \to \mathbb{R}^2$ is defined as

$$G(z, x) = \left(x, \phi(z, x)\right),$$

where ϕ is the operator associated with the iterative method with memory.

Definition 8. *A point (z, x) is a fixed point of G if $z = x$ and $x = \phi(z, x)$. If (z, x) is a fixed point of operator G that does not satisfy $p(x) = 0$, it is called a strange fixed point.*

Definition 9. *The basin of attraction of a fixed point (z^*, x^*) is defined as the set of pre-images of any order such that*

$$\mathcal{A}(z^*, x^*) = \{(z_1, x_1) \in \mathbb{R}^2 \colon G^m(z_1, x_1) \to (z^*, x^*),\ m \to \infty\}.$$

To study the character of the fixed points, we use the following result from [13].

Theorem 2.2.4. *Let $G\colon \mathbb{R}^2 \to \mathbb{R}^2$ be of class $\mathcal{C}^2$ and y a fixed point. Let λ_1 and λ_2 be the eigenvalues of $G'(y)$, where G' is the Jacobian matrix of operator G.*

- *If $|\lambda_j| < 1$, for $j = 1, 2$, then y is an attractor.*
- *If $|\lambda_j| = 0$, for $j = 1, 2$, then y is a superattractor.*
- *If one eigenvalue λ_{j_0} has $|\lambda_{j_0}| > 1$, then y is repelling or saddle.*
- *If $|\lambda_j| > 1$, for $j = 1, 2$, then y is repelling.*

If one eigenvalue λ of $G'(y)$ satisfies $|\lambda| = 1$, then y is not hyperbolic and we cannot conclude anything about the character of this fixed point.

Another relevant concept in a dynamical study is the critical point. In this case, we use the following definition of these type of points.

Definition 10. *The point (z, x) is a critical point of $G(z, x)$ if all the eigenvalues of $G'(z, x)$ are 0. It is called free critical point if $p(x) \neq 0$.*

As in the previous section, for the design of the dynamical planes, dynamical lines and parameter planes have been resorted to [18, 19], where these graphical representations of the dynamical iterative methods are exposed and explained in more detail.

Chapter 3

Multi-step iterative methods with derivatives to solve nonlinear equations

Based on [Cordero, A.; Torregrosa, JR.; Triguero-Navarro, P. (2021). A General Optimal Iterative Scheme with Arbitrary Order of Convergence. Symmetry (Basel). 13(5):1-17. https://doi.org/10.3390/sym13050884]

3.1 Introduction

A well-known iterative method is Newton's method, due to its efficiency and simplicity, whose scheme is as follows

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \qquad k = 0, 1, 2, \ldots \tag{3.1}$$

where the derivative of function f evaluated at iteration number k, denoted by $f'(x_k)$, must be non-zero.
Besides being simple and efficient, Newton's method has quadratic convergence under certain conditions and is optimal, in the sense of Kung-Traub's conjecture.
Composing Newton's method n-times, it defines an n-step method by repeating its structure from an initial estimation x_0

$$\begin{cases} y_1 = x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ y_2 = y_1 - \dfrac{f(y_1)}{f'(y_1)}, \\ \qquad \vdots \\ y_{j+1} = y_j - \dfrac{f(y_j)}{f'(y_j)}, \qquad j = 1, 2, \ldots, n-2 \\ \qquad \vdots \\ x_{k+1} = y_{n-1} - \dfrac{f(y_{n-1})}{f'(y_{n-1})}, \quad k = 0, 1, 2, \ldots \end{cases} \tag{3.2}$$

The order of Newton's method for n steps is 2^n. We perform 2 functional evaluations per step, so when 2 or more steps are performed it is not an optimal method, because $2^n \neq 2^{2n-1}$, unless $n = 1$.
In the following, we are going to modify Newton's n-step method to obtain an optimal method. To do this, we reduce the number of functional evaluations by approximating the derivatives, which appear in (3.2), after the first step by polynomials satisfying certain conditions. There are many other families of iterative methods that have been designed using interpolation techniques such as the classes designed in [20] and [21].

This chapter is structured as follows. In Section 3.2, the iterative expression of the proposed schemes is obtained, and their convergence order is proven in Section 3.3. In Section 3.4, we analyse the set of initial estimations that converge to the solution of the proposed and other known methods when they are applied to different nonlinear equations. In Section 3.5 of this chapter, several numerical experiments are carried out and, finally, in Section 3.6, we discuss some conclusions derived from this work.

3.2 Design of the family of iterative methods

In this section, we design the family of iterative methods we propose. From expression (3.2) and approximating the derivatives after the first step, we obtain the following iterative expression

$$\begin{cases} y_1 = x_k - \dfrac{f\left(x_k\right)}{f'\left(x_k\right)}, \\ y_2 = y_1 - \dfrac{f\left(y_1\right)}{P_2'\left(y_1\right)}, \\ \qquad \vdots \\ y_{j+1} = y_j - \dfrac{f\left(y_j\right)}{P_{j+1}'\left(y_j\right)}, \quad j = 1, 2, \ldots, n-2 \\ \qquad \vdots \\ x_{k+1} = y_{n-1} - \dfrac{f\left(y_{n-1}\right)}{P_n'\left(y_{n-1}\right)}, \quad k = 0, 1, \ldots \end{cases} \tag{3.3}$$

where $y_0 = x_k$ and $P_l\left(x\right) = \sum\limits_{i=0}^{l} a_i \left(x - y_{l-1}\right)^i$ for $l = 1, 2, \ldots, n$, where P_l are polynomials that satisfy

1. $P_l\left(y_i\right) = f\left(y_i\right)$ con $i = 0, \ldots, l-1$.
2. $P_l{}'\left(y_0\right) = f'\left(y_0\right)$.

We look for an explicit expression for $P_l{}'\left(y_{l-1}\right)$ since this is what we use to obtain the approximation in step l. To simplify the expression, we look for $P_{n+1}'(y_n)$. Thus,

$$P_{n+1}\left(x\right) = a_{n+1}\left(x - y_n\right)^{n+1} + a_n\left(x - y_n\right)^n + \ldots + a_1\left(x - y_n\right) + a_0.$$

From the interpolating conditions, it is easy to deduce that
$P_{n+1}\left(y_n\right) = a_0 = f\left(y_n\right)$. Term a_1 is the one we are interested in, because $P_{n+1}'\left(y_n\right) = a_1$. Denoting $a = \left(a_{n+1}, a_n, \ldots, a_2, a_1, a_0\right)^T$. Thus, to obtain term a_1, we have to solve

$$\begin{pmatrix} \left(y_{n-1} - y_n\right)^{n+1} & \left(y_{n-1} - y_n\right)^n & \ldots & \left(y_{n-1} - y_n\right)^2 & \left(y_{n-1} - y_n\right) & 1 \\ \left(y_{n-2} - y_n\right)^{n+1} & \left(y_{n-2} - y_n\right)^n & \ldots & \left(y_{n-2} - y_n\right)^2 & \left(y_{n-2} - y_n\right) & 1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ \left(y_0 - y_n\right)^{n+1} & \left(y_0 - y_n\right)^n & \ldots & \left(y_0 - y_n\right)^2 & \left(y_0 - y_n\right) & 1 \\ \left(n+1\right)\left(y_0 - y_n\right)^n & n\left(y_0 - y_n\right)^{n-1} & \ldots & 2\left(y_0 - y_n\right) & 1 & 0 \end{pmatrix} a = \begin{pmatrix} f\left(y_{n-1}\right) \\ f\left(y_{n-2}\right) \\ \vdots \\ f\left(y_0\right) \\ f'\left(y_0\right) \end{pmatrix}.$$

The product between the above matrix and the vector of coefficients a is equivalent to

$$\begin{pmatrix} (y_{n-1}-y_n)^{n+1} & (y_{n-1}-y_n)^{n} & \dots & (y_{n-1}-y_n)^{2} & (y_{n-1}-y_n) \\ (y_{n-2}-y_n)^{n+1} & (y_{n-2}-y_n)^{n} & \dots & (y_{n-2}-y_n)^{2} & (y_{n-2}-y_n) \\ \vdots & \vdots & & \vdots & \vdots \\ (y_0-y_n)^{n+1} & (y_0-y_n)^{n} & \dots & (y_0-y_n)^{2} & (y_0-y_n) \\ (n+1)(y_0-y_n)^{n} & n(y_0-y_n)^{n-1} & \dots & 2(y_0-y_n) & 1 \end{pmatrix} \begin{pmatrix} a_{n+1} \\ a_n \\ \vdots \\ a_2 \\ a_1 \end{pmatrix} + \begin{pmatrix} a_0 \\ a_0 \\ \vdots \\ a_0 \\ 0 \end{pmatrix}.$$

If we subtract a_0 from each of the rows, except the last one, from both parts of the equality and then divide each of the rows by its respective term $y_j - y_n$, we obtain that the solution of the previous system is equivalent to the solution of

$$\begin{pmatrix} (y_{n-1}-y_n)^{n} & (y_{n-1}-y_n)^{n-1} & \dots & (y_{n-1}-y_n) & 1 \\ (y_{n-2}-y_n)^{n} & (y_{n-2}-y_n)^{n-1} & \dots & (y_{n-2}-y_n) & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ (y_0-y_n)^{n} & (y_0-y_n)^{n-1} & \dots & (y_0-y_n) & 1 \\ (n+1)(y_0-y_n)^{n} & n(y_0-y_n)^{n-1} & \dots & 2(y_0-y_n) & 1 \end{pmatrix} a' = \begin{pmatrix} \frac{f(y_{n-1})-a_0}{y_{n-1}-y_n} \\ \frac{f(y_{n-2})-a_0}{y_{n-2}-y_n} \\ \vdots \\ \frac{f(y_0)-a_0}{y_0-y_n} \\ f'(y_0) \end{pmatrix},$$

where $a' = (a_{n+1}, a_n, \dots, a_2, a_1)^T$.

Let us remember that the term a_0 is equivalent to $f(y_n)$, so that

$$\begin{pmatrix} (y_{n-1}-y_n)^{n} & (y_{n-1}-y_n)^{n-1} & \dots & (y_{n-1}-y_n) & 1 \\ (y_{n-2}-y_n)^{n} & (y_{n-2}-y_n)^{n-1} & \dots & (y_{n-2}-y_n) & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ (y_0-y_n)^{n} & (y_0-y_n)^{n-1} & \dots & (y_0-y_n) & 1 \\ (n+1)(y_0-y_n)^{n} & n(y_0-y_n)^{n-1} & \dots & 2(y_0-y_n) & 1 \end{pmatrix} \begin{pmatrix} a_{n+1} \\ a_n \\ \vdots \\ a_2 \\ a_1 \end{pmatrix} = \begin{pmatrix} f[y_{n-1}, y_n] \\ f[y_{n-2}, y_n] \\ \vdots \\ f[y_0, y_n] \\ f'(y_0) \end{pmatrix},$$

where $f[y_j, y_n]$ is the first order divided difference of f at points y_j and y_n, that is,

$$f[y_j, y_n] = \frac{f(y_j) - f(y_n)}{y_j - y_n}.$$

If we subtract the penultimate row to the last one and divide the resulting row by $y_0 - y_n$, we obtain the following system to be solved

$$\begin{pmatrix} (y_{n-1}-y_n)^{n} & (y_{n-1}-y_n)^{n-1} & \dots & (y_{n-1}-y_n) & 1 \\ (y_{n-2}-y_n)^{n} & (y_{n-2}-y_n)^{n-1} & \dots & (y_{n-2}-y_n) & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ (y_0-y_n)^{n} & (y_0-y_n)^{n-1} & \dots & (y_0-y_n) & 1 \\ n(y_0-y_n)^{n-1} & (n-1)(y_0-y_n)^{n-2} & \dots & 1 & 0 \end{pmatrix} \begin{pmatrix} a_{n+1} \\ a_n \\ \vdots \\ a_2 \\ a_1 \end{pmatrix} = \begin{pmatrix} f[y_{n-1}, y_n] \\ f[y_{n-2}, y_n] \\ \vdots \\ f[y_0, y_n] \\ \frac{f'(y_0)-f[y_0,y_n]}{y_0-y_n} \end{pmatrix}.$$

The coefficient matrix of the above system is a confluent Vandermonde matrix, so it is invertible and the system can be solved.

We are going to see how to solve a system with the confluent Vandermonde matrix, which can be found in reference [22], explained with more detail. First we define

$$P(x) = (x-b_0)^2 (x-b_1)\dots(x-b_{n-1}),$$

where $b_i = y_i - y_n$, $i = 0, 1, \dots, n-1$, and the confluent matrix of Vandermonde

$$V = \begin{pmatrix} 1 & 0 & 1 & \dots & 1 \\ b_0 & 1 & b_1 & \dots & b_{n-1} \\ b_0^2 & 2b_0 & b_1^2 & \dots & b_{n-1}^2 \\ \vdots & \vdots & & \vdots & \\ b_0^n & nb_0^{n-1} & b_1^n & \dots & b_{n-1}^n \end{pmatrix}.$$

We denote by $P_j(x) = \frac{P(x)}{(x-b_j)^{m_j}}$, where m_j is the maximum exponent of b_j, that is,

$$P_0(x) = \frac{P(x)}{(x-b_0)^2} = \prod_{i=1}^{n-1} (x-b_i),$$

$$P_j(x) = (x-b_0) \prod_{i=0, i\neq j}^{n-1} (x-b_i), \quad j = 1, \dots, n-1.$$

Let us define $g_j(x) = \frac{1}{P_j(x)}$ and

$$L_{j,k_j}(x) = P_j(x) \left(x-b_j\right)^{k_j} \sum_{i=0}^{m_j-k_j-1} \frac{1}{i!} g_j^{(i)}\left(b_j\right)\left(x-b_j\right)^i, \qquad 0 \leq k_j \leq m_j - 1.$$

Then, one has

$$V^{-1} = \begin{pmatrix} L_{0,0}(0) & {L_{0,0}}'(0) & \dots & \frac{1}{(n-1)!} L_{0,0}^{(n-1)}(0) \\ L_{0,1}(0) & {L_{0,1}}'(0) & \dots & \frac{1}{(n-1)!} {L_{0,1}}^{(n-1)}(0) \\ L_{1,0}(0) & {L_{1,0}}'(0) & \dots & \frac{1}{(n-1)!} {L_{1,0}}^{(n-1)}(0) \\ \vdots & \vdots & & \vdots \\ L_{n-1,0}(0) & {L_{n-1,0}}'(0) & \dots & \frac{1}{(n-1)!} {L_{n-1,0}}^{(n-1)}(0) \end{pmatrix}.$$

Since the system uses matrix V^T, then we use the inverse transpose to obtain the values of a_i, $i = 1, \dots, n+1$, and we obtain

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} L_{0,0}(0) & L_{0,1}(0) & L_{1,0}(0) & \dots & L_{n-1,0}(0) \\ {L_{0,0}}'(0) & {L_{0,1}}'(0) & {L_{1,0}}'(0) & \dots & {L_{n-1,0}}'(0) \\ \frac{{L_{0,0}}''(0)}{2!} & \frac{{L_{0,1}}''(0)}{2!} & \frac{{L_{1,0}}''(0)}{2!} & \dots & \frac{{L_{n-1,0}}''(0)}{2!} \\ & & \vdots & & \\ \frac{{L_{0,0}}^{(n-1)}(0)}{(n-1)!} & \frac{{L_{0,1}}^{(n-1)}(0)}{(n-1)!} & \frac{{L_{1,0}}^{(n-1)}(0)}{(n-1)!} & \dots & \frac{{L_{n-1,0}}^{(n-1)}(0)}{(n-1)!} \end{pmatrix} \begin{pmatrix} f[y_0, y_n] \\ \frac{f'(y_0) - f[y_0, y_n]}{y_0 - y_n} \\ f[y_1, y_n] \\ \vdots \\ f[y_{n-1}, y_n] \end{pmatrix}.$$

It follows from the above equality that

$$a_1 = L_{0,1}(0)\,\frac{f'(y_0) - f[y_0, y_n]}{y_0 - y_n} + \sum_{j=0}^{n-1} L_{j,0}(0)\, f[y_j, y_n].$$

Let us now determine $L_{0,1}(0)$ and $L_{j,0}(0)$ for $j = 0, \ldots, n-1$.
We have

$$g_0(x) = \frac{1}{\prod\limits_{i=1}^{n-1}(x - b_i)},$$

$$g_0{}'(x) = -\frac{1}{\prod\limits_{i=1}^{n-1}(x - b_i)} \sum_{i=1}^{n-1} \frac{1}{x - b_i},$$

$$g_j(x) = \frac{1}{(x - b_0)\prod_{i=0, i\neq j}^{n-1}(x - b_i)}.$$

Thus,

$$g_0(b_0) = \frac{1}{\prod\limits_{i=1}^{n-1}(b_0 - b_i)},$$

$$g_0{}'(b_0) = -\frac{1}{\prod\limits_{i=1}^{n-1}(b_0 - b_i)} \sum_{i=1}^{n-1} \frac{1}{b_0 - b_i},$$

$$g_j\left(b_j\right) = \frac{1}{\left(b_j - b_0\right)\prod_{i=0, i\neq j}^{n-1}\left(b_j - b_i\right)}.$$

Since we have

$$\begin{aligned}
L_{0,0}(x) &= P_0(x)\left(g_0(b_0) + g_0{}'(b_0)(x - b_0)\right),\\
L_{0,1}(x) &= P_0(x)(x - b_0)\, g_0(b_0),\\
L_{j,0}(x) &= P_j(x)\, g_j\left(b_j\right),
\end{aligned}$$

then

$$L_{0,0}(0) = (-1)^{n-1} \prod_{i=1}^{n-1} \frac{b_i}{b_0 - b_i} \left(1 + b_0 \sum_{i=1}^{n-1} \frac{1}{b_0 - b_i}\right).$$

Furthermore,

$$L_{0,1}(0) = (-1)^n\, b_0 \prod_{i=1}^{n-1} \frac{b_i}{b_0 - b_i}.$$

Finally, we calculate $L_{j,0}(0)$ for $j = 1, \dots, n-1$,

$$L_{j,0}(0) = (-1)^n \frac{b_0}{b_j - b_0} \prod_{i=0, i\neq j}^{n-1} \frac{b_i}{b_j - b_i}.$$

Thus, if the terms are grouped properly, we obtain

$$\begin{aligned} a_1 = &(-1)^n \prod_{i=1}^{n-1} \frac{b_i}{b_0 - b_i} \left(f'(y_0) - \left(2 + b_0 \sum_{i=1}^{n-1} \frac{1}{b_0 - b_i} \right) f[y_0, y_n] \right) \\ &+ \sum_{j=1}^{n-1} (-1)^n \frac{b_0}{b_j - b_0} \prod_{i=0, i\neq j}^{n-1} \frac{b_i}{b_j - b_i} f[y_j, y_n]. \end{aligned}$$

Thus, the term a_1 is explicit in each step.
We comment on some of the methods obtained from this family. The first one is the method obtained by performing two steps

$$\begin{cases} y_1 = x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ x_{k+1} = y_1 - \dfrac{f(y_1)}{2f[x_k, y_1] - f'(x_k)}, \end{cases} \tag{3.4}$$

Let us remember that this method has order 4, since it is Ostrowski's method, [1]. Method (3.4) is denoted by $M4$.
Another of the methods belonging to this family is the following three-step method

$$\begin{cases} y_1 = x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ y_2 = y_1 - \dfrac{f(y_1)}{2f[x_k, y_1] - f'(x_k)}, \\ x_{k+1} = y_2 - \dfrac{f(y_2)}{a_1}, \end{cases} \tag{3.5}$$

where

$$a_1 = \frac{f[y_1, y_2](x_k - y_2)^2 + (y_1 - y_2)\left(f'(x_k)(x_k - y_1) + f[x_k, y_2](-3x_k + 2y_1 + y_2) \right)}{(x_k - y_1)^2}.$$

Next, we prove that this method has order 8. We denote method (3.5) by $M8$ to simplify the notation.

3.3 Convergence analysis

Let $f : D \to \mathbb{R}$ be a sufficiently differentiable function on an interval $D \subset \mathbb{R}$ containing α, solution of the nonlinear equation $f(x) = 0$. We consider the divided difference operator

$$f[x+h,x] = \int_0^1 f'(x+th)\,dt, \tag{3.6}$$

defined by Genochi-Hermite in [3]. Using Taylor's development of $f'(x+th)$ around the point x and integrating, we obtain

$$f[x+h,x] = f'(x) + \frac{1}{2}f''(x)\,h + \frac{1}{6}f'''(x)\,h^2 + O\left(h^3\right), \tag{3.7}$$

which we use to prove the following result, where it is deduced that the order of the n-step method defined in (3.3) is 2^n.

Theorem 3.3.1. *Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a sufficiently differentiable function on an interval D satisfying $\alpha \in D$, such that $f(\alpha) = 0$. We assume that $f'(\alpha) \neq 0$. Then, taking an estimation x_0 close enough to α, the sequence of iterates $\{x_k\}_{k\geq 0}$, generated by the n-step method (3.3) converges to α with order 2^n.*

Proof. We perform this proof by induction on the number of steps.
The method of one step defined in (3.3) is Newton's scheme, so we know that the method of one step has order of convergence 2.
Suppose that any method of i steps has order 2^i when $i \leq j$. Let us proof that the $j+1$-step method has order 2^{j+1}.
We denote by $y_0 = x_k$ and y_i defined in (3.3). We note that

$$P_{j+1}(x) = P_j(x) + \left(f\left(y_j\right) - P_j\left(y_j\right)\right)\left(\prod_{i=0}^{j-1} \frac{x - y_i}{y_j - y_i}\right)\frac{x - y_0}{y_j - y_0}.$$

Moreover, there exists an $\epsilon \in D$ that satifies

$$f(x) - P_{j+1}(x) = \frac{f^{(j+2)}(\epsilon)}{(j+2)!}\left(x - y_j\right)\cdots\left(x - y_1\right)\left(x - y_0\right)^2,$$

from this expression we deduce that

$$f'(x) - P'_{j+1}(x) = \frac{f^{(j+2)}(\epsilon)}{(j+2)!}\left(\left(x - y_0\right)\left(\sum_{r=0}^{j}\prod_{i=0,i\neq r}^{j}\left(x - y_i\right)\right) + 2\prod_{i=0}^{j}\left(x - y_i\right)\right).$$

Evaluating the above expression in y_j, we obtain

$$f'\left(y_j\right) - P'_{j+1}\left(y_j\right) = \frac{f^{(j+2)}(\alpha)}{(j+2)!}\left(y_j - y_0\right)\prod_{i=0}^{j-1}\left(y_j - y_i\right).$$

Since $C_i = \frac{1}{i!}\frac{f^{(i)}(\alpha)}{f'(\alpha)}$, one has that

$$P'_{j+1}\left(y_j\right) = f'\left(y_j\right) - C_{j+2}f'\left(\alpha\right)\left(y_j - y_0\right)\prod_{i=0}^{j-1}\left(y_j - y_i\right).$$

Let us consider Taylor's development of $f\left(y_i\right)$ around α

$$f\left(y_i\right) = f'\left(\alpha\right)\left(y_i - \alpha + C_2\left(y_i-\alpha\right)^2 + O\left(\left(y_i-\alpha\right)^3\right)\right). \tag{3.8}$$

Then, the development of $f'\left(y_i\right)$ around α has the following expression

$$f'\left(y_i\right) = f'\left(\alpha\right)\left(1 + 2C_2\left(y_i - \alpha\right)\right) + O\left(\left(y_i-\alpha\right)^2\right). \tag{3.9}$$

Substituting (3.9) into $P_{j+1}(y_j)$, it follows that

$$P'_{j+1}\left(y_j\right) = f'\left(\alpha\right)\left(1+2C_2\left(y_j-\alpha\right)\right) + O\left(\left(y_j-\alpha\right)^2\right) - C_{j+2}f'\left(\alpha\right)\left(y_j-y_0\right)\prod_{i=0}^{j-1}\left(y_j-y_i\right).$$

Assuming that the i-step method has order of convergence 2^i for $i \leq j$, that is, $y_i - \alpha = M_i\left(y_0-\alpha\right)^{2^i} + O\left(\left(y_0-\alpha\right)^{2^i+1}\right)$ and denoting by $e_k = y_0 - \alpha$, we obtain

$$y_i - \alpha = M_i e_k^{2^i} + O\left(e_k^{2^i+1}\right).$$

We now calculate the expression that $\prod\limits_{i=0}^{j-1}\left(y_j - y_i\right)$ has from the previous result. Since

$$y_j - y_i = \left(y_j-\alpha\right) - \left(y_i-\alpha\right) = M_j e_k^{2^j} - M_i e_k^{2^i} + O\left(e_k^{2^i+1}\right) = -M_i e_k^{2^i} + O\left(e_k^{2^i+1}\right),$$

is given, then it follows from the above expression that

$$\begin{aligned}\prod_{i=0}^{j-1}\left(y_j-y_i\right) &= \prod_{i=0}^{j-1}\left(-M_i e_k^{2^i} + O\left(e_k^{2^i+1}\right)\right)\\ &= \left(\prod_{i=0}^{j-1} -M_i\right) e_k^{\sum\limits_{i=0}^{j-1} 2^i} + O\left(e_k^{\sum\limits_{i=0}^{j-1} 2^i+1}\right)\\ &= (-1)^j\, e_k^{2^j-1}\prod_{i=0}^{j-1} M_i + O\left(e_k^{2^j}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}P'_{j+1}\left(y_j\right) =& f'\left(\alpha\right)\left(1+2C_2\left(y_j-\alpha\right)\right) + O\left(\left(y_j-\alpha\right)^2\right)\\ &- C_{j+2}f'\left(\alpha\right)\left(y_j-y_0\right)\left((-1)^j\, e_k^{2^j-1}\prod_{i=0}^{j-1} M_i + O\left(e_k^{2^j}\right)\right).\end{aligned}$$

As it happens that $y_j - y_0 = -e_k + O\left(e_k^2\right)$ and $y_j - \alpha = M_j e_k^{2^j} + O\left(e_k^{2^j+1}\right)$, then

$$\begin{aligned}
P'_{j+1}\left(y_j\right) &= f'\left(\alpha\right)\left(1 + 2C_2 M_j e_k^{2^j}\right) - C_{j+2} f'\left(\alpha\right)\left(-1\right)^{j+1} e_k^{2^j} \prod_{i=0}^{j-1} M_i + O\left(e_k^{2^j+1}\right) \\
&= f'\left(\alpha\right)\left(1 + e_k^{2^j}\left(2C_2 M_j + C_{j+2}\left(-1\right)^j \prod_{i=0}^{j-1} M_i\right)\right) + O\left(e_k^{2^j+1}\right).
\end{aligned}$$

As $y_{j+1} - \alpha = y_j - \alpha - \frac{f(y_j)}{P'_{j+1}(y_j)}$, the Taylor development of $\frac{f(y_j)}{P'_{j+1}(y_j)}$ is

$$\begin{aligned}
\frac{f\left(y_j\right)}{P'_{j+1}\left(y_j\right)} &= \frac{\left(y_j - \alpha + C_2\left(y_j - \alpha\right)^2 + O\left(\left(y_j - \alpha\right)^3\right)\right)}{\left(1 + e_k^{2^j}\left(2C_2 M_j + C_{j+2}\left(-1\right)^j \prod_{i=0}^{j-1} M_i\right)\right) + O\left(e_k^{2^j+1}\right)} \\
&= \frac{\left(y_j - \alpha\right) + C_2 M_j^2 e_k^{2^{j+1}} + O\left(e_k^{2^{j+1}+1}\right)}{1 + e_k^{2^j}\left(2C_2 M_j + C_{j+2}\left(-1\right)^j \prod_{i=0}^{j-1} M_i\right) + O\left(e_k^{2^j+1}\right)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
y_{j+1} - \alpha &= y_j - \alpha - \frac{f\left(y_j\right)}{P'_{j+1}\left(y_j\right)} \\
&= y_j - \alpha - \frac{\left(y_j - \alpha\right) + C_2 M_j^2 e_k^{2^{j+1}} + O\left(e_k^{2^{j+1}+1}\right)}{1 + e_k^{2^j}\left(2C_2 M_j + C_{j+2}\left(-1\right)^j \prod_{i=0}^{j-1} M_i\right) + O\left(e_k^{2^j+1}\right)} \\
&= \frac{\left(e_k^{2^{j+1}}\left(2C_2 M_j^2 + C_{j+2}\left(-1\right)^j \prod_{i=0}^{j} M_i\right)\right) - C_2 M_j^2 e_k^{2^{j+1}} + O\left(e_k^{2^{j+1}+1}\right)}{1 + e_k^{2^j}\left(2C_2 M_j + C_{j+2}\left(-1\right)^j \prod_{i=0}^{j-1} M_i\right) + O\left(e_k^{2^j+1}\right)} \\
&= \frac{e_k^{2^{j+1}}\left(C_2 M_j^2 + C_{j+2}\left(-1\right)^j \prod_{i=0}^{j} M_i\right) + O\left(e_k^{2^{j+1}+1}\right)}{1 + e_k^{2^j}\left(2C_2 M_j + C_{j+2}\left(-1\right)^j \prod_{i=0}^{j-1} M_i\right) + O\left(e_k^{2^j+1}\right)}.
\end{aligned}$$

From the above expression we obtain

$$y_{j+1} - \alpha = e_k^{2^{j+1}}\left(C_2 M_j^2 + C_{j+2}\left(-1\right)^j \prod_{i=0}^{j} M_i\right) + O\left(e_k^{2^{j+1}+1}\right).$$

Therefore, it is proven that the method of $j+1$ steps (3.3) has order of convergence 2^{j+1} and, thus, by induction, the proposed method of n steps has order of convergence 2^n. $\square$

According to the Kung and Traub conjecture, defined in [7], for an iterative method without memory that performs d distinct functional evaluations in each iteration to be optimal, it must be satisfied that 2^{d-1} coincides with the order of the method.
In this case, the n-step method (3.3) performs $n+1$ functional evaluations, since we perform the derivative of f at y_0 and also the image of f at the approximations $y_0, y_1, \ldots, y_{n-1}$. For this reason, we have that the proposed family of methods (3.3) is optimal because $2^n = 2^{n+1-1}$.
Thus, we have a family of optimal methods that is based on Newton's composition. We now analyse the stability of the family members using tools from complex dynamics.

3.4 Complex dynamics

When analyzing an iterative method, convergence order is not the only important criterion. A method's validity is also determined by other aspects such as the behaviour of the initial estimations, which is why it is necessary to introduce tools that allow a more thorough investigation. In the study of iterative methods, the dynamical analysis is becoming one of the most studied parts. It allows us to classify iterative schemes, based not only on their speed of convergence, also analysing their behaviour based on the initial estimation taken. Both analytical and graphical aspects permit to analyse the behaviour of the method and to visualise the solutions and the convergence regions. Moreover, it provides important information on the stability and reliability of the iterative method. We focus on studying the dynamics of methods $M4$ and $M8$ of the proposed family (3.3), defined in (3.4) and (3.5), respectively.
To compare the results with methods of similar order, it is necessary to focus on the elements of the 4-th and 8-th order family. We start by introducing the Cayley test, see [23, 24].

Theorem 3.4.1. *(Cayley quadratic test (CQT)). Let $Op\left(z\right)$ be the rational operator obtained from a general iterative scheme applied to a quadratic polynomial $q\left(z\right) = \left(z-\alpha_1\right)\left(z-\alpha_2\right)$, with $\alpha_1 \neq \alpha_2$. Assume that $Op\left(z\right)$ is conjugated to the operator $z \to z^p$, by a Möbius transformation $M\left(z\right) = \frac{z-\alpha_1}{z-\alpha_2}$.*

By calculating the rational operators of $M4$ and $M8$ on a quadratic polynomial, we can easily check that these operators satisfy the quadratic Cayley test, that is, that the operator associated to each of them can be transformed by means of a Möbius transformation into the operator z^p, where p is the order of the method.
Therefore, it follows that they are stable methods, where the only superattracting fixed points are the roots of the quadratic polynomial. If there are strange fixed points they have a repulsor character and there are no free critical points in a basin of attraction different from those of the roots. Moreover, by satisfying Cayley's quadratic test we also know that the Julia set of the transformed operator on a quadratic polynomial is a straight line which separates the Riemann sphere into two semiplanes.

In the following, the dynamical planes of the $M4$ and $M8$ methods applied on some nonlinear equations are compared with the dynamical planes of other methods of order 4 and 8. We define below the methods used for this comparative study.

On the one hand, we compare the proposed methods with Newton's method, denoted by N, from which we have generated our methods. The two methods of order 4 with which we compare the proposed methods are Jarratt's method, which can be found in [9], which is defined as follows

$$\begin{cases} y_1 = x_k - \dfrac{2}{3}\dfrac{f(x_k)}{f'(x_k)}, \\ x_{k+1} = y_1 - \dfrac{1}{2}\dfrac{f(x_k)}{f'(x_k)}\dfrac{3f'(y_1)+f'(x_k)}{3f'(y_1)-f'(x_k)}, \quad k=0,1,2\dots \end{cases} \tag{3.10}$$

and the King's family method for $\beta = 1$, which can be found in [8], which is defined as follows

$$\begin{cases} y_1 = x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ x_{k+1} = y_1 - \dfrac{f(y_1)}{f'(x_k)}\dfrac{f(x_k)+f(y_1)}{f(x_k)-f(y_1)}, \quad k=0,1,2\dots \end{cases} \tag{3.11}$$

Methods (3.10) and (3.11) are denoted by $J4$ and $K4$, respectively.

The 8.th order schemes with which we compare the proposed method are the scheme, denoted by $J8$, defined in [25], which is defined as follows

$$\begin{cases} y_1 = x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ y_2 = x_k - \dfrac{1}{8}\dfrac{f(x_k)}{f'(x_k)} - \dfrac{3}{8}\dfrac{f(x_k)}{f'(y_1)}, \\ y_3 = x_k - 6\dfrac{f(x_k)}{f'(x_k)+f'(y_1)+4f'(y_2)}, \\ x_{k+1} = y_3 - \dfrac{f(y_3)}{f'(x_k)}\dfrac{f'(x_k)+f'(y_1)-f'(y_2)}{2f'(y_1)-f'(y_2)}, \quad k=0,1,2\dots \end{cases} \tag{3.12}$$

and also the method, which we denote by $K8$, that can be found in [26], which is deduced from $K4$

$$\begin{cases} y_1 = x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ y_2 = y_1 - \dfrac{f(y_1)}{f'(x_k)}\dfrac{f(x_k)+f(y_1)}{f(x_k)-f(y_1)}, \\ x_{k+1} = y_2 - \dfrac{H_3(y_2)}{f'(y_2)}, \quad k=0,1,2\dots \end{cases} \tag{3.13}$$

where

$$\begin{aligned} H_3(y_2) =& f(x_k) + f'(x_k)\frac{(y_2-y_1)^2(y_2-x_k)}{(y_1-x_k)(x_k+2y_1-3y_2)} + f'(y_2)\frac{(y_2-y_1)(x_k-y_2)}{x_k+2y_1-3y_2} \\ &- f[x_k,y_1]\frac{(y_2-x_k)^3}{(y_1-x_k)(x_k+2y_1-3y_2)}. \end{aligned}$$

We define now the equations we have chosen to compare the dynamical planes. These are the same equations that we are going to use to carry out the numerical experiments.

1. $f_1(x) = \cos(x) - x = 0$, whose root is $\alpha \approx 0.739085133215$.
2. $f_2(x) = (x-1)^6 - 1 = 0$, whose roots are $\alpha_1 = 2$, $\alpha_2 = 0$, $\alpha_3 = \frac{3}{2} + \frac{\sqrt{3}}{2}i$, $\alpha_4 = \frac{3}{2} - \frac{\sqrt{3}}{2}i$, $\alpha_5 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\alpha_6 = \frac{1}{2} - \frac{\sqrt{3}}{2}i$.
3. $f_3(x) = \arctan(x) = 0$, whose root is $\alpha = 0$.
4. $f_4(x) = \arctan(x) - \frac{2x}{x^2+1} = 0$, whose roots are $\alpha_1 = 0$. $\alpha_2 \approx 1.3917452$ and $\alpha_3 \approx -1.3917452$.

The algorithms used to generate the dynamical planes are similar to those that can be found in [14]. To implement the dynamical planes, we denote by z a complex initial estimation. Every point z in the plane is considered as the starting point of the iterative method, and is represented in different colours according to the point to which it converges. These dynamical planes shown in this section were generated with a grid of 400×400 points with a maximum of 80 iterations per point. The fixed points are represented as white circles, and the superattracting fixed points are identified by a white star.

We begin by analysing the dynamical planes for equation $\cos(x) - x = 0$, shown in Figure 3.1, for the different methods discussed above.

We represent in orange the points of the plane that converge to the root 0.73908513. In blue are represented the points that tend to infinity, which are determined as the points whose absolute value is greater than 800, and in black the points that do not converge to α before reaching the maximum number of iterations. For this equation, all methods have only one fixed point. The dynamical planes with wider bassins of attraction are those associated with the $K4$, $K8$ and $J8$ methods, (3.1c,3.1f,3.1g).

We comment on what happens in the case of the equation $(x-1)^6 - 1 = 0$. Figure 3.2 shows the dynamical planes associated with this equation.

In this case, there are more noticeable changes between the different dynamical planes. We associate the roots of the equation, which are superattracting fixed points, with one of the colours represented in the planes, except for black, which is associated with the initial points that do not converge to any root, and blue, which is associated with the initial estimations whose iteration in absolute value is greater than 800.

The centres of the images are points of slow convergence to the attracting fixed points. By increasing the maximum number of iterations, the non-convergence areas at the centre of the plane would decrease, and if we increase the infinity limit, the area of convergence to infinity would decrease. It can be seen that the planes that stand out most for their non-convergence zones in the centre are those associated with the $K4$ and $K8$ methods, as can be seen in Figures

3.2c and 3.2f, which is why this methods would not be recommended to be used for initial estimations in this zone.

On the other hand, we see that Newton's method has a larger non-convergence area than the methods derived from Jarratt or the methods proposed in the paper.

As can be seen in the dynamical planes associated to the equation $(x-1)^6 - 1 = 0$, we have not studied the strange repelling fixed points, since the complexity of some methods requires a high computational cost to calculate these points, but these repelling fixed points will be on the Julia set.

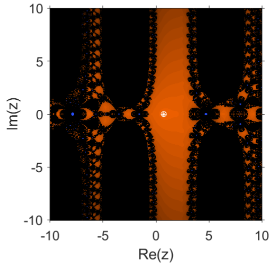
In conclusion, from these images it is clear that it is more efficient to use $M8$ versus $K4$ or $K8$ in this example.

We discuss the dynamical planes associated to the equation $\arctan(x) = 0$ represented in Figure 3.3. The initial estimations that converge to the solution 0 are represented in orange and the initial estimations that converge to ∞ are represented in blue, also with the same criteria as previously.
Here we observe that most of the planes are similar, although some of them have strange repulsive fixed points and some of them do not. We observe that in the planes associated with methods $M4$ and $M8$, the basin of attraction for the solution $\alpha = 0$ is much larger than in the rest of the cases, for this reason it is more convenient in this case to select one of these two methods since we have a larger number of initial estimations that converge to the solution we are looking for.

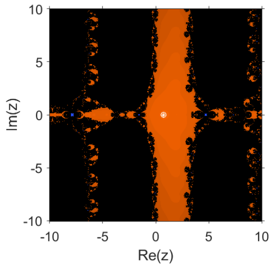
We analyse the dynamical planes associated with the equation $\arctan(x) - \frac{2x}{x^2+1} = 0$ which are shown in Figure 3.4. Here the initial estimations that converge to the solution 0 are represented in purple, the initial estimations that converge to the approximate solution -1.3917452 are represented in green, the initial estimations that converge to 1.3917452 are represented in orange and are represented in blue the initial estimations whose iteration in absolute value is greater than 800.
We observe in this case that most of the planes are similar, except for those associated with the $M4$, $J4$ and $M8$ methods, but among these three it can be seen that the one with the largest convergence area to ∞ would be the $J4$ method, so it would be less advisable to use this method.

Figure 3.1: Dynamical planes for $\cos(x) - x = 0$

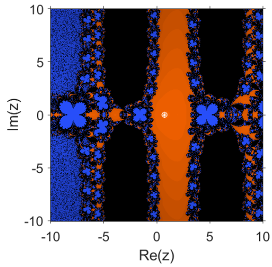
(a) N



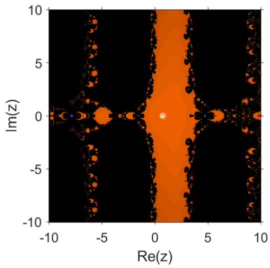
(b) $M4$



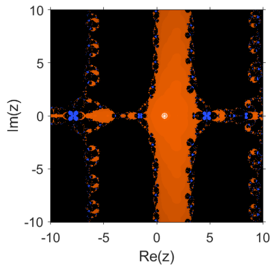
(c) $K4$



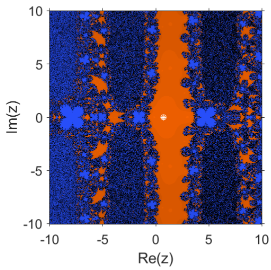
(d) $J4$



(e) $M8$



(f) $K8$



(g) $J8$

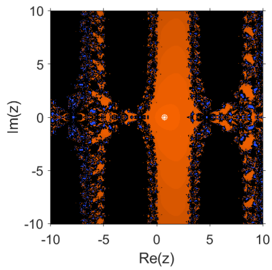
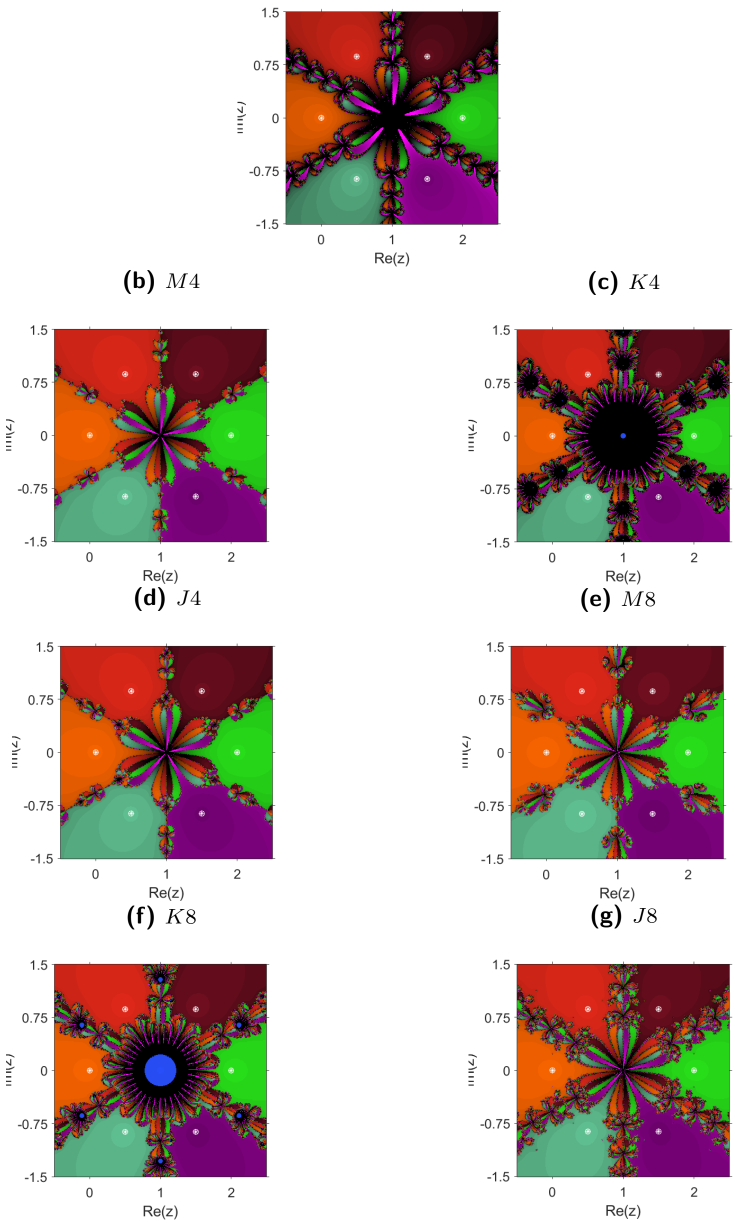


Figure 3.2: Dynamical planes for $(x-1)^6-1=0$



(a) N

(b) $M4$

(c) $K4$

(d) $J4$

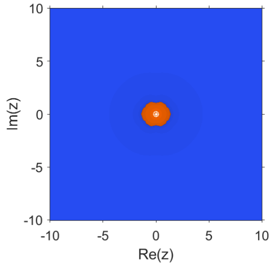
(e) $M8$

(f) $K8$

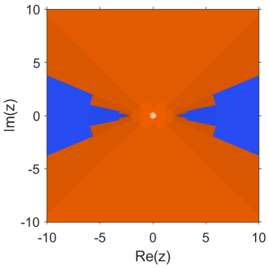
(g) $J8$

Figure 3.3: Dynamical planes for $\arctan(x) = 0$

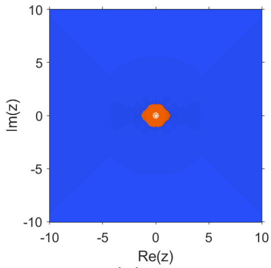
(a) N



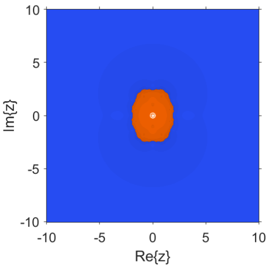
(b) $M4$



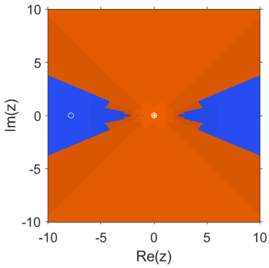
(c) $K4$



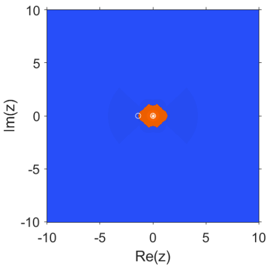
(d) $J4$



(e) $M8$



(f) $K8$



(g) $J8$

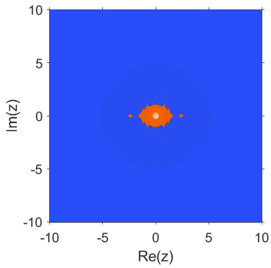
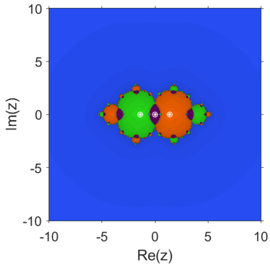
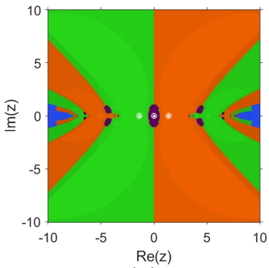


Figure 3.4: Dynamical planes for $\arctan(x) - \frac{2x}{x^2+1} = 0$

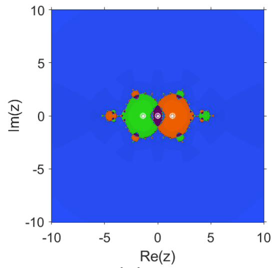
(a) N



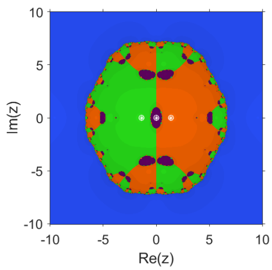
(b) $M4$



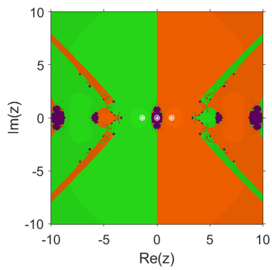
(c) $K4$



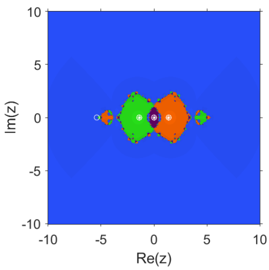
(d) $J4$



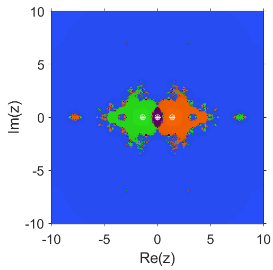
(e) $M8$



(f) $K8$



(g) $J8$



We note that, for these examples, methods $M4$ and $M8$ methods have been among the most prominent methods in the different nonlinear equations.

3.5 Numerical Experiments

In this section, we are going to solve some nonlinear equations to compare our proposed methods of order 4 and order 8 with the same known methods of order 4 and order 8 that we compared our methods with in Section 3.4.
We use Matlab R2020b with variable precision arithmetics with 1000 digits for the computational calculations, iterating from an initial estimation x_0 until the following stopping criterion is satisfied

$$|x_{k+1} - x_k| + |f(x_{k+1})| < 10^{-100}.$$

The numerical results we are going to compare the methods in these examples are as follows

- the approximation to the solution obtained,
- the absolute value of the nonlinear function evaluated in that approximation (which we denote by $|f(x_{k+1})|$ in the tables),
- the absolute value of the distance between the last two approximations (which we denote by $|x_{k+1} - x_k|$ in the tables),
- the number of iterations needed to satisfy the required tolerance (which we denote by Iteration in the tables),
- the computational time (which we denote by Time in the tables),
- and the approximate computational order of convergence (ACOC).

The equations we use, which coincide with those used in the previous section, are as follows

- Function $\cos(x) - x$, which has a root $\alpha \approx 0.73908513$. We take as an initial estimation for all methods $x_0 = 1$.
- Function $(x-1)^6 - 1$, which has a root $\alpha = 2$. We take as an initial estimation for all methods $x_0 = 1.5$.
- Function $\arctan(x)$, which has a root $\alpha = 0$. We take as an initial estimation for all methods $x_0 = 1.5$.
- Function $\arctan(x) - \frac{2x}{x^2+1}$, which has a root $\alpha = 0$. We take as an initial estimation for all methods $x_0 = 0.4$.

Let us consider the results, which have been obtained by solving the equation $\cos(x) - x = 0$ which are shown in Table 3.1.

Not surprisingly, Newton's method performs more iterations than the others because it has the lowest order of convergence and, for this reason, also requires more computational time.
Among the methods of order 4, similar results can be seen in terms of number of iterations, computational time and the value of the function at the last iteration.
Analysing the results obtained by the 8-th order methods, it can be seen that they perform the same number of iterations, although in this case the $M8$ method has less computational time, as happens with the results of the third column, that is, in this case $M8$ method give us better results than methods $J8$ and $K8$.
In conclusion, it is obtained that all the methods give similar results, although it would be advisable to use $M8$ method in this case, as it is the one that requires the least computational time, is one of the methods with the fewest iterations to satisfy the stopping criterion and the one that obtains the highest order, and is by far the method whose distance between the last iterations is the smallest, as can be seen in the second column of Table 3.1.

Table 3.1: Results for the function $\cos(x) - x$

Method	$\lvert x_{k+1} - x_k \rvert$	$\lvert f(x_{k+1}) \rvert$	Iteration	ACOC	Time
N	7.11815$\times 10^{-167}$	1.8724$\times 10^{-333}$	8	2	0.2856
$M4$	4.21403$\times 10^{-296}$	1.41767$\times 10^{-1008}$	5	4	0.1984
$K4$	1.90125$\times 10^{-279}$	1.41767$\times 10^{-1008}$	5	4	0.1922
$J4$	1.6318$\times 10^{-299}$	1.41767$\times 10^{-1008}$	5	4	0.1906
$M8$	5.27514$\times 10^{-640}$	1.41767$\times 10^{-1008}$	4	8	0.1875
$K8$	9.74433$\times 10^{-270}$	9.92368$\times 10^{-1008}$	4	6	0.2344
$J8$	6.51848$\times 10^{-608}$	1.41767$\times 10^{-1008}$	4	8	0.2031

Let us consider the equation $(x-1)^6 - 1 = 0$ whose results are shown in the Table 3.2. Not all methods converge to the solution in this case, since $K4$ and $K8$ methods do not converge considering $x_0 = 1.5$ as the initial estimation, as can be seen in the dynamical planes associated with these methods for equation $(x-1)^6 - 1 = 0$ (Figure 3.2).
We can appreciate differences between Newton's method and the rest of the converging methods, since the number of iterations has grown considerably, as has the computational time with respect to the previous case.
Among the methods of order 4, we can see that the results are similar in all aspects.
Observing the results of the 8-th order methods, it can be seen that they perform the same number of iterations, but the $M8$ method has a shorter computational time. The most notable feature of this table is that the value of the third column of $M8$ method is smaller than in the other methods, and considerably smaller than in the case of $J8$ method.
As a conclusion of this numerical experiment we obtain that between the converging methods we have similar results in most cases, although we emphasise $M8$ method, thanks to the fact that it performs fewer iterations, obtains higher order and, as can be seen in the second column of Table 3.2, the distance between iterations is smaller.

Table 3.2: Results for the function $(x-1)^6-1$

Method	$\lvert x_{k+1}-x_k \rvert$	$\lvert f(x_{k+1}) \rvert$	Iteration	ACOC	Time
N	2.72448×10^{-119}	1.11342×10^{-236}	19	2	0.5234
$M4$	2.83553×10^{-271}	0	9	4	0.3250
$K4$	n.c.	n.c.	n.c.	n.c.	n.c.
$J4$	2.02789×10^{-263}	0	9	4	0.3328
$M8$	3.7096×10^{-468}	0	7	8	0.3281
$K8$	n.c.	n.c.	n.c.	n.c.	n.c.
$J8$	3.10018×10^{-130}	0	7	7.9992	0.3641

Now, we comment on the results that have been obtained by solving the equation $\arctan(x)=0$ which are shown in Table 3.3. Not all methods converge to the solution in this case, since Newton, $K4$ and $K8$ methods do not converge considering as initial estimation $x_0=1.5$, as can be seen in the dynamical planes associated to these methods for the equation $\arctan(x)=0$.
Among the 4-th order methods, we can see that the results for the number of iterations, computational time and the value of the function at the last iteration are similar.
By comparing the results of the 8-th order methods, we can see that $M8$ method performs fewer iterations and also has the shortest computational time. Also, the value of the ACOC of the $M8$ method has increase to 11 instead of 8, although it is true that $J8$ method also increases to 9 and has the smallest value of the absolute value of the equation evaluated in the last iteration, although by performing one more iteration than the $M8$ method it is reasonable that this happens.
In conclusion, we obtain similar results between the converging methods, although we would like to highlight methods $M4$ and $M8$ due to the dynamical planes associated with both methods.

Table 3.3: Results for the function $\arctan(x)$

Method	$\lvert x_{k+1}-x_k \rvert$	$\lvert f(x_{k+1}) \rvert$	Iteration	ACOC	Time
N	n.c.	n.c.	n.c.	n.c.	n.c.
$M4$	2.55693×10^{-252}	2.42873×10^{-1259}	6	5	0.2250
$K4$	n.c.	n.c.	n.c.	n.c.	n.c.
$J4$	7.27099×10^{-263}	1.80085×10^{-1270}	6	5	0.2594
$M8$	5.654×10^{-126}	1.11859×10^{-1379}	4	10.9979	0.1891
$K8$	n.c.	n.c.	n.c.	n.c.	n.c.
$J8$	1.98863×10^{-777}	0	5	9	0.2672

The results that have been obtained by solving the equation $\arctan(x)-\frac{2x}{x^2+1}=0$ are shown in Table 3.4. As we can see, Newton's method is the one that performs the most iterations to reach the same tolerance, although in this example the ACOC increases by one unit.

Similarly to the previous problem, examining the results obtained by the methods of order 4, we can see that in all of them the value of the ACOC is 5, when the expected value would be 4.

Comparing the results of schemes $M4$ and $J4$, we see that the results are similar although $M4$ method obtains a better approximation in this case with the same number of iterations, and also takes less computational time, so that for this example, the most convenient method of order 4 would be $M4$ method.

We observe in the results obtained for the 8 order methods that method $K8$ performs more iterations than the rest of the methods, as well as being one of the methods that requires more computational time. The results of the $M8$ method show that it requires less computational time, although what stands out most in this table is the value of the ACOC of the $M8$ method, since for this case it increases to 11 instead of 8, although it is true that methods $J8$ and $K8$ also increase to 9. We also see that $M8$ method obtains a more accurate approximation than the rest of the 8-th order methods.

As a conclusion of this numerical experiment, we obtain that there are notable differences between the methods, the most convenient methods we emphasise for their results, and for their dynamical planes, methods $M4$ and $M8$, but especially $M8$ method.

Table 3.4: Results for the function $\arctan(x) - \frac{2x}{x^2+1}$

Method	$\lvert x_{k+1} - x_k \rvert$	$\lvert f(x_{k+1}) \rvert$	Iteration	ACOC	Time
N	9.24306×10^{-282}	2.63224×10^{-843}	14	3	0.5891
$M4$	4.96455×10^{-427}	1.67544×10^{-2131}	6	5	0.3359
$K4$	9.73169×10^{-441}	4.84918×10^{-2200}	11	5	0.6359
$J4$	9.05734×10^{-438}	2.27536×10^{-1445}	6	5	0.4297
$M8$	1.33304×10^{-219}	6.46592×10^{-2102}	4	11	0.2797
$K8$	2.25726×10^{-269}	3.25561×10^{-1343}	7	9	0.5328
$J8$	4.84986×10^{-136}	5.74708×10^{-1218}	4	9.00019	0.3141

After performing these experiments we conclude that the most recommendable methods in these cases are $M4$ and $M8$ because they are the only ones, together with $J4$, that converged to the solution in all the cases, as well as the ones that performed remarkably well for their numerical results in all the examples, although the one that stands out most is the $M8$ method.

3.6 Conclusions

In this chapter, we have designed a family of optimal iterative methods with n steps and convergence order 2^n for $n = 1, 2, ...$ From this class, we are able to select infinitely optimal iterative schemes with the desired order of convergence. Also, some classical methods are members of this family such as Newton's and Ostrowski's method. We worked, from the dynamical and numerical point of view, with the elements of this family of orders 4 and 8, comparing the results obtained with those of other known methods. The results provided by the two elements of the family are very satisfying, both numerically (number of iterations, error bounds, etc.) and in terms of stability (width of the convergence basins, existence of strange fixed points, etc.).

Chapter 4

Derivative-free iterative methods for solving nonlinear equations

Based on [Cordero, A.; Garrido, N.; Torregrosa, JR.; Triguero-Navarro, P. (2023). Memory in the iterative processes for non-linear problems. Mathematical Methods in Applied Science 46 (4), 4145- 4158. https://doi.org/10.1002/mma.8746] and on [Cordero, A.; Garrido, N.; Torregrosa, JR.; Triguero-Navarro, P. (2023). Three-step iterative weight function scheme with memory for solving nonlinear problems. Mathematical Methods in Applied Science. Accepted]

4.1 Introduction

As mentioned above in Chapter 3, one of the most frequently used methods for solving nonlinear equations is Newton's method (3.1). Based on Newton's scheme, a large number of other iterative methods have emerged. A well-known one is Traub's procedure, which is structured as

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ x_{k+1} = y_k - \dfrac{f(y_k)}{f'(x_k)}, \quad k = 0, 1, 2, \ldots \end{cases} \tag{4.1}$$

When we use Newton's or Traub's scheme, one of the problems we can encounter is not being able to obtain the derivative of the function f. For this reason, derivative-free methods have arisen.

When the derivative in Newton's scheme is replaced by the divided difference $f[x_k + f(x_k), x_k]$, we obtain the Steffensen's method, see [11], which is a derivative-free and also optimal scheme. Many other optimal schemes appearing in the literature can be found, for example, [27] (Kumar et al) and [28] (Cordero et al) the references therein. The existence of derivatives in the iterative expression of a method can be a drawback when the function to be studied cannot be derived or its derivative is too costly to calculate. For this reason, derivative-free methods have arisen in the literature; see, for example, (Chun and Neta) [26] and (Kumar et al) [29].

In this chapter, we modify Traub's scheme to obtain a derivative-free method, and to increase the order of the method and obtain a parametric family of optimal schemes, we introduce a parameter and a weight function.
We can generate other procedures by modifying the original procedure in several ways. One of them is to modify the schemes that use derivatives in their iterative expression in order to obtain derivative-free methods, as we have already mentioned, which is what we are going to do in this chapter. Another way is to introduce memory to the iterative scheme. When we introduce memory in a procedure, the aim is to improve the order of convergence of the original procedure without introducing new functional evaluations. One way to obtain iterative schemes with memory is to start from a parametric family of iterative schemes without memory and replace the parameter, depending on the error equation of the iterative family, by an expression that is a combination of the iterations and functional evaluations already performed. In this chapter, we propose different approximations for the proposed parametric family and study the order of convergence for the obtained methods with memory.

Not only is it important to study the convergence of a scheme, also it is important to know how the method behave according to an initial estimation. This is why it is important to analyse the dynamical behaviour of iterative procedures. It allows to visualise graphically the neighbourhood of initial approximations that converge to a given root of the equation. It provides information on the stability and reliability of the iterative method. Usually, this analysis is performed by applying complex dynamical concepts, but in our case we are going to study the real dynamics in order to be able to compare the results of the proposed schemes without memory and their variants with memory. By using at least two previous iterates in the iterative expression of a method with memory, in order to be able to graphically represent the behaviour of methods with memory, it is necessary to study the real dynamics.

In this chapter, we discuss the following sections. In Section 4.2, we derive the iterative expression of the proposed parametric family and prove its convergence order depending on the parameter. In Section 4.3, we add memory to the parametric family proposed in the previous section and analyse the convergence of the schemes with memory obtained with the parameter approximations. In Section 4.4, we study the real dynamical behaviour of the proposed family on the basins of quadratic polynomials $x^2 - c$, when $c \in \{0, 1\}$. We also study the real dynamical behaviour of the memory procedures to be proposed and the quadratic polynomial $x^2 - c$ where $c \in \mathbb{R}$, $c \geq 0$. In Section 4.5, we add an iterative step to the proposed family and study how to add memory to the new parametric family of derivative-free iterative methods, and we analyse the order of convergence without memory and of the schemes obtained by introducing memory. In Section 4.6, several numerical experiments are carried out in order to show the behaviour of the different procedures discussed and, finally, in Section 4.7, some conclusions are drawn.

4.2 Design of the family of methods and convergence analysis

To obtain a parametric family of derivative-free iterative methods from Traub's scheme, the first thing to do is to replace the derivative of the function by another expression. One way to do this is to replace the derivative by a divided difference operator. In Traub's first step, instead of using the derivative we use the divided difference operator $f[w_k, x_k]$ where $w_k = x_k + \beta f(x_k)$. We can observe that this operator has a parameter β which must be different from 0.
To avoid the derivative in the second step, we replace it with the divided difference operator $f[y_k, x_k]$. We use this divided difference operator instead of the previous one in order to obtain a higher order of convergence.
We obtain the following parametric family with the above changes

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f[w_k, x_k]}, \quad \text{where} \quad w_k = x_k + \beta f(x_k), \\ x_{k+1} = y_k - \dfrac{f(y_k)}{f[y_k, x_k]} \quad k = 0, 1 \dots \end{cases} \tag{4.2}$$

This family has order 3, it is not a family of optimal methods because we perform 3 functional evaluations. For this reason we have introduced a weight function in the second iterative step, thus proposing the following parametric family of derivative-free schemes

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f[w_k, x_k]}, \quad \text{where} \quad w_k = x_k + \beta f(x_k), \\ x_{k+1} = y_k - H(\mu_k)\dfrac{f(y_k)}{f[y_k, x_k]}, \quad \text{where} \quad \mu_k = \dfrac{f(y_k)}{f(w_k)}, \quad k = 0, 1 \dots \end{cases} \tag{4.3}$$

being $H(t)$ a real function.
We denote by M_4 the parametric family (4.3), and prove below that this family of schemes has order 4.

Convergence analysis

We prove next that the order of the parametric family M_4 is 4 under certain conditions.

Theorem 4.2.1. *Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a sufficiently differentiable function in a neighbourhood of α, denoted by $D \subset \mathbb{R}$, such that $f(\alpha) = 0$. We assume $f'(\alpha) \neq 0$. Let $H(t)$ be a real function satisfying $H(0) = 1$, $H'(0) = 1$ and $|H''(0)| < \infty$. Then, taking an estimation x_0 sufficiently close to α, the sequence of iterates $\{x_k\}_{k\geq 0}$ generated by proposed family (4.3) converges to α with order 4, and its error equation is*

$$e_{k+1} = \frac{1}{2}c_2\left(1+\beta f'(\alpha)\right)\left(-2c_3\left(1+\beta f'(\alpha)\right)+c_2^2\left(6+4\beta f'(\alpha)-H_2\right)\right)e_k^4+O\left(e_k^5\right), \tag{4.4}$$

being $c_j = \frac{1}{j!}\frac{f^{(j)}(\alpha)}{f'(\alpha)}$ for $j = 2, 3, \ldots$, where $e_k = x_k - \alpha$ and denoting by $H_2 = H''(0)$ to simplify the notation.

Proof. We use the Taylor development of $f(x_k)$ around α,

$$f(x_k) = f'(\alpha)\left(e_k + c_2e_k^2 + c_3e_k^3 + c_4e_k^4 + c_5e_k^5 + O\left(e_k^6\right)\right). \tag{4.5}$$

We also consider the Taylor development of $f(w_k)$ around α,

$$f(w_k) = f'(\alpha)\left(e_w + c_2e_w^2 + c_3e_w^3 + c_4e_w^4 + c_5e_w^5 + O\left(e_w^6\right)\right), \tag{4.6}$$

being $e_w = w_k - \alpha$.
Now, we calculate the expansion of the divided difference operator $f[w_k, x_k]$ by definition using equations (4.5) and (4.6)

$$\begin{aligned}
f[w_k, x_k] &= \frac{f(w_k) - f(x_k)}{w_k - x_k} = \frac{f(w_k) - f(x_k)}{w_k - \alpha + \alpha - x_k} = \frac{f(w_k) - f(x_k)}{e_w - e_k}\\
&= \frac{f'(\alpha)\left((e_w - e_k) + c_2\left(e_w^2 - e_k^2\right) + c_3\left(e_w^3 - e_k^3\right) + c_4\left(e_w^4 - e_k^4\right) + O_5(e_k, e_w)\right)}{e_w - e_k}\\
&= f'(\alpha)\left(1 + c_2(e_w + e_k) + c_3\frac{(e_w^3 - e_k^3)}{e_w - e_k} + c_4\frac{(e_w^4 - e_k^4)}{e_w - e_k} + O_4(e_k, e_w)\right),
\end{aligned}$$

where $O_s(e_k, e_w)$ denotes all terms in where the sum of exponents of e_k and e_w is at least s.
Given that $w_k = x_k + \beta f(x_k)$, then we have

$$\begin{aligned}
e_w &= e_k + \beta f(x_k) = e_k + \beta f'(\alpha)\left(e_k + c_2e_k^2 + c_3e_k^3 + c_4e_k^4 + O\left(e_k^5\right)\right)\\
&= \left(1 + \beta f'(\alpha)\right)e_k + \beta f'(\alpha)\left(c_2e_k^2 + c_3e_k^3\right) + O\left(e_k^4\right).
\end{aligned}$$

It follows

$$\begin{aligned}
f[w_k, x_k] =& f'(\alpha)\left(1 + c_2\left(2 + \beta f'(\alpha)\right)e_k + \left(\beta c_2^2 f'(\alpha) + c_3\left(3 + 3\beta f'(\alpha) + \beta^2 f'(\alpha)^2\right)\right)e_k^2\right.\\
&\left. + \left(2 + \beta f'(\alpha)\right)\left(2\beta c_2c_3f'(\alpha) + c_4\left(2 + 2\beta f'(\alpha) + \beta^2 f'(\alpha)^2\right)\right)e_k^3\right) + O\left(e_k^4\right).
\end{aligned}$$

From this, we obtain

$$
\begin{aligned}
y_k - \alpha &= e_k - \frac{f(x_k)}{f[w_k, x_k]} \\
&= \frac{c_2\left(1+\beta f'(\alpha)\right)e_k^2 + \left(\beta c_2^2 f'(\alpha) + c_3\left(2+3\beta f'(\alpha)+\beta^2 f'(\alpha)^2\right)\right)e_k^3 + O\left(e_k^4\right)}{f[w_k, x_k]} \\
&= c_2\left(1+\beta f'(\alpha)\right)e_k^2 + \left(-c_2^2\left(1+(1+\beta f'(\alpha))^2\right) + c_3\left(2+3\beta f'(\alpha)+\beta^2 f'(\alpha)^2\right)\right)e_k^3 \\
&+ O\left(e_k^4\right).
\end{aligned}
$$

We consider the Taylor development of $f(y_k)$ around α

$$
f(y_k) = f'(\alpha)\left(e_y + c_2 e_y^2 + c_3 e_y^3 + c_4 e_y^4 + c_5 e_y^5 + O\left(e_y^6\right)\right), \tag{4.7}
$$

being $e_y = y_k - \alpha$.
So, the Taylor expansion of divided difference operator $f[y_k, x_k]$ is

$$
f[y_k, x_k] = \frac{f'(\alpha)\left((e_y - e_k) + c_2\left(e_y^2 - e_k^2\right) + c_3\left(e_y^3 - e_k^3\right) + c_4\left(e_y^4 - e_k^4\right) + O_5(e_k, e_y)\right)}{e_y - e_k}.
$$

As $e_y = c_2\left(1+\beta f'(\alpha)\right)e_k^2 + O\left(e_k^3\right)$, then

$$
f[y_k, x_k] = f'(\alpha)\left(1 + c_2 e_k + \left(c_3 + c_2^2\left(1+\beta f'(\alpha)\right)\right)e_k^2\right) + O\left(e_k^3\right).
$$

We now calculate $\mu_k = \frac{f(y_k)}{f(w_k)}$,

$$
\frac{f(y_k)}{f(w_k)} = c_2 e_k + \left(c_3\left(2+\beta f'(\alpha)\right) - c_2^2\left(3+2\beta f'(\alpha)\right)\right)e_k^2 + O\left(e_k^3\right).
$$

From which we obtain

$$
\begin{aligned}
H(\mu_k) &= H_0 + H_1\mu_k + \frac{1}{2}H_2\mu_k^2 + O(\mu_k^3) = 1 + \mu_k + \frac{H_2}{2}\mu_k^2 + O(\mu_k^3) \\
&= 1 + c_2 e_k + \left(c_3\left(2+\beta f'(\alpha)\right) - c_2^2\left(3+2\beta f'(\alpha)\right)\right)e_k^2 + \frac{H_2}{2}c_2^2 e_k^2 + O\left(e_k^3\right) \\
&= 1 + c_2 e_k + \left(c_3\left(2+\beta f'(\alpha)\right) + \frac{1}{2}c_2^2\left(-6-4\beta f'(\alpha)+H_2\right)\right)e_k^2 + O\left(e_k^3\right).
\end{aligned}
$$

We then calculate the error equation

$$
e_{k+1} = \frac{1}{2}c_2\left(1+\beta f'(\alpha)\right)\left(-2c_3\left(1+\beta f'(\alpha)\right) + c_2^2\left(6+4\beta f'(\alpha)-H_2\right)\right)e_k^4 + O\left(e_k^5\right).
$$

Thus it is proven that parametric family (4.3) has order of convergence 4. □

According to the Kung and Traub conjecture, defined in [7], in order to be optimal, an iterative method without memory that performs d functional evaluations per iteration must satisfy that the order of convergence of the method coincides with 2^{d-1}. In this case, our proposed family performs 3 evaluations, since we compute the image of f in the approximations x_k, w_k and y_k. For this reason, we have that the proposed family of schemes (4.3) is optimal.
Then, we have a family of optimal procedures whatever parameter β, other than $\beta \neq 0$.

4.3 Introducing memory

The first iterative class with memory that includes accelerating parameters in its iterative expression was designed by Traub in [2] from transformations on Steffensen's method [11]. There are many other iterative methods with memory, such as those presented in [30].
We note that if $\beta = -\frac{1}{f'(\alpha)}$, then our family of methods would increase its order by at least one unit. Since the value of α is unknown, we approximate the value of $f'(\alpha)$ in order to increase the order of the iterative scheme. We approximate this derivative by an expression in which we use the previous iterates and their functional evaluations. In this way we obtain a method with memory.
If we consider Newton's interpolating polynomial of degree 1 at nodes x_k and x_{k-1}, then we can approximate the derivative of f evaluated at the solution as follows

$$f'(\alpha) \approx N_1'(x_k) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}.$$

Therefore, we choose

$$\beta_k = -\frac{1}{N_1'(x_k)},$$

and thus, substituting the parameter for this expression we obtain a method with memory, which we denote by M_4N_1.

Theorem 4.3.1. *Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a sufficiently differentiable function in an neighbourhood of α denoted by $D \subset \mathbb{R}$ such that $f(\alpha) = 0$. We assume $f'(\alpha) \neq 0$. Let $H(t)$ be a real function satisfying $H(0) = 1$, $H'(0) = 1$, $H''(0) = 2$ and $|H'''(0)| < \infty$. Then, taking an estimation x_0 close enough to α, the sequence of iterates $\{x_k\}_{k\geq 0}$, generated by method M_4N_1 converges to α with order $p = 2 + \sqrt{6} \approx 4.4495$.*

Proof. From the error equation (4.4) and $H_2 = H''(0) = 2$,

$$e_{k+1} \sim \left(1 + \beta f'(\alpha)\right)^2 c_2 \left(2c_2^2 - c_3\right) e_k^4 + O\left(e_k^5\right).$$

Using Taylor developments around α, we obtain

$$f(x_k) = f'(\alpha)\left(e_k + c_2 e_k^2 + O\left(e_k^3\right)\right), \tag{4.8}$$

$$f(x_{k-1}) = f'(\alpha)\left(e_{k-1} + c_2 e_{k-1}^2 + O\left(e_{k-1}^3\right)\right). \tag{4.9}$$

This results in

$$\begin{aligned}
\beta_k &= -\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} \\
&= -\frac{e_k - e_{k-1}}{f'(\alpha)(e_k - e_{k-1} + c_2\left(e_k^2 - e_{k-1}^2\right) + O_3(e_{k-1}, e_k)} \\
&= -\frac{1}{f'(\alpha)\left(1 + c_2\left(e_k + e_{k-1}\right) + O_2(e_{k-1}, e_k)\right)}.
\end{aligned}$$

Therefore

$$
\begin{aligned}
1+\beta_k f'(\alpha) &= 1-\frac{1}{1+c_2\left(e_k+e_{k-1}\right)+O_2(e_{k-1},e_k)}\\
&= \frac{c_2\left(e_k+e_{k-1}\right)+O_2(e_k,e_{k-1})}{1+c_2\left(e_k+e_{k-1}\right)+O_2(e_{k-1},e_k)}.
\end{aligned}
$$

Thus, $1+\beta_k f'(\alpha)\sim c_2 e_{k-1}$. From the error equation (4.4) and the above relation we get

$$
e_{k+1}\sim\left(c_2 e_{k-1}\right)^2 c_2\left(2c_2^2-c_3\right)e_k^4\sim e_{k-1}^2 e_k^4. \tag{4.10}
$$

On the other hand, suppose that the R-order of the method is at least p. Therefore, it is satisfied that

$$
e_{k+1}\sim D_{k,p}e_k^p,
$$

where $D_{k,p}$ tends to the asymptotic error constant, D_p, when $k\to\infty$. Analogously,

$$
e_k\sim D_{k-1,p}e_{k-1}^p.
$$

Then one has

$$
e_{k+1}\sim D_{k,p}\left(D_{k-1,p}e_{k-1}^p\right)^p = D_{k,p}D_{k-1,p}^p e_{k-1}^{p^2}. \tag{4.11}
$$

In the same way as relation (4.10) is obtained,

$$
e_{k+1}\sim e_{k-1}^2\left(D_{k-1,p}e_{k-1}^p\right)^4 = D_{k-1,p}^4 e_{k-1}^{4p+2}. \tag{4.12}
$$

Then by equating the exponents of e_{k-1} of (4.11) and (4.12), one has

$$
p^2 = 4p+2,
$$

whose only positive solution is the order of convergence of method M_4N_1, that is $p\approx 4.4495$, according to Theorem 2.1.1. □

In the previous case, we approximated the parameter using the divided difference operator $f[x_k,x_{k-1}]$. What we do next is to use the Kurchatov's divided difference operator at nodes x_k and x_{k-1}. Therefore, we choose

$$
\beta_k = -\frac{1}{f[2x_k-x_{k-1},x_{k-1}]},
$$

and thus, substituting the parameter for this expression we obtain a method with memory, which we denote by M_4K_1.

Theorem 4.3.2. *Let $f: D\subset\mathbb{R}\longrightarrow\mathbb{R}$ be a sufficiently differentiable function in an neighbourhood of α denoted by $D\subset\mathbb{R}$ such that $f(\alpha)=0$. We assume $f'(\alpha)\neq 0$. Let $H(t)$ be a real function satisfying $H(0)=1$, $H'(0)=1$, $H''(0)=2$ and $|H'''(0)|<\infty$. Then, taking an estimation x_0 close enough to α, the sequence of iterates $\{x_k\}_{k\geq 0}$, generated by method M_4K_1 converges to α with order $p=2+2\sqrt{2}\approx 4.83$.*

Proof. From the error equation (4.4) and $H_2 = H''(0) = 2$,

$$e_{k+1} \sim \left(1+\beta f'(\alpha)\right)^2 c_2 \left(2c_2^2 - c_3\right) e_k^4 + O\left(e_k^5\right).$$

Using Taylor developments of $f(x_{k-1})$ around α, we obtain

$$f(x_{k-1}) = f'(\alpha)\left(e_{k-1} + c_2 e_{k-1}^2 + c_3 e_{k-1}^3 + O\left(e_{k-1}^4\right)\right). \tag{4.13}$$

By definition of divided difference operator, $f[2x_k - x_{k-1}, x_{k-1}]$ has the following expansion

$$f[2x_k - x_{k-1}, x_{k-1}] = f'(\alpha)\left(1 + 2c_2 e_k + 4c_3 e_k^2 + c_3 e_{k-1}^2 - 2c_3 e_k e_{k-1}\right) + O_3(e_k, e_{k-1}).$$

This results in

$$\begin{aligned} 1 + \beta_k f'(\alpha) &= 1 - \frac{1}{1 + 2c_2 e_k + 4c_3 e_k^2 + c_3 e_{k-1}^2 - 2c_3 e_k e_{k-1} + O_3(e_k, e_{k-1})} \\ &= \frac{2c_2 e_k + 4c_3 e_k^2 + c_3 e_{k-1}^2 - 2c_3 e_k e_{k-1} + O_3(e_k, e_{k-1})}{1 + 2c_2 e_k + 4c_3 e_k^2 + c_3 e_{k-1}^2 - 2c_3 e_k e_{k-1} + O_3(e_k, e_{k-1})}. \end{aligned}$$

Thus, $1 + \beta_k f'(\alpha) \sim 2c_2 e_k + 4c_3 e_k^2 + c_3 e_{k-1}^2 - 2c_3 e_k e_{k-1}$.

It is known that e_k converges faster to 0 than e_k^2 and $e_k e_{k-1}$, hence the behaviour of $1+\beta_k f'(\alpha)$ is either like the behaviour of e_k or like the behaviour of e_{k-1}^2.
Suppose that the R-order of the method is at least p. Therefore, it is satisfied that

$$e_{k+1} \sim D_{k,p} e_k^p,$$

where $D_{k,p}$ tends to the asymptotic error constant, D_p, when $k \to \infty$. Analogously,

$$\frac{e_k}{e_{k-1}^2} \sim D_{k-1,p} e_{k-1}^{p-2}.$$

Thus we obtain that $1 + \beta_k f'(\alpha) \sim e_{k-1}^2$ if $p > 2$.
From the error equation (4.4) and the above relation we obtain

$$e_{k+1} \sim \left(e_{k-1}^2\right)^2 e_k^4 \sim e_{k-1}^4 e_k^4. \tag{4.14}$$

On the other hand, assuming that the R-order of the method is at least p, we have

$$e_{k+1} \sim D_{k,p}\left(D_{k-1,p} e_{k-1}^p\right)^p = D_{k,p} D_{k-1,p}^p e_{k-1}^{p^2}. \tag{4.15}$$

In the same way as relation (4.14) is obtained,

$$e_{k+1} \sim e_{k-1}^4 \left(D_{k-1,p} e_{k-1}^p\right)^4 = D_{k-1,p}^4 e_{k-1}^{4p+4}. \tag{4.16}$$

Then by equating the exponents of e_{k-1} of (4.15) and (4.16) one has

$$p^2 = 4p + 4,$$

whose only positive solution is the order of convergence of method M_4K_1, that is $p \approx 4.83$, according to Theorem 2.1.1. $\square$

For the first approximation of the parameter we chose to use a Newton interpolating polynomial of degree 1. In this case we also choose another Newton interpolating polynomial of degree 1 at nodes x_k and y_{k-1}. Then we can approximate the derivative of f evaluated at the solution in the following way

$$f'(\alpha) \approx N'_{1y}(x_k) = \frac{f(x_k) - f(y_{k-1})}{x_k - y_{k-1}}.$$

Therefore, we choose

$$\beta_k = -\frac{1}{f[x_k, y_{k-1}]},$$

and thus, substituting the parameter for this expression we obtain a method with memory, which we denote by M_4N_{1Y}.

Theorem 4.3.3. *Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a sufficiently differentiable function in an neighbourhood of α denoted by $D \subset \mathbb{R}$ such that $f(\alpha) = 0$. We assume $f'(\alpha) \neq 0$. Let $H(t)$ be a real function satisfying $H(0) = 1$, $H'(0) = 1$, $H''(0) = 2$ and $|H'''(0)| < \infty$. Then, taking an estimation x_0 close enough to α, the sequence of iterates $\{x_k\}_{k\geq 0}$, generated by method M_4N_{1Y} converges to α with order $p = 5$.*

Proof. From the error equation (4.4) and $H_2 = H''(0) = 2$,

$$e_{k+1} \sim \left(1 + \beta f'(\alpha)\right)^2 c_2 \left(2c_2^2 - c_3\right) e_k^4 + O\left(e_k^5\right).$$

Using Taylor developments around α, we obtain

$$f(x_k) = f'(\alpha)\left(e_k + c_2 e_k^2 + O\left(e_k^3\right)\right), \tag{4.17}$$

$$f(y_{k-1}) = f'(\alpha)\left(e_{y,k-1} + c_2 e_{y,k-1}^2 + O\left(e_{y,k-1}^3\right)\right). \tag{4.18}$$

This results in

$$\begin{aligned}
\beta_k &= -\frac{x_k - y_{k-1}}{f(x_k) - f(y_{k-1})} \\
&= -\frac{e_k - e_{y,k-1}}{f'(\alpha)\left(e_k - e_{y,k-1} + c_2\left(e_k^2 - e_{y,k-1}^2\right)\right) + O_3(e_{y,k-1}, e_k)} \\
&= -\frac{1}{f'(\alpha)\left(1 + c_2\left(e_k + e_{y,k-1}\right) + O_2(e_{y,k-1}, e_k)\right)}.
\end{aligned}$$

Therefore,

$$\begin{aligned} 1+\beta_k f'(\alpha) &= 1-\frac{1}{1+c_2(e_k+e_{y,k-1})+O_2(e_{y,k-1},e_k)} \\ &= \frac{c_2(e_k+e_{y,k-1})+O_2(e_k,e_{y,k-1})}{1+c_2(e_k+e_{y,k-1})+O_2(e_{y,k-1},e_k)}. \end{aligned}$$

Thus, $1+\beta_k f'(\alpha)\sim e_k+e_{y,k-1}$.
Assume that the R-order of the method is at least p. Consider sequence $\{y_k\}_{k\geq 0}$ generated by the first step of the method, and suppose that the R-order is at least p_1. Therefore, it is satisfied

$$e_{k+1}\sim D_{k,p}e_k^p \qquad \text{and} \qquad e_{y,k}\sim D_{k,p_1}e_k^{p_1},$$

where $D_{k,p}$ tends to the asymptotic error constant, D_p, and where D_{k,p_1} tends to the asymptotic error constant, D_{p_1}, when $k\to\infty$. Since $e_k\sim D_{k-1,p}e_{k-1}^p$, then

$$\frac{e_k}{e_{y,k-1}}\sim\frac{D_{k-1,p}e_{k-1}^p}{e_{y,k-1}}\sim\frac{D_{k-1,p}e_{k-1}^p}{D_{k-1,p_1}e_{k-1}^{p_1}}.$$

Then if $p\geq p_1$, we have

$$1+\beta_k f'(\alpha)\sim e_{y,k-1}. \tag{4.19}$$

From error equation (4.4) and the above relation we get

$$e_{k+1}\sim e_{y,k-1}^2 e_k^4\sim e_{y,k-1}^2 e_k^4. \tag{4.20}$$

We assume that the R-order of the method is at least p we obtain relation (4.11).
Assuming that sequence $\{y_k\}_{k\geq 0}$ converges to R-order at least p_1, we obtain the following relation

$$e_{y,k}\sim D_{k,p_1}e_k^{p_1}\sim D_{k,p_1}\left(D_{k-1,p}e_{k-1}^p\right)^{p_1}\sim D_{k,p_1}D_{k-1,p}^{p_1}e_{k-1}^{pp_1}. \tag{4.21}$$

Therefore,

$$e_{k+1}\sim\left(D_{k-1,p_1}e_{k-1}^{p_1}\right)^2\left(D_{k-1,p}e_{k-1}^p\right)^4=D_{k-1,p_1}^2D_{k-1,p}^4e_{k-1}^{2p_1}e_{k-1}^{4p}. \tag{4.22}$$

On the other hand, we have

$$e_{k,y}\sim\left(1+\beta_k f'(\alpha)\right)e_k^2\sim e_{y,k-1}e_k^2\sim e_{k-1}^{p_1}\left(e_{k-1}^p\right)^2\sim e_{k-1}^{2p+p_1}. \tag{4.23}$$

Then by equating the exponents of e_{k-1} of (4.11) and (4.22), and by equating the exponents of (4.21) and (4.23), one has

$$\begin{aligned} p^2 &= 4p+2p_1, \\ pp_1 &= 2p+p_1, \end{aligned}$$

whose only positive solution is the order of convergence of method M_4N_{1Y}, that is $p=5$ and $p_1=2.5$, according to Theorem 2.1.1. □

In the previous case, we approximated the parameter by the divided difference operator $f[x_k, y_{k-1}]$. What we do next is to use the Kurchatov divided difference operator at nodes x_k and y_{k-1}. Therefore, we choose

$$\beta_k = -\frac{1}{f[2x_k - y_{k-1}, y_{k-1}]},$$

and thus, substituting the parameter for this expression we obtain a method with memory, which we denote by M_4K_{1Y}.

Theorem 4.3.4. *Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a sufficiently differentiable function in an neighbourhood of α denoted by $D \subset \mathbb{R}$ such that $f(\alpha) = 0$. We assume $f'(\alpha) \neq 0$. Let $H(t)$ be a real function satisfying $H(0) = 1$, $H'(0) = 1$, $H''(0) = 2$ and $|H'''(0)| < \infty$. Then, taking an estimation x_0 close enough to α, the sequence of iterates $\{x_k\}_{k\geq 0}$, generated by method M_4K_{1Y} converges to α with order 6.*

Proof. From the error equation (4.4) and $H_2 = H''(0) = 2$,

$$e_{k+1} \sim \left(1 + \beta f'(\alpha)\right)^2 c_2 \left(2c_2^2 - c_3\right) e_k^4 + O\left(e_k^5\right).$$

Using Taylor's series developments of $f(x_{k-1})$ around α we obtain

$$f(y_{k-1}) = f'(\alpha) \left(e_{e,k-1} + c_2 e_{e,k-1}^2 + c_3 e_{e,k-1}^3 + O\left(e_{e,k-1}^4\right)\right). \tag{4.24}$$

By definition of divided difference operator, $f[2x_k - y_{k-1}, y_{k-1}]$ has the following expansion

$$f[2x_k - y_{k-1}, y_{k-1}] = f'(\alpha) \left(1 + 2c_2 e_k + 4c_3 e_k^2 + c_3 e_{y,k-1}^2 - 2c_3 e_k e_{y,k-1}\right) + O_3(e_k, e_{y,k-1}).$$

Thus, $1 + \beta_k f'(\alpha) \sim 2c_2 e_k + 4c_3 e_k^2 + c_3 e_{y,k-1}^2 - 2c_3 e_k e_{y,k-1}$.
It follows that e_k converges faster to 0 than e_k^2 and $e_k e_{y,k-1}$, hence the behaviour of $1+\beta_k f'(\alpha)$ is either like the behaviour of e_k or like the behaviour of $e_{y,k-1}^2$.
Suppose that the R-order of the method is at least p. Consider sequence $\{y_k\}_{k\geq 0}$ generated by the first step of the method, and suppose that the R-order is at least p_1. Therefore, it is satisfied

$$e_{k+1} \sim D_{k,p} e_k^p \qquad \text{and} \qquad e_{y,k} \sim D_{k,p_1} e_k^{p_1},$$

where $D_{k,p}$ tends to the asymptotic error constant, D_p, and where D_{k,p_1} tends to the asymptotic error constant, D_{p_1}, when $k \to \infty$.
Since $e_k \sim D_{k-1,p} e_{k-1}^p$, then

$$\frac{e_k}{e_{y,k-1}} \sim \frac{D_{k-1,p} e_{k-1}^p}{e_{y,k-1}^2} \sim \frac{D_{k-1,p} e_{k-1}^p}{D_{k-1,p_1} e_{k-1}^{2p_1}}.$$

Then if $p \geq 2p_1$, we have therefore

$$1 + \beta_k f'(\alpha) \sim e_{y,k-1}^2. \tag{4.25}$$

From error equation (4.4) and the above relation we obtain

$$e_{k+1} \sim \left(e_{y,k-1}^2\right)^2 e_k^4 \sim e_{y,k-1}^4 e_k^4. \tag{4.26}$$

We assume that the R-order of the method is at least p we obtain relation (4.11).
Assuming that sequence $\{y_k\}_{k\geq 0}$ converges to R-order at least p_1, we obtain the relation (4.21).
In the same way as relation (4.26) is obtained,

$$e_{k+1} \sim \left(D_{k-1,p_1} e_{k-1}^{p_1}\right)^4 \left(D_{k-1,p} e_{k-1}^{p}\right)^4 = D_{k-1,p_1}^2 D_{k-1,p}^4 e_{k-1}^{4p_1} e_{k-1}^{4p}. \tag{4.27}$$

On the other hand, we have

$$e_{k,y} \sim (1+\beta_k f'(\alpha))e_k^2 \sim e_{y,k-1}^2 e_k^2 \sim \left(e_{k-1}^{p_1}\right)^2 \left(e_{k-1}^{p}\right)^2 \sim e_{k-1}^{2p+2p_1}. \tag{4.28}$$

Then by equating the exponents of e_{k-1} of (4.11) and (4.27), and by equating the exponents of (4.21) and (4.28), one has

$$\begin{aligned} p^2 &= 4p + 4p_1, \\ pp_1 &= 2p + 2p_1, \end{aligned}$$

whose only positive solution is the order of convergence of method M_4K_{1Y}, that is $p = 6$ and $p_1 = 3$, according to Theorem 2.1.1. □

In the case of M_4N_1, we applied a Newton interpolating polynomial of degree 1 at nodes x_k and x_{k-1} to approximate the parameter.
In what follows, we approximate $f'(\alpha)$ to increase the degree of Newton's interpolating polynomial, in this case to an interpolating polynomial of degree 2, at nodes x_k, x_{k-1} and y_{k-1}.
If we define $N_2(t) = f(x_k) + f[x_k, x_{k-1}](t - x_k) + f[x_k, x_{k-1}, y_{k-1}](t - x_k)(t - x_{k-1})$, an approximation of the derivative would be

$$f'(\alpha) \approx N_2'(x_k).$$

Therefore, we choose

$$\beta_k = -\frac{1}{N_2'(x_k)},$$

and thus, substituting the parameter for this expression we obtain a method with memory, which we denote by M_4N_2.

Theorem 4.3.5. *Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a sufficiently differentiable function in an neighbourhood of α denoted by $D \subset \mathbb{R}$ such that $f(\alpha) = 0$. We assume $f'(\alpha) \neq 0$. Let $H(t)$ be a real function satisfying $H(0) = 1$, $H'(0) = 1$, $H''(0) = 2$ and $|H'''(0)| < \infty$. Then, taking an estimation x_0 close enough to α, the sequence of iterates $\{x_k\}_{k\geq 0}$, generated by method M_4N_2 converges to α with order $p = \frac{1}{2}(5 + \sqrt{33}) \approx 5.37228$.*

Proof. From the error equation (4.4) and $H_2 = H''(0) = 2$,

$$e_{k+1} \sim (1+\beta f'(\alpha))^2 c_2 \left(2c_2^2 - c_3\right) e_k^4 + O\left(e_k^5\right).$$

Using Taylor's series developments around α, we obtain

$$f(x_k) = f'(\alpha)\left(e_k + c_2 e_k^2 + O\left(e_k^3\right)\right), \tag{4.29}$$

$$f(y_{k-1}) = f'(\alpha)\left(e_{y,k-1} + c_2 e_{y,k-1}^2 + O\left(e_{y,k-1}^3\right)\right), \tag{4.30}$$

$$f(x_{k-1}) = f'(\alpha)\left(e_{k-1} + c_2 e_{k-1}^2 + O\left(e_{k-1}^3\right)\right). \tag{4.31}$$

This results in

$$f[x_k, x_{k-1}] = f'(\alpha)(1 + c_2(e_k + e_{k-1}) + O_2(e_{k-1}, e_k)).$$
$$f[y_{k-1}, x_{k-1}] = f'(\alpha)(1 + c_2(e_{y,k-1} + e_{k-1}) + O_2(e_{y,k-1}, e_{k-1})).$$

Therefore,

$$f[x_k, x_{k-1}, y_{k-1}] = \frac{f[y_{k-1}, x_{k-1}] - f[x_k, x_{k-1}]}{y_{k-1} - x_k}.$$

From this we obtain

$$f[x_k, x_{k-1}, y_{k-1}] = f'(\alpha)\frac{c_2(e_{y,k-1} - e_k) + O_2(e_{y,k-1}, e_{k-1}, e_k)}{e_{y,k-1} - e_k}.$$

Thus,

$$\begin{aligned} N_2'(x_k) &= f[x_k, x_{k-1}] + f[x_k, x_{k-1}, y_{k-1}](x_k - x_{k-1}) \\ &= f'(\alpha) + 2c_2 f'(\alpha) e_k + c_3 f'(\alpha) e_k e_y + c_3 f'(\alpha)(e_k - e_{y,k-1}) e_{k-1} + O\left(e_{k-1}^2\right) \\ &+ O_2(e_{y,k-1}, e_k, e_{k-1}). \end{aligned}$$

This means that $1 + \beta_k f'(\alpha)$ can behave in the same way as e_k, as $e_k e_{y,k-1}$, as $e_{k-1} e_k$ or as $e_{k-1} e_{y,k-1}$.
We have that $e_k e_{y,k-1}$ converges faster to zero than e_k when $k \to \infty$, and we also have that $e_{k-1} e_k$ converges faster to zero than $e_{k-1} e_{y,k-1}$. For this reason we have to look at if it behaves like e_k or like $e_{k-1} e_{y,k-1}$.
Suppose that the R-order of the method is at least p. Consider sequence $\{y_k\}_{k \geq 0}$ generated by the first step of the method, and suppose that the R-order is at least p_1. Therefore, it is satisfied

$$e_{k+1} \sim D_{k,p} e_k^p \qquad \text{and} \qquad e_{y,k} \sim D_{k,p_1} e_k^{p_1},$$

where $D_{k,p}$ tends to the asymptotic error constant, D_p, and where D_{k,p_1} tends to the asymptotic error constant, D_{p_1}, when $k \to \infty$.
Since $e_k \sim D_{k-1,p} e_{k-1}^p$, then

$$\frac{e_k}{e_{k-1} e_{y,k-1}} \sim \frac{D_{k-1,p} e_{k-1}^p}{e_{k-1} e_{y,k-1}} \sim \frac{D_{k-1,p} e_{k-1}^p}{D_{k-1,p_1} e_{k-1} e_{k-1}^{p_1}}.$$

Then if $p \geq p_1 + 1$, we have therefore

$$1 + \beta_k f'(\alpha) \sim -c_3 e_{k-1} e_{y,k-1}. \tag{4.32}$$

From error equation (4.4) and the above relation we obtain

$$e_{k+1} \sim (-c_3 e_{k-1} e_{y,k-1})^2 c_2 \left(2c_2^2 - c_3\right) e_k^4 \sim e_{k-1}^2 e_{y,k-1}^2 e_k^4. \tag{4.33}$$

Assuming that the R-order of the method is at least p we obtain relation (4.11).
We assume that sequence $\{y_k\}_{k\geq 0}$ converges to R-order at least p_1, we obtain

$$e_{y,k} \sim D_{k,p_1} e_k^{p_1} \sim D_{k,p_1} (D_{k-1,p} e_{k-1}^p)^{p_1} \sim D_{k,p_1} D_{k-1,p}^{p_1} e_{k-1}^{pp_1}. \tag{4.34}$$

In the same way as relation (4.33) is obtained,

$$e_{k+1} \sim e_{k-1}^2 \left(D_{k-1,p_1} e_{k-1}^{p_1}\right)^2 \left(D_{k-1,p} e_{k-1}^p\right)^4 = D_{k-1,p_1}^2 D_{k-1,p}^4 e_{k-1}^{2p_1} e_{k-1}^{4p+2}. \tag{4.35}$$

On the other hand, we have

$$e_{k,y} \sim (1+\beta_k f'(\alpha)) e_k^2 \sim e_{k-1} e_{y,k-1} e_k^2 \sim e_{k-1} \left(D_{k-1,p_1} e_{k-1}^{p_1}\right) \left(D_{k-1,p} e_{k-1}^p\right)^2 \sim e_{k-1}^{2p+1+p_1}. \tag{4.36}$$

Then by equating the exponents of e_{k-1} of (4.11) and (4.35), and by equating the exponents of (4.34) and (4.36), one has

$$\begin{aligned} p^2 &= 4p + 2 + 2p_1, \\ pp_1 &= 2p + 1 + p_1, \end{aligned}$$

whose only positive solution is the order of convergence of method M_4N_2, that is $p \approx 5.37228$ and $p_1 \approx 2.68614$, according to Theorem 2.1.1. □

Finally, we increase by another unit the degree of the polynomial with which we approximate the derivative of the equation to be solved. In this case, to approximate $f'(\alpha)$ we use the following Newton interpolating polynomial of degree 3. If we define $N_3(t) = f(x_k) + f[x_k, x_{k-1}](t - x_k) + f[x_k, x_{k-1}, y_{k-1}](t - x_k)(t - x_{k-1}) + f[x_k, x_{k-1}, y_{k-1}, w_{k-1}](t - x_k)(t - x_{k-1})(t - y_{k-1})$, then an approximation of the derivative would be

$$f'(\alpha) \approx N_3'(x_k).$$

Therefore, we choose

$$\beta_k = -\frac{1}{N_3'(x_k)},$$

and thus, substituting the parameter for this expression we obtain a method with memory, which we denote by M_4N_3.

Theorem 4.3.6. *Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a sufficiently differentiable function in an neighbourhood of α denoted by $D \subset \mathbb{R}$ such that $f(\alpha) = 0$. We assume $f'(\alpha) \neq 0$. Let $H(t)$ be a real function satisfying $H(0) = 1$, $H'(0) = 1$, $H''(0) = 2$ and $|H'''(0)| < \infty$. Then, taking an estimation x_0 close enough to α, the sequence of iterates $\{x_k\}_{k\geq 0}$, generated by method M_4N_3 converges to α with order $p = 6$.*

Proof. From the error equation (4.4) and $H_2 = H''(0) = 2$,

$$e_{k+1} \sim (1 + \beta f'(\alpha))^2 c_2 \left(2c_2^2 - c_3\right) e_k^4 + O\left(e_k^5\right).$$

Using Taylor's developments of $f(x_k)$, $f(y_{k-1})$, $f(x_{k-1})$ and $f(w_{k-1})$ around α, we can calculate $N_3'(x_k)$, which has the following expression

$$\begin{aligned} N_3'(x_k) &= f[x_k, x_{k-1}] + f[x_k, x_{k-1}, y_{k-1}](x_k - x_{k-1}) \\ &\quad + f[x_k, x_{k-1}, y_{k-1}, w_{k-1}](x_k - x_{k-1})(x_k - y_{k-1}). \end{aligned}$$

We therefore have

$$1 + \beta_k \sim 2c_2 e_k + c_4 e_{y,k-1} e_{k-1} e_{w,k-1}.$$

This means that $1 + \beta_k f'(\alpha)$ will behave as e_k or as $e_k e_{y,k-1} e_{w,k-1}$, as the other terms converge faster than these two. We now check that the behaviour of $1 + \beta_k f'(\alpha)$ is like that of $e_k e_{y,k-1} e_{w,k-1}$.
Suppose that the R-order of the method is at least p. Therefore, it is satisfied

$$e_{k+1} \sim D_{k,p} e_k^p,$$

where $D_{k,p}$ tends to the asymptotic error constant, D_p, when $k \to \infty$. Moreover, if we assume that sequence $\{y_k\}_{k\geq 0}$ generated by the first step of the method and sequence $\{w_k\}_{k\geq 0}$, converge with R-order at least p_1 and at least p_2, respectively. Then

$$\frac{e_k}{e_{k-1} e_{y,k-1} e_{w,k-1}} \sim \frac{D_{k-1,p} e_{k-1}^p}{D_{k-1,p_1} D_{k-1,p_2} e_{k-1} e_{k-1}^{p_1} e_{k-1}^{p_2}},$$

where D_{k,p_1} and D_{k,p_2} tend to the asymptotic error constants, D_{p_1} and D_{p_2}, respectively, when $k \to \infty$. Then if $p \geq p_1 + p_2 + 1$, we have therefore

$$1 + \beta_k f'(\alpha) \sim c_4 e_{k-1} e_{y,k-1} e_{w,k-1}.$$

From error equation (4.4) and the above relation we obtain

$$e_{k+1} \sim \left(c_4 e_{k-1} e_{y,k-1} e_{w,k-1}\right)^2 c_2 \left(2c_2^2 - c_3\right) e_k^4 \sim e_{k-1}^2 e_{y,k-1}^2 e_{w,k-1}^2 e_k^4. \tag{4.37}$$

We assume that the R-order of the method is at least p we obtain relation (4.11). Assuming that sequence $\{y_k\}_{k\geq 0}$ and sequence $\{w_k\}_{k\geq 0}$ converge with R-order at least p_1 and at least p_2, respectively. Then, we obtain the relation defined in (4.34) and the following relation

$$e_{w,k} \sim D_{k,p_2} e_k^{p_2} \sim D_{k,p_2} \left(D_{k-1,p} e_{k-1}^p\right)^{p_2} \sim D_{k,p_2} D_{k-1,p}^{p_2} e_{k-1}^{pp_2}. \tag{4.38}$$

In the same way as relation (4.37) is obtained,

$$e_{k+1} \sim e_{k-1}^{2p_1+2p_2+4p+2}. \tag{4.39}$$

On the other hand, we have

$$e_{k,y} \sim (1+\beta_k f'(\alpha))e_k^2 \sim e_{k-1}^{2p+1+p_1+p_2}. \tag{4.40}$$

And we also have

$$e_{w,y} \sim (1+\beta_k f'(\alpha))e_k \sim e_{k-1}^{p+1+p_1+p_2}. \tag{4.41}$$

Then by equating the exponents of e_{k-1} of (4.11) and (4.39), by equating those of (4.34) and (4.40) and by equating the exponents of (4.38) and (4.41), one has

$$\begin{aligned} p^2 &= 4p+2+2p_1+2p_2 \\ pp_1 &= 2p+1+p_1+p_2 \\ pp_2 &= p+1+p_1+p_2, \end{aligned}$$

whose only positive solution is the order of convergence of method M_4N_3, that is $p=6$, $p_1=3$ and $p_2=2$, according to Theorem 2.1.1. □

4.4 Dynamical analysis

In this section, we analyze the dynamics of the proposed parametric family (4.3) and of methods M_4N_1 and M_4N_2. We are going to study only the dynamics of these methods with memory because the rational operators of the rest of the schemes with memory coincide with the rational operator of some of the procedures that we are going to study. We study the stability of these schemes depending on the initial approximations, that is, we apply the methods to simple nonlinear functions and analyse the fixed points of operator obtained. In this case, we choose the quadratic polynomials $p_c(x)=x^2-c$, for the case of the methods with memory, and for the methods without memory we use the same polynomial when $c\in\{0,1\}$. The weight function for all them used is the polynomial $H(\mu_k)=\mu_k^2+\mu_k+1$.

Parametric family

We study the operator obtained by applying parametric family (4.3) to the polynomial $p_1(x)=x^2-1$. We denote the rational operator by O_1, which has the following expression

$$\begin{aligned} O_1(x,\beta) = & \frac{\beta^4\left(x^2-1\right)^4\left(\beta^2\left(x^2-1\right)^2\left(x^2+1\right)+\beta\left(11x^4+2x^2-13\right)x+49x^4+72x^2-1\right)}{\left(\beta\left(x^2-1\right)+2x\right)^5\left(2\beta\left(x^2-1\right)x+3x^2+1\right)} \\ & + \frac{2\beta^2\left(x^2-1\right)^2\left(\beta\left(57x^6+49x^4-109x^2+3\right)x+73x^6+175x^4-9x^2+1\right)}{\left(\beta\left(x^2-1\right)+2x\right)^5\left(2\beta\left(x^2-1\right)x+3x^2+1\right)} \\ & + \frac{-1+8x^2-18x^4+112x^6+27x^8+\beta x(-6+32x^2-332x^4+208x^6+98x^8)}{\left(\beta\left(x^2-1\right)+2x\right)^5\left(2\beta\left(x^2-1\right)x+3x^2+1\right)}. \end{aligned}$$

Next, we calculate the fixed points of operator depending on the parameter β and analyse for which values of the parameter we obtain real strange fixed points and their asymptotic behavior. Recall that the fixed points are obtained by solving $O_1(x,\beta) = x$. We define

$$\begin{aligned} r(x) =& \beta^6 x^{12} + 12\beta^5 x^{11} + (61\beta^4 - 6\beta^6)x^{10} + (166\beta^3 - 60\beta^5)x^9 + (15\beta^6 - 245\beta^4 + 254\beta^2)x^8 \\ &+ (120\beta^5 - 504\beta^3 + 206\beta)x^7 + (-20\beta^6 + 370\beta^4 - 524\beta^2 + 69)x^6 \\ &+ (-120\beta^5 + 516\beta^3 - 226\beta)x^5 + (15\beta^6 - 250\beta^4 + 288\beta^2 - 11)x^4 \\ &+ (60\beta^5 - 184\beta^3 + 26\beta)x^3 + (-6\beta^6 + 65\beta^4 - 20\beta^2 + 7)x^2 \\ &+ (-12\beta^5 + 6\beta^3 - 6\beta)x + \beta^6 - \beta^4 + 2\beta^2 - 1. \end{aligned}$$

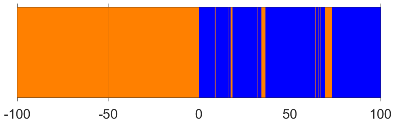
Proposition 4.4.0.1. *The fixed points of operator $O_1(x,\beta)$ are the roots of the polynomial $p_1(x)$, that is, -1 and 1, with superattracting behaviour, for any value of β and*

- *if $|\beta| \leq 0.73847$, two roots of $r(x)$ will be strange fixed points.*
- *if $0.73847 < |\beta| < 16.1039$, four roots of $r(x)$ will be strange fixed points.*
- *if $|\beta| \geq 16.1039$, six roots of $r(x)$ will be strange fixed points.*

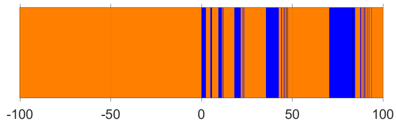
In the following, we illustrate by means of dynamical lines the basins of attraction of the roots of polynomial $p_1(x)$ for the values of the parameter that satisfy $|\beta| = 0.73847$ and $|\beta| = 16.1039$. In Figure 4.1a we show the dynamical line for $\beta = -16.1039$, in Figure 4.1b we show the dynamical line for $\beta = -0.73847$, in Figure 4.1d we show the dynamical line for $\beta = 0.73847$ and in Figure 4.1c we show the dynamical line for $\beta = 16.1039$. In this case, in blue are represented the initial guesses that converge to the root 1 and in orange the initial points that converge to the root -1. In Figure 4.1a, it seems that there are no initial points attracted by strange fixed points.

Figure 4.1: Some dynamical lines

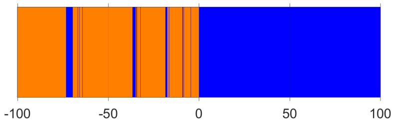
(a) $\beta = -16.1039$



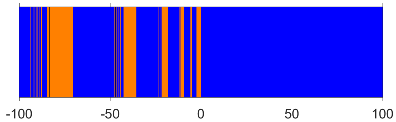
(b) $\beta = -0.73847$



(c) $\beta = 16.1039$

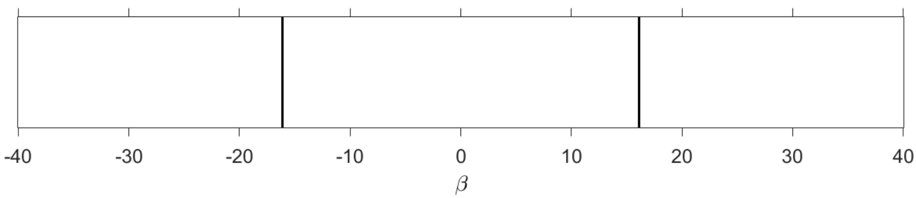


(d) $\beta = 0.73847$



Studying the character of the strange fixed points obtained is complicated, for this reason we have drawn a stability line in Matlab, Figure 4.2 for the roots of $r(x)$ in the case that they are real values. On the abscissa axis we have represented the parameter β. On this line we represent in black the values of the parameter β for which some of the roots of $r(x)$ is a strange attracting fixed point, and in white the values of the parameter for which all the roots are repulsive or are not strange fixed points. We obtain the stability line, Figure 4.2

Figure 4.2: Stability of the roots of $r(x)$

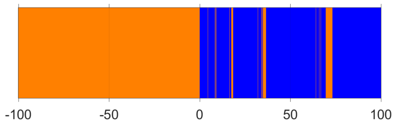


As can be seen in Figure 4.2, the only values for which any of the roots of $r(x)$ have an attracting character are $\beta = -16.14039$ and $\beta = 16.14039$. We calculate the strange fixed points for these values and draw the dynamical lines to see how they behave in these cases.

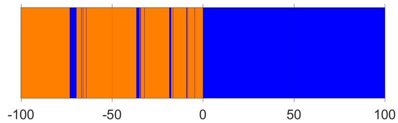
- If $\beta = -16.14039$, we have that operator $O_1(z,x)$ has 5 fixed strange repelling points, which are as follows $\{-0.949066, -0.938044, -0.924546, 1.04658, 1.06196\}$, and an attracting strange fixed point $\{-0.924061\}$. We see the dynamical line for this parameter value in Figure 4.3a.
- If $\beta = 16.14039$, we have that operator $O_1(z,x)$ has 5 fixed strange repelling points, which are as follows $\{0.949066, 0.938044, 0.924546, -1.04658, -1.06196\}$, and an attracting strange fixed point $\{0.924061\}$. We see the dynamical line for this parameter value in Figure 4.3b. In this case, in blue are represented the initial guesses that converge to the root 1 and in orange the initial points that converge to the root -1.

Figure 4.3: Dynamical lines for $|\beta| = 16.14039$

(a) $\beta = -16.14039$



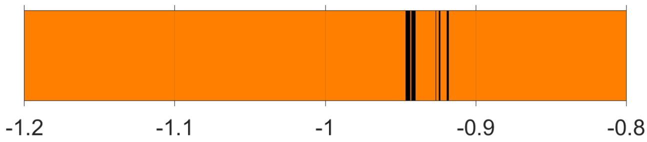
(b) $\beta = 16.14039$



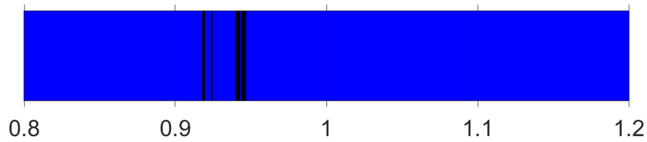
In these dynamical lines no other basins of attraction are visible, but if we zoom in near the attracting strange fixed points we can see how the dynamics change. In Figures 4.4a and 4.4b it can be seen that the basins of attraction for the strange fixed points are small.

Figure 4.4: Zoom on certain region of the dynamical lines for $|\beta| = 16.14039$

(a) $\beta = -16.14039$



(b) $\beta = 16.14039$



We study now the critical points of operator, in order to analyse to which basins of attraction they belong depending on the parameter β. If we define

$$\begin{aligned} q(x) =& 2\beta^8x^{12} + 30\beta^7x^{11} + (197\beta^6 - 12\beta^8)x^{10} + (724\beta^5 - 150\beta^7)x^9 \\ &+ (30\beta^8 - 793\beta^6 + 1620\beta^4)x^8 + (300\beta^7 - 2240\beta^5 + 2244\beta^3)x^7 \\ &+ (-40\beta^8 + 1202\beta^6 - 3612\beta^4 + 1868\beta^2)x^6 + (-300\beta^7 + 2376\beta^5 - 3276\beta^3 + 852\beta)x^5 \\ &+ (30\beta^8 - 818\beta^6 + 2364\beta^4 - 1548\beta^2 + 162)x^4 + \\ &+ (150\beta^7 - 928\beta^5 + 1068\beta^3 - 328\beta)x^3 + (-12\beta^8 + 217\beta^6 - 372\beta^4 + 180\beta^2 - 24)x^2 + \\ &+ (-30\beta^7 + 68\beta^5 - 36\beta^3 - 12\beta)x + 2\beta^8 - 5\beta^6 + 12\beta^2 - 10. \end{aligned}$$

The derivative of operator $O_1(x, \beta)$ is

$$O_1'(x, \beta) = \frac{(x^2-1)^3 q(x)}{(\beta(x^2-1) + 2x)^6 (2\beta(x^2-1)x + 3x^2 + 1)^2}.$$

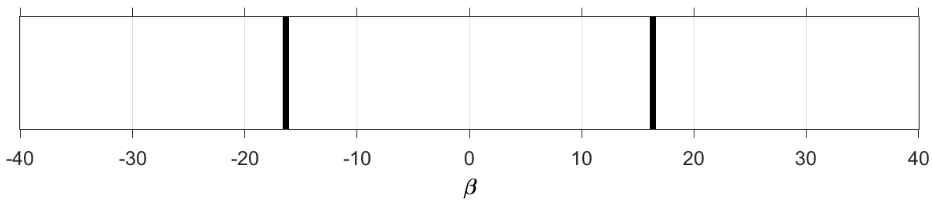
We study now the critical points, which we remember are those that, by evaluating them in the derivative of the rational operator, we obtain the value 0. It is obvious that the roots of p_1, that is, -1 and 1, are critical points. According to the values of β we have the following free critical points, that is to say, the critical points that are not roots of $p_1(x)$.

Proposition 4.4.0.2. *The number of critical points of operator $O_1(x, \beta)$ depending on parameter β are*

- *If* $0 < |\beta| < 0.327212$, *then four roots of* $q(x)$ *will be free critical points.*
- *If* $|\beta| = 0.327212$, *then five roots of* $q(x)$ *will be free critical points.*
- *If* $0.327212 < |\beta| < 0.528315$ *and* $|\beta| \neq \frac{1}{2}$, *then six roots of* $q(x)$ *will be free critical points.*
- *If* $|\beta| = \frac{1}{2}$, *then four roots of* $q(x)$ *will be free critical points.*
- *If* $|\beta| = 0.528135$, *then five roots of* $q(x)$ *will be free critical points.*
- *If* $0.528315 < |\beta|$ *and* $|\beta| \neq 2.20183$, *then four roots of* $q(x)$ *will be free critical points.*
- *If* $|\beta| = 2.20183$, *then three roots of* $q(x)$ *will be free critical points.*

To analyse these free critical points we observe Figure 4.5. It shows when any of the free critical points does not converge to any of the roots of $p_1(x)$. On the abscissa axis we have represented the parameter β. On this line we represent in black the values of the parameter β for which some of the critical points do not converge to 1 or -1, and in white the values of the parameter for which all the critical points converge to the roots of $p_1(x)$. We obtain the following line

Figure 4.5: Behaviour of the critical points



As in the case of the strange fixed points, it can be seen that the value of the parameter for which some of the critical points do not converge to the roots of $p_1(x)$ is -16.14039 and 16.14039. In this case, one of the critical point converges to the attracting strange fixed point seen above.
We now study the operator obtained by applying to the polynomial $p_0(x) = x^2$ the parametric family (4.3). We denote the rational operator by $O_0(x)$, which has the following expression

$$O_0(x, \beta) = \frac{x(\beta x + 1)\left(\beta^5 x^5 + 10\beta^4 x^4 + 39\beta^3 x^3 + 75\beta^2 x^2 + 71\beta x + 27\right)}{(\beta x + 2)^5 (2\beta x + 3)}.$$

Proposition 4.4.0.3. *For operator* $O_0(x)$ *we obtain a single real fixed point, which is* 0, *that is, the root of the polynomial, so there are no strange fixed points.*

Proof. We calculate the fixed points of operator $O_0(x, \beta)$ depending on β. The fixed points are obtained by solving $O_0(x, \beta) = x$, and in this case, the only fixed point is $x = 0$. □

Proposition 4.4.0.4. *The fixed point $z = 0$ of operator $O_0(x)$ is an attractor.*

Proof. We calculate the derivative of operator $O_0(x, \beta)$ to analyse the character of the fixed point 0 and then obtain the critical points.
If we denote

$$q_0(x) = 162+852\beta x+1868\beta^2x^2+2244\beta^3x^3+1620\beta^4x^4+724\beta^5x^5+197\beta^6x^6+30\beta^7x^7+2\beta^8x^8,$$

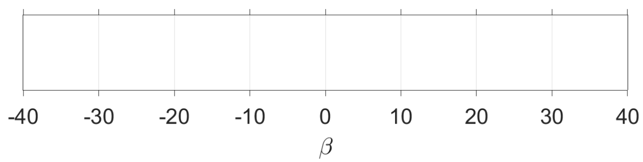
then

$$O_0'(x, \beta) = \frac{q_0(x)}{(\beta x + 2)^6(2\beta x + 3)^2}.$$

Since $O_0'(0, \beta) = \frac{9}{32} < 1$, it follows that $x = 0$ is an attracting fixed point. □

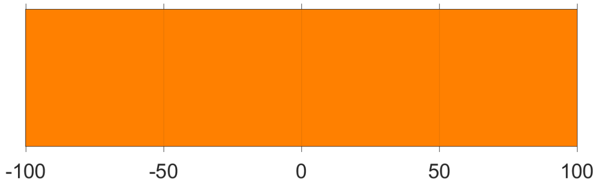
Let us now look at the real free critical points. In this case, we have that the free critical points are the roots of $q_0(x)$. We have that only two of them are real, which we denote by $E_{q_0,1}$ and $E_{q_0,2}$. Therefore we study the asymptotic behaviour of the free critical points.
We draw a line showing whether the free critical points $E_{q_0,1}$ and $E_{q_0,2}$ converge to the fixed point 0 or not. As can be seen in the following image, see Figure 4.6, in this case we have convergence of the critical points to the fixed point for any value of the parameter β shown. In black are represented the values of the parameter when one of the free critical points do not converge to fixed point 0, and in white are represented the values when all the critical points converge to 0.

Figure 4.6: Behaviour of critical points of O_0



We also show the dynamical line obtained for operator $O_0(x,\beta)$ when $\beta=-0.1$, see Figure 4.7. In orange are represented the initial guesses that converge to the root 0.

Figure 4.7: Real dynamical line of $O_0(x,\beta)$ when $\beta=-0.1$



Methods with memory

We are now going to study the real dynamics of the method with memory M_4N_1. This study is multidimensional since it is a method with memory. In this case, the polynomial $p_c(x)=x^2-c$ is chosen, when c is a positive real value.
The operator obtained by applying M_4N_1 method to this polynomial is denoted by O_{N_1}, and has the following expression, where $x_{k-1}=z$ and $x_k=x$

$$O_{N_1}(z,x)=\left(x,\phi_{N_1}(z,x)\right),$$

where

$$\phi_{N_1}(z,x)=\frac{\left(c-x^2\right)^3\left(c-z^2\right)(x+z)\left(\frac{\left(c-x^2\right)^2(x+z)^4}{(c+x(x+2z))^4}-\frac{\left(c-x^2\right)(x+z)^2}{(c+x(x+2z))^2}+1\right)}{(c+x(x+2z))^3\left(\frac{\left(c(2x+z)+x^2z\right)^2}{(c+x(x+2z))^2}-x^2\right)}+\frac{2cx+cz+x^2z}{c+x^2+2xz}.$$

We calculate the fixed points of operator $O_{N_1}(z,x)$.

Proposition 4.4.0.5. *The only fixed points of operator $O_{N_1}(z,x)$ are vectors whose components are the roots of the polynomial $p_c(x)$, that is, $(\sqrt{c},\sqrt{c})$ and $(-\sqrt{c},-\sqrt{c})$, and both are superattracting fixed points.*

Proof. To calculate the fixed points we simultaneously do $z=x$ and $O_{N_1}(z,x)=(x,x)$, which gives us the following operator

$$O_{N_1}(x,x)=\left(x,\ \frac{2x\left(\frac{16x^4\left(c-x^2\right)^2}{(c+3x^2)^4}-\frac{4x^2\left(c-x^2\right)}{(c+3x^2)^2}+1\right)\left(c-x^2\right)^4}{\left(c+3x^2\right)^3\left(\frac{(3cx+x^3)^2}{(c+3x^2)^2}-x^2\right)}+\frac{3cx+x^3}{c+3x^2}\right).$$

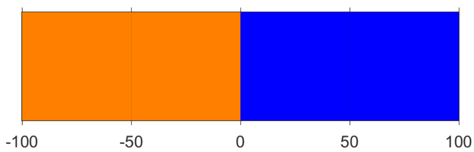
From this, the fixed points are $z = x = \sqrt{c}$ and $z = x = -\sqrt{c}$. Now, we are going to study the character of the fixed points. To do this, we evaluate the two fixed points in the Jacobian matrix and obtain that for both, the matrix obtained is

$$O'_{N_1}(\pm\sqrt{c}, \pm\sqrt{c}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

thus, obtaining that their associated eigenvalues are both 0, and therefore, they are superattracting fixed points. $\square$

It can also be checked that for any value of $c > 0$ operator O_{N_1} will have no free critical points. We now draw the dynamical line when $c = 1$, Figure 4.8, in order to compare it with the dynamical lines of the family. In this case, in blue are represented the initial guesses that converge to the root 1 and in orange the initial points that converge to the root -1.

Figure 4.8: Dynamical line of M_4N_1 for $c = 1$



We are now going to study the real dynamics of scheme M_4N_2. This study is also multidimensional since it is a method with memory.
The operator obtained by applying M_4N_2 method to $p_c(x) = x^2 - c$ is denoted by O_{N_2}. If we define

$$\phi_{N_2}(x) = 2659c^2x^{10} + 2291c^3x^8 + 781c^4x^6 + 189c^5x^4 + 17c^6x^2 + c^7 + 2063cx^{12} + 191x^{14},$$

then operator O_{N_2} has the following expression, being $x_{k-1} = z$, $y_{k-1} = zy$, $x_k = x$ and $y_k = xy$

$$O_{N_2}(z, zy, x) = \left(x, xy, \frac{\phi_{N_2}(x)}{4x\,(c + x^2)\,(c + 3x^2)^5} \right).$$

We calculate the fixed points of the resulting operator.

Proposition 4.4.0.6. *The only fixed points of $O_{N_2}(z, zy, x)$ are vectors whose components are the roots of the polynomial $p_c(x)$, that is, $(\sqrt{c}, \sqrt{c}, \sqrt{c})$ and $(-\sqrt{c}, -\sqrt{c}, -\sqrt{c})$, and both are superattracting fixed points.*

Proof. To calculate the fixed points we simultaneously do $z = x$, $zy = x$, which gives us the following operator

$$O_{N_2}(x, x, x) = \left(x, x, \frac{\phi_{N_2}(x)}{4x\,(c + x^2)\,(c + 3x^2)^5} \right).$$

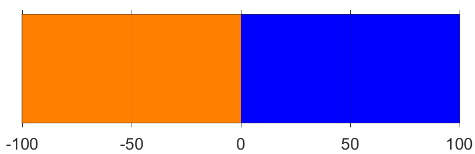
Now, we are going to study the character of the fixed points. To do this, we evaluate the two fixed points in the Jacobian matrix and we obtain that for both the matrix obtained is

$$O'_{N_2}(\pm\sqrt{c}, \pm\sqrt{c}, \pm\sqrt{c}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

thus obtaining that their associated eigenvalues are both 0, and therefore, they are superattractors. □

It can also be checked that for any value of $c > 0$ operator O_{N_2} will have no free critical points. We now draw the dynamical line when $c = 1$, Figure 4.9, in order to compare it with the dynamical lines of the family. In this case, in blue are represented the initial guesses that converge to the root 1 and in orange the initial points that converge to the root -1.

Figure 4.9: Dynamical line of M_4N_2 for $c = 1$



In this case, we do not study the dynamics of methods M_4N_3, M_4K_1 and M_4K_{1Y}, since the rational operator of these methods coincides with method M_4N_2 for the polynomial $p_c(x) = x^2 - c$, so the study is the same for those schemes.

If we compare the dynamics of the procedures with memory to that of the proposed M_4 parametric family, we can clearly see that the real dynamics of the methods with memory is much simpler than in the case without memory.

4.5 Adding a new step to the family of iterative methods

In this section we add a step to the parametric family (4.3). We want this new step to be similar to the second step of the already proposed family, that is why the parametric family of 3 steps that we propose is the following

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f[w_k, x_k]}, & \text{where } w_k = x_k + \beta f(x_k), \quad k = 0, 1, \ldots, \\ z_k = y_k - H(\mu_k)\dfrac{f(y_k)}{f[y_k, x_k]}, & \text{where } \mu_k = \dfrac{f(y_k)}{f(w_k)}, \\ x_{k+1} = z_k - G(\nu_k)\dfrac{f(z_k)}{f[z_k, y_k]}, & \text{where } \nu_k = \dfrac{f(z_k)}{f(y_k)}, \end{cases} \tag{4.42}$$

where $H(t)$ and $G(t)$ are real functions.
We denote this parametric family by M_6. We have proven that the parametric family (4.3) has order 4 under certain conditions, see Theorem 4.2.1. What we are going to see next is that the family (4.42) has order 6 under the same conditions and adding others on function G.

Theorem 4.5.1. *Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a sufficiently differentiable function in an neighbourhood of α denoted by $D \subset \mathbb{R}$ such that $f(\alpha) = 0$. We assume $f'(\alpha) \neq 0$. Let $H(t)$ be a real function satisfying $H(0) = 1$, $H'(0) = 1$, and $|H''(0)| < \infty$ and $G(t)$ be a real function satisfying $G(0) = 1$ and $|G'(0)| < \infty$. Then, taking an estimation x_0 close enough to α, the sequence of iterates $\{x_k\}_{k\geq 0}$, generated by family of iterative methods (4.42) converges to α with order 6.*

Proof. From Theorem 4.3.1 it follows that

$$e_z = \frac{1}{2}c_2\left(1 + \beta f'(\alpha)\right)\left(-2c_3\left(1 + \beta f'(\alpha)\right) + c_2^2\left(6 + 4\beta f'(\alpha) - H_2\right)\right)e_k^4 + O\left(e_k^5\right), \tag{4.43}$$

where $e_z = z_k - \alpha$ and $H_2 = H''(0)$.
We consider the Taylor development of $f(y_k)$ and of $f(z_k)$ around α

$$f(y_k) = f'(\alpha)\left(e_y + c_2 e_y^2 + c_3 e_y^3 + c_4 e_y^4 + c_5 e_y^5 + O\left(e_y^6\right)\right), \tag{4.44}$$

$$f(z_k) = f'(\alpha)\left(e_z + c_2 e_z^2 + c_3 e_z^3 + c_4 e_z^4 + c_5 e_z^5 + O\left(e_z^6\right)\right), \tag{4.45}$$

where $e_y = y_k - \alpha$ and $e_z = z_k - \alpha$.
We now calculate the expansion of $f[z_k, y_k]$ using (4.44) and (4.45),

$$\begin{aligned} f[z_k, y_k] &= \frac{f(z_k) - f(y_k)}{z_k - y_k} = \frac{f(z_k) - f(y_k)}{z_k - \alpha + \alpha - y_k} = \frac{f(z_k) - f(y_k)}{e_z - e_y} \\ &= \frac{f'(\alpha)\left((e_z - e_y) + c_2\left(e_z^2 - e_y^2\right) + c_3\left(e_z^3 - e_y^3\right) + c_4\left(e_z^4 - e_y^4\right) + O_5(e_z, e_y)\right)}{e_z - e_y}. \end{aligned}$$

Substituting the known expressions for e_y and e_z, it follows that

$$\begin{aligned} f[z_k, y_k] =& f'(\alpha)\Big(1 + c_2^2\left(1 + \beta f'(\alpha)\right) e_k^2 \\ &+ c_2 f'(\alpha)\left(-c_2^2\left(2 + 2\beta f'(\alpha) + \beta^2 f'(\alpha)^2\right) + c_3\left(2 + 3\beta f'(\alpha) + \beta^2 f'(\alpha)^2\right)\right)\Big) e_k^3 \\ &+ \left(e_k^4\right). \end{aligned}$$

We compute now the expansion of $\nu_k = \frac{f(z_k)}{f(y_k)}$,

$$\frac{f(z_k)}{f(y_k)} = \left(-c_3\left(1 + \beta f'(\alpha)\right) + c_2^2\left(3 + 2\beta f'(\alpha) - \frac{H_2}{2}\right)\right) e_k^2 + O\left(e_k^3\right).$$

We denote $G_0 = G(0)$, $G_1 = G'(0)$ and $G_2 = G''(0)$. It follows that

$$\begin{aligned} G(\nu_k) &= G_0 + G_1\nu_k + \frac{1}{2}G_2\nu_k^2 + O\left(\nu_k^3\right) = 1 + G_1\nu_k + \frac{G_2}{2}\nu_k^2 + O\left(\nu_k^3\right) \\ &= 1 + G_1\left(-c_3\left(1 + \beta f'(\alpha)\right) + c_2^2\left(3 + 2\beta f'(\alpha) - \frac{H_2}{2}\right)\right) e_k^2 + O(e_k^3). \end{aligned}$$

Let us then calculate $e_{k+1} = e_z - G(\nu_k)\frac{f(z_k)}{f[z_k, y_k]}$ using the above results

$$e_{k+1} = \frac{-c_2}{4}\left(1 + \beta f'(\alpha)\right)\left(-2c_3(1 + \beta f'(\alpha)) + c_2^2(6 + 4\beta f'(\alpha) - H_2)\right) \tag{4.46}$$

$$\left(-2c_3(1 + \beta f'(\alpha))G_1 + c_2^2(-2 + 6G_1 + 2\beta f'(\alpha)(-1 + 2G_1) - G_1H_2)\right) e_k^6 + O\left(e_k^7\right). \tag{4.47}$$

Thus it is proven that method (4.42) has order 6 under these conditions.
In particular, if $H_2 = H''(0) = 2$, then

$$e_{k+1} = c_2\left(2c_2^2 - c_3\right)\left(c_2^2\left(1 - 2G_1\right) + c_3G_1\right)\left(1 + \beta f'(\alpha)\right)^3 e_k^6 + O\left(e_k^7\right). \tag{4.48}$$

□

Now, we approximate the parameter in order to obtain a new method with memory. In the first case, we apply the same approximation that was done for the M_4 family, that is, we choose

$$\beta_k = -\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})},$$

and replace the parameter of family (4.42) by this expression, thus obtaining a method with memory, which we denote by M_6N_1.

Theorem 4.5.2. *Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a sufficiently differentiable function in an neighbourhood of α denoted by $D \subset \mathbb{R}$ such that $f(\alpha) = 0$. We assume $f'(\alpha) \neq 0$. Let $H(t)$ be a real function satisfying $H(0) = 1$, $H'(0) = 1$, $H''(0) = 2$ and $|H'''(0)| < \infty$ and $G(t)$ be a real function satisfying $G(0) = 1$ and $|G'(0)| < \infty$. Then, taking an estimation x_0 close enough to α, the sequence of iterates $\{x_k\}_{k\geq 0}$, generated by method M_6N_1 converges to α with order $p = 3 + 2\sqrt{3} \approx 6.4641$.*

Proof. From the error equation (4.48)

$$e_{k+1} \sim (1 + \beta_k f'(\alpha))^3 e_k^6.$$

Using the Taylor series developments of $f(x_k)$ and $f(x_{k-1})$ in the same way as in Theorem 4.3.1 we obtain

$$1 + \beta_k f'(\alpha) \sim c_2 e_{k-1}.$$

From the error equation (4.48) and the above relation we obtain

$$e_{k+1} \sim (c_2 e_{k-1})^3 e_k^6 \sim e_{k-1}^3 e_k^6. \tag{4.49}$$

Assuming that the R-order of the method is at least p, the relation (4.11) is obtained. In the same way as relation (4.49) is obtained,

$$e_{k+1} \sim e_{k-1}^3 (D_{k-1,p} e_{k-1}^p)^6 = D_{k-1,p}^6 e_{k-1}^{6p+3}. \tag{4.50}$$

Then by equating the exponents of e_{k-1} of (4.11) and (4.50), one has

$$p^2 = 6p + 3,$$

whose only positive solution is the order of convergence of the M_6N_1 method, that is $p \approx 6.4641$, according to Theorem 2.1.1. □

Just as the M_4K_1 method is defined, we now define the M_6K_1 method, that is, we choose

$$\beta_k = -\frac{1}{f[2x_k - x_{k-1}, x_{k-1}]},$$

and replace the parameter of family (4.42) by this expression, thus obtaining a method with memory, which we denote by M_6K_1.

Theorem 4.5.3. *Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a sufficiently differentiable function in an neighbourhood of α denoted by $D \subset \mathbb{R}$ such that $f(\alpha) = 0$. We assume $f'(\alpha) \neq 0$. Let $H(t)$ be a real function satisfying $H(0) = 1$, $H'(0) = 1$, $H''(0) = 2$ and $|H'''(0)| < \infty$ and $G(t)$ be a real function satisfying $G(0) = 1$ and $|G'(0)| < \infty$. Then, taking an estimation x_0 close enough to α, the sequence of iterates $\{x_k\}_{k\geq 0}$, generated by method M_6K_1 converges to α with order $p = 3 + \sqrt{15} \approx 6.873$.*

Proof. From the error equation (4.48)

$$e_{k+1} \sim (1+\beta_k f'(\alpha))^3 e_k^6.$$

Using the Taylor series developments of $f(x_k)$ and $f(x_{k-1})$ in the same way as in Theorem 4.3.2, one has

$$1+\beta_k f'(\alpha) \sim e_{k-1}^2.$$

From error equation (4.48) and the above relation, the following is obtained

$$e_{k+1} \sim \left(e_{k-1}^2\right)^3 e_k^6 \sim e_{k-1}^6 e_k^6. \tag{4.51}$$

Assuming that the R-order of the method is at least p yields relation (4.11). In the same way as relation (4.51) is obtained,

$$e_{k+1} \sim \left(e_{k-1}^2\right)^3 \left(D_{k-1,p} e_{k-1}^p\right)^6 = D_{k-1,p}^6 e_{k-1}^{6p+6}. \tag{4.52}$$

Then by equating the exponents of e_{k-1} of (4.11) and (4.52), one has

$$p^2 = 6p+6,$$

whose only positive solution is the order of convergence of the M_6K_1 method, that is $p = 3+\sqrt{15} \approx 6.873$, according to Theorem 2.1.1. □

Just as the M_4N_{1Y} method is defined, we now define the M_6N_{1Y} method, that is, we choose

$$\beta_k = -\frac{1}{f[x_k, y_{k-1}]},$$

and we substitute the parameter of family (4.42) by this expression, thus obtaining a method with memory, which we denote by M_6N_{1Y}.

Theorem 4.5.4. *Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a sufficiently differentiable function in an neighbourhood of α denoted by $D \subset \mathbb{R}$ such that $f(\alpha) = 0$. We assume $f'(\alpha) \neq 0$. Let $H(t)$ be a real function satisfying $H(0) = 1$, $H'(0) = 1$, $H''(0) = 2$ and $|H'''(0)| < \infty$ and $G(t)$ be a real function satisfying $G(0) = 1$ and $|G'(0)| < \infty$. Then, taking an estimation x_0 close enough to α, the sequence of iterates $\{x_k\}_{k\geq 0}$, generated by method M_6N_{1Y} converges to α with order $p = 7$.*

Proof. From the error equation (4.48)

$$e_{k+1} \sim (1+\beta_k f'(\alpha))^3 e_k^6.$$

Using the Taylor series developments of $f(x_k)$, $f(x_{k-1})$ and $f(y_{k-1})$ in the same way as in Theorem 4.3.3, we obtain

$$1+\beta_k f'(\alpha) \sim e_{y,k-1}.$$

From error equation (4.48) and the above relation we obtain

$$e_{k+1} \sim (e_{y,k-1})^3 e_k^6 \sim e_{y,k-1}^3 e_k^6. \tag{4.53}$$

Assuming that the R-order of the method is at least p yields relation (4.11), and assuming that sequence $\{y_k\}_{k\geq 0}$ converges with R-order at least p_1 yields relation (4.34).
In the same way as relation (4.67) is obtained, we get

$$e_{k+1} \sim \left(D_{k-1,p_1} e_{k-1}^{p_1}\right)^3 \left(D_{k-1,p} e_{k-1}^{p}\right)^6 \sim e_{k-1}^{3p_1} e_{k-1}^{6p}. \tag{4.54}$$

On the other hand, we have

$$e_{k,y} \sim (1+\beta_k f'(\alpha)) e_k^2 \sim e_{y,k-1} e_k^2 \sim \left(e_{k-1}^{p_1}\right) \left(e_{k-1}^{p}\right)^2 \sim e_{k-1}^{2p+p_1}. \tag{4.55}$$

Then by equating the exponents of e_{k-1} of (4.11) and (4.54), and by equating the exponents of (4.34) and (4.55), one has

$$\begin{aligned} p^2 &= 6p + 3p_1, \\ pp_1 &= 2p + p_1, \end{aligned}$$

whose only positive solution is the order of convergence of method M_6N_{1Y}, that is $p = 7$ and $p_1 \approx 2.333$, according to Theorem 2.1.1. □

Just as the M_4K_{1Y} method is defined, we now define the M_6K_{1Y} method, that is, we choose

$$\beta_k = -\frac{1}{f[2x_k - y_{k-1}, y_{k-1}]},$$

and we replace the parameter of family (4.42) by this expression, thus obtaining a method with memory, which we denote by M_6K_{1Y}.

Theorem 4.5.5. *Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a sufficiently differentiable function in an neighbourhood of α denoted by $D \subset \mathbb{R}$ such that $f(\alpha) = 0$. We assume $f'(\alpha) \neq 0$. Let $H(t)$ be a real function satisfying $H(0) = 1$, $H'(0) = 1$, $H''(0) = 2$ and $|H'''(0)| < \infty$ and $G(t)$ be a real function satisfying $G(0) = 1$ and $|G'(0)| < \infty$. Then, taking an estimation x_0 close enough to α, the sequence of iterates $\{x_k\}_{k\geq 0}$, generated by method M_6K_{1Y} converges to α with order $p = 8$.*

Proof. From the error equation (4.48)

$$e_{k+1} \sim (1+\beta_k f'(\alpha))^3 e_k^6.$$

Using the Taylor series developments of $f(x_k)$, $f(x_{k-1})$ and $f(y_{k-1})$ in the same way as in Theorem 4.3.4, one has

$$1+\beta_k f'(\alpha) \sim e_{y,k-1}^2.$$

By error equation (4.48) and the above relation we obtain

$$e_{k+1} \sim \left(e_{y,k-1}^2\right)^3 e_k^6 \sim e_{y,k-1}^6 e_k^6. \tag{4.56}$$

Assuming that the R-order of the method is at least p we obtain relation (4.11), and assuming that sequence $\{y_k\}_{k\geq 0}$ converges to R-order of at least p_1 we obtain the relation (4.34).
In the same way as relation (4.56) is obtained, we get

$$e_{k+1} \sim \left(D_{k-1,p_1} e_{k-1}^{2p_1}\right)^3 \left(D_{k-1,p} e_{k-1}^{p}\right)^6 \sim e_{k-1}^{6p_1} e_{k-1}^{6p}. \tag{4.57}$$

On the other hand, we have

$$e_{k,y} \sim (1+\beta_k f'(\alpha)) e_k^2 \sim e_{y,k-1}^2 e_k^2 \sim \left(e_{k-1}^{2p_1}\right)\left(e_{k-1}^{p}\right)^2 \sim e_{k-1}^{2p+2p_1}. \tag{4.58}$$

Then by equating the exponents of e_{k-1} of (4.11) and (4.57), and by equating the exponents (4.34) and (4.58), one has

$$\begin{aligned} p^2 &= 6p + 6p_1, \\ pp_1 &= 2p + 2p_1, \end{aligned}$$

whose only positive solution is the order of convergence of method M_6K_{1Y}, that is $p = 8$ and $p_1 \approx 2.666$, according to Theorem 2.1.1. $\square$

Now, we approximate the parameter by a Newton interpolating polynomial of degree 1 at nodes x_k and z_{k-1}, we choose

$$\beta_k = -\frac{1}{f[x_k, z_{k-1}]},$$

and we replace the parameter of family (4.42) by this expression, thus obtaining a method with memory, which we denote by M_6N_{1Z}.

Theorem 4.5.6. *Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a sufficiently differentiable function in an neighbourhood of α denoted by $D \subset \mathbb{R}$ such that $f(\alpha) = 0$. We assume $f'(\alpha) \neq 0$. Let $H(t)$ be a real function satisfying $H(0) = 1$, $H'(0) = 1$, $H''(0) = 2$ and $|H'''(0)| < \infty$ and $G(t)$ be a real function satisfying $G(0) = 1$ and $|G'(0)| < \infty$. Then, taking an estimation x_0 close enough to α, the sequence of iterates $\{x_k\}_{k\geq 0}$, generated by method M_6N_{1Z} converges to α with order $p = 8$.*

Proof. From error equation (4.48)

$$e_{k+1} \sim (1+\beta_k f'(\alpha))^3 e_k^6.$$

Using the Taylor series developments of $f(x_k)$, $f(x_{k-1})$ and $f(z_{k-1})$ in the same way as in the Theorem 4.3.3, we obtain

$$1 + \beta_k f'(\alpha) \sim e_{z,k-1},$$

provided that $p > p_1$ where p is the R-order of the method and p_1 is the R-order of sequence $\{z_k\}_{k\geq 0}$.
From the error equation (4.48) and the above relation we obtain

$$e_{k+1} \sim (e_{z,k-1})^3 e_k^6 \sim e_{z,k-1}^3 e_k^6. \tag{4.59}$$

Let us assume that the R-order of the method is at least p gives the relation (4.11), and assuming that sequence $\{z_k\}_{k\geq 0}$ converges to R-order of at least p_1 gives the relation

$$e_{z,k} \sim D_{k,p_1} e_k^{p_1} \sim D_{k,p_1}\left(D_{k-1,p} e_{k-1}^p\right)^{p_1} \sim D_{k,p_1} D_{k-1,p}^{p_1} e_{k-1}^{pp_1}. \tag{4.60}$$

In the same way as relation (4.59) is obtained,

$$e_{k+1} \sim \left(D_{k-1,p_1} e_{k-1}^{p_1}\right)^3 \left(D_{k-1,p} e_{k-1}^p\right)^6 \sim e_{k-1}^{3p_1} e_{k-1}^{6p}. \tag{4.61}$$

On the other hand, we have

$$e_{z,k} \sim (1+\beta_k f'(\alpha))^2 e_k^4 \sim e_{z,k-1}^2 e_k^4 \sim e_{k-1}^{4p+2p_1}. \tag{4.62}$$

Then by equating the exponents of e_{k-1} of (4.11) and (4.61), and by equating the exponents of (4.60) and (4.62), one has

$$\begin{aligned} p^2 &= 6p + 3p_1, \\ pp_1 &= 4p + 2p_1, \end{aligned}$$

whose only positive solution is the order of convergence of method M_6N_{1Z}, that is $p = 8$ and $p_1 \approx 5.333$, according to Theorem 2.1.1. □

Instead of using the previous divided difference operator, which was the divided difference operator at nodes x_k and z_{k-1}, we apply Kurchatov's divided difference operator at same nodes, therefore we choose

$$\beta_k = -\frac{1}{f[2x_k - z_{k-1}, z_{k-1}]},$$

and we substitute the parameter of family (4.42) by this expression, thus obtaining a method with memory, which we denote by M_6K_{1Z}.

Theorem 4.5.7. *Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a sufficiently differentiable function in an neighbourhood of α denoted by $D \subset \mathbb{R}$ such that $f(\alpha) = 0$. We assume $f'(\alpha) \neq 0$. Let $H(t)$ be a real function satisfying $H(0) = 1$, $H'(0) = 1$, $H''(0) = 2$ and $|H'''(0)| < \infty$ and $G(t)$ be a real function satisfying $G(0) = 1$ and $|G'(0)| < \infty$. Then, taking an estimation x_0 close enough to α, the sequence of iterates $\{x_k\}_{k\geq 0}$, generated by method M_6K_{1Z} converges to α with order $p = 9$.*

Proof. From error equation (4.48)

$$e_{k+1} \sim (1+\beta_k f'(\alpha))^3 e_k^6.$$

Using the Taylor series developments of $f(x_k)$, $f(x_{k-1})$ and $f(z_{k-1})$ in the same way as in Theorem 4.3.4, we obtain

$$1+\beta_k f'(\alpha) \sim e_{z,k-1}^2,$$

provided that $p > 2p_1$ where p is the R-order of the method and p_1 is the R-order of sequence $\{z_k\}_{k\geq 0}$. Otherwise we have that

$$1+\beta_k f'(\alpha) \sim e_k.$$

From error equation (4.48) and the above relation we obtain

$$e_{k+1} \sim \left(e_{z,k-1}^2\right)^3 e_k^6 \sim e_{z,k-1}^6 e_k^6. \tag{4.63}$$

Let us assume that the R-order of the method is at least p we obtain relation (4.11), and assuming that sequence $\{z_k\}_{k\geq 0}$ converges to R-order of at least p_1 we obtain the relation 4.71.
In the same way as the relation (4.63) is obtained, we get

$$e_{k+1} \sim (D_{k-1,p_1} e_{k-1}^{2p_1})^3 (D_{k-1,p} e_{k-1}^{p})^6 \sim e_{k-1}^{6p_1} e_{k-1}^{6p}. \tag{4.64}$$

On the other hand, we have

$$e_{z,k} \sim (1+\beta_k f'(\alpha))^2 e_k^4 \sim (e_{z,k-1}^2)^2 e_k^4 \sim e_{k-1}^{4p+4p_1}. \tag{4.65}$$

Then by equating the exponents of e_{k-1} of (4.11) and (4.64), and by equating the exponents of (4.60) and (4.65), one has

$$p^2 = 6p + 6p_1,$$
$$pp_1 = 4p + 4p_1,$$

whose only positive solution is $p = 10$ and $p_1 \approx 6.67$, therefore that $p > 2p_1$ is not satisfied, and thus

$$1+\beta_k f'(\alpha) \sim e_k.$$

From this it follows from the error equation that

$$e_{k+1} \sim e_k^3 e_k^6 \sim e_k^9, \tag{4.66}$$

and therefore method M_6K_{1Z} has order of convergence 9, according to Theorem 2.1.1. □

In the same way as M_4N_2 method is defined, we define the M_6N_2 method, that is to say, we choose

$$\beta_k = -\frac{1}{N_2'(x_k)},$$

and we substitute the parameter of family (4.42) by this expression, thus obtaining a method with memory, which we denote by M_6N_2.

Theorem 4.5.8. *Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a sufficiently differentiable function in an neighbourhood of α denoted by $D \subset \mathbb{R}$ such that $f(\alpha) = 0$. We assume $f'(\alpha) \neq 0$. Let $H(t)$ be a real function satisfying $H(0) = 1$, $H'(0) = 1$, $H''(0) = 2$ and $|H'''(0)| < \infty$ and $G(t)$ be a real function satisfying $G(0) = 1$ and $|G'(0)| < \infty$. Then, taking an estimation x_0 close enough to α, the sequence of iterates $\{x_k\}_{k\geq 0}$, generated by method M_6N_2 converges to α with order $p = \frac{1}{2}(7 + \sqrt{61}) \approx 7.4051$.*

Proof. From the error equation (4.48)

$$e_{k+1} \sim (1 + \beta_k f'(\alpha))^3 e_k^6.$$

Using the Taylor series developments of $f(x_k)$, $f(x_{k-1})$ and $f(y_{k-1})$ in the same way as in Theorem 4.3.5, we obtain

$$\begin{aligned} N_2'(x_k) =& f'(\alpha) + 2c_2 f'(\alpha)e_k + c_3 f'(\alpha)e_k e_y + c_3 f'(\alpha)(e_k - e_{y,k-1})e_{k-1} \\ &+ O_2(e_{y,k-1}, e_k, e_{k-1}). \end{aligned}$$

Thus, $1 + \beta_k f'(\alpha)$ can behave as e_k, as $e_k e_{y,k-1}$, as $e_{k-1}e_k$ or as $e_{k-1}e_{y,k-1}$.
It is obvious that $e_k e_{y,k-1}$ ends faster to zero than e_k when $k \to \infty$, and that $e_{k-1}e_k$ tends faster to zero than $e_{k-1}e_{y,k-1}$. For this reason we have to look at if e_k converges faster to zero than $e_{k-1}e_{y,k-1}$ does.
Suppose the R-order of the method is at least p. Consider sequence $\{y_k\}_{k\geq 0}$ generated by the first step of the method, and suppose that it converges with R-order at least p_1.
Therefore, it is satisfied

$$e_{k+1} \sim D_{k,p} e_k^p \qquad \text{and} \qquad e_{y,k} \sim D_{k,p_1} e_k^{p_1},$$

where $D_{k,p}$ tends to the asymptotic error constant, D_p, and where D_{k,p_1} tends to the asymptotic error constant, D_{p_1}, when $k \to \infty$.
Then,

$$\frac{e_k}{e_{k-1}e_{y,k-1}} \sim \frac{D_{k-1,p} e_{k-1}^p}{D_{k-1,p_1} e_{k-1} e_{k-1}^{p_1}}.$$

Then if $p \geq p_1 + 1$, it follows that

$$1 + \beta_k f'(\alpha) \sim e_{k-1}e_{y,k-1}.$$

From error equation (4.48) and the above relation, we obtain

$$e_{k+1} \sim (e_{k-1}e_{y,k-1})^3 e_k^6 \sim e_{k-1}^3 e_{y,k-1}^3 e_k^6. \tag{4.67}$$

Assuming that the R-order of the method is at least p we obtain relation (4.11), and assuming that sequence $\{y_k\}_{k\geq 0}$ converges to R-order of at least p_1 we obtain the relation (4.34).
In the same way as relation (4.67) is obtained, we get

$$e_{k+1} \sim e_{k-1}^3 \left(D_{k-1,p_1} e_{k-1}^{p_1}\right)^3 \left(D_{k-1,p} e_{k-1}^p\right)^6 \sim e_{k-1}^{3p_1} e_{k-1}^{6p+3}. \tag{4.68}$$

On the other hand, we have

$$e_{k,y} \sim (1+\beta_k f'(\alpha))e_k^2 \sim e_{k-1}e_{y,k-1}e_k^2 \sim e_{k-1}^{2p+1+p_1}. \tag{4.69}$$

Then by equating the exponents of e_{k-1} of (4.11) and (4.68), and by equating those of (4.34) and (4.69), one has

$$\begin{aligned} p^2 &= 6p+3+3p_1, \\ pp_1 &= 2p+1+p_1, \end{aligned}$$

whose only positive solution is the order of convergence of method M_6N_2, that is $p \approx 7.4051$ and $p_1 \approx 2.468$, according to Theorem 2.1.1. □

We now apply Newton's interpolating polynomial of degree 2 at nodes x_k, x_{k-1} and z_{k-1}, which is

$$N_{2z}(t) = f(x_k) + f[x_k, x_{k-1}](t-x_k) + f[x_k, x_{k-1}, z_{k-1}](t-x_k)(t-x_{k-1}).$$

Therefore, we choose

$$\beta_k = -\frac{1}{N'_{2z}(x_k)},$$

and we replace the parameter of family (4.42) by this expression, thus obtaining a method with memory, which we denote by M_6N_{2Z}.

Theorem 4.5.9. *Let $f: D \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a sufficiently differentiable function in an neighbourhood of α denoted by $D \subset \mathbb{R}$ such that $f(\alpha) = 0$. We assume $f'(\alpha) \neq 0$. Let $H(t)$ be a real function satisfying $H(0) = 1$, $H'(0) = 1$, $H''(0) = 2$ and $|H'''(0)| < \infty$ and $G(t)$ be a real function satisfying $G(0) = 1$ and $|G'(0)| < \infty$. Then, taking an estimation x_0 close enough to α, the sequence of iterates $\{x_k\}_{k\geq 0}$, generated by method M_6N_{2Z} converges to α with order $p = 4+\sqrt{19} \approx 8.3589$.*

Proof. From the error equation (4.48)

$$e_{k+1} \sim (1+\beta_k f'(\alpha))^3 e_k^6.$$

Using the Taylor series developments of $f(x_k)$, $f(x_{k-1})$ and $f(z_{k-1})$ in the same way as in Theorem 4.3.5, we obtain

$$\begin{aligned} N'_{2z}(x_k) =& f'(\alpha) + 2c_2 f'(\alpha)e_k + c_3 f'(\alpha)e_k e_{z,k-1} + c_3 f'(\alpha)(e_k - e_{z,k-1})e_{k-1} \\ &+ O_2(e_{z,k-1}, e_k, e_{k-1}). \end{aligned}$$

Thus, $1+\beta_k f'(\alpha)$ will behave like e_k, as $e_k e_{z,k-1}$, as $e_{k-1}e_k$ or as $e_{k-1}e_{z,k-1}$.
It is obvious that $e_k e_{z,k-1}$ tends faster to zero than e_k when $k \to \infty$, and that $e_{k-1}e_k$ tends faster to zero than $e_{k-1}e_{z,k-1}$. For this reason we have to look at if e_k converges faster to zero than $e_{k-1}e_{z,k-1}$ does.
Suppose the R-order of the method is at least p. Consider sequence $\{z_k\}_{k\geq 0}$ generated by the

second step of the method, and suppose that it converges to R-order at least p_1. Therefore, it satisfies

$$e_{k+1} \sim D_{k,p} e_k^p \quad \text{and} \quad e_{z,k} \sim D_{k,p_1} e_k^{p_1},$$

where $D_{k,p}$ tends to the asymptotic error constant, D_p, and where D_{k,p_1} tends to the asymptotic error constant, D_{p_1}, when $k \to \infty$.
Then

$$\frac{e_k}{e_{k-1}e_{z,k-1}} \sim \frac{D_{k-1,p}e_{k-1}^p}{D_{k-1,p_1}e_{k-1}e_{k-1}^{p_1}}.$$

Then if $p \geq p_1 + 1$, we have that

$$1 + \beta_k f'(\alpha) \sim e_{k-1}e_{z,k-1}.$$

From the error equation (4.48) and the above relation we obtain

$$e_{k+1} \sim (e_{k-1}e_{z,k-1})^3 e_k^6 \sim e_{k-1}^3 e_{z,k-1}^3 e_k^6. \tag{4.70}$$

Assuming that the R-order of the method is at least p gives relation (4.11), and assuming that sequence $\{z_k\}_{k\geq 0}$ converges to R-order of at least p_1 gives the relation

$$e_{z,k} \sim D_{k,p_1} e_k^{p_1} \sim D_{k,p_1}\left(D_{k-1,p}e_{k-1}^p\right)^{p_1} \sim D_{k,p_1} D_{k-1,p}^{p_1} e_{k-1}^{pp_1}. \tag{4.71}$$

. In the same way as relation (4.70) is obtained,

$$e_{k+1} \sim e_{k-1}^3 \left(D_{k-1,p_1}e_{k-1}^{p_1}\right)^3 \left(D_{k-1,p}e_{k-1}^p\right)^6 \sim e_{k-1}^{3p_1} e_{k-1}^{6p+3}. \tag{4.72}$$

On the other hand, we have

$$e_{z,k} \sim (1 + \beta_k f'(\alpha))^2 e_k^4 \sim e_{k-1}^2 e_{z,k-1}^2 e_k^4 \sim e_{k-1}^{4p+2+2p_1}. \tag{4.73}$$

Then by equating the exponents of e_{k-1} of (4.11) and (4.72), and by equating the exponents of (4.71) and (4.73), one has

$$\begin{aligned} p^2 &= 6p + 3 + 3p_1, \\ pp_1 &= 4p + 2 + 2p_1, \end{aligned}$$

whose only positive solution is the order of convergence of method M_6N_{2Z}, that is $p \approx 8.3589$ and $p_1 \approx 5.5726$, according to Theorem 2.1.1. $\square$

In the same way as M_4N_3 method is defined, we define the M_6N_3 method, that is, we choose

$$\beta_k = -\frac{1}{N_3'(x_k)},$$

and replace the parameter of family (4.42) by this expression, thus obtaining a method with memory, which we denote by M_6N_3.

Theorem 4.5.10. *Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a sufficiently differentiable function in an neighbourhood of α denoted by $D \subset \mathbb{R}$ such that $f(\alpha) = 0$. We assume $f'(\alpha) \neq 0$. Let $H(t)$ be a real function satisfying $H(0) = 1$, $H'(0) = 1$, $H''(0) = 2$ and $|H'''(0)| < \infty$ and $G(t)$ be a real function satisfying $G(0) = 1$ and $|G'(0)| < \infty$. Then, taking an estimation x_0 close enough to α, the sequence of iterates $\{x_k\}_{k\geq 0}$, generated by method M_6N_3 converges to α with order $p = 8$.*

Proof. From the error equation (4.48)

$$e_{k+1} \sim (1 + \beta_k f'(\alpha))^3 e_k^6.$$

Using the Taylor series developments of $f(x_k)$, $f(y_{k-1})$, $f(x_{k-1})$ and $f(w_{k-1})$ as was done in Theorem 4.3.6 we have then that

$$1 + \beta_k \sim 2c_2 e_k + c_4 e_{y,k-1} e_{k-1} e_{w,k-1}.$$

Thus $1 + \beta_k f'(\alpha)$ may behave as e_k or as $e_{k-1}e_{y,k-1}e_{w,k-1}$, since the other terms converge faster than these two. We now prove that the behaviour of $1 + \beta_k f'(\alpha)$ is like that of $e_{k-1}e_{y,k-1}e_{w,k-1}$.
Suppose that the R-order of the method is at least p. Therefore, it is satisfied

$$e_{k+1} \sim D_{k,p} e_k^p,$$

where $D_{k,p}$ tends to the asymptotic error constant, D_p, when $k \to \infty$. Moreover, if we assume that sequence $\{y_k\}_{k\geq 0}$ generated by the first step of the method and sequence $\{w_k\}_{k\geq 0}$, converge with R-order at least p_1 and at least p_2, respectively. Then

$$\frac{e_k}{e_{k-1}e_{y,k-1}e_{w,k-1}} \sim \frac{D_{k-1,p} e_{k-1}^p}{D_{k-1,p_1} D_{k-1,p_2} e_{k-1} e_{k-1}^{p_1} e_{k-1}^{p_2}},$$

where D_{k,p_1} and D_{k,p_2} tend to asymptotic error constants, D_{p_1} and D_{p_2}, respectively, when $k \to \infty$.
Then if $p \geq p_1 + p_2 + 1$, therefore

$$1 + \beta_k f'(\alpha) \sim c_4 e_{k-1} e_{y,k-1} e_{w,k-1}.$$

From error equation (4.48) and the above relation we obtain

$$e_{k+1} \sim (c_4 e_{k-1} e_{y,k-1} e_{w,k-1})^3 e_k^6 \sim e_{k-1}^3 e_{y,k-1}^3 e_{w,k-1}^3 e_k^6. \tag{4.74}$$

Assuming that the R-order of the method is at least p we obtain relation (4.11). We assume that sequence $\{y_k\}_{k\geq 0}$ and sequence $\{w_k\}_{k\geq 0}$ converge with R-order at least p_1 and at least p_2, respectively. Then, we obtain the relations defined in (4.34) and (4.38).
In the same way as relation (4.74) is obtained, we get

$$e_{k+1} \sim e_{k-1}^3 \left(D_{k-1,p_1} e_{k-1}^{p_1}\right)^3 \left(D_{k-1,p_2} e_{k-1}^{p_2}\right)^3 \left(D_{k-1,p} e_{k-1}^{p}\right)^6 \sim e_{k-1}^{3p_1+3p_2} e_{k-1}^{6p+3}. \tag{4.75}$$

On the other hand, we have

$$e_{y,k} \sim (1+\beta_k f'(\alpha))e_k^2 \sim e_{k-1}e_{y,k-1}e_{w,k-1}e_k^2 \sim e_{k-1}^{2p+1+p_1+p_2}. \tag{4.76}$$

And we also have that

$$e_{w,k} \sim (1+\beta_k f'(\alpha))e_k \sim e_{k-1}e_{y,k-1}e_{w,k-1}e_k \sim e_{k-1}^{p+1+p_1+p_2}. \tag{4.77}$$

Then by equating the exponents of e_{k-1} of (4.11) and (4.75), and by equating those of (4.34) and (4.76) and by equating the expons of (4.38) and (4.77), one has

$$\begin{aligned} p^2 &= 6p+3+3p_1+3p_2 \\ pp_1 &= 2p+1+p_1+p_2 \\ pp_2 &= p+1+p_1+p_2, \end{aligned}$$

whose only positive solution is the order of convergence of method M_6N_3, that is $p = 8$, $p_1 \approx 2.6667$ and $p_2 \approx 1.6667$, according to Theorem 2.1.1. □

As in the previous case a Newton interpolating polynomial of degree 3 was used, we now apply another interpolating polynomial, in this case also of degree 3, but at nodes x_k, x_{k-1}, z_{k-1} and w_{k-1}. Thus the polynomial is $N_{3z}(t) = N_{2z}(t) + f[x_k, x_{k-1}, z_{k-1}, w_{k-1}](t-x_k)(t-x_{k-1})(t-z_{k-1})$ In the following case we choose

$$\beta_k = -\frac{1}{N'_{3z}(x_k)},$$

and we replace the parameter of family (4.42) by this expression, thus obtaining a method with memory, which we denote by M_6N_{3Z}.

Theorem 4.5.11. *Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a sufficiently differentiable function in an neighbourhood of α denoted by $D \subset \mathbb{R}$ such that $f(\alpha) = 0$. We assume $f'(\alpha) \neq 0$. Let $H(t)$ be a real function satisfying $H(0) = 1$, $H'(0) = 1$, $H''(0) = 2$ and $|H'''(0)| < \infty$ and $G(t)$ be a real function satisfying $G(0) = 1$ and $|G'(0)| < \infty$. Then, taking an estimation x_0 close enough to α, the sequence of iterates $\{x_k\}_{k\geq 0}$, generated by method M_6N_{3Z} converges to α with order 9.*

Proof. From the error equation (4.48)

$$e_{k+1} \sim (1+\beta_k f'(\alpha))^3 e_k^6.$$

Using the Taylor series developments of $f(x_k)$, $f(x_{k-1})$, $f(z_{k-1})$ and $f(w_{k-1})$ around α in the same way as in Theorem 4.3.6, we obtain

$$1+\beta_k \sim 2c_2 e_k + c_4 e_{k-1}e_{z,k-1}e_{w,k-1}.$$

Thus $1+\beta_k f'(\alpha)$ may behave as e_k or as $e_{k-1}e_{z,k-1}e_{w,k-1}$, since the other terms converge faster than these two. We now prove that the behaviour of $1+\beta_k f'(\alpha)$ is like the behaviour of

$e_{k-1}e_{z,k-1}e_{w,k-1}$.
Suppose that the R-order of the method is at least p. Therefore, it is satisfied

$$e_{k+1} \sim D_{k,p}e_k^p,$$

where $D_{k,p}$ tends to the asymptotic error constant, D_p, when $k \to \infty$. Moreover, we assume that sequence $\{z_k\}_{k\geq 0}$ generated by the second step of the method and sequence $\{w_k\}_{k\geq 0}$, converge with R-order at least p_1 and at least p_2, respectively. Then

$$\frac{e_k}{e_{k-1}e_{z,k-1}e_{w,k-1}} \sim \frac{D_{k-1,p}e_{k-1}^p}{D_{k-1,p_1}D_{k-1,p_2}e_{k-1}e_{k-1}^{p_1}e_{k-1}^{p_2}},$$

where D_{k,p_1} and D_{k,p_2} tend to the asymptotic error constants, D_{p_1} and D_{p_2}, respectively, when $k \to \infty$.
Then if $p \geq p_1 + p_2 + 1$, one has

$$1 + \beta_k f'(\alpha) \sim c_4 e_{k-1}e_{z,k-1}e_{w,k-1}.$$

From error equation (4.48) and the above relation we obtain

$$e_{k+1} \sim (e_{k-1}e_{z,k-1}e_{w,k-1})^3 e_k^6 \sim e_{k-1}^3 e_{z,k-1}^3 e_{w,k-1}^3 e_k^6. \tag{4.78}$$

We assume that the R-order of the method is at least p we obtain relation (4.11). Assuming that sequence $\{z_k\}_{k\geq 0}$ and sequence $\{w_k\}_{k\geq 0}$ converge with R-order at least p_1 and at least p_2, respectively. Then, we obtain the relations defined in (4.71) and (4.38).
In the same way as relation (4.78) is obtained,

$$e_{k+1} \sim e_{k-1}^3 \left(D_{k-1,p_1}e_{k-1}^{p_1}\right)^3 \left(D_{k-1,p_2}e_{k-1}^{p_2}\right)^3 \left(D_{k-1,p}e_{k-1}^{p}\right)^6 \sim e_{k-1}^{3p_1+3p_2} e_{k-1}^{6p+3}. \tag{4.79}$$

On the other hand, we have

$$e_{z,k} \sim (1 + \beta_k f'(\alpha))^2 e_k^4 \sim e_{k-1}^2 e_{z,k-1}^2 e_{w,k-1}^2 e_k^4 \sim e_{k-1}^{4p+2+2p_1+2p_2}. \tag{4.80}$$

And we also have that

$$e_{w,k} \sim (1 + \beta_k f'(\alpha))e_k \sim e_{k-1}e_{z,k-1}e_{w,k-1}e_k \sim e_{k-1}^{p+1+p_1+p_2}. \tag{4.81}$$

Then by equating the exponents of e_{k-1} of (4.11) and (4.79), and by equating those of (4.71) and (4.80) and by equating the exponents of (4.38) and (4.81), one has

$$\begin{aligned} p^2 &= 6p + 3 + 3p_1 + 3p_2 \\ pp_1 &= 4p + 2 + 2p_1 + 2p_2 \\ pp_2 &= p + 1 + p_1 + p_2, \end{aligned}$$

whose only positive solution is the order of convergence of method M_6N_{3Z}, that is $p = 9$, $p_1 = 6$ and $p_2 = 2$, according to Theorem 2.1.1. $\square$

Next, we show Table 4.1 where we have a collection of the different convergence orders obtained by introducing memory to families M_4 and M_6.

Table 4.1: Collection of the different orders of convergence

Parameter approximation	Method	Order
	M_4	4
$f[x_k, x_{k-1}]$	M_4N_1	$2+\sqrt{6} \approx 4.4495$
$f[2x_k - x_{k-1}, x_{k-1}]$	M_4K_1	$2+2\sqrt{2} \approx 4.8284$
$f[x_k, y_{k-1}]$	M_4N_{1Y}	5
$f[2x_k - y_{k-1}, y_{k-1}]$	M_4K_{1Y}	6
$N_2'(x_k)$	M_4N_2	$\frac{1}{2}(5+\sqrt{33}) \approx 5.37228$
$N_3'(x_k)$	M_4N_3	6
	M_6	6
$f[x_k, x_{k-1}]$	M_6N_1	$3+2\sqrt{3} \approx 6.4641$
$f[2x_k - x_{k-1}, x_{k-1}]$	M_6K_1	$2+\sqrt{15} \approx 6.873$
$f[x_k, y_{k-1}]$	M_6N_{1Y}	7
$f[2x_k - y_{k-1}, y_{k-1}]$	M_6K_{1Y}	8
$f[x_k, z_{k-1}]$	M_6N_{1Z}	8
$f[2x_k - z_{k-1}, z_{k-1}]$	M_6K_{1Z}	9
$N_2'(x_k)$	M_6N_2	$\frac{1}{2}(7+\sqrt{61}) \approx 7.4051$
$N_{2Z}'(x_k)$	M_6N_{2Z}	$4+\sqrt{19} \approx 8.3589$
$N_3'(x_k)$	M_6N_3	8
$N_{3Z}'(x_k)$	M_6N_{3Z}	9

Let us remark that the highest order of convergence is reached by M_6K_{1Z} and M_6N_{3Z}, but the computational cost of the first one is much lower than those of the later one.

4.6 Numerical experiments

In this section, we perform several numerical experiments in order to show the behaviour of the methods proposed in the chapter.
We use Matlab R2020b with variable precision arithmetic with 2000 digits for the computational calculations, iterating from an initial estimation x_0 until the following stopping criterion is satisfied

$$|x_{k+1} - x_k| + |f(x_{k+1})| < 10^{-100}.$$

The numerical results we are going to compare the methods in these examples are as follows

- the estimation to the solution obtained,
- the absolute value of the nonlinear function evaluated in that estimation (which we denote by $|f(x_{k+1})|$ in the tables),

- the absolute value of the distance between the last two estimations (which we denote by $|x_{k+1} - x_k|$ in the tables),
- the number of iterations needed to satisfy the required tolerance (which we denote by Iteration in the tables) in seconds,
- the computational time (which we denote by Time in the tables)
- and the approximate computational order of convergence (ACOC).

The functions we used are as follows

- $f_1(x) = \cos(x) - x$, which has a root $\alpha \approx 0.73908513$.
- $f_2(x) = \arctan(x)$, which has a root $\alpha = 0$.
- $f_3(x) = \arctan(x) - \frac{2x}{x^2+1}$, which has a root $\alpha \approx -1.39175$.
- $f_4(x) = (x-1)^3 - 1$, which has a root $\alpha = 2$.

We use the quadratic polynomial $H(\mu_k) = \mu_k^2 + \mu_k + 1$ as the weight function for all methods because is the easiest polynomial that satisfy the conditions to ensure the convergence of the different methods. As the parameter β we use in all the cases $\beta = -1$. Table 4.2 lists the initial estimates that are used for each equation.

Table 4.2: Initial estimations

Function	x_0	x_{-1}	w_{-1}	y_{-1}	z_{-1}
$f_1(x)$	1	2	1.75	1.5	1.3
$f_2(x)$	0.75	1.5	1.25	1	0.9
$f_3(x)$	-1	-0.25	-0.5	-0.75	-0.85
$f_4(x)$	1.5	0	0.5	1.1	1.3

We show the results obtained for the different methods presented in the chapter for equation $\cos(x) - x = 0$ in Table 4.3. As can be seen in the ACOC column, in this case, the theoretical convergence order coincides with the ACOC.
Among the methods of two steps, similar results can be seen in terms of number of iterations and the value of the function at the last iteration. We observe that the method without memory, in this case, requires one more iteration to satisfy the stopping criterion.

Among the methods of three steps, we can see that same number of iterations is required to satisfy the stopping criterion, but the computational time required by each method differs considerably. We observe that the method with memory that requires the least time is also the one that obtains the best approximation and the highest ACOC, that is, method M_6K_{1Z}, although time

of M_6N_{3Z} us better with similar ACOC.
As a conclusion of this numerical experiment it is obtained that all the methods obtain similar results, although it would be advisable in this case to use method M_6N_{3Z}, since it is one of the methods with the highest ACOC and obtains a great approximation as we can see in Table 4.3.

The recommended method among the two step methods, is M_4K_{1Y} because it obtains a better approximation and higher ACOC with the same number of iterations as the other schemes, with similar performance as M_4N_3, with the same order of convergence.

Table 4.3: Results for the equation $\cos(x) - x = 0$

Method	$\lvert x_{k+1} - x_k\rvert$	$\lvert f(x_{k+1})\rvert$	Iteration	ACOC	Time
M_4	7.98384×10^{-225}	1.76651×10^{-897}	5	4	0.2547
M_4N_1	6.36668×10^{-125}	6.52553×10^{-555}	4	4.4822	0.2172
M_4K_1	3.88156×10^{-171}	3.96378×10^{-824}	4	4.95099	0.5188
M_4N_{1Y}	9.72756×10^{-165}	3.46507×10^{-823}	4	4.99744	0.2813
M_4K_{1Y}	6.84252×10^{-270}	2.18741×10^{-1618}	4	6.00073	0.3875
M_4N_2	2.43352×10^{-190}	8.10487×10^{-1022}	4	5.3755	0.2500
M_4N_3	8.22231×10^{-257}	5.63335×10^{-1540}	4	5.99732	0.2891
M_6	2.42699×10^{-192}	2.75185×10^{-1151}	4	6.0	0.2625
M_6N_1	3.48199×10^{-391}	2.99111×10^{-2527}	4	6.48815	0.3281
M_6K_1	6.00966×10^{-530}	2.4162×10^{-3638}	4	6.97266	0.6937
M_6N_{1Y}	5.69681×10^{-476}	3.77689×10^{-3331}	4	6.99725	0.6750
M_6K_{1Y}	1.61818×10^{-721}	6.53861×10^{-5772}	4	7.98899	0.6594
M_6N_{1Z}	1.32316×10^{-627}	1.93001×10^{-5019}	4	7.99805	0.7203
M_6K_{1Z}	1.89884×10^{-855}	4.48316×10^{-7697}	4	9.0	0.6828
M_6N_2	6.99999×10^{-533}	4.91149×10^{-3945}	4	7.41158	0.4219
M_6N_{2Z}	3.23046×10^{-693}	5.69868×10^{-5793}	4	8.36077	0.3250
M_6N_3	4.52655×10^{-681}	4.6994×10^{-5448}	4	7.99385	0.3844
M_6N_{3Z}	4.45028×10^{-838}	6.87813×10^{-7544}	4	8.99977	0.3172

We show now the results obtained by the different methods proposed in the chapter for the equation $\arctan(x) = 0$ in Table 4.4. As can be seen in the ACOC column, in this case, the theoretical order of convergence is lower than the ACOC for all methods.
Among the methods of two steps, we can see that the methods with memory increase their order by one unit, but the method without memory increases it by 5 units, which makes this method more suitable in this case, and also needs less time and requires the fewest iterations to satisfy the tolerance, as can be seen in Table 4.4.
Among the methods of three steps, we can see that the method without memory increases its ACOC order by one unit for this numerical example, and that the methods with memory increase it by more than one unit, resulting in method M_6K_{1Z} being the method with the highest ACOC. In this case all the methods perform the same number of iterations. The method that gives the best approximation is method M_6K_{1Z} followed by methods M_6N_{1Z} and M_6N_{3Z}, although the

rest of the methods are not very different from them.
As a conclusion of this numerical experiment, the recommended methods in this case, with three and two steps, would be method M_6K_{1Z} and method M_4, respectively.

Table 4.4: Results for the equation $\arctan(x) = 0$

Method	$\lvert x_{k+1} - x_k\rvert$	$\lvert f(x_{k+1})\rvert$	Iteration	ACOC	Time
M_4	1.73764×10^{-292}	1.783×10^{-2628}	4	9.0	0.3266
M_4N_1	6.26995×10^{-560}	2.31762×10^{-3190}	5	5.6937	0.4484
M_4K_1	4.87295×10^{-556}	3.55754×10^{-3168}	5	5.69382	0.4469
M_4N_{1Y}	1.92116e$\times10^{-156}$	4.14261×10^{-1135}	4	7.2191	0.2625
M_4K_{1Y}	3.09301×10^{-163}	1.72471×10^{-1184}	4	7.25542	0.3031
M_4N_2	6.4109×10^{-126}	2.48132×10^{-811}	4	6.44746	0.4531
M_4N_3	9.83907×10^{-160}	1.2967×10^{-1158}	4	7.2503	0.5250
M_6	9.63399×10^{-303}	2.88138×10^{-2122}	4	7	0.3594
M_6N_1	2.40435×10^{-233}	6.78309×10^{-1810}	4	7.81347	0.3203
M_6K_1	7.26769×10^{-255}	5.77411×10^{-1977}	4	7.82898	0.3422
M_6N_{1Y}	1.10005×10^{-357}	4.06924×10^{-3367}	4	9.36642	0.2656
M_6K_{1Y}	1.54723×10^{-383}	1.25996×10^{-3610}	4	9.39834	0.3141
M_6N_{1Z}	3.96266×10^{-481}	2.54191×10^{-4789}	4	11.1413	0.2766
M_6K_{1Z}	2.60934×10^{-544}	1.81813×10^{-4890}	4	11.1632	0.3187
M_6N_2	2.76956×10^{-308}	2.72641×10^{-2642}	4	8.5653	0.3531
M_6N_{2Z}	7.34124×10^{-375}	8.011×10^{-2937}	4	9.40829	0.3312
M_6N_3	1.13781×10^{-375}	7.5513×10^{-2939}	4	9.3993	0.4094
M_6N_{3Z}	2.53673×10^{-448}	7.51537×10^{-4609}	4	10.2653	0.3438

We show the results obtained by the different methods proposed for the equation $f_3(x) = 0$ in Table 4.5. As can be seen in the ACOC column, in this case, the theoretical convergence order coincides with the ACOC.
Among the methods of two steps, we can see that there are differences in the number of iterations required and the computational time taken. If we analyse which method obtain better approximation with fewer iterations, we have that in this case it is method M_4N_3.
Among the methods of three steps, the only method that performs one more iteration to satisfy the stopping criterion is method M_6N_1. We observe that the method that obtains a better approximation in this case is method M_6N_{3Z}.
As a conclusion, it is obtained that in this case it is more advisable to use method M_4N_3 than the rest of the two step methods and it would be advisable to use method M_6N_{3Z} since it is one of the methods that obtains the highest order, and by far the method that obtains the closest approximation to the solution as can be seen in the third column of Table 4.5.

Table 4.5: Results for the equation $\arctan(x) - \frac{2x}{(x^2+1)} = 0$

Method	$\lvert x_{k+1} - x_k\rvert$	$\lvert f(x_{k+1})\rvert$	Iteration	ACOC	Time
M_4	3.83003×10^{-287}	4.31925×10^{-1149}	5	4	0.2484
M_4N_1	4.49563×10^{-177}	2.0329×10^{-788}	6	4.4329	0.3578
M_4K_1	1.08803×10^{-451}	2.69748×10^{-2008}	5	4.82737	0.3422
M_4N_{1Y}	3.16171×10^{-164}	7.17119×10^{-823}	4	4.95602	0.2625
M_4K_{1Y}	1.46333×10^{-220}	6.63869×10^{-1325}	4	6.07599	0.2844
M_4N_2	1.69774×10^{-171}	6.23524×10^{-921}	4	5.38703	0.2734
M_4N_3	7.11925×10^{-222}	9.37588×10^{-1330}	4	5.99633	0.3063
M_6	1.81148×10^{-213}	7.02595×10^{-1281}	4	6	0.2344
M_6N_1	6.11018×10^{-297}	1.72607×10^{-1919}	5	6.48618	0.3625
M_6K_1	8.81864×10^{-256}	2.90803×10^{-1756}	4	6.87588	0.5313
M_6N_{1Y}	5.10933×10^{-416}	1.31614×10^{-2914}	4	6.95424	0.5047
M_6K_{1Y}	7.28593×10^{-552}	2.87494×10^{-4416}	4	7.96108	0.5781
M_6N_{1Z}	4.60816×10^{-521}	8.7338×10^{-4170}	4	7.98726	0.5047
M_6K_{1Z}	1.076×10^{-684}	2.39326×10^{-6162}	4	9.00003	0.5406
M_6N_2	2.04398×10^{-453}	6.47836×10^{-3357}	4	7.41241	0.3063
M_6N_{2Z}	3.6637×10^{-692}	4.99673×10^{-5784}	4	8.37841	0.3109
M_6N_3	8.71194×10^{-555}	2.15171×10^{-4436}	4	7.98899	0.3391
M_6N_{3Z}	2.88326×10^{-675}	4.75202×10^{-6180}	4	8.99721	0.3297

We show the results obtained by the different methods for the equation $(x-1)^3 - 1 = 0$ in Table 4.6. As can be seen in the ACOC column, in this case, the theoretical order of convergence coincides with the ACOC in all methods except in M_6N_3 where the order increases by one unit. Among the two step methods, the methods M_4K_{1Y} and M_4N_3 require one less iteration to satisfy the tolerance. If we compare which of them obtains a better approximation, we can see that method M_4N_2 is the one that obtains a better approximation, but it performs one more iteration than M_4K_{1Y}, so the recommended one in this case is M_4K_{1Y} as we can see on Table 4.6.

Among the three step methods, almost all of them perform 5 iterations. We observe in column 3 that the method that obtains a better approximation is method M_6K_{1Z}.
The conclusion of this numerical experiment is that the most recommended methods are method M_4K_{1Y} and M_6K_{1Z}, since they obtain a better approximation and are ones of the methods that require the fewest iterations to satisfy the stopping criteria.

Table 4.6: Results for the equation $(x-1)^3 - 1 = 0$

Method	$\lvert x_{k+1} - x_k \rvert$	$\lvert f(x_{k+1}) \rvert$	Iteration	ACOC	Time
M_4	1.067×10^{-181}	2.59228×10^{-723}	6	4	0.2906
M_4N_1	2.03723×10^{-285}	7.96453×10^{-1267}	6	4.44956	0.3859
M_4K_1	2.69898×10^{-346}	4.85264×10^{-1669}	6	4.8361	0.4453
M_4N_{1Y}	4.33709×10^{-318}	4.60379×10^{-1587}	6	5.00014	0.3719
M_4K_{1Y}	1.39248×10^{-181}	7.13286×10^{-1085}	5	6.00027	0.3750
M_4N_2	1.5561×10^{-416}	1.49505×10^{-2234}	6	5.4087	0.3875
M_4N_3	6.26407×10^{-151}	4.83317×10^{-901}	5	6	0.3625
M_6	1.28999×10^{-225}	1.2288×10^{-1348}	5	6	0.3016
M_6N_1	1.07719×10^{-218}	4.0404×10^{-1409}	5	6.47927	0.3063
M_6K_1	3.78521×10^{-242}	1.62794×10^{-1660}	5	6.85364	0.5469
M_6N_{1Y}	8.79934×10^{-361}	1.22538×10^{-2520}	6	7.0162	0.6266
M_6K_{1Y}	1.38495×10^{-584}	1.35351×10^{-4672}	5	7.98531	0.5766
M_6N_{1Z}	7.76951×10^{-186}	1.65981×10^{-1480}	5	7.98009	0.5500
M_6K_{1Z}	8.21326×10^{-537}	2.26765×10^{-4824}	5	8.99996	0.5609
M_6N_2	2.44389×10^{-303}	2.62354×10^{-2242}	6	7.36596	0.4422
M_6N_{2Z}	2.2108×10^{-192}	1.54585×10^{-1602}	5	8.53031	0.3656
M_6N_3	9.73317×10^{-479}	1.04526×10^{-4301}	5	9.0	0.4281
M_6N_{3Z}	9.73317×10^{-479}	1.04526×10^{-4301}	5	9.0	0.3984

4.7 Conclusions

In this work, two parametric families of iterative methods with orders of convergence 4 and 6, respectively, for solving nonlinear equations, have been designed from Traub's scheme.

Memory has been introduced, in different ways, to these two families in order to obtain iterative methods with higher order of convergence without the need to increase the number of functional evaluations per iteration. These methods with memory have managed to increase the order by up to 2 units for the family of order 4 and up to 3 units for the family of order 6.

But not only does the introduction of memory improve the order of convergence, we study the stability of these schemes with memory for the sake of comparison. We conclude that, in general, the behaviour of these methods is similar, and that wide convergence zones are obtained for the function analyzed. Finally, we also perform numerical experiments, and it can be seen that the introduction of memory helps to obtain better results in general, than those obtained by their partners without memory.

Chapter 5

Iterative methods to obtain solutions simultaneously

Based on [Cordero, A.; Garrido, N.; Torregrosa, JR.; Triguero-Navarro, P. (2022). Iterative schemes for finding all roots simultaneously of nonlinear equations. Applied Mathematics Letters. https://doi.org/10.1016/j.aml.2022.108325]

5.1 Introduction

In the most of applied mathematics problems we analytically solve nonlinear equations of the form $f(x) = 0$, $f : D \subseteq \mathbb{C} \rightarrow \mathbb{C}$, but in general, solving these equations is usually not possible. Iterative methods are useful because, given an initial estimation of the solution, they generate a sequence of iterations that, under certain conditions, converge to a solution of the nonlinear equation $f(x) = 0$.
Usually iterative methods focus on obtaining a single solution to the problem, but sometimes we need to obtain more than one solution. This is the reason why, iterative methods that obtain roots simultaneously arise, which, given a set of initial estimations, obtain a set of sequences of iterations that, under certain conditions, converge to all the roots of the equation simultaneously. We want to emphasise that when we say that we obtain all the roots simultaneously, we are referring to obtaining as many roots as we wish or as possible depending on the problem, since sometimes we can have infinite solutions or not as many as we imagine. Some iterative methods for simultaneous roots are designed by Proinov et al. [31, 32, 33, 34] and Petković et al. [35, 36]. Most methods that obtain roots simultaneously found in the literature focus on polynomial problems rather than arbitrary problems.

In [37], Ehrlich presented an iterative method of order 3 for polynomials that simultaneously computes all the zeros of a polynomial $p(x)$. It is defined by the fixed point iteration

$$x^{(k+1)} = \Psi(x^{(k)}) = \left(\psi_1(x^{(k)}), \psi_2(x^{(k)}), \ldots, \psi_n(x^{(k)})\right),$$

where $\Psi : \mathcal{D} \subseteq \mathbb{C}^n \rightarrow \mathbb{C}^n$ and

$$\psi_i(x^{(k)}) = x_i^{(k)} - \frac{p(x_i^{(k)})}{p'(x_i^{(k)}) - p(x_i^{(k)}) \sum_{j \neq i} 1/\left(x_i^{(k)} - x_j^{(k)}\right)}, \quad i = 1, 2, \ldots, n.$$

and where $x^{(k)}$ denotes $x^{(k)} = (x_i^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)})$.

The chapter is structured as follows. In Section 5.2, given a fixed point iterative scheme of order p and by using an Ehrlich-type method, we set out the structure of the iterative method and prove that $2p$ convergence order is obtained when solving nonlinear equations, which is not usually done, and that $3p$ convergence order is obtained when solving polynomial equations. In Section 5.3, several numerical experiments are performed to compare the results obtained by methods with the proposed structure and other known methods that obtain all roots simultaneously. This section also analyses the dynamical planes obtained by different known iterative methods and compare their behaviour with that of their modified partner that obtains the roots simultaneously.

5.2 Design and convergence analysis

Let us consider a nonlinear equation $f(x) = 0$ with n simple roots, which we denote by α_i for $i = 1, \ldots, n$.

We consider the iterative fixed point method of the form $x^{(k+1)} = \phi(x^{(k)})$.
From the idea of Ehrlich's scheme, we consider the following two-step iterative method to simultaneously approximate the roots of $f(x) = 0$. This is denoted by ϕ_S, using ϕ as a predictor, whose iterative expression is the following

$$\begin{cases} y_i^{(k)} = \phi(x_i^{(k)}), \quad i = 1, \ldots, n \\ x_i^{(k+1)} = y_i^{(k)} - \dfrac{f(y_i^{(k)})}{f'(y_i^{(k)}) - f(y_i^{(k)}) \sum_{j=1, j\neq i}^{n} \frac{1}{y_i^{(k)} - y_j^{(k)}}}, \quad i = 1, 2, \ldots, n. \end{cases} \tag{5.1}$$

Next, we prove that, if the iterative method ϕ has order of convergence p, then the iterative method ϕ_S has order of convergence $2p$, being f an arbitrary nonlinear function.

Theorem 5.2.1. *Let $f : D \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a sufficiently differentiable function in a neighbourhood D of α_i for $i = 1, \ldots, n$, such that $f(\alpha_i) = 0$ for $i = 1, \ldots, n$. We also assume that $f'(\alpha_i) \neq 0$ for $i = 1, \ldots, n$. If ϕ is an iterative method with order p, then, taking an initial estimation $x^{(0)}$ close enough to $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, the sequence of iterates $\{x^{(k)}\}_{k\geq 0}$ generated by method ϕ_S, with iterative expression (5.1), converges to α with order $2p$.*

Proof. We denote by $e_{i,k} = x_i^{(k)} - \alpha_i$, the error of ith component of iterate $x^{(k)}$ and by $e_{y,i,k} = y_i^{(k)} - \alpha_i$, the error of ith component of iterate $y^{(k)}$. Since ϕ is an iterative scheme that has order of convergence p, then we know that $e_{y,i,k+1} \sim e_{i,k}^p$.
Applying Taylor's developments of $f(y_i^{(k)})$ and $f'(y_i^{(k)})$ around α_i, we obtain

$$f(y_i^{(k)}) = f'(\alpha) \left(e_{y,i,k+1} + C_2 e_{y,i,k+1}^2 \right) + O\left(e_{y,i,k+1}^3 \right).$$

$$f'(y_i^{(k)}) = f'(\alpha) \left(1 + 2C_2 e_{y,i,k+1} \right) + O\left(e_{y,i,k+1}^2 \right).$$

To simplify the expressions, we denote $S_i(y^{(k)})$ as

$$S_i(y^{(k)}) = \sum_{j=1, j\neq i}^{n} \frac{1}{y_i^{(k)} - y_j^{(k)}}.$$

From the above expressions

$$f'(y_i^{(k)}) - f(y_i^{(k)}) S_i(y^{(k)}) \sim f'(\alpha) \left(1 + \left(2C_2 - S_i(y^{(k)}) \right) e_{y,i,k+1} \right).$$

Then,

$$\begin{aligned}
x_i^{(k+1)} - \alpha_i = y_i^{(k)} - \alpha_i - & \frac{f(y_i^{k+1})}{f'(y_i^{(k)}) - f(y_i^{(k)})S_i(y^{(k)})} \\
&= e_{y,i,k+1} - \frac{e_{y,i,k+1} + C_2 e_{y,i,k+1}^2 + O\left(e_{y,i,k+1}^3\right)}{1 + \left(2C_2 - S_i(y^{(k)})\right) e_{y,i,k+1} + O\left(e_{y,i,k+1}^2\right)} \\
&= \frac{e_{y,i,k+1}\left(1 + \left(2C_2 - S_i(y^{(k)})\right) e_{y,i,k+1}\right) - \left(e_{y,i,k+1} + C_2 e_{y,i,k+1}^2\right)}{1 + \left(2C_2 - S_i(y^{(k)})\right) e_{y,i,k+1} + O\left(e_{y,i,k+1}^2\right)} \\
&+ O\left(e_{y,i,k+1}^3\right) \\
&= \frac{\left(C_2 - S_i(y^{(k)})\right) e_{y,i,k+1}^2 + O\left(e_{y,i,k+1}^3\right)}{1 + \left(2C_2 - S_i(y^{(k)})\right) e_{y,i,k+1} + O\left(e_{y,i,k+1}^2\right)} \\
&\sim \left(C_2 - S_i(y^{(k)})\right) e_{y,i,k+1}^2 + O\left(e_{y,i,k+1}^3\right).
\end{aligned}$$

Therefore,

$$e_{i,k+1} \sim \left(C_2 - S_i(y^{(k)})\right) e_{y,i,k+1}^2 + O\left(e_{y,i,k+1}^3\right). \tag{5.2}$$

Thus, by relation (5.2) and since ϕ has order p, we obtain that

$$e_{i,k+1} \sim e_{y,i,k+1}^2 \sim \left(e_{i,k}^p\right)^2 \sim e_{i,k}^{2p}.$$

Therefore, it is proven that method ϕ_S has order of convergence $2p$. □

The above theorem has been proved for any nonlinear function, being, to the best of our knowledge, the first result in this line. However, the order of convergence obtained can be increased if the function is polynomial. We prove in the following result that the order increases to $3p$ when we are dealing with polynomial equations.

Theorem 5.2.2. *Let $f : D \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a polynomial function in a neighbourhood D of α_i for $i = 1, \ldots, n$, such that $p(\alpha_i) = 0$ for $i = 1, \ldots, n$, and $p(x)$ is a polynomial function. We assume that $p'(\alpha_i) \neq 0$ for $i = 1, \ldots, n$. If ϕ is an iterative method with order p, then, taking an estimate $x^{(0)}$ close enough to $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, the sequence of iterates $\{x^{(k)}\}_{k\geq 0}$ generated by method ϕ_S converges to α with order $3p$.*

Proof. Since ϕ is an iterative scheme that has order of convergence p, then we know that $e_{y,i,k+1} \sim e_{i,k}^p$, being $e_{i,k} = x_i^{(k)} - \alpha_i$ and $e_{y,i,k} = y_i^{(k)} - \alpha_i$.
If the iterates $y_i^{(k)}$ are close to α_i, for $i = 1, \ldots, n$, we can approximate function $p(x)$ by the

following expression

$$p(y_i^{(k)}) \approx \prod_{j=1}^{n} \left(y_i^{(k)} - \alpha_j\right).$$

It also follows that the derivative of $p(x)$ can be approximated by

$$p'(y_i^{(k)}) \approx \sum_{r=1}^{n} \prod_{j=1, j\neq r}^{n} \left(y_i^{(k)} - \alpha_j\right).$$

Thus, the following expression is obtained

$$\frac{p'(y_i^{(k)})}{p(y_i^{(k)})} \approx \frac{\sum_{r=1}^{n} \prod_{j=1, j\neq r}^{n} \left(y_i^{(k)} - \alpha_j\right)}{\prod_{j=1}^{n} \left(y_i^{(k)} - \alpha_j\right)} = \sum_{j=1}^{n} \frac{1}{y_i^{(k)} - \alpha_j}.$$

Thus,

$$\begin{aligned}
\frac{p'(y_i^{(k)})}{p(y_i^{(k)})} - \sum_{j=1, j\neq i}^{n} \frac{1}{y_i^{(k)} - y_j^{(k)}} &\approx \sum_{j=1}^{n} \frac{1}{y_i^{(k)} - \alpha_j} - \sum_{j=1, j\neq i}^{n} \frac{1}{y_i^{(k)} - y_j^{(k)}} \\
&\approx \frac{1}{y_i^{(k)} - \alpha_i} + \sum_{j=1, j\neq i}^{n} \left(\frac{1}{y_i^{(k)} - \alpha_j} - \frac{1}{y_i^{(k)} - y_j^{(k)}} \right) \\
&\approx \frac{1}{y_i^{(k)} - \alpha_i} + \sum_{j=1, j\neq i}^{n} \frac{y_i^{(k)} - y_j^{(k)} - \left(y_i^{(k)} - \alpha_j\right)}{\left(y_i^{(k)} - \alpha_j\right)\left(y_i^{(k)} - y_j^{(k)}\right)} \\
&\approx \frac{1}{y_i^{(k)} - \alpha_i} + \sum_{j=1, j\neq i}^{n} \frac{\alpha_j - y_j^{(k)}}{\left(y_i^{(k)} - \alpha_j\right)\left(y_i^{(k)} - y_j^{(k)}\right)}.
\end{aligned} \tag{5.3}$$

Since method $y_j^{(k)} = \phi(x_j^{(k)})$ has order of convergence p, this means that $y_k^{(k)}$ satisfies $y_j^{(k)} - \alpha_j = M_{j,k} e_{j,k}^{p} + O\left(e_{j,k}^{p+1}\right)$, where $M_{j,k}$ is a constant, for $j = 1, \ldots, n$.
If we replace this error in (5.3), we obtain

$$\begin{aligned}
\frac{p'(y_i^{(k)})}{p(y_i^{(k)})} - \sum_{j=1, j\neq i}^{n} \frac{1}{y_i^{(k)} - y_j^{(k)}} &\approx \frac{1}{y_i^{(k)} - \alpha_i} + \sum_{j=1, j\neq i}^{n} \frac{\alpha_j - y_j^{(k)}}{\left(y_i^{(k)} - \alpha_j\right)\left(y_i^{(k)} - y_j^{(k)}\right)} \\
&\approx \frac{1}{e_{y,i,k+1}} + \sum_{j=1, j\neq i}^{n} \frac{M_{j,k} e_{j,k}^{p} + O\left(e_{j,k}^{p+1}\right)}{\left(y_i^{(k)} - \alpha_j\right)\left(y_i^{(k)} - y_j^{(k)}\right)}.
\end{aligned}$$

If we denote by $E_{j,k} = \left(y_i^{(k)} - \alpha_j\right)\left(y_i^{(k)} - y_j^{(k)}\right)$, then

$$\frac{p'(y_i^{(k)})}{p(y_i^{(k)})} - \sum_{j=1,j\neq i}^{n} \frac{1}{y_i^{(k)} - y_j^{(k)}} \& \approx \frac{1}{e_{y,i,k+1}} + \sum_{j=1,j\neq i}^{n} \frac{M_{j,k}e_{j,k}^{p} + O\left(e_{j,k}^{p+1}\right)}{E_{j,k}}.$$

To simplify the notation, we denote $R_i(y^{(k)})$ as

$$R_i(y^{(k)}) = \sum_{j=1,j\neq i}^{n} \frac{M_{j,k}e_{j,k}^{p} + O\left(e_{j,k}^{p+1}\right)}{E_{j,k}}.$$

Thus, the error equation can be expressed as

$$\begin{aligned}
x_i^{(k+1)} - \alpha_i = y_i^{(k)} - \alpha_i - \frac{1}{\frac{p'(y_i^{(k)})}{p(y_i^{(k)})} - \sum_{j=1,j\neq i}^{n} \frac{1}{y_i^{(k)} - y_j^{(k)}}} \\
= e_{y,i,k+1} - \frac{1}{\frac{1}{e_{y,i,k+1}} + R_i(y^{(k)})} \\
= e_{y,i,k+1} - \frac{e_{y,i,k+1}}{1 + e_{y,i,k+1}R_i(y^{(k)})} \\
= \frac{e_{y,i,k+1}\left(1 + e_{y,i,k+1}R_i(y^{(k)})\right) - e_{y,i,k+1}}{1 + M_{i,k}e_{i,k}^{p}R_i(y^{(k)})} \\
= \frac{e_{y,i,k+1}^{2}R_i(y^{(k)})}{1 + M_{i,k}e_{i,k}^{p}R_i(y^{(k)})}.
\end{aligned}$$

By applying that $e_{y,i,k+1} = M_{i,k}e_{i,k}^{p} + O\left(e_{i,k}^{p+1}\right)$, we have

$$\begin{aligned}
x_i^{(k+1)} - \alpha_i = \frac{\left(e_{i,k}^{2p} + O\left(e_{i,k}^{2p+1}\right)\right)R_i(y^{(k)})}{1 + M_{i,k}e_{i,k}^{p}R_i(y^{(k)})} \\
= \frac{e_{i,k}^{2p}R_i(y^{(k)}) + O_{3p+1}\left(e_k\right)}{1 + M_{i,k}e_{i,k}^{p}R_i(y^{(k)})},
\end{aligned}$$

where $O_{3p+1}\left(e_k\right)$ denotes terms where the sum of the orders of the error product of e_k is at least $3p+1$, since the order of $R_i(y^{(k)})$ is p.

Then,

$$\begin{aligned} x_i^{(k+1)} - \alpha_i &\sim e_{i,k}^{2p} R_i(y^{(k)}) \\ &\sim e_{i,k}^{2p} \sum_{j=1, j\neq i}^{n} \frac{M_{j,k} e_{j,k}^{p}}{E_{j,k}} \\ &\sim e_{i,k}^{2p} \sum_{j=1, j\neq i}^{n} e_{j,k}^{p}. \end{aligned}$$

Thus, it is proven that ϕ_S method has order of convergence $3p$ when $p(x) = 0$ is a polynomial equation. □

5.3 Numerical experiments

In this section, we perform different numerical test in order to observe the behavior of the proposed methods. In this case, we use as predictors, as discussed in the previous section, Newton', Steffensen' [11], N_4 and N_8 methods designed in [28], and M_4 and M_6 schemes constructed in [38]. We denote these procedures in the same way as in the previous section, that is, if the method is denoted by ϕ, then its variant with the added step is denoted by ϕ_S. Furthermore, we compare the results obtained by these modified schemes with those of the following well-known methods for simultaneous roots: Ehrlich's method [37] (denoted by E), Shams' method [39] (denoted by SH) and Petkovic's method [35] (denoted by P), all with order of convergence 3 when they are applied on polynomials.

Matlab 2020b has been used to carry out the numerical experiments, with variable precision arithmetics with 6000 digits. As stopping criterion we choose

$$\left\| x^{(k+1)} - x^{(k)} \right\|_2 + \left\| F(x^{(k+1)}) \right\|_2 < 10^{-200},$$

where $F(x^{(k+1)}) = \left(f(x_1^{(k+1)}), \ldots, f(x_n^{(k+1)}) \right)$.
We use also a maximum of 100 iterations.
In the different tables we show the following data

- the norm of the function evaluated in the last iteration, $\left\| F(x^{(k+1)} \right\|_2$,
- the norm of the distance between the last two approximations, $\left\| x^{(k+1)} - x^{(k)} \right\|_2$,
- the number of iterations needed to satisfy the required tolerance,
- and the approximated computational order of convergence (ACOC).

The first numerical experiment we perform is to solve all the roots of the polynomial of degree 10, $x^{10} - 1 = 0$. As initial estimate we choose vector

$$x^{(0)} = (-2, 2, 0.5 + i, 0.5 - i, -0.5 + i, -0.5 - i, -1 + 0.5i, -1 - 0.5i, 1 + 0.5i, 1 - 0.5i)\,.$$

In Table 5.1, we show the results obtained by proposed and known iterative methods for the polynomial of 10th-degree. We can see that P scheme is the only method that does not converge to the roots for this set of initial estimations. In addition, ACOC matches the expected results for the designed methods.

According to the number of iterations, most of the proposed methods require fewer iterations than known ones, with method $N_{8,S}$ performing the fewest iterations. In the second and third columns, we can see that methods $N_{8,S}$ and $M_{6,S}$ provide the best results taking into account the number of iterations needed.

Table 5.1: Results for equation $x^{10} - 1 = 0$

Method	$\left\lVert x^{(k+1)} - x^{(k)} \right\rVert_2$	$\left\lVert F(x^{(k+1)}) \right\rVert_2$	Iteration	ACOC
N_S	1.5372×10^{-1001}	4.5113×10^{-6016}	6	6.025
S_S	3.5248×10^{-530}	4.6789×10^{-3177}	15	6.020
$N_{4,S}$	2.5614×10^{-207}	3.4721×10^{-2482}	4	12.026
$N_{8,S}$	5.0785×10^{-710}	1.1834×10^{-8007}	4	23.988
$M_{4,S}$ $\beta = 0.01$	3.2734×10^{-1775}	7.3494×10^{-8008}	5	12.087
$M_{6,S}$ $\beta = 0.01$	3.3994×10^{-327}	9.0694×10^{-5945}	4	18.504
P	n.c.	n.c	n.c	n.c
SH	7.9468×10^{-511}	2.0186×10^{-2007}	6	4.9999
E	2.9015×10^{-553}	3.1822×10^{-1657}	8	3.0

Now, we calculate all the roots of the nonlinear equation $e^{x^2} - x = 0$. As initial estimations we choose $x^{(0)} = (-i,\, i)$. The results obtained by the proposed and known iterative methods are shown in Table 5.2.

We see that for this initial estimation all the methods converge to the roots. Furthermore, that the ACOC of the designed schemes is identical to the expected one, which is twice as high as the original method. According to the number of iterations, all the proposed methods require significantly fewer iterations than known ones. In Table 5.2, we can see that the best results are obtained by $N_{8,S}$ and $M_{6,S}$ methods, taking into account how many iterations they perform.

Table 5.2: Results for equation $e^{x^2} - x = 0$

Method	$\left\|x^{(k+1)} - x^{(k)}\right\|_2$	$\left\|F(x^{(k+1)})\right\|_2$	Iteration	ACOC
N_S	1.2767×10^{-427}	1.3179×10^{-1708}	6	4
S_S	1.0824×10^{-224}	1.9281×10^{-896}	6	4
$N_{4,S}$	6.4008×10^{-215}	6.6603×10^{-1716}	4	8.0
$N_{8,S}$	8.9419×10^{-1739}	1.9074×10^{-2008}	4	16.0
$M_{4,S}$ $\beta = 0.01$	7.756×10^{-274}	1.9074×10^{-2008}	4	8
$M_{6,S}$ $\beta = 0.001$	4.4876×10^{-829}	1.9074×10^{-2008}	4	12
P	1.8968×10^{-217}	5.0848×10^{-434}	12	2.0
SH	2.3099×10^{-274}	7.5408×10^{-548}	12	2.0
E	2.6495×10^{-371}	9.9211×10^{-742}	12	2.0

An example of dynamics

In the following, we generate several dynamical planes for some of the methods discussed to compare them from the additional perspective of the wideness of their basins of attraction.
What we draw in these cases is whether or not the initial points converge to the roots of our problems. We apply this idea on Newton', Steffensen's and M_4 method, both the original and the variants by adding the step to obtain all the roots simultaneously.

In this case, we only show the dynamical planes associated with each of the methods when they are applied to a simple quadratic polynomial, $p(x) = x^2 - 1$, whose roots are 1 and -1.
To generate the dynamical planes, we have chosen a mesh of 400×400 points, and what we do is apply our methods to each of these points, taking them as the initial estimate.
For the non-simultaneous methods, one of the axes is the real part of the initial point, and the other is the imaginary part. For methods that are simultaneous, one of the axes is the initial estimate $x_1^{(0)}$ and the other is the initial estimate $x_2^{(0)}$, being both real.
We have also defined that the maximum number of iterations each initial estimate must do is 80, and that we determine that the initial point converges to one of the solutions if the distance to that solution is lower than 10^{-3}.

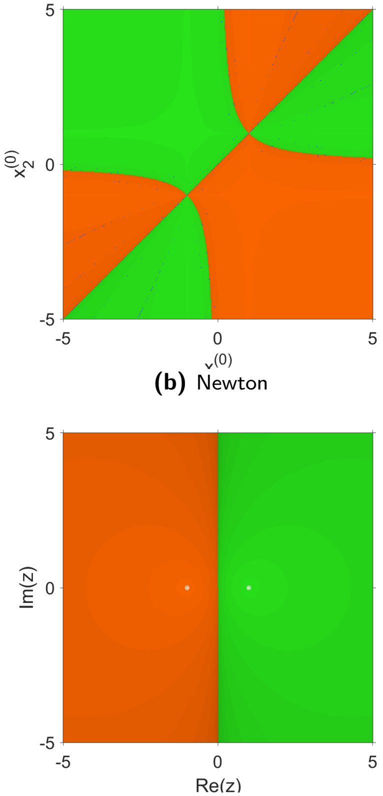
For the original methods, we represent in orange the initial points that converge to the root -1, in green the initial points that converge to the root 1 and in blue the initial points that do not converge to any root.
For the modified schemes, we represent the initial point green if the part of the point on the $x_1^{(0)}$ axis converges to the -1 roots and the part on the $x_2^{(0)}$ axis converges to the 1 root, we represent the point orange if the part on the $x_1^{(0)}$ axis converges to the 1 root and the part on the $x_2^{(0)}$ axis converges to the -1 root. In case of non-convergence we represent the initial point blue.

In Figure 5.1, we show the dynamical planes obtained for the quadratic polynomial of Newton' and N_S methods. As we can see, the basins of attraction show global convergence in Newton's procedure, as they do for its variant to find roots simultaneously.

Figure 5.1: Dynamical planes of N_S ans Newton' methods

(a) N_S

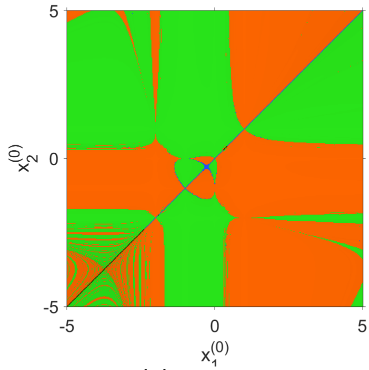
(b) Newton



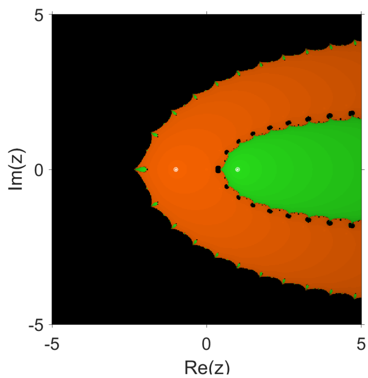
In Figure 5.2, we show the dynamical planes obtained for the quadratic polynomial of Steffensen and S_S methods. In this case, Steffensen's scheme does not converge in some areas, as for example at the point $z = -5$, although we can observe that its variant S_S does converge to the roots at any point of this mesh, except in a small area around $x_1^{(0)} = x_2^{(0)} = 0$.

Figure 5.2: Dynamical planes of S_S and Steffensen' methods

(a) S_S



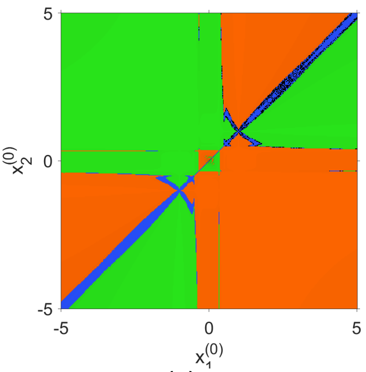
(b) Steffensen



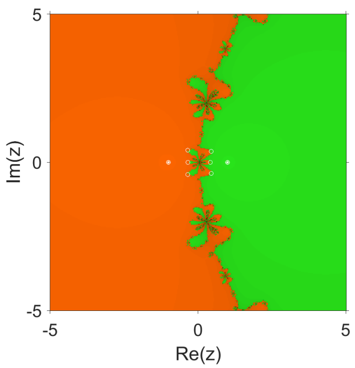
Also,in Figure 5.3, we show the dynamical planes obtained for the quadratic polynomial of M_4 and $M_{4,S}$ methods. As we can observe, in this case we obtain that the dynamical plane of $M_{4,S}$ has blue zones of non-convergence to the roots. This behavior corresponds to the higher order of convergence, as the denominator of the second step is closer to zero. This is solved by using more digits in the calculation, but the conditions have been held, for the sake of consistency.

Figure 5.3: Dynamical planes of $M_{4,S}$ and M_4' methods

(a) $M_{4,S}$



(b) M_4



In the figures shown above we can see, for example, that in the non-simultaneous case, the initial estimations $z = 2$ and $z = 5$ both converge to the same root, 1, but if we take the simultaneous variant where one of the components of the estimation is 2 and the other is 5, we obtain convergence to both roots simultaneously.

5.4 Conclusions

In this chapter, we have defined a general procedure that can be used in any iterative scheme for scalar nonlinear problems. This process introduces an iterative step to any iterative method in such a way that the new iterative scheme is able to find the roots simultaneously. Moreover, this new method increases the order of convergence of the original scheme twice and even three times when a polynomial equation is being solved.

We have selected a number of known iterative methods to which we applied this procedure, and we performed different numerical experiments to test the behavior of these new iterative methods. We found that the ACOC is similar to the theoretical order of convergence. The results obtained from these iterative methods have also been compared with other methods that find the roots simultaneously, and it has been observed that the proposed methods generally perform less iterations, because the order is higher, therefore they reach the stopping criterion earlier, particularly for the non-polynomial nonlinear equation.

It has also been concluded, by means of a qualitative study, that adding this iterative step modifies the basins of attraction. In general, however, the basins of attraction generally are similar or better than in the original schemes, in terms of the width of the basins of attraction of the roots.

Chapter 6

Iterative methods for multiple roots

Based on [Cordero, A.; Garrido, N.; Torregrosa, JR.; Triguero-Navarro, P. (2022). Modifying Kurchatov's method to find multiple roots of nonlinear equations. Applied Numerical Mathematics. Submitted] and on [Cordero, A.; Garrido, N.; Torregrosa, JR.; Triguero-Navarro, P. (2023). An iterative scheme to obtain multiple roots simultaneously. Applied Mathematics Letters. Submitted]

6.1 Introduction

Many problems in engineering or applied mathematics, require to solve nonlinear equations $f(x) = 0$. They cannot always be solved exactly, which is why iterative methods appear to solve them. As we have discussed in previous chapters, one of the most well-known methods is Newton's method.

To ensure the convergence of Newton's method, the derivative of the function evaluated in the solution must be non-zero, that is, the solution must be a simple root of $f(x) = 0$. This is not always the case. If the root is multiple instead of simple, what usually happens to this method is divergence or linear convergence instead of quadratic one. For this reason, iterative methods appear that allow us to obtain roots with a multiplicity greater than 1.

Numerous iterative schemes, without memory, involving or not derivatives, are designed for approximating the multiple roots of a nonlinear equation $f(x) = 0$. For this reason, there appear iterative methods that allow us to obtain solutions with a multiplicity greater than 1, for example [40, 29, 27, 41, 42, 43, 44] contain a collection of iterative schemes created to approximate the multiple roots of a nonlinear equation $f(x) = 0$. In the most of them, the authors assume that the multiplicity is known and it appears in the iterative expression of the method. In order to be able to apply this methods, we must know the multiplicity of the solution in advance. But the multiplicity is not always known in advance, and for this reason, iterative methods for multiple roots that do not use the multiplicity in their iterative expression are designed.

It is known that Schröder scheme, see [45], whose scheme is

$$x_{k+1} = x_k - \frac{f(x_k)f'(x_k)}{f'(x_k)^2 - f(x_k)f''(x_k)}, \text{ for } k = 0, 1, \ldots$$

has order of convergence 2 for multiple roots of the $f(x) = 0$. This method was designed from Newton's scheme applied to $g(x) = \dfrac{f(x)}{f'(x)}$. Its main feature is that you do not need to know in advance the multiplicity of the solution, which does not appear in the iterative expression.

In a similar way, in paper [46], an iterative method with memory is constructed to approximate the multiple roots, which avoids the need to know the multiplicity in advance.

In this chapter, we modify Kurchatov's method to obtain an iterative scheme for finding multiple roots of nonlinear equations. Kurchatov's procedure is an iterative scheme second-order of convergence, obtained from Newton's scheme by replacing the derivative by the divided difference of Kurchatov at nodes x_k and x_{k-1}, that is $f[2x_k - x_{k-1}, x_{k-1}]$.

The iterative method has the following structure

$$x_{k+1} = x_k - \frac{f(x_k)}{f[2x_k - x_{k-1}, x_{k-1}]}, k = 1, 2, \ldots$$

Applying the same idea that is applied in [45] and [46], we define the following method, denoted by KM, to estimate the solutions of $f(x) = 0$

$$x_{k+1} = x_k - \frac{g(x_k)}{g[2x_k - x_{k-1}, x_{k-1}]}, \qquad k = 0, 1, 2, \dots$$

where $g(x) = \frac{f(x)}{f'(x)}$ and $g[y, z] = \dfrac{g(y) - g(z)}{y - z}$.

To calculate the expression of $g(x)$ in the previous method we use the derivative of the function to be solved. We can replace this derivative by a divided difference operator, so that to estimate the solutions of $f(x) = 0$, we define the following method, denoted by KMD,

$$x_{k+1} = x_k - \frac{g(x_k)}{g[2x_k - x_{k-1}, x_{k-1}]}, \qquad k = 0, 1, 2, \dots$$

where $g(x) = \frac{f(x)}{f[x+f(x),x]}$ and $g[y, z] = \dfrac{g(y) - g(z)}{y - z}$.

This chapter is structured as follows. In Section 6.2, we perform the convergence analysis of the iterative method KM with memory, to find multiple roots without the knowledge of its multiplicity. A dynamical analysis of the rational function obtained by applying the proposed scheme on low-degree polynomials is presented in Section 6.3. In Section 6.4, we perform the convergence analysis of the iterative scheme KMD with memory, to find multiple roots without the knowledge of its multiplicity. Finally, in Section 6.5, we apply the iterative step presented in Chapter 5, to approximate all the solutions simultaneously to the iterative procedure KM to obtain an iterative method that calculates all the solutions with multiplicity equal or greater than 1 simultaneously. In Section 6.6, we perform several numerical experiments with the Kurchatov scheme for multiple roots and compare the results obtained by this scheme with other known ones designed of the same kind. We conclude this chapter in Section 6.7 with some conclusions and future work.

6.2 Convergence analysis

We are going to prove that the scheme KM maintains the order of convergence of Kurchatov's scheme, that is, its order of convergence is 2.

Theorem 6.2.1. *Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a sufficiently differentiable function in a neighbourhood of α which we denote by $D \subset \mathbb{R}$ such that α is a multiple root of $f(x) = 0$ with unknown multiplicity $m \in \mathbb{N}\backslash\{1\}$. Then, taking an estimate x_0 close enough to α, the sequence of iterates $\{x_k\}_{k\geq 0}$ generated by method KM converges to α with order 2, and the error equation is*

$$e_{k+1} = \left(\frac{-1}{m} C_1 e_k^2 + \frac{(m+1)C_1^2 - 2mC_2}{m^2} \left(-5e_k^3 + 2e_k^2 e_{k-1} - e_k e_{k-1}^2 \right) \right) + O_4(e_k, e_{k-1}),$$

being $C_j = \frac{m!}{(m+j)!} \frac{f^{(m+j)}(\alpha)}{f^{(m)}(\alpha)}$ for $j = 1, 2, \dots$

Proof. We first obtain the Taylor development of $f(x_k)$ around α,

$$f(x_k) = \frac{f^{(m)}(\alpha)}{m!}\left(e_k^m + C_1 e_k^{m+1} + C_2 e_k^{m+2} + C_3 e_k^{m+3}\right) + O(e_k^{m+4}),$$

where $e_k = x_k - \alpha$. Now we obtain the Taylor development of $f'(x_k)$ around α,

$$f'(x_k) = \frac{f^{(m)}(\alpha)}{m!}\left(m e_k^{m-1} + (m+1)C_1 e_k^m + (m+2)C_2 e_k^{m+1} + (m+3)C_3 e_k^{m+2}\right) + O(e_k^{m+3}).$$

Then,

$$\begin{aligned} g(x_k) &= \frac{f(x_k)}{f'(x_k)} \\ &= \frac{1}{m}\left(e_k - \frac{1}{m}C_1 e_k^2 + \frac{(m+1)C_1^2 - 2mC_2}{m^2} e_k^3\right) + O(e_k^4). \end{aligned}$$

Similarly, it is proven

$$\begin{aligned} g(x_{k_1}) &= \frac{f(x_{k-1})}{f'(x_{k-1})} \\ &= \frac{1}{m}\left(e_{k-1} - \frac{1}{m}C_1 e_{k-1}^2 + \frac{(m+1)C_1^2 - 2mC_2}{m^2} e_{k-1}^3\right) + O(e_{k-1}^4), \end{aligned}$$

and

$$\begin{aligned} g(2x_k - x_{k-1}) &= \frac{f(x_k)}{f'(2x_k - x_{k-1})} \\ &\sim \frac{1}{m}\left(2e_k - e_{k-1} - \frac{1}{m}C_1(2e_k - e_{k-1})^2 + O_3(e_k, e_{k-1})\right), \end{aligned}$$

with $e_{k-1} = x_{k-1} - \alpha$ and $e_k = x_k - \alpha$.
Thus

$$\begin{aligned} g[2x_k - x_{k-1}, x_{k-1}] &= \frac{g(2x_k - x_{k-1}) - g(x_{k-1})}{2(x_k - x_{k-1})} \\ &\sim \frac{1}{m}\left(1 - \frac{2}{m}C_1 e_k + \frac{(m+1)C_1^2 - 2mC_2}{m^2}\left(4e_k^2 - 2e_j e_{k-1} + e_{k-1}^2\right)\right). \end{aligned}$$

Therefore,

$$\begin{aligned} x_{k+1} - \alpha &= x_k - \alpha - \frac{g(x_k)}{g[2x_k - x_{k-1}, x_{k-1}]} \\ &= e_k - \frac{\left(e_k - \frac{1}{m}C_1 e_k^2 + \frac{(m+1)C_1^2 - 2mC_2}{m^2} e_k^3\right) + O\left(e_k^4\right)}{\left(1 - \frac{2}{m}C_1 e_k + \frac{(m+1)C_1^2 - 2mC_2}{m^2}\left(4e_k^2 - 2e_j e_{k-1} + e_{k-1}^2\right)\right) + O_3(e_k, e_{k-1})} \\ &= \left(\frac{-1}{m}C_1 e_k^2 + \frac{(m+1)C_1^2 - 2mC_2}{m^2}\left(-5e_k^3 + 2e_k^2 e_{k-1} - e_k e_{k-1}^2\right)\right) + O_4(e_k, e_{k-1}). \end{aligned}$$

Then, we have several possibilities for the term that determines the convergence order of the method, e_{k+1}, with respect to e_k and e_{k-1}. By its expression we only consider if the behaviour is as e_k^2 or as $e_k e_{k-1}^2$, since e_k^3 and $e_k^2 e_{k-1}$ tend faster to 0 than e_k^2.

- If $e_{k+1} \sim e_k^2$, then the order of convergence is 2.
- We assume now that $e_{k+1} \sim e_k e_{k-1}^2$. Then, assuming that the method has R-order p, this means that
$$e_{k+1} \sim D_{k,p} e_k^p,$$
where $D_{k,p}$ tends to the asymptotic error constant, D_p, when $k \to \infty$.
Analogously, $e_k \sim e_{k-1}^p$. Thus, we obtain
$$e_{k+1} \sim e_{k-1}^{p^2}.$$
From the error equation it is also obtained
$$e_{k+1} \sim e_k e_{k-1}^2 \sim e_{k-1}^{p+2}.$$
If we equate simultaneously the exponents of these last two equations, using Theorem 2.1.1 what we obtain is
$$p^2 - p - 2 = 0,$$
whose only positive solution is $p = 2$, which is the order of convergence of the iterative method KM, so it is proven that the order of the method is 2.

□

6.3 Dynamical analysis

In this section, we study the stability of the fixed points of the rational operator obtained when KM scheme is applied on the polynomial $p_m(x) = (x+1)(x-1)^m$, when m is a positive integer greater than 1.

The theoretical concepts to perform the dynamical analysis of an iterative method with memory are explained in Chapter 2. First, we calculate the auxiliar vectorial operator $Op(z, x)$ where $z = x_{k-1}$ and $x = x_k$

$$Op(z,x) = \left(x, x - \frac{\left(x^2-1\right)(mz+m+z-1)(2mx-mz+m+2x-z-1)}{(mx+m+x-1)(m(z+1)(2x-z+1)+(z-1)(2x-z-1))} \right).$$

Theorem 6.3.1. *The fixed points of operator $Op(z,x)$ are $(1,1)$, $(-1,-1)$ and $\left(\dfrac{1-m}{1+m}, \dfrac{1-m}{1+m}\right)$. The fixed points coming from the solutions of polynomial $p_m(x)$ have superattracting character and $\left(\dfrac{1-m}{1+m}, \dfrac{1-m}{1+m}\right)$ is an unstable strange fixed point.*

Proof. To calculate the fixed points we simultaneously do $z = x$ and $Op(z,x) = (x,x)$. First, we compute $Op(x,x)$

$$Op(x,x) = \left(x, \frac{m(x+1)^2 - (x-1)^2}{m(x+1)^2 + (x-1)^2}\right).$$

By equating $Op(x,x) = (x,x)$, we obtain that the fixed points satisfy

$$\begin{aligned}
\frac{m(x+1)^2 - (x-1)^2}{m(x+1)^2 + (x-1)^2} &= x, \\
m(x+1)^2 - (x-1)^2 &= xm(x+1)^2 + x(x-1)^2, \\
m(1-x)(x+1)^2 &= (x+1)(x-1)^2.
\end{aligned}$$

If $x = 1$ or $x = -1$, then it is obvious that the above equation is satisfied.

Suppose that $x \neq 1$ and $x \neq -1$. Then, the above equation can be rewritten as

$$\begin{aligned}
-m(x-1)(x+1)^2 &= (x+1)(x-1)^2, \\
-m(x+1) &= x-1, \\
(-m-1)x &= -1+m, \\
x &= \frac{-1+m}{-m-1} = \frac{1-m}{1+m}.
\end{aligned}$$

So, we obtain two fixed point coming from the solutions of the equation, that is, $z = x = 1$ and $z = x = -1$, and one strange fixed point whose components are defined by $z = x = \dfrac{1-m}{1+m}$.

Now, we analyze the stability of the fixed points coming from the solutions are superattractors. First, we calculate the Jacobian matrix $Op'(z,x)$.

$$Op'(z,x) = \begin{pmatrix} 0 & 1 \\ dOp_z(z,x) & dOp_x(z,x) \end{pmatrix},$$

where

$$
\begin{aligned}
dOp_z(z,x) &= -\frac{8m(m+1)\left(x^2-1\right)(x-z)}{(mx+m+x-1)(m(z+1)(2x-z+1)+(z-1)(2x-z-1))^2},\\
dOp_x(z,x) &= -\frac{4m^3(z+1)\left(x^2(5z+1)+x\left(-4z^2+2z-2\right)+z^3-z^2-2\right)}{(mx+m+x-1)^2(m(z+1)(2x-z+1)+(z-1)(2x-z-1))^2}\\
&+\frac{8m^2\left(x^2\left(5z^2-3\right)-4xz^3+z^4-z^2+2\right)}{(mx+m+x-1)^2(m(z+1)(2x-z+1)+(z-1)(2x-z-1))^2}\\
&-\frac{4m(z-1)\left(x^2(5z-1)-2x\left(2z^2+z+1\right)+z^3+z^2+2\right)}{(mx+m+x-1)^2(m(z+1)(2x-z+1)+(z-1)(2x-z-1))^2}.
\end{aligned}
$$

The eigenvalues of $Op'(x,x)$ are 0 and $-\dfrac{8m\left(x^2-1\right)}{\left(m(x+1)^2+(x-1)^2\right)^2}$.

Then, both eigenvalues are 0 when $x^2-1=0$, that is, $x=1$ or $x=-1$, so we find that the fixed points coming from the solutions are superattracting fixed points.

In the case $x=\dfrac{1-m}{1+m}$, we obtain that the second eigenvalue is 2, the strange fixed point has an unstable character (repelling or saddle). □

Theorem 6.3.2. *Operator $Op(z,x)$ does not have free critical points, that is, it has only two critical points that are the superattracting fixed points.*

Proof. First, we calculate the determinant of $Op'(z,x)$, because when the determinant is 0, it means that at least one of the eigenvalues is 0,

$$
\det(Op'(z,x))=\frac{8m(m+1)\left(x^2-1\right)(x-z)}{(mx+m+x-1)(m(z+1)(2x-z+1)+(z-1)(2x-z-1))^2}.
$$

By equating that expression to 0, we obtain 3 types of possible critical points

- Points (z,x) where $x=-1$. The eigenvalues of $Op'(z,-1)$ are 0 and

$$
-\frac{m(1+m)(1+z)^2}{-3+2z+z^2+m(1+z)^2}.
$$

  The second eigenvalue is 0 if $z=-1$. Therefore, there is only one critical point with this structure which is the fixed point $(-1,-1)$.

- The points (z,x) where $x=1$. The eigenvalues of $Op'(z,1)$ are 0 and

$$
-\frac{(1+m)(-1+z)^2}{m((z-1)^2+m(z^2-2z-3))}.
$$

  The second eigenvalue is 0 if $z=1$. Therefore, $(1,1)$ is only one critical point with this structure which.

- The points (z, x) where $z = x$. The eigenvalues of $Op'(z, z)$ are 0 and
$$-\frac{8m(-1+z^2)}{((-1+z)^2+m(1+z)^2)^2}.$$
The second eigenvalue is 0 if $z = \pm 1$. Therefore, the critical points that satisfy this structure are the non strange fixed points, that is, $(1, 1)$ and $(-1, -1)$.

Then, the operator does not have free critical points. □

From Theorem 6.3.2, the only feasible performance is to converge to the roots, both simple and multiple.

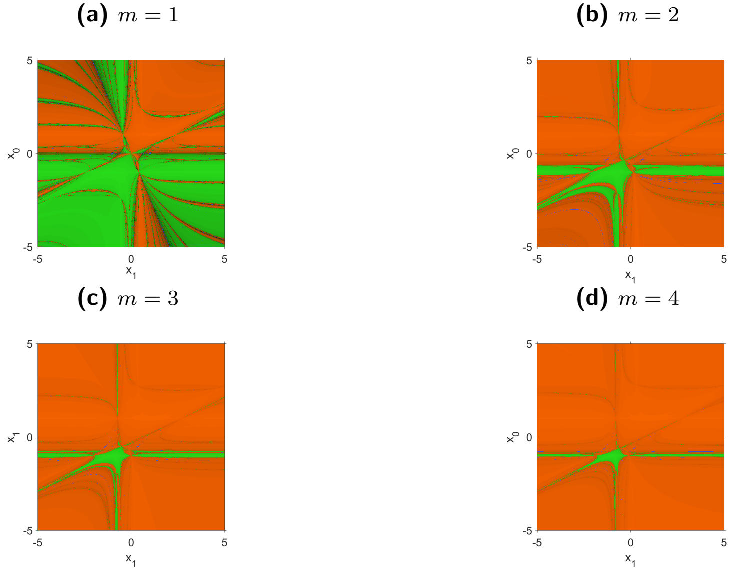
Below we show some real dynamical planes to see the behaviour of the method and the basins of attraction for the function p_m by varying the value of m.

These planes have been generated by making a mesh of 400 points by 400 points, where each point of the mesh is considered as the set of initial estimations of the iterative method, on the abscissa axis we have the component x_1 and on the ordinate axis the component x_0.

If the distance between iterations of the method to one of the solutions of the function is less than 10^{-3}, then we say that the initial point converges to that solution. Moreover, this convergence must happen before 100 iterations.

We represent the initial point in different colours according to its convergence. In orange the initial points that converge to the fixed point $(1, 1)$ and in green the initial points that converge to the fixed point $(-1, -1)$. We would also represent in black those initial points that do not converge to any of the solutions, but in this case, that does not happens.

Figure 6.1: Real dynamical planes with different values of m



(a) $m = 1$ **(b)** $m = 2$ **(c)** $m = 3$ **(d)** $m = 4$

As we can see in Figure 6.1, the wideness of the basin of attraction of $(1, 1)$ increases if we increase the value of m, which is the multiplicity of the root 1. As can be seen in all the dynamical planes, all the initial points converge to one of the solutions. With this study we show what happens with a family of polynomials with one simple solution and one multiple solution. Now we perform a dynamical analysis to see what happens when we have two multiple roots.

The polynomial is $f_{m,n}(x) = (x+1)^n(x-1)^m$ where $m > 1$ and $n > 1$.
Now, we calculate the auxiliar vectorial operator

$$Of(z, x) = \left(x, \Phi_f(z, x)\right),$$

where

$$\Phi_f(z, x) = \frac{m^2(x+1)(z+1)(2x-z+1) + 2mn\left(2xz - z^2 - 1\right) - n^2(x-1)(z-1)(2x-z-1)}{(m(x+1)+n(x-1))(m(z+1)(2x-z+1)+n(z-1)(2x-z-1))}.$$

Theorem 6.3.3. *The fixed points of operator $Of(z, x)$ are those coming from the solutions of polynomial $f_{m,n}(x)$, that is, $(1, 1)$ and $(-1, -1)$, and $\left(\frac{n-m}{n+m}, \frac{n-m}{n+m}\right)$. Those fixed points coming from the solutions of $f_{m,n}(x)$ are superattracting fixed points and the third is an unstable strange fixed point.*

Proof. We simultaneously do $z = x$ and $Of(z,x) = (x,x)$ to calculate the fixed points. First, we compute $Of(x,x)$

$$Of(x,x) = \left(x, \frac{m(x+1)^2 - n(x-1)^2}{m(x+1)^2 + n(x-1)^2}\right).$$

By solving $Of(x,x) = (x,x)$, we obtain that the fixed points are those that satisfy

$$\begin{aligned} \frac{m(x+1)^2 - n(x-1)^2}{m(x+1)^2 + n(x-1)^2} &= x, \\ m(x+1)^2 - n(x-1)^2 &= xm(x+1)^2 + xn(x-1)^2, \\ m(1-x)(x+1)^2 &= n(x+1)(x-1)^2. \end{aligned}$$

It is obvious that the equation is satisfied if $x = 1$ or $x = -1$. Suppose that $x \neq \pm 1$. Therefore, the above equation is

$$\begin{aligned} -m(x-1)(x+1)^2 &= n(x-1)^2, \\ -m(x+1) &= n(x-1), \\ (-m-n)x &= -n+m, \\ x &= \frac{-n+m}{-m-n} = \frac{n-m}{n+m}. \end{aligned}$$

Therefore, there are two fixed point coming from the solutions of the equation and one strange fixed point when $z = x = \dfrac{n-m}{n+m}$.

Now, we check that the fixed points coming from the solutions are superattractors. The eigenvalues of the Jacobian matrix $Of'(x,x)$, that are 0 and $-\dfrac{8mn\left(z^2-1\right)}{\left(m(z+1)^2 + n(z-1)^2\right)^2}$.
Both eigenvalues are 0 when $x^2 - 1 = 0$, therefore, the fixed points coming from the solutions are superattractors.

If $z = x = \dfrac{n-m}{n+m}$, the second eigenvalue is 2, therefore is a point with an unstable character (repelling or saddle). $\square$

Theorem 6.3.4. *Operator $Of(z,x)$ does not have free critical points, that is, it has only two critical points that are the superattracting fixed points.*

Proof. First, we analyze the determinant of $Of'(z,x)$, because when the determinant is 0, it means that at least one of the eigenvalues is 0,

$$\det\left(Of'(z,x)\right) = \frac{8mn\left(x^2-1\right)(m+n)(x-z)}{(m(x+1)+n(x-1))(m(z+1)(2x-z+1)+n(z-1)(2x-z-1))^2}.$$

By equating that expression to 0, we obtain 3 types of possible critical points

- The points (z,x) where $x=-1$. The eigenvalues of $Of'(z,-1)$ are 0 and

$$-\frac{m(z+1)^2(m+n)}{n\left(m(z+1)^2+n\left(z^2+2z-3\right)\right)}.$$

  The second eigenvalue is 0 if $z=-1$. Therefore, there is only one critical point with this structure which is the fixed point $(-1,-1)$.

- The points (z,x) where $x=1$. The eigenvalues of $Of'(z,1)$ are 0 and

$$-\frac{n(z-1)^2(m+n)}{m\left(m\left(z^2-2z-3\right)+n(z-1)^2\right)}.$$

  The second eigenvalue is 0 if $z=1$. Therefore, $(1,1)$ is the only critical point with this structure.

- The points (z,x) where $z=x$. The eigenvalues of $Of'(z,z)$ are 0 and

$$-\frac{8mn\left(z^2-1\right)}{\left(m(z+1)^2+n(z-1)^2\right)^2}.$$

  The second eigenvalue is 0 if $z=\pm 1$, therefore the critical points that satisfy this structure are $(1,1)$ and $(-1,-1)$.

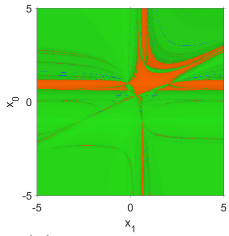
Then, the operator does not have free critical points. □

Below we show some real dynamical planes to see the behaviour of the method and the basins of attraction for the function $f_{m,n}$ varying the value of m and n.

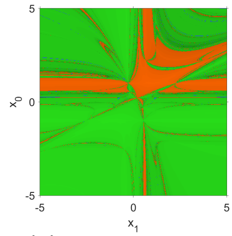
Under the same conditions and criteria that the previous dynamical planes are performed, we have been generated in Figure 6.2. Remember that we represent in orange the initial points that converge to $(1,1)$ and in green those that converge to $(-1,-1)$.

Figure 6.2: Real dynamical planes with different values of n and m

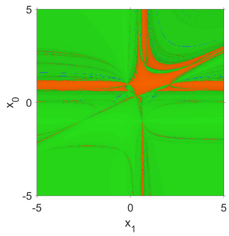
(a) $m = 1$ and $n = 2$



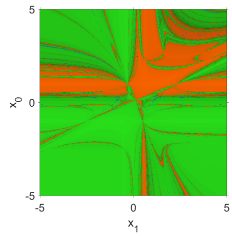
(b) $m = 2$ and $n = 3$



(c) $m = 2$ and $n = 4$



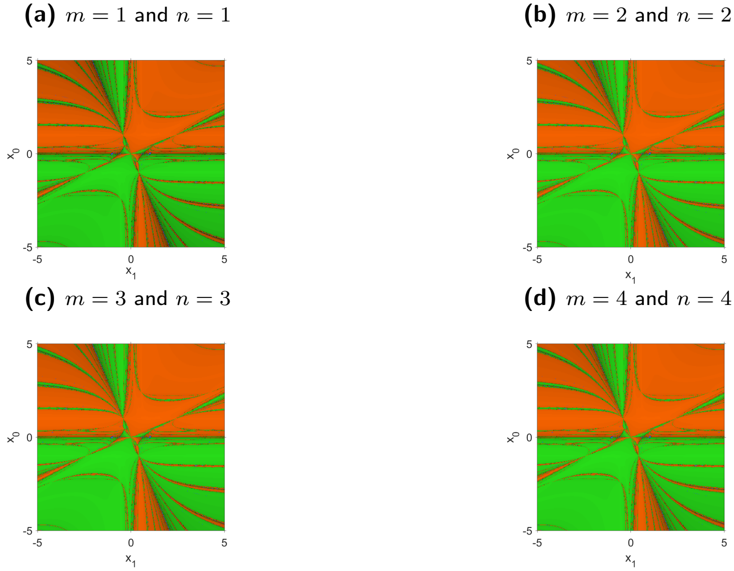
(d) $m = 3$ and $n = 4$



As we can see in Figures 6.2 and 6.3, if the value of n is greater than the value of m, the area of convergence to $(-1, -1)$ is greater than the zone of convergence to $(1, 1)$. If both values are equal, then the basin of attraction do not change if we increase the multiplicity value.

As can be seen in all the dynamical planes, all the initial points coming from the mesh converge to one of the solutions. With this study we show that the method is stable for that family of polynomials that have two multiple roots.

Figure 6.3: Real dynamical planes with different values of n and m



6.4 Convergence analysis of KMD

Now, we prove that the order of convergence of the method KMD is 2, and therefore, it is the same order of convergence of method KM.

Theorem 6.4.1. *Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a sufficiently differentiable function in an neighbourhood of α which we denote by $D \subset \mathbb{R}$ such that α is a multiple root of $f(x) = 0$ with unknown multiplicity $m \in \mathbb{N}\backslash\{1\}$. Then, taking an estimate x_0 close enough to α, the sequence of iterates $\{x_k\}_{k\geq 0}$ generated by method KMD converges to α with order 2.*

Proof. We first obtain the Taylor development of $f(x_k)$ around α, where $e_k = x_k - \alpha$,

$$f(x_k) = \frac{f^{(m)}(\alpha)}{m!}\left(e_k^m + C_1 e_k^{m+1}\right) + O\left(e_k^{m+2}\right),$$

being $C_j = \frac{m!}{(m+j)!}\frac{f^{(m+j)}(\alpha)}{f^{(m)}(\alpha)}$ for $j = 1, 2, \ldots$
In the same way,

$$f(x_k + f(x_k)) = \frac{f^{(m)}(\alpha)}{m!}\left((e_k + f(x_k))^m + C_1\left(e_k + f(x_k)\right)^{m+1}\right) + O\left(e_k^{m+2}\right).$$

Then,

$$f(x_k + f(x_k)) - f(x_k) \sim \frac{f^{(m)}(\alpha)}{m!}\left((e_k + f(x_k))^m - e_k^m + C_1\left((e_k + f(x_k))^{m+1} - e_k^{m+1}\right)\right).$$

Using Newton's binomial and the Taylor expansion of $f(x_k)$ around α we obtain that

$$\frac{f(x_k + f(x_k)) - f(x_k)}{x_k + f(x_k) - x_k} = \frac{f^{(m)}(\alpha)}{m!}\left(me_k^{m-1} + (m+1)C_1 e_k^m\right) + O\left(e_k^{m+1}\right).$$

Then,

$$\begin{aligned} g(x_k) &= \frac{f(x_k)}{f[x_k + f(x_k), x_k]} \\ &= \frac{e_k^m + C_1 e_k^{m+1} + O\left(e_k^{m+2}\right)}{me_k^{m-1} + (m+1)C_1 e_k^m + O\left(e_k^{m+1}\right)} \\ &= \frac{1}{m}\left(e_k - \frac{1}{m}C_1 e_k^2\right) + O\left(e_k^3\right). \end{aligned}$$

Similarly, it is proven that

$$\begin{aligned} g(x_{k-1}) &= \frac{f(x_{k-1})}{f[x_{k-1} + f(x_{k-1}), x_{k-1}]} \\ &= \frac{1}{m}\left(e_{k-1} - \frac{1}{m}C_1 e_{k-1}^2\right) + O\left(e_{k-1}^3\right). \end{aligned}$$

And also

$$g(2x_k - x_{k-1}) = \frac{1}{m}\left(2e_k - e_{k-1} - \frac{1}{m}C_1(2e_k - e_{k-1})^2\right) + O_3(e_k, e_{k-1}),$$

with $e_{k-1} = x_{k-1} - \alpha$ and $e_k = x_k - \alpha$.
Thus,

$$\begin{aligned} g[2x_k - x_{k-1}, x_{k-1}] &= \frac{g(2x_k - x_{k-1}) - g(x_{k-1})}{2(x_k - x_{k-1})} \\ &= \frac{2e_k - 2e_{k-1} - \frac{1}{m}C_1\left((2e_k - e_{k-1})^2 - e_{k-1}^2\right)}{2m(e_k - e_{k-1})} + O_3(e_k, e_{k-1}) \\ &= \frac{1}{m}\left(1 - \frac{2}{m}C_1 e_k\right) + O_2(e_k, e_{k-1}). \end{aligned}$$

Therefore,

$$\begin{aligned} x_{k+1} - \alpha &= x_k - \alpha - \frac{g(x_k)}{g[2x_k - x_{k-1}, x_{k-1}]} \\ &= e_k - \frac{\left(e_k - \frac{1}{m}C_1 e_k^2\right) + O\left(e_k^3\right)}{\left(1 - \frac{2}{m}C_1 e_k\right) + O_2(e_k, e_{k-1})} \\ &= e_k - \frac{2}{m}C_1 e_k^2 + e_k O_2(e_k, e_{k-1}) - e_k + \frac{1}{m}C_1 e_k^2 + O\left(e_k^3\right) \\ &= -\frac{1}{m}C_1 e_k^2 + e_k O_2(e_k, e_{k-1}). \end{aligned}$$

Then, we have several possibilities for the term that determines the convergence order of the method, e_{k+1}, with respect to e_k and e_{k-1}. By its expression we only consider if the behaviour is as e_k^2 or as $e_k e_{k-1}^2$, since e_k^3 and $e_k^2 e_{k-1}$ tend faster to 0 than e_k^2.

- If $e_{k+1} \sim e_k^2$, then the order of convergence is 2.
- We assume $e_{k+1} \sim e_k e_{k-1}^2$.
  Then, assuming that the method has R-order p, this means that
  $$e_{k+1} \sim D_{k,p} e_k^p,$$
  where $D_{k,p}$ tends to the asymptotic error constant, D_p, when $k \to \infty$.
  Analogously, $e_k \sim e_{k-1}^p$. Thus, we obtain
  $$e_{k+1} \sim e_{k-1}^{p^2}.$$
  By the error equation we also obtain
  $$e_{k+1} \sim e_k e_{k-1}^2 \sim e_{k-1}^{p+2}.$$
  If we equate simultaneously the exponents of these last two equations, using Theorem 2.1.1 what we obtain is
  $$p^2 - p - 2 = 0,$$
  whose only positive solution is $p = 2$, which is the order of convergence of the iterative method KMD, so it is proven that the order of the method is 2.

□

6.5 Solving multiple roots simultaneously

In the previous cases we have assumed that we search an only multiple solution, but what happens if we want to get more than one multiple solution simultaneously, maybe with different and unknown multiplicities? Let us remark that the iterative step presented in Chapter 5 (based on [47]) assumes that the solutions are simple.

For this reason, we combine method KM, with the iterative step defined in (5.1), in order to find as many solutions as we wish, and if it is possible, with independence of their multiplicity.

The proposed method, which we denote by KMS, has the following expression

$$\begin{cases} y_i^{(k)} = x_i^{(k)} - \dfrac{g(x_i^{(k)})}{g\left[2x_i^{(k)} - x_i^{(k-1)}, x_i^{(k-1)}\right]}, & i = 1, \ldots, n \qquad k = 0, 1, \ldots, \\ x_i^{(k+1)} = y_i^{(k)} - \dfrac{g(y_i^{(k)})}{g'(y_i^{(k)}) - g(y_i^{(k)}) \sum_{j=1, j\neq i}^{n} \dfrac{1}{y_i^{(k)} - y_j^{(k)}}}, & i, j = 1, \ldots, n \end{cases} \tag{6.1}$$

where $g(x) = \frac{f(x)}{f'(x)}$.
We prove below that it has order of convergence 4 for arbitrary nonlinear equations and increases to order 6 when we are working with polynomial equations.

Theorem 6.5.1. *Let $f : D \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be a sufficiently differentiable function in a neighbourhood D of α_i for $i = 1, \ldots, n$, such that $f(\alpha_i) = 0$ for $i = 1, \ldots, n$ with unknown multiplicity $m_i \in \mathbb{N}\backslash\{1\}$. for $i = 1, \ldots, n$. Then, taking an initial estimation $x^{(0)}$ close enough to $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, the sequence of iterates $\{x^{(k)}\}_{k \geq 0}$ generated by method KMS converges to α with order 4 for arbitrary nonlinear equations and increases to order 6 for polynomial equations.*

Proof. We denote $C_{i,j} = \dfrac{m_i!}{(m_i + j)!} \dfrac{f^{(m_i+j)}(\alpha_i)}{f^{(m_i)}(\alpha_i)}$ for $j = 1, 2, \ldots$ and $i = 1, 2, \ldots, n$.

On the one hand, in Theorem 6.2.1 we have proven that the first step has order 2, that is,

$$e_{i,y,k} \sim \left(\frac{-1}{m_i} C_{i,1} e_{i,k}^2 + \frac{(m_i+1)C_{i,1}^2 - 2m_i C_{i,2}}{m_i^2} \left(-5e_{i,k}^3 + 2e_{i,k}^2 e_{i,k-1} - e_{i,k} e_{i,k-1}^2 \right) \right).$$

We first obtain the Taylor development of $f(y_i^{(k)})$ around α_i where $e_{i,y,k} = y_i^{(k)} - \alpha_i$,

$$f(y_i^{(k)}) = \frac{f^{(m_i)}(\alpha_i)}{m_i!} \left(e_{i,y,k}^{m_i} + C_{i,1} e_{i,y,k}^{m_i+1} + C_{i,2} e_{i,y,k}^{m_i+2} \right) + O\left(e_{i,y,k}^{m_i+3} \right).$$

In the same way, we obtain the Taylor development of $f'(y_i^{(k)})$ around α_i

$$f'(y_i^{(k)}) = \frac{f^{(m_i)}(\alpha)}{m_i!} \left(m_i e_{i,y,k}^{m_i-1} + (m_i+1)C_{i,1} e_{i,y,k}^{m_i} + (m_i+2)C_{i,2} e_{i,y,k}^{m_i+1} \right) + O\left(e_{i,y,k}^{m_i+2} \right).$$

Then,

$$\begin{aligned} g(y_i^{(k)}) &= \frac{f(y_i^{(k)})}{f'(y_i^{(k)})} \\ &= \frac{1}{m_i} \left(e_{i,y,k} - \frac{1}{m_i} C_{i,1} e_{i,y,k}^2 + \frac{(m_i+1)C_{i,1}^2 - 2m_i C_{i,2}}{m_i^2} e_{i,y,k}^3 \right) + O\left(e_{i,y,k}^4 \right). \end{aligned}$$

Analogously,

$$g'(y_i^{(k)}) = \frac{1}{m_i} \left(1 - \frac{2}{m_i} C_{i,1} e_{i,y,k} + 3 \frac{(m_i+1)C_{i,1}^2 - 2m_i C_{i,2}}{m_i^2} e_{i,y,k}^2 \right) + O\left(e_{i,y,k}^3 \right).$$

On the other hand,

$$y_i^{(k)} - y_j^{(k)} = y_i^{(k)} - \alpha_i + \alpha_i - y_j^{(k)} + \alpha_j - \alpha_j = e_{i,y,k} - e_{j,y,k} + \alpha_i - \alpha_j.$$

Moreover, $\sum_{j=1,j\neq i}^{n} \frac{1}{y_i^{(k)} - y_j^{(k)}}$ can then be rewritten as follows

$$\sum_{j=1,j\neq i}^{n} \frac{1}{y_i^{(k)} - y_j^{(k)}} = \sum_{j=1,j\neq i}^{n} \frac{1}{e_{i,y,k} - e_{j,y,k} + \alpha_i - \alpha_j}.$$

To simplify the notation, we denote by $S_i(y^{(k)})$ the following expression

$$S_i(y^{(k)}) = \sum_{j=1,j\neq i}^{n} \frac{1}{e_{i,y,k} - e_{j,y,k} + \alpha_i - \alpha_j}.$$

Then,

$$\begin{aligned}
&g'(y_i^{(k)}) - g(y_i^{(k)}) \sum_{j=1,j\neq i}^{n} \frac{1}{y_i^{(k)} - y_j^{(k)}} = \\
&= \frac{1}{m_i}\left(1 - \frac{2}{m_i}C_{i,1}e_{i,y,k}\right) - \frac{1}{m_i}\left(e_{i,k,j}\right) S_i(y^{(k)}) + O\left(e_{i,y,k}^2\right) \\
&= \frac{1}{m_i} - \frac{1}{m_i}\left(\frac{2}{m_i}C_{i,1} + S_i(y^{(k)})\right) e_{i,y,k} + O\left(e_{i,y,k}^2\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{g(y_i^{(k)})}{g'(y_i^{(k)}) - g(y_i^{(k)}) \sum_{j=1,j\neq i}^{n} \frac{1}{y_i^{(k)} - y_j^{(k)}}} = \\
&= \frac{m_i g(y_i^{(k)})}{1 - \left(\frac{2}{m_i}C_{i,1} + S_i(y^{(k)})\right) e_{i,y,k} + O\left(e_{i,y,k}^2\right)}.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
e_{i,k+1} &= e_{i,y,k} - \frac{g(y_i^{(k)})}{g'(y_i^{(k)}) - g(y_i^{(k)}) \displaystyle\sum_{j=1, j\neq i}^{n} \frac{1}{y_i^{(k)} - y_j^{(k)}}} = \\
&= e_{i,y,k} - \frac{m_i g(y_i^{(k)})}{1 - \left(\frac{2}{m_i} C_{i,1} + S_i(y^{(k)})\right) e_{i,y,k} + O_2(e_{i,y,k})} \\
&= \frac{e_{i,y,k} \cdot \left(1 - \left(\frac{2}{m_i} C_{i,1} + S_i(y^{(k)})\right) e_{i,y,k}\right) - m_i g(y_i^{(k)})}{1 - \left(\frac{2}{m_i} C_{i,1} + S_i(y^{(k)})\right) e_{i,y,k} + O_2(e_{i,y,k})} \\
&= \frac{e_{i,y,k} - \left(\frac{2}{m_i} C_{i,1} + S_i(y^{(k)})\right) e_{i,y,k}^2 - \left(e_{i,y,k} - \frac{1}{m_i} C_{i,1} e_{i,y,k}^2\right) + O_3(e_{i,y,k})}{1 - \left(\frac{2}{m_i} C_{i,1} + S_i(y^{(k)})\right) e_{i,y,k} + O_2(e_{i,y,k})} \\
&= \frac{-\left(\frac{2}{m_i} C_{i,1} - \frac{1}{m_i} C_{i,1} + S_i(y^{(k)})\right) e_{i,y,k}^2 + O_3(e_{i,y,k})}{1 - \left(\frac{2}{m_i} C_{i,1} + S_i(y^{(k)})\right) e_{i,y,k} + O_2(e_{i,y,k})} \\
&= \frac{-\left(\frac{1}{m_i} C_{i,1} + S_i(y^{(k)})\right) e_{i,y,k}^2 + O_3(e_{i,y,k})}{1 - \left(\frac{2}{m_i} C_{i,1} + S_i(y^{(k)})\right) e_{i,y,k} + O_2(e_{i,y,k})}.
\end{aligned}
$$

Thus,

$$e_{i,k+1} \sim e_{i,y,k}^2.$$

And since $e_{i,y,k}$ has order of convergence 2, by Theorem 6.2.1, it is proven that KMS method has order of convergence 4 for nonlinear equations.

In a similar way as in the case of Theorem 5.2.2, we can prove that the KMS method has order of convergence 6 for polynomial nonlinear equations. □

6.6 Numerical experiments

We use Matlab R2020b with variable precision arithmetics of 500 digits for the computational calculations. As a stopping criterion we use that the absolute value of the function at the last iteration is less than a tolerance of 10^{-25}, that is,

$$|f(x_{k+1})| < 10^{-25}.$$

Also, is used as a stopping criterion a maximum number of iterations that can be done, in this case is 100. We compare the proposed methods with the method (2) from [46], which is denoted by gTM.

The numerical elements we are going to compare in the different examples are

- the initial estimations chosen, x_0, x_{-1} and x_{-2},
- the approximation obtained, x_{k+1},
- the absolute value of the equation evaluated in that approximation, $f(x_{k+1})$,
- the distance between the last two approximations, x_k and x_{k+1},
- the number of iterations needed to satisfy the required tolerance,
- the computational time and the approximate computational convergence order (ACOC).

We are going to solve three nonlinear equations:

1. $f_1(x) = (x^3 - 1)^4 = 0$, has three solutions with multiplicity four.
2. In [48], the authors considered the isothermal CSTR problem, with the following equation for the transfer function of the reactor

$$K_C 2.98(x + 2.25)/((x + 1.45)(x + 2.85)^2(x + 4.35)) = -1,$$

   where K_C is the gain of the proportional controller. If we choose $K_C = 0$, the nonlinear equation to solve is the following one

$$f_2(x) = x^4 + 11.50x^3 + 47.49x^2 + 86.0325x + 51.23266875 = 0.$$

   There is one multiple root with multiplicity 2.
3. $f_3(x) = (x^2 - 1)e^{x-1} = 0$, has two solutions with different multiplicities.

Table 6.1: Numerical results for equation $f_1(x) = 0$

Method	x_0	x_{-1}	x_{-2}	$\lvert x_{k+1} - x_k \rvert$	$\lvert f_1(x_{k+1}) \rvert$	Iteration	ACOC
KM	0.5	0.1		1.5776×10^{-13}	0	8	1.9994
KMD	0.5	0.1		6.1173×10^{-14}	0	6	1.8434
gTM	0.5	0.1	-0.1	1.7764×10^{-15}	0	42	1.5850

As we can see in Table 6.1, all the methods obtain good results for the chosen initial points. The approximate computational convergence order coincides with the theoretical one. For the initial points chosen, we see that the KMD method performs less iterations to satisfy the stopping criterion than KM, but both perform far less iterations than the gTM method.

Table 6.2: Numerical results for equation $f_2(x) = 0$

Method	x_0	x_{-1}	x_{-2}	$\|x_{k+1} - x_k\|$	$\|f_2(x_{k+1})\|$	Iteration	ACOC
KM	-3	-3.25		1.9884×10^{-09}	1.6566×10^{-30}	4	2.2725
KMD	-3	-3.25		2.4269×10^{-08}	2.0293×10^{-29}	4	2.0649
gTM	-3	-3.25	-3.5	2.5116×10^{-11}	1.0354×10^{-29}	5	1.7914

From Table 6.2, it can be seen that the schemes perform well for the chosen initial points. The ACOC coincides with the theoretical one and the number of iterations needed to satisfy the stopping criterion is almost the same for all the methods.

Table 6.3: Numerical results for equation $f_3(x) = 0$

Method	x_0	x_{-1}	x_{-2}	$\|x_{k+1} - x_k\|$	$\|f_3(x_{k+1})\|$	Iteration	ACOC
KM	0.8	0.6		6.4120×10^{-12}	0	6	1.9956
KMD	0.8	0.6		9.9960×10^{-11}	0	8	1.9999
gTM	0.8	0.6	0.4	1.1768×10^{-14}	0	14	1.5989

As shown in Table 6.3, for the chosen points, each scheme produces satisfactory results. We see that the KM method performs less iterations to satisfy the stopping criterion than KMD, but both perform less iterations than the gTM method.

Now we are going to perform some numerical experiments for the method KMS to find simultaneously all the roots of a nonlinear function, with independence of their multiplicity. These specifications are different from the previous ones, since these methods obtain roots simultaneously. We use Matlab R2020b with arithmetic precision of 500 digits for the computational calculations.

As a stopping criterion we use that

$$\left\| F(x^{(k+1)}) \right\|_2 < 10^{-200},$$

where $F(x^{(k+1)}) = \left(f(x_1^{(k+1)}), \ldots, f(x_n^{(k+1)}) \right)$.

Also, is used as a stopping criterion a maximum number of iterations that can be done, in this case is 100.

In the different tables we show the following data

- the initial set of approximations used, $x^{(0)}$,

- the norm of the function evaluated in the last iteration, $\left\| F(x^{(k+1)}) \right\|_2$,
- the norm of the distance between the last two approximations, $\left\| x^{(k+1)} - x^{(k)} \right\|_2$,
- the number of iterations needed to satisfy the required tolerance,
- and the approximated computational order of convergence (ACOC).

We are going to solve three nonlinear equations

1. $g_1(x) = (x^2-1)^2 = 0$, which has two solutions with multiplicity two.
2. $g_2(x) = (x-1)^4(x-3)^2(x+2) = 0$, which has three solutions with different multiplicities.
3. $g_3(x) = (e^{x^2-1} - e^{x^3-2x^2-x+2})^2 = 0$, which has three solutions with multiplicity two.

We also choose $x^{(-1)} = 0.95x^{(0)}$ for all the equations.

Table 6.4: Numerical results for KMS for different equations

Function	$x^{(0)}$	$\left\| x^{(k+1)} - x^{(k)} \right\|_2$	$\left\| G(x^{(k+1)}) \right\|_2$	Iteration	ACOC
$g_1(x)$	(-1.5,1.5)	3.1386$\times 10^{-22}$	3.9569$\times 10^{-69}$	4	4.0326
$g_2(x)$	(0.8,3.5,-1.5)	5.1263$\times 10^{-10}$	1.2125$\times 10^{-28}$	4	5.6266
$g_3(x)$	(-1.2,1.2,2.8)	4.0863$\times 10^{-12}$	2.6753$\times 10^{-33}$	4	8.9077

As we can see in Table 6.4, the method performs correctly, getting the solutions in a small number of iterations. If we choose a smaller tolerance, the ACOC will coincides with the theoretical one, but with the chosen one, the ACOC is at least 4 in all cases.

We do not compare with another method because, at the moment, we do not know of any other method in the literature that can find multiple roots simultaneously without using multiplicity in its iterative expression.

6.7 Conclusions

In this chapter, we have modified Kurchatov's method to make it applicable to obtain multiple roots while maintaining its quadratic order of convergence.

We have modified this scheme so that it does not use the multiplicity of the solution in its expression, so that it is not necessary to know this value before applying the iterative method and also can approach different solutions with different multiplicity.

We have performed the dynamical analysis of the iterative method for two family of functions, one of the polynomials with one simple solution and one multiple solution, and another with two multiple roots, showing that the method is stable in both cases.

We have also modified the method we propose to obtain the KMD method, which is a method with memory derivative-free, with the same characteristics as the KM method, that is, it can be applied to obtain solutions with multiplicity greater than one, and does not involve the value of this multiplicity in its iterative expression.

We have also added the simultaneity step explained in Chapter 5 so we obtain a method for solutions with multiplicity greater or equal to 1 that obtains all the solutions simultaneously.

Chapter 7

Family of iterative methods with Jacobian matrices for nonlinear systems

Based on [Cordero, A.; Villalba, E.G.; Torregrosa, J.R.; Triguero-Navarro, P. (2021). Convergence and Stability of a Parametric Class of Iterative Schemes for Solving Nonlinear Systems. Mathematics, 9, 86. https://doi.org/10.3390/math9010086]

7.1 Introduction

The need to solve nonlinear equations has already been discussed in previous chapters. Just as this need arises, so does the need to solve systems of nonlinear equations, which cannot always be solved exactly.
For this reason, iterative methods are also used to approximate the solution of nonlinear problems of the form

$$F(x) = 0 \quad \text{where} \quad F : \mathbb{R}^n \to \mathbb{R}^n, \;\; n \geq 1. \tag{7.1}$$

It is well known that one of the most classic iterative methods for its simplicity and effectiveness is Newton's method (2.3).

In addition to its simplicity, this method has quadratic convergence under certain conditions and great accessibility, that is, it has a wide region of initial estimations $x^{(0)}$ for which the method converges.
In this chapter, a new parametric family of iterative schemes whose first step is Newton's method is designed. The scheme of this parametric family for nonlinear equations arises from performing the convex combination of the iterative methods that are presented in [49] and in [50].

This chapter is structured as follow. In Section 7.2, is explained the way to extend the unidimensional case to the multidimensional one. Also, the family of iterative methods in the multidimensional case is proposed. In Section 7.3, the order of convergence of the new classes of iterative methods for solving nonlinear systems is analysed. In Section 7.4, the dynamical behaviour of the proposed family depending on a parameter is studied, and we also perform some dynamical planes to visualize the dynamical behaviour for some methods of the family. In Section 7.5, we perform some numerical experiments for confirming the theoretical results and this chapter ends with some conclusions in Section 7.6.

7.2 Design of the parametric family

The first thing to note is that one of the most common ways to generate iterative methods that solve multidimensional problems is to extend a unidimensional method, but in order for this to be done, the expressions we use must be adapted.

For example, we can use $f'(x) + f(x)$ for the scalar case, but in the case of systems of equations the sum is not possible since $F'(x)$ is a matrix and $F(x)$ is a vector and they do not have the same size.

As a consequence, the expressions must be adapted previously, if is possible. There are several references in the literature that extend the applicability to systems, such as [51].

The class of iterative method that we want to adapt is:

$$\begin{cases} z_k = x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ x_{k+1} = x_k - \left(\gamma \dfrac{1}{2} \left(3 - \dfrac{f'(z_k)}{f'(x_k)} \right) - (1-\gamma) \left(\dfrac{1}{\frac{f(z_k)}{f(x_k)} - 1} - \dfrac{f(z_k)^2}{f(x_k)^2} \right) \right) \dfrac{f(x_k)}{f'(x_k)} \quad k = 0, 1, \ldots \end{cases} \tag{7.2}$$

In this case, to extend its applicability to nonlinear systems, we must obtain an expression of $\dfrac{f(y)}{f(x)}$ with the appropriate dimension.

Since the first step is Newton's scheme, we can write $(x-y)F'(x) = F(x)$.

On the other hand, $F(y) = F(x) + [x, y; F](y - x)$.

Thus $\dfrac{F(y)}{F(x)}$ is as follows

$$\begin{aligned} \frac{F(y)}{F(x)} &= \frac{F(x) + [x, y; F](y-x)}{F(x)} = 1 + \frac{[x, y; F](y-x)}{F(x)} \\ &= 1 + \frac{[x, y; F](y-x)}{(x-y)F'(x)} = 1 - \frac{[x, y; F]}{F'(x)}. \end{aligned}$$

Therefore,

$$\frac{F(y^{(k)})}{F(x^{(k)})} = 1 - \frac{[x^{(k)}, y^{(k)}; F]}{F'(x^{(k)})},$$

which, rewritten for the multidimensional case, is $F(y^{(k)})^{-1}F(x^{(k)}) = I - F'(x^{(k)})^{-1}[x^{(k)}, y^{(k)}; F]$. Therefore, the parametric family of iterative iterative methods for systems of nonlinear equations that we are going to study is described by the following algorithm

$$\begin{aligned} y^{(k)} &= x^{(k)} - [F'(x^{(k)})]^{-1}F(x^{(k)}), \quad k = 0, 1, 2, \ldots, \\ x^{(k+1)} &= x^{(k)} - H(x^{(k)}, y^{(k)}, \gamma)[F'(x^{(k)})]^{-1}F(x^{(k)}), \end{aligned} \tag{7.3}$$

for $\gamma \in \mathbb{R}$ and being

$$\begin{aligned} &P_k = [x^{(k)}, y^{(k)}; F], \\ &B_k = F'(x^{(k)})^{-1}P_k, \\ &H(x^{(k)}, y^{(k)}, \gamma) = I + \frac{\gamma}{2}I + (1-\gamma)B_k^{-1} - (1-\gamma)B_k(2I - B_k) - \frac{\gamma}{2}F'(x^{(k)})^{-1}F'(y^{(k)}), \end{aligned}$$

where F is a sufficiently differentiable Fréchet function in a neighborhood of α, which we denote by $D \subset \mathbb{R}^n$, such that $F(\alpha) = 0$ and the Jacobian matrix of the function F evaluated at iteration $x^{(k)}$, denoted by $F'(x^{(k)})$, is non-singular.

7.3 Convergence analysis

Let $F: D \subset \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable function on a convex set $D \subset \mathbb{R}^n$ containing α, such that $F(\alpha) = 0$. We use Genochi-Hermite formula (2.13) to prove the following result, where we deduce the order of the family of methods (7.3) for any $\gamma \in \mathbb{R}$.

Theorem 7.3.1. *Let $F : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a sufficiently differentiable function on a convex neighbourhood of α, which we denote by $D \subset \mathbb{R}^n$, such that $F(\alpha) = 0$. We assume that the Jacobian matrix $F'(x)$ is continuous and non-singular in α. Then, taking an initial estimation $x^{(0)}$ enough close to α, the sequence of iterates $\{x^{(k)}\}_{k\geq 0}$ generated by the parametric family (7.3) converges to α with the following error equation*

$$\begin{aligned} e_{k+1} &= \frac{\gamma}{2}\left(C_3 + 4C_2^2\right)e_k^3 \\ &\quad + \left(\gamma C_4 + (4-13\gamma)\,C_2^3 + 3\gamma C_2 C_3 + \left(-1 + \frac{5}{2}\gamma\right)C_3 C_2\right)e_k^4 + O\left(e_k^5\right). \end{aligned} \tag{7.4}$$

where $C_j = \frac{1}{j!}[F'(\alpha)]^{-1}F^{(j)}(\alpha) \in L_j\left(\mathbb{R}^n, \mathbb{R}^n\right)$, where $L_j\left(\mathbb{R}^n, \mathbb{R}^n\right)$ is the set of j-linear bounded functions, for $j \geq 2$.

Proof. We consider the Taylor development of $F(x^{(k)})$ around α

$$F(x^{(k)}) = F'(\alpha)\left(e_k + C_2 e_k^2 + C_3 e_k^3 + C_4 e_k^4 + C_5 e_k^5 + O\left(e_k^6\right)\right). \tag{7.5}$$

Then, calculating the Taylor development of the derivatives of $F(x^{(k)})$ around α, one has

$$\begin{aligned} F'(x^{(k)}) &= F'(\alpha)\left(I + 2C_2 e_k + 3C_3 e_k^2 + 4C_4 e_k^3 + 5C_5 e_k^4\right) + O\left(e_k^5\right), \\ F''(x^{(k)}) &= F'(\alpha)\left(2C_2 + 6C_3 e_k + 12C_4 e_k^2 + 20C_5 e_k^3\right) + O\left(e_k^4\right), \\ F'''(x^{(k)}) &= F'(\alpha)\left(6C_3 + 24C_4 e_k + 60C_5 e_k^2\right) + O\left(e_k^3\right). \end{aligned} \tag{7.6}$$

We calculate the expansion of the inverse

$$F'(x^{(k)})^{-1} = \left(I + X_2 e_k + X_3 e_k^2 + X_4 e_k^3\right)F'(\alpha)^{-1} + O\left(e_k^4\right), \tag{7.7}$$

with X_2, X_3, X_4 and X_5 satisfying $[F'(x^{(k)})]^{-1}F'(x^{(k)}) = I$. Therefore, we obtain

$$\begin{aligned} X_2 &= -2C_2, \\ X_3 &= -3C_3 - 2C_2X_2 = 4C_2^2 - 3C_3, \\ X_4 &= -4C_4 - 3C_3X_2 - 2C_2X_3 = -8C_2^3 + 6C_2C_3 + 6C_3C_2 - 4C_4. \end{aligned}$$

Applying (7.5) and (7.7)

$$\begin{aligned} [F'(x^{(k)})]^{-1}F(x^{(k)}) &= e_k - C_2 e_k^2 + \left(-2C_3 + 2C_2^2\right)e_k^3 + \left(-3C_4 + 4C_2C_3 + 3C_3C_2 - 4C_2^3\right)e_k^4 \\ &\quad + O\left(e_k^5\right). \end{aligned} \tag{7.8}$$

Then we obtain the error equation of the first step of the parametric family (7.3)

$$y^{(k)} - \alpha = x^{(k)} - \alpha - F'(x^{(k)})^{-1}F(x^{(k)}) =$$
$$= C_2 e_k^2 + \left(2C_3 - 2C_2^2\right) e_k^3 + \left(3C_4 - 4C_2C_3 - 3C_3C_2 + 4C_2^3\right) e_k^4 + O\left(e_k^5\right).$$

Replacing this expression into the Taylor expansion of $F(y^{(k)})$ around α, we obtain

$$F(y^{(k)}) = F'(\alpha)\left(C_2 e_k^2 + \left(2C_3 - 2C_2^2\right) e_k^3 + \left(3C_4 - 4C_2C_3 - 3C_3C_2 + 5C_2^3\right) e_k^4\right) + O\left(e_k^5\right).$$

Furthermore

$$F'(y^{(k)}) \sim F'(\alpha)\left(I + 2C_2^2 e_k^2 + \left(4C_2C_3 - 4C_2^3\right) e_k^3 + O(e_k^4)\right). \tag{7.9}$$

From (7.7) and (7.9), we have

$$\begin{aligned} F'(x^{(k)})^{-1}F'(y^{(k)}) \sim & I - 2C_2 e_k + \left(-3C_3 + 6C_2^2\right) e_k^2 \\ & + \left(-4C_4 + 10C_2C_3 + 6C_3C_2 - 16C_2^3\right) e_k^3 + O\left(e_k^4\right). \end{aligned} \tag{7.10}$$

To obtain the development of the divided difference operator of (7.3), we use the Taylor development of (2.13). Considering in this case $x + h = y$, then $h = y - x = -[F'(x^{(k)})]^{-1}F(x^{(k)})$. Therefore, replacing (7.8) and (7.6) in (2.13), we obtain

$$\begin{aligned} [x^{(k)}, y^{(k)}; F] = & F'(\alpha)\left(I + C_2 e_k + \left(C_3 + C_2^2\right) e_k^2 + \left(2C_4 + C_3C_2 + 2C_2C_3 - 2C_2^3\right) e_k^3\right) \\ & + O\left(e_k^4\right). \end{aligned} \tag{7.11}$$

To calculate the inverse of this operator, we are looking for

$$[x^{(k)}, y^{(k)}; F]^{-1} = \left(I + Y_2 e_k + Y_3 e_k^2 + Y_4 e_k^3\right)[F'(\alpha)]^{-1} + O\left(e_k^4\right), \tag{7.12}$$

with Y_2, Y_3 and Y_4 satisfying $[x^{(k)}, y^{(k)}; F]^{-1}[x^{(k)}, y^{(k)}; F] = I$.
Then, if we denote by

$$\begin{aligned} P_2 &= C_2, \\ P_3 &= C_3 + C_2^2, \\ P_4 &= 2C_4 + C_3C_2 + 2C_2C_3 - 2C_2^3, \end{aligned}$$

we can write $[x^{(k)}, y^{(k)}; F] = F'(\alpha)\left(I + P_2 e_k + P_3 e_k^2 + P_4 e_k^3 +\right) + O\left(e_k^4\right)$, and then

$$\begin{aligned} Y_2 &= -P_2 = -C_2, \\ Y_3 &= -P_3 - Y_2P_2 = -C_3, \\ Y_4 &= -P_4 - Y_3P_2 - Y_2P_3 = -2C_4 - C_2C_3 + 3C_2^3. \end{aligned}$$

Now, using (7.7) and (7.11), we obtain the expansion of B_k

$$\begin{aligned} B_k &= F'(x^{(k)})^{-1}P_k \\ &= I - C_2e_k + \left(-2C_3 + 3C_2^2\right)e_k^2 + \left(-2C_4 + 6C_2C_3 + 4C_3C_2 - 8C_2^3\right)e^3 + O\left(e_k^4\right), \end{aligned} \tag{7.13}$$

and using (7.6) and (7.12), we calculate

$$B_k^{-1} = I + C_2e_k + \left(2C_3 - 2C_2^2\right)e_k^2 + \left(2C_4 - 4C_2C_3 - 2C_3C_2 + 3C_2^3\right)e_k^3 + O\left(e_k^4\right). \tag{7.14}$$

Replacing the expressions (7.8), (7.10), (7.13) and (7.14) in (7.3), we obtain that the error equation of the parametric family is

$$\begin{aligned} e_{k+1} = x^{(k+1)} - \alpha \sim \frac{\gamma}{2}\left(C_3 + 4C_2^2\right)e_k^3 \\ + \left(\gamma C_4 + (4-13\gamma)C_2^3 + 3\gamma C_2C_3 + \left(\frac{5}{2} - 1\gamma\right)C_3C_2\right)e_k^4. \end{aligned}$$

Finally, from the error equation we conclude that the parametric family (7.3) has order 3 for all $\gamma \neq 0$ and in the particular case where $\gamma = 0$, it has order of convergence 4, being the error equation

$$e_{k+1} = \left(4C_2^3 - C_3C_2\right)e_k^4 + O\left(e_k^5\right).$$

□

7.4 Dynamical analysis

In this chapter, we focus on the study of the complex dynamics of the parametric family (7.3) in the case of quadratic polynomials of the form $p(z) = (z-a)(z-b)$, where $a, b \in \mathbb{C}$. For this study, we use the Scaling Theorem, since it allows us to equate the dynamical behaviour of one operator with the behaviour associated to another, conjugated by means of an affine transformation. This result will be of great use to us since we can apply the Möbius transformation on the operator $R_{p,\gamma}$ associated with our parametric family acting on $p(z)$. We want to use the Möbius transformation to eliminate the operator's dependence on parameters a and b.

Theorem 7.4.1. *Let $f(z)$ be an analytic function on the Riemann sphere and let $T(z) = \alpha z + \beta$ be an affine transformation with $\alpha \neq 0$. We consider $g(z) = \lambda(f \circ T)(z)$, $\lambda \neq 0$. Let $R_{f,\gamma}$ and $R_{g,\gamma}$ be the fixed point operators of the family (7.3) associated to the functions f and g, respectively, that is,*

$$R_{f,\gamma}(z) = z + \left(-\frac{\gamma}{2}\left(3 - \frac{f'(y)}{f'(z)}\right) + (1-\gamma)\left(\frac{1}{\frac{f(y)}{f(z)} - 1} - \left(\frac{f(y)}{f(z)}\right)^2\right)\right)\frac{f(z)}{f'(z)},$$

$$R_{g,\gamma}(z) = z + \left(-\frac{\gamma}{2}\left(3 - \frac{g'(y)}{g'(z)}\right) + (1-\gamma)\left(\frac{1}{\frac{g(y)}{g(z)} - 1} - \left(\frac{g(y)}{g(z)}\right)^2\right)\right)\frac{g(z)}{g'(z)},$$

where $y = z - \frac{f(z)}{f'(z)}$ and $z \in \mathbb{C}$. Then $R_{f,\gamma}$ is conjugate analytically to $R_{g,\gamma}$ through T, that is,

$$(T \circ R_{g,\gamma} \circ T^{-1}) = R_{f,\gamma}(z).$$

Proof. Taking into account that $T(x-y) = T(x) - T(y) + \beta$, $T(x+y) = T(x) + T(y) - \beta$ and $g'(z) = \alpha\lambda f'(T(z))$, then

$$(R_{g,\gamma} \circ T^{-1})(z) = R_{g,\gamma}(T^{-1}(z)) =$$
$$= T^{-1}(z) + \left((1-\gamma)\left(\frac{1}{\frac{g(T^{-1}(y))}{g(T^{-1}(z))} - 1} - \left(\frac{g(T^{-1}(y))}{g(T^{-1}(z))}\right)^2\right) - \frac{\gamma}{2}\left(3 - \frac{g'(T^{-1}(y))}{g'(T^{-1}(z))}\right)\right)\frac{g(T^{-1}(z))}{g'(T^{-1}(z))},$$

where $y = z - \frac{g(z)}{g'(z)}$, where $T(T^{-1}(z)) = z$ and

$$\begin{aligned} T\left(T^{-1}(y)\right) &= T\left(T^{-1}(z) - \frac{g(T^{-1}(z))}{g'(T^{-1}(z))}\right) \\ &= T\left(T^{-1}(z) - \frac{f(z)}{\alpha f'(z)}\right) \\ &= z - T\left(\frac{f(z)}{\alpha f'(z)}\right) + \beta = z - \frac{f(z)}{f'(z)} = y. \end{aligned}$$

Therefore, replacing these equalities and simplifying, we have

$$\begin{aligned} &(T \circ R_{g,\gamma} \circ T^{-1}) = \\ &= T\left(T^{-1}(z) + \left(-\frac{\gamma}{2}\left(3 - \frac{f'(y)}{f'(z)}\right) + (1-\gamma)\left(\frac{1}{\frac{f(y)}{f(z)} - 1} - \left(\frac{f(y)}{f(z)}\right)^2\right)\right)\frac{f(z)}{\alpha f'(z)}\right) \\ &= z + T\left(-\frac{\gamma}{2}\left(3 - \frac{f'(y)}{f'(z)}\right) + (1-\gamma)\left(\frac{1}{\frac{f(y)}{f(z)} - 1} - \left(\frac{f(y)}{f(z)}\right)^2\right)\frac{f(z)}{\alpha f'(z)}\right) - \beta \\ &= z + T\left(-\frac{\gamma}{2}\left(3 - \frac{f'(y)}{f'(z)}\right) + (1-\gamma)\left(\frac{1}{\frac{f(y)}{f(z)} - 1} - \left(\frac{f(y)}{f(z)}\right)^2\right)\right)\frac{f(z)}{\alpha f'(z)}, \end{aligned}$$

then $(T \circ R_{g,\gamma} \circ T^{-1})(z) = R_{f,\gamma}(z)$, that is, $R_{f,\gamma}$ and $R_{g,\gamma}$ are analytically conjugated by $T(z)$. ☐

Now, we can apply the Möbius transformation on the operator associated to the parametric family (7.3) and the polynomial $p(z) = (z-a)(z-b)$. The Möbius transformation is $h(z) = \frac{z-a}{z-b}$. The rational operator that we obtain after the Möbius transformation is as follows

$$O_\gamma(z) = (h \circ R_{p,\gamma} \circ h^{-1})(z) = \frac{z^3\left(2\gamma z^2 + 3\gamma z + 2\gamma + z^5 + 5z^4 + 10z^3 + 9z^2 + 4z\right)}{2\gamma z^5 + 3\gamma z^4 + 2\gamma z^3 + 4z^4 + 9z^3 + 10z^2 + 5z + 1}. \quad (7.15)$$

We can see from the rational function (7.15) that the order of methods for quadratic polynomials is 3 when $\gamma \neq 0$ and the order is 4 when $\gamma = 0$.
Now, we are going to obtain the fixed points of $O_\gamma(z)$.

We are going to study which are the fixed points of the operator O_γ and its character depending on the value of the parameter γ.

Proposition 7.4.1.1. *Fixed points of operator $O_\gamma(z)$ are:*

- $z = 0$ *and* $z = \infty$ *are fixed points for any value of* γ.
- $z = 1$ *is a strange fixed point when* $\gamma \neq -\frac{29}{7}$.
- *the roots of*

$$k(t) = 1 + 6t + (16 - 2\gamma)t^2 + (21 - 3\gamma)t^3 + (16 - 2\gamma)t^4 + 6t^5 + t^6, \quad (7.16)$$

  denoted by $Ex_i(\gamma)$, *where* $i = 1, \ldots, 6$, *are strange fixed points also for any value of* γ.

We need the expression of the derivative of the operator to analyse the stability of the fixed points and to obtain the critical points,

$$O'_\gamma(z) = \frac{z^2(z+1)^4\left(\gamma\left(6z^6 + 8z^5 + 7z^4 + 7z^2 + 8z + 6\right) + z\left(16z^4 + 41z^3 + 60z^2 + 41z + 16\right)\right)}{\left(2\gamma z^5 + (3\gamma + 4)z^4 + (2\gamma + 9)z^3 + 10z^2 + 5z + 1\right)^2}.$$

It is clear that 0 and ∞ are always superattracting fixed points because they come from the roots of the polynomial and the order of the iterative methods is greater than 2, but the stability of the other fixed points may change depending on the parameter γ.

Proposition 7.4.1.2. *The character of the strange fixed point $z = 1$ depending on the value of γ is as follows*

- $z = 1$ *cannot be a superattractor.*
- *If* $\gamma = -\frac{29}{7}$, *then* $z = 1$ *is not an strange fixed point.*
- *If* $\gamma \in \{\gamma = \gamma_1 + \gamma_2 I \in \mathbb{C}$ *such that* $0 < 49(\gamma_1^2 + \gamma_2^2) + 406\gamma_1 - 8375\}$, *then* $z = 1$ *is an attractor.*
- *If* $\gamma \in \{\gamma = \gamma_1 + \gamma_2 I \in \mathbb{C}$ *such that* $49(\gamma_1^2 + \gamma_2^2) + 406\gamma_1 = 8375\}$, *then* $z = 1$ *is parabolic.*
- *Otherwise,* $z = 1$ *is a repelling.*

Proof. We obtain

$$|O'_\gamma(1)| = \left|\frac{96}{7\gamma + 29}\right|.$$

It is not difficult to check that $|O'_\gamma(1)|$ cannot be 0, therefore $z = 1$ cannot be a superattractor, and when $\gamma = -\frac{29}{7}$, $z = 1$ is not a fixed point.

Let us now study when $z = 1$ is an attractor. It is easy to check that $|O'_\gamma(1)| < 1$ is equivalent to $96^2 < |29 + 7\gamma|^2$, and this expression gives the following inequation

$$8375 < 406Re(\gamma) + 49Re(\gamma)^2 + 49Im(\gamma)^2. \tag{7.17}$$

When inequality (7.17) is satisfied, then $z = 1$ is an attractor.

On the other hand, $z = 1$ is a parabolic point when $8375 - 406Re(\gamma) - 49Re(\gamma)^2 = 49Im(\gamma)^2$.

In other cases, $z = 1$ is a repelling. □

Now, we establish the stability of the strange fixed points that are roots of the polynomial (7.16). Let us note that this polynomial is a symmetric polynomial of sixth degree, that is, a polynomial reducible to a polynomial of third degree and that satisfies the following properties

- $t = 0$ is not a root;
- if α is a root, $\frac{1}{\alpha}$ will also be a root.

Performing the reduction of (7.16), we obtain

$$\begin{aligned}
&1 + 6t + (16 - 2\gamma)t^2 + (21 - 3\gamma)t^3 + (16 - 2\gamma)t^4 + 6t^5 + t^6 = 0\\
\leftrightarrow &\left(\frac{1}{t^3} + t^3\right) + 6\left(\frac{1}{t^2} + t^2\right) + (16 - 2\gamma)\left(\frac{1}{t} + t\right) + 21 - 3\gamma = 0\\
\leftrightarrow &z^3 + 6z^2 + (13 - 2\gamma)z + 9 - 3\gamma = 0,
\end{aligned}$$

where $z = \frac{1}{t} + t$, $z^2 - 2 = \frac{1}{t^2} + t^2$ and $z^3 - 3z = \frac{1}{t^3} + t^3$.
Now, we calculate the roots of this polynomial and we obtain

$$\begin{aligned}
z_1(\gamma) &= \frac{\sqrt[3]{\frac{2}{3}}(2\gamma - 1)}{z_4(\gamma)} + \frac{z_4(\gamma)}{\sqrt[3]{2}3^{2/3}} - 2,\\
z_2(\gamma) &= \frac{\sqrt[3]{-\frac{2}{3}}(1 - 2\gamma)}{z_4(\gamma)} + \frac{z_4(\gamma)}{\sqrt[3]{2}3^{2/3}} - 2,\\
z_3(\gamma) &= \frac{\sqrt[3]{\frac{2}{3}}(2\gamma - 1)}{z_4(\gamma)} - \frac{z_4(\gamma)}{\sqrt[3]{-18}} - 2,
\end{aligned}$$

where $z_4(\gamma) = \sqrt[3]{-9\gamma + \sqrt{3\gamma((75-32\gamma)\gamma - 78) + 93} + 9}$.

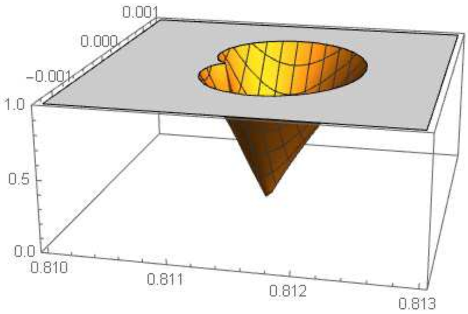
To calculate the roots of the polynomial (7.16) from $z_i(\gamma)$, $i = 1, 2, 3,$, we undo the change of variable since $t = \frac{z_i(\gamma) \pm \sqrt{z_i(\gamma)^2 - 4}}{2}$. Therefore, we obtain the roots of the polynomial of sixth degree, which are conjugated two by two

$$
\begin{aligned}
Ex_1(\gamma) &= \frac{z_1(\gamma) + \sqrt{z_1(\gamma)^2 - 4}}{2}, & Ex_2(\gamma) &= \frac{z_1(\gamma) - \sqrt{z_1(\gamma)^2 - 4}}{2}, \\
Ex_3(\gamma) &= \frac{z_2(\gamma) + \sqrt{z_2(\gamma)^2 - 4}}{2}, & Ex_4(\gamma) &= \frac{z_2(\gamma) - \sqrt{z_2(\gamma)^2 - 4}}{2}, \\
Ex_5(\gamma) &= \frac{z_3(\gamma) + \sqrt{z_3(\gamma)^2 - 4}}{2}, & Ex_6(\gamma) &= \frac{z_3(\gamma) - \sqrt{z_3(\gamma)^2 - 4}}{2}.
\end{aligned}
$$

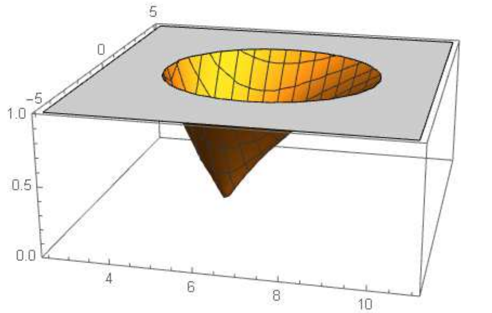
Now, we study when the roots of the polynomial (7.16) are superattractors. To do so, we solve $|O'_\gamma(Ex_i(\gamma))| = 0$ for all $i = 1, \ldots, 6$ and obtain the following relevant values of γ: $\gamma_1 \approx 0.8114608$, $\gamma_2 \approx 5.5908453$, $\gamma_3 \approx 0.7671009 + 0.7784254i$ and $\gamma_4 \approx 0.7671009 - 0.7784254i$.

Next, we study the character of the fixed points by analysing those values of γ close to the parameter values for which some $Ex_i(\gamma)$ is a superattractor. To do this, we study how $|O'_\gamma(Ex_i(\gamma))|$ behaves near the above four values and obtain regions where some of the roots will be attractors. These regions are represented in Figure 7.1.

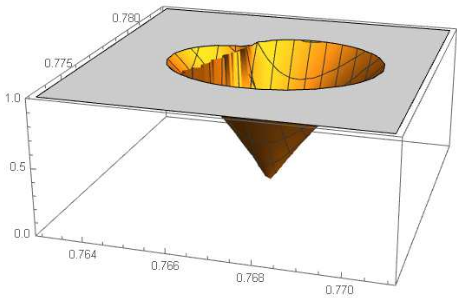
Figure 7.1: Character of the strange fixed points $Ex_i(\gamma)$ in the neighbourhood of γ_i



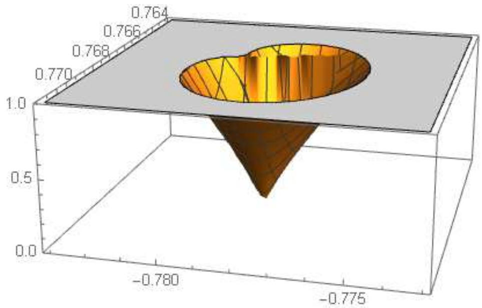
(a) γ_1



(b) γ_2



(c) γ_3



(d) γ_4

As shown in Figure 7.1, the areas where these are attractors are small.

Proposition 7.4.1.3. *For the parametric family (7.3), the critical points are $z = 0$, $z = -1$, $z = \infty$, and the roots of the polynomial*

$$q(t) = 6\gamma + (16 + 8\gamma)t + (41 + 7\gamma)t^2 + 60t^3 + (41 + 7\gamma)t^4 + (16 + 8\gamma)t^5 + 6\gamma t^6,$$

denoted by $Zx_i(\gamma)$, where $i = 1, \ldots, 6$.

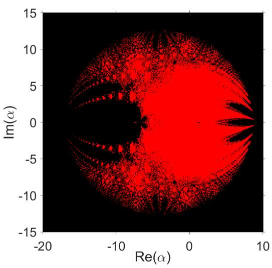
Proof. Let us observe that $z = -1$ is a preimage of the fixed point $z = 1$. We can see that $q(t)$ is a symmetric polynomial, so we can obtain the roots of $q(t)$ by obtaining roots of a polynomial of degree 3. The reduced polynomial of $q(t)$ is the following polynomial, which we obtain analogously to the polynomial (7.16)

$$\hat{q}(t) = 6\gamma t^3 + (16 + 8\gamma)t^2 + (41 - 11\gamma)t + 28 - 16\gamma.$$

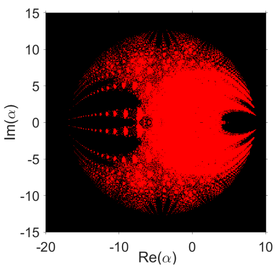
To obtain the roots of $q(t)$, we need to obtain the roots of $\hat{q}(t)$ and apply the following expression $\frac{z \pm \sqrt{z^2-4}}{2}$. Therefore, we have that the roots of $q(t)$ are conjugate. $\square$

Now let us study the asymptotical behaviour of the critical points to establish whether there are convergence basins other than those generated by the roots.
For the free critical point -1 we have $O_\gamma(-1) = 1$, which is an strange fixed point, so the parameter plane associated to this critical point is not significant, since we know the stability of $z = 1$.
The other free critical points are roots of a polynomial that depends on γ, so for them we use the parameter planes. Since we have that the free critical points are conjugate we only draw three parameter planes generated using as an initial estimation a free critical point that depends on γ. We establish a mesh in the complex plane of 500×500 points. Each point of the mesh corresponds to a value of the parameter. At each point, the rational function is iterated to obtain the orbit of the free critical point as a function of γ. If that orbit converges to $z = 0$ or $z = \infty$ in less than 40 iterations, that point on the grid is represented in red; otherwise, the point is black.

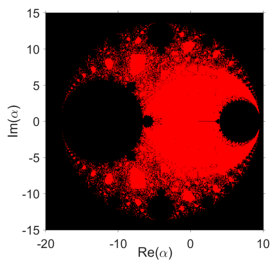
Figure 7.2: Parameter planes associated the free critical points of $O_\gamma(z)$



(a)



(b)



(c)

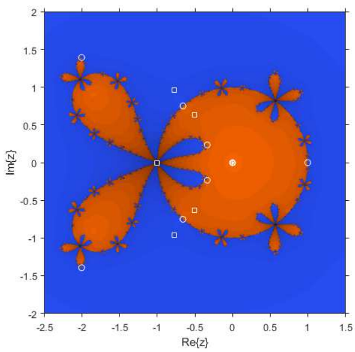
As we can see in Figure 7.2, there are many values of γ parameter that would give rise to a method of the family in which the free critical points converge to one of the two roots. These methods are located in the red area on the right of the plane. In addition, some black areas can be identified as the stability regions of those strange fixed points that can be attractors.
Now, we select some stable (in red in the parameter planes) and unstable values of γ (in black) to show the behaviour of the associated methods of the family using the dynamical planes.

In the case of dynamical planes, the value of the parameter γ is fixed. Each point of the complex plane is considered as a starting point of the iterative scheme, and is represented in different colours depending on the point it converged to. In this case, we represent in blue the points that converged to ∞, and in orange the points that converged to 0. The dynamical planes of Figures 7.3 to 7.6 have been generated with a mesh of 500×500 points and a maximum of 80 iterations per point. We mark the strange fixed points with white circles and the free critical points with white squares.

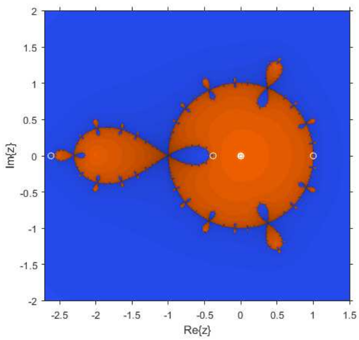
One value of the parameter that would be an interesting value is $\gamma = 0$, because it is the only the only value whose corresponding scheme has order 4. In that case, we obtain the dynamical plane shown in Figure 7.3a. Another value for the parameter we study is $\gamma = 1$ (Figure 7.3b). As we can see, this dynamical plane is similar to $\gamma = 0$, but in this case we get 3 critical points instead of 5 critical points and 4 fixed points instead of 7 fixed points.

Figure 7.3: Dynamical planes of $O_\gamma(z)$ for $\gamma = 0$ and $\gamma = 1$

(a) $\gamma = 0$



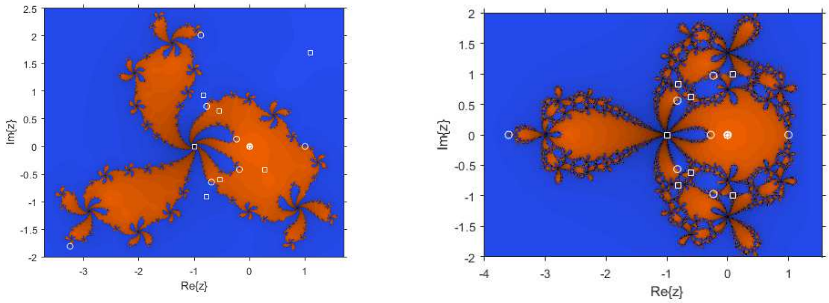
(b) $\gamma = 1$



By Theorem 7.3.1 and Propositions 7.4.1.1, 7.4.1.3 and 7.4.1.2, the simplest dynamical methods are those corresponding to $\gamma = 0$ and $\gamma = 1$. In Figures 7.4 to 7.6, we see the dynamical planes associated with other values of the parameter γ. Some of these planes are not as simple as the previous ones. This is the case of $\gamma = 2$ in Figure 7.4b, or the case of $\gamma = 2i$ in Figure 7.4a.

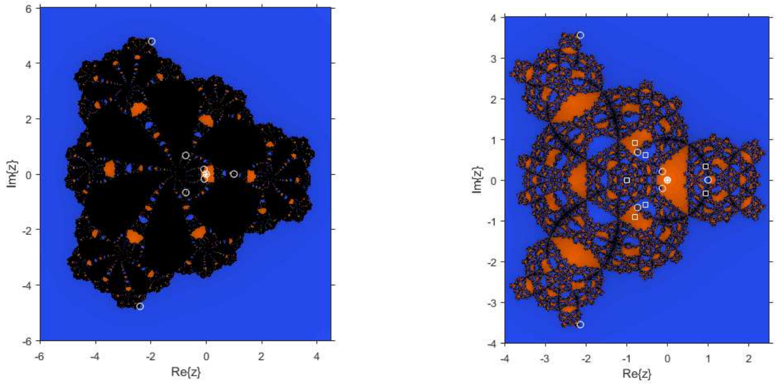
Figure 7.4: Dynamical planes of $O_\gamma(z)$ for $\gamma = 2i$ and $\gamma = 2$

(a) $\gamma = 2i$ **(b)** $\gamma = 2$



However, there exist values of the parameter that present a dynamical plane but with more complex dynamics and with a larger number of free critical points. We can see some of these dynamical planes in the Figures 7.5a, 7.5b and 7.6a, corresponding to $\gamma = -10+i$, $\gamma = -5$ and $\gamma = -\frac{29}{7}$. here are also parameter values with superattracting strange fixed points, for example, $\gamma = 5$ (Figure 7.6b). These cases should be avoided as the associated method may not converge to the roots and may end up converging to other points.

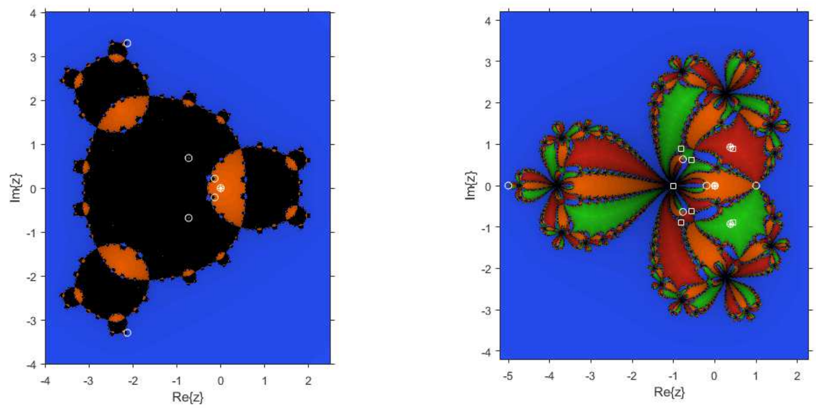
Figure 7.5: Dynamical planes of $O_\gamma(z)$ for $\gamma = -10+i$ and $\gamma = -5$



(a) $\gamma = -10+i$ **(b)** $\gamma = -5$

Figure 7.6: Dynamical planes for $\gamma = -\frac{29}{7}$ and $\gamma = 5$

(a) $\gamma = -\frac{29}{7}$ **(b)** $\gamma = 5$



7.5 Numerical experiments

In this section, we are going to compare different iterative methods of the parametric family studied throughout this chapter, solving two classical problems of applied mathematics: a Hammerstein integral equation and the Fisher partial derivative equation. The methods we are going to use for solving the nonlinear problems are those corresponding to the values of γ studied in the dynamical planes.

Matlab 2020b has been used to carry out the numerical experiments, with an arithmetical precision of 1000 digits. As stopping criterion we use

$$\left\| x^{(k+1)} - x^{(k)} \right\|_2 + \left\| F(x^{(k)}) \right\|_2 < \xi,$$

where ξ is the chosen tolerance in each numerical experiment. We use also a maximum of 100 iterations.
In the tables presented in this section we show the following data

- the approximation to the solution obtained,
- the norm of the function evaluated in the last approximation, $\left\| F(x^{(k+1)}) \right\|_2$,
- the norm of the distance between the last two approximations, $\left\| x^{(k+1)} - x^{(k)} \right\|_2$,
- the number of iterations necessary to satisfy the required tolerance,
- the computational time in seconds,

- and the approximated computational order of convergence (ACOC).

First, we consider the well-known Hammerstein integral equation (see [3]), which is given as follows

$$x(s) = 1 + \frac{1}{5}\int_0^1 F(s,t)x(t)^3 dt, \tag{7.18}$$

where $x \in \mathbb{C}[0,1]$, $s, t \in [0,1]$ and the kernel F is

$$F(s,t) = \begin{cases} (1-s)t & t \leq s, \\ s(1-t) & s \leq t. \end{cases}$$

We transform the above equation into a finite-dimensional problem using the Gauss-Legendre quadrature formula given as $\int_0^1 f(t)dt = \sum_{j=1}^{7} \omega_j f(t_j)$, where the abscissae t_j and the weights ω_j are determined for $n = 7$ (see Table 7.1).

Table 7.1: Abscissae and weights by Gauss-Legendre quadrature

i	Weight ω_i	Abscissa t_i
1	0.0647424831	0.0254460438
2	0.1398526957	0.1292344072
3	0.1909150252	0.2970774243
4	0.2089799185	0.5000000000
5	0.1909150252	0.7029225757
6	0.1398526955	0.8707655928
7	0.0647424831	0.9745539561

Denoting the approximations of $x(t_i)$ by x_i ($i = 1, \dots, 7$), we obtain the nonlinear equation system

$$5x_i - 5 - \sum_{j=1}^{7} a_{ij} x_j^3 = 0,$$

where $i = 1, \dots, 7$ and

$$a_{ij} = \begin{cases} w_j t_j (1-t_i) & j \leq i, \\ w_j t_i (1-t_j) & i < j. \end{cases}$$

Starting from an initial approximation $x^{(0)} = (-1, \dots, -1)^T$ and with a tolerance of 10^{-15}, we execute the schemes of the parametric family obtained for different values of the parameter γ. The results are shown in Table 7.2.

Table 7.2: Results for a Hammerstein's equation for different values of γ

γ	$\left\lVert x^{(k+1)} - x^{(k)} \right\rVert_2$	$\left\lVert F(x^{(k+1)}) \right\rVert_2$	Iteration	ACOC	Time
0	5.40317×10^{-46}	1.82600×10^{-184}	4	3.99753	38.0469
1	1.10606×10^{-20}	7.36657×10^{-63}	4	2.85884	33.8594
-10+i	4.02251×10^{-45}	3.70484×10^{-135}	6	2.98801	84.8594
$-29/7$	1.73829×10^{-32}	1.18363×10^{-97}	5	2.98095	44.0781
-5	8.18771×10^{-29}	1.48807×10^{-86}	5	2.97987	46.2500
5	6.98712×10^{-28}	9.02414×10^{-84}	5	2.97222	36.3281
2i	5.87285×10^{-47}	2.22194×10^{-141}	5	2.98606	35.3281
2	5.36968×10^{-17}	8.93118×10^{-52}	4	2.93508	25.8750

In all cases we obtain as an approximation of the solution of the equation (7.18) the following vector

$$x^{(k+1)} \approx (1.0026875, 1.0122945, 1.0229605, 1.0275616, 1.0229605, 1.0122945, 1.0026875)^T.$$

In the case of the Hammerstein integral equation, we see that the numerical results of the parametric family (7.3) for different values of γ are quite similar. The main difference observed between the methods is that the ACOC for $\gamma = 0$ is 4 and for the rest of the methods it is about 3, which was theoretically expected.
On the other hand, we observe that the method with $\gamma = -10 + i$ needs to perform a greater number of iterations than the rest of the methods to satisfy the required tolerance, so the approximation time to the solution is also greater. This is consistent with what has been obtained in the dynamical analysis of the different elements of the class of iterative methods, given that the best performing methods correspond to methods that have performed well in the dynamical analysis.
Finally, taking into account the columns that measure the error of the approximation, that is, the columns $\left\lVert F(x^{(k+1)}) \right\rVert_2$ and $\left\lVert x^{(k+1)} - x^{(k)} \right\rVert_2$, we see that the iterative methods that commit a smaller error are those associated with the parameters $\gamma = 0$ and $\gamma = 2$.

The second example we study is the Fisher equation proposed in [52] by Fisher to model the diffusion process in population dynamics. The analytical expression of this partial derivatives equation is as follows

$$u_t(x,t) = Du_{xx}(x,t) + ru(x,t)\left(1 - \frac{u(x,t)}{p}\right), \qquad x \in [a,b], \quad t \geq 0, \tag{7.19}$$

where $D \leq 0$ is the diffusion constant, r is the growth rate of the species and p is the carrying capacity. In this case, we study the Fisher equation for the values $p = 1$, $r = 1$ and $D = 1$ in the spatial interval $[0,1]$ and with the initial condition $u(x,0) = sech^2(\pi x)$ and $u(0,t) = u(1,t) = 0$.

We transform the problem just described into a set of nonlinear systems by applying an implicit finite difference method, providing the estimated solution at time t_k from the one estimated at

t_{k-1}.
We denote the spatial step by $h = \frac{1}{n_x}$ and the time step by $k = \frac{T_{max}}{n_t}$, where T_{max} is the final instant and n_x and n_t are the number of subintervals in x and t, respectively. Therefore, we define a mesh of the domain $[0,1] \times [0, T_{max}]$, consisting of points (x_i, t_j), as follows

$$x_i = 0 + ih, \quad i = 0, \ldots, n_x, \qquad t_j = 0 + jk, \quad j = 0, \ldots, n_t.$$

Our aim is to approximate the solution of the problem (7.19) at these points of the grid, solving as many nonlinear systems as there are t_j time nodes in the grid. To do this, we use the following finite differences to approximate the partial derivatives

$$\begin{aligned} u_t(x,t) &\approx \frac{u(x,t) - u(x,t-k)}{k} \\ u_{xx}(x,t) &\approx \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2}. \end{aligned}$$

We note that for the time step we use first-order backward divided differences and for the spatial step we use second-order centred divided differences.

We denote $u_{i,j}$ as the approximation of the solution in (x_i, t_j), and replacing it into the Cauchy problem, we obtain the system

$$ku_{i+1,j} + \left(kh^2 - 2k - h^2\right) u_{i,j} - kh^2 u_{i,j}^2 + ku_{i-1,j} = -h^2 u_{i,j-1},$$

for $i = 1, 2, \ldots, n_x - 1$ and $j = 1, 2, \ldots, n_t$.
The unknowns of this system are $u_{1,j}, u_{2,j}, \ldots, u_{n_x-1,j}$, that is, the approximations of the solution at each spatial node for the fixed time t_j.

In this example, we are going to work with the parameters $T_{max} = 10$, $n_x = 10$ and $n_t = 50$. As we have said, it is necessary to solve as many systems as t_j time nodes, for each of these systems we use the parametric family (7.3) to approximate its solution.

Thus, starting from the initial condition $u_{i,0} = sech^2(\pi x_i)$, $i = 0, \ldots, n_x$, with a tolerance of 10^{-6}, we execute the parametric family for different values of γ so that we obtain Table 7.3.

Table 7.3: Results for Fisher's equation with different values of γ

γ	$\left\|x^{(k+1)} - x^{(k)}\right\|_2$	$\left\|F(x^{(k+1)})\right\|_2$	Iteration	ACOC	Time
0	1.00166×10^{-8}	1.12488×10^{-35}	3	4.21099	213.4219
1	1.9199×10^{-16}	5.88036×10^{-50}	4	2.99609	248.7344
-10+i	8.08037×10^{-9}	4.65282×10^{-26}	5	3.01506	352.6563
$-29/7$	1.8002×10^{-7}	2.00583×10^{-22}	4	2.86978	247.9844
-5	1.89574×10^{-19}	2.9985×10^{-58}	5	2.99569	267.2969
5	2.4177×10^{-17}	6.2774×10^{-52}	5	2.99654	275.7344
2i	2.27659×10^{-11}	1.96645×10^{-34}	4	2.97846	252.8438
2	9.67264×10^{-12}	1.50906×10^{-35}	4	3.00948	231.2188

In all cases, we obtain as an approximation to the solution of the problem (7.19) the following vector

$$x^{(k+1)} \approx (0, 0.4326, 0.7087, 0.8534, 0.9188, 0.9373, 0.9188, 0.8534, 0.7087, 0.4326, 0)^T.$$

It can be observed in Table 7.3 that the results are very similar, although there are some differences. For example, method $\gamma = 0$ uses a smaller number of iterations than the rest to satisfy the required tolerance, although this does not make it much faster than the rest of the methods as the time difference is seconds, due to the fact that this method has order 4. On the other hand, if we look at the time column, we can see that there is one method that stands out for its slowness, this is the case of $\gamma = -10 + i$. This is consistent with what has been obtained in the dynamical analysis of the different elements of the class of iterative methods. Again, we can see that the ACOC of the methods is approximately the theoretical one.
Looking at the error columns we also find similar results and that in this case, having a larger tolerance than in the first example, we do not observe large differences in these results.

7.6 Conclusions

In this chapter, we have presented a parametric family of iterative methods for solving nonlinear systems based on the iterative methods that are presented in [49] and in [50], obtaining a family with order 3, with one element of order 4.

A dynamical analysis is performed on quadratic polynomials in order to determine which members of the parametric family have the best stability properties. It is demonstrated that there is a wide range of values of the parameter both real and complex for which the corresponding methods are stable.

The theoretical results concerning the convergence and stability of the proposed class have been confirmed by numerical examples related to the Hammerstein equation and the Fisher equation.

Chapter 8

Jacobian-free iterative methods

Based on [Cordero, A.; Garrido, N.; Torregrosa, JR.; Triguero-Navarro, P. (2023). Design of iterative methods with memory for solving nonlinear systems. Mathematical Methods in Applied Science, 4145- 4158. https://doi.org/10.1002/mma.9182]

8.1 Introduction

As mentioned above in Chapter 7 and in other parts of the Thesis, one of the most frequently used methods for solving nonlinear systems $F(x) = 0$, $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$, is Newton's method (2.3). The inverse of the Jacobian matrix in the iterative expression of a method can be a drawback when the function to be studied cannot be derived, its derivative is too expensive to calculate or the Jacobian matrix is singular. When the derivative in (2.3) is replaced by the divided difference $[x^{(k)} + F(x^{(k)}), x^{(k)}; F]$ we obtain the Steffensen's scheme [11], which is Jacobian-free and also has quadratic convergence.

Different techniques have been used to design Newton-like iterative schemes, as direct composition, weight functions, estimations of the Jacobian matrix by means of the divided difference operator, etc. So, some high-order methods for computing the solutions of $F(x) = 0$ have been proposed in the literature. These new schemes are proposed with the aim of accelerating the convergence or improving the computational efficiency. For example, recently Cordero et al., Amiri et al. and Chicharro et al. proposed in [53, 54, 55], respectively, new parametric families of iterative methods and a fast algorithm for solving nonlinear systems. Other researchers have published iterative methods that avoid the Jacobian matrix with interesting orders of convergence, see, for instance [56, 57, 58]. In these manuscripts the Jacobian matrix is replaced by the divided difference operator $[\cdot, \cdot; F]$. The procedure of weight functions (in this case, matrix functions) plays also an important role for designing schemes for solving systems $F(x) = 0$, as we can see in [59, 60].

All the papers cited, and many others that appear in the literature, present iterative methods with high order of convergence but considerably increasing the computational cost. To avoid this increase, we may resort to schemes with memory, that is, iterative schemes in which one iteration is obtained from several of the previous ones. Iterative processes with memory for systems are also beginning to appear in the literature. These are methods in which the new iteration is obtained from at least the previous two. In general, the convergence order is increased without adding functional evaluations, [61, 62, 63].

In order to increase the quadratic convergence of Newton's method, Traub [2] proposed the following scheme

$$\begin{cases} y^{(k)} = x^{(k)} - [F'(x^{(k)})]^{-1} F(x^{(k)}), \\ x^{(k+1)} = y^{(k)} - [F'(x^{(k)})]^{-1} F(y^{(k)}), \quad k = 0, 1, \ldots \end{cases} \tag{8.1}$$

In Chapter 4, the unidimensional Traub method was modified in order to obtain a derivative-free method family where the elements of the family were optimal. In this chapter, we extend this scalar family to the multidimensional case. We also add memory to this family in order to increase the order of convergence without the need to perform new functional evaluations as was done in the unidimensional variant.

This chapter is structured as follows. In Section 8.2, we explain the way to extend the scalar case to the vectorial one. Also, we propose the families of iterative methods in the multidimensional case and analyse their order of convergence. We also study in Section 8.2 how to introduce memory to these parametric families in order to increase the order of convergence without performing new functional evaluations. In Section 8.3, we perform some numerical experiments for confirming the theoretical results, including some dynamical planes to illustrate the behaviour of the different methods on a given polynomial system. This chapter ends with some conclusions.

8.2 Design of iterative scheme and convergence analysis

In Chapter 4, we studied two parametric families of iterative methods for nonlinear equations. We now develop how we have modified these two families of iterative methods so that they can be applied to nonlinear systems as well.

It is easy to extend $\dfrac{f(x_k)}{f[w_k, x_k]}$ to systems, since the divided difference operator is a matrix in the case of systems, so the modification would be $[w^{(k)}, x^{(k)}; F]^{-1}F(x^{(k)})$. The same applies to terms $\dfrac{f(y_k)}{f[y_k, x_k]}$ and $\dfrac{f(z_k)}{f[z_k, y_k]}$ which can be extended to systems with expressions $[y^{(k)}, x^{(k)}; F]^{-1}F(y^{(k)})$ and $[z^{(k)}, y^{(k)}; F]^{-1}F(z^{(k)})$, respectively.

The terms that cause some problems with extension to systems are $\dfrac{f(y_k)}{f(w_k)}$ and $\dfrac{f(z_k)}{f(y_k)}$. Using the expression of the iterates y_k, z_k and w_k as well as the divided difference operators, we obtain compatible expressions for systems of nonlinear equations. We are going to do this in a similar way as it was firstly done in [51] and [64].

Let us then calculate $\frac{F(y_k)}{F(w_k)}$

$$\frac{F(y_k)}{F(w_k)} = \frac{F(w_k)}{F(w_k)} + \frac{[w_k, y_k; F](y_k - w_k)}{F(w_k)} = 1 + \frac{[w_k, y_k; F](y_k - w_k)}{F(w_k)}.$$

Then, $y_k - w_k$ can be expressed as

$$y_k - w_k = -(1 + \gamma[w_k, x_k; F])\frac{F(x_k)}{[w_k, x_k; F]}.$$

It then follows

$$\begin{aligned}\frac{F(y_k)}{F(w_k)} &= 1 + \frac{[w_k, y_k; F](y_k - w_k)}{F(w_k)} \\ &= 1 - \frac{[w_k, y_k; F](1 + \gamma[w_k, x_k; F])F(x_k)}{[w_k, x_k; F]F(w_k)}.\end{aligned}$$

On the other hand, one has

$$\begin{aligned}F(w_k) &= F(x_k) + [w_k, x_k; F](w_k - x_k) \\ &= (1 + F[w_k, x_k]\gamma)F(x_k).\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{F(y_k)}{F(w_k)} &= 1 - \frac{[w_k, y_k; F](1+\gamma[w_k, x_k; F])F(x_k)}{[w_k, x_k; F]F(w_k)} \\ &= 1 - [w_k, x_k; F]^{-1}[w_k, y_k; F].\end{aligned}$$

Then,

$$\mu^{(k)} = I - [w^{(k)}, x^{(k)}; F]^{-1}[w^{(k)}, y^{(k)}; F]. \tag{8.2}$$

In a similar way, we calculate $\frac{F(z_k)}{F(y_k)}$

$$\begin{aligned}\frac{F(z_k)}{F(y_k)} &= 1 + \frac{[z_k, y_k; F](z_k - y_k)}{F(y_k)} \\ &= 1 - \frac{[z_k, y_k; F]H(\mu)}{[w_k, x_k; F]}.\end{aligned}$$

Then,

$$\nu^{(k)} = I - [w^{(k)}, x^{(k)}; F]^{-1}[z^{(k)}, y^{(k)}; F]H(\mu^{(k)}). \tag{8.3}$$

Modifying the expression of μ by expression (8.2) in the parametric family M_4 of Chapter 4, we obtain a Jacobian-free variant of Traub's method which we denote by M_4

$$\begin{cases} y^{(k)} = x^{(k)} - [w^{(k)}, x^{(k)}; F]^{-1}F(x^{(k)}), \\ x^{(k+1)} = y^{(k)} - H(\mu^{(k)})[y^{(k)}, x^{(k)}; F]^{-1}F(y^{(k)}), \quad k = 0, 1, \ldots \end{cases} \tag{8.4}$$

where $w^{(k)} = x^{(k)} + \gamma F(x^{(k)})$, $\gamma \neq 0$, $\gamma \in \mathbb{R}$ and the variable of the weight function is $\mu^{(k)} = I - [w^{(k)}, x^{(k)}; F]^{-1}[y^{(k)}, w^{(k)}; F]$. The first step of this method corresponds to Steffensen's scheme when $\gamma = 1$.

On the other hand, the following parametric family is obtained, starting from the parametric family M_6 of Chapter 4 by modifying the expressions of μ and ν by expressions (8.2) and (8.3), which as we see below is a class of iterative methods of seventh order, which we denote by M_7.

$$\begin{cases} y^{(k)} = x^{(k)} - [w^{(k)}, x^{(k)}; F]^{-1}F(x^{(k)}), \\ z^{(k)} = y^{(k)} - H(\mu^{(k)})[y^{(k)}, x^{(k)}; F]^{-1}F(y^{(k)}), \\ x^{(k+1)} = z^{(k)} - G(\mu^{(k)}, \nu^{(k)})[z^{(k)}, y^{(k)}; F]^{-1}F(z^{(k)}), \quad k = 0, 1, \ldots \end{cases} \tag{8.5}$$

where the new variable of weight function G is $\nu^{(k)} = I - [w^{(k)}, x^{(k)}; F]^{-1}[z^{(k)}, y^{(k)}; F]H(\mu^{(k)})$.

Now, we remember and introduce some theoretical concepts necessary for the development of the chapter. Let $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable function on a convex set $D \subset \mathbb{R}^n$ containing α, such that $F(\alpha) = 0$. We will use Genochi-Hermite formula (2.13) to prove the order of convergence of designed class.

Let $X = \mathbb{R}^{n\times n}$ be the Banach space of real square matrices of size $n \times n$, and $H : X \to X$ a function defined in the way that its Fréchet derivatives satisfy

- $H'(u)(v) = H_1uv$, where $H' : X \to \mathcal{L}(X)$ and $H_1 \in \mathbb{R}$,
- $H''(u,v)(w) = H_2uvw$, where $H'' : X \times X \to \mathcal{L}(X)$ and $H_2 \in \mathbb{R}$,

where $\mathcal{L}(X)$ denotes the set of linear operators defined in X. When k tends to infinity, variable $\mu^{(k)}$ tends to the zero matrix 0. So, there exist real numbers H_0, H_1, H_2 such that H can be expanded around 0 as

$$H(\mu^{(k)}) = H_0I + H_1\mu^{(k)} + \frac{1}{2}H_2(\mu^{(k)})^2 + O\left((\mu^{(k)})^3\right),$$

where I is the identity matrix. In the same way, we define a multivariable matrix function $G(\mu^{(k)}, \nu^{(k)})$, so, there exist real numbers G_0, G_{11}, G_{12}, G_{2i} for $i = 1, 2, 3$ and G_{3j} for $j = 1, 2, 3, 4$ such that G can be expanded around $(0, 0)$ as

$$\begin{aligned}G(\mu^{(k)}, \nu^{(k)}) &= G_0I + G_{11}\mu^{(k)} + G_{12}\nu^{(k)} + \frac{1}{2}\left(G_{21}(\mu^{(k)})^2 + G_{22}\mu^{(k)}\nu^{(k)} + G_{23}(\nu^{(k)})^2\right)\\ &+ \frac{G_{31}(\mu^{(k)})^3 + G_{32}(\mu^{(k)})^2\nu^{(k)} + G_{33}\mu^{(k)}(\nu^{(k)})^2 + G_{34}(\nu^{(k)})^3}{6} + O_4(\mu^{(k)}, \nu^{(k)}),\end{aligned}$$

where $O_4\left(\mu^{(k)}, \nu^{(k)}\right)$ denotes all terms in where the sum of exponents of $\mu^{(k)}$ and $\nu^{(k)}$ is at least 4.

Convergence analysis of M_4

Now we prove that the order of convergence of the parametric family M_4 is four for each $\gamma \neq 0$ under certain conditions for the weight function.

Theorem 8.2.1. *Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a sufficiently differentiable function in a convex neighbourhood of α which we denote by $D \subset \mathbb{R}^n$ such that $F(\alpha) = 0$. We assume that $F'(\alpha)$ is non singular. Let $H(t)$ be a real matrix function satisfying that $H_0 = 1$, $H_1 = 1$ and $\|H_2\| < \infty$, where I is the identity matrix of size $n \times n$. Then, taking an estimation $x^{(0)}$ close enough to α, the sequence of iterates $\{x^{(k)}\}_{k\geq 0}$ generated by proposed family (8.4) converges to α with order 4, and its error equation is*

$$\begin{aligned}e_{k+1} = &\left(-C_3(I + \gamma F'(\alpha)) + C_2\left(3I - \frac{H_2}{2}I + \gamma F'(\alpha)\right)C_2 + \gamma C_2^2F'(\alpha)\right)C_2\left(I + \gamma F'(\alpha)\right)e_k^4\\ &+ O\left(e_k^5\right),\end{aligned} \tag{8.6}$$

where $C_j = \frac{1}{j!}F'(\alpha)^{-1}F^{(j)}(\alpha)$ for $j = 2, 3, \ldots$, where $e_k = x^{(k)} - \alpha$.

Proof. Let us consider the Taylor development of $F(x^{(k)})$ and $F(w^{(k)})$ around α

$$F(x^{(k)}) = F'(\alpha)\left(e_k + C_2e_k^2 + C_3e_k^3 + C_4e_k^4 + O\left(e_k^5\right)\right), \tag{8.7}$$

$$F(w^{(k)}) = F'(\alpha)\left(e_w + C_2e_w^2 + C_3e_w^3 + C_4e_w^4 + O\left(e_w^5\right)\right), \tag{8.8}$$

being $e_w = w^{(k)} - \alpha$.
By using Genochi-Hermite formula,

$$\begin{aligned}
[w^{(k)}, x^{(k)}; F] &= F'(x^{(k)}) + \frac{1}{2}F''(x^{(k)})h + \frac{1}{6}F'''(x^{(k)})h^2 + O\left(h^3\right) \\
&= F'(\alpha)\left(I + Y_2e_k + Y_3e_k^2\right) + O\left(e_k^3\right),
\end{aligned}$$

being

$$\begin{aligned}
Y_2 &= C_2\left(2I + \gamma F'(\alpha)\right), \\
Y_3 &= \gamma C_2F'(\alpha)C_2 + C_3\left(3I + 3\gamma F'(\alpha) + \gamma^2F'(\alpha)F'(\alpha)\right).
\end{aligned}$$

We obtain below the inverse of the divided difference operator $[w^{(k)}, x^{(k)}; F]$.
The inverse of the operator has the following expression

$$[w^{(k)}, x^{(k)}; F]^{-1} = \left(I + X_2e_k + X_3e_k^2 + O\left(e_k^3\right)\right)F'(\alpha)^{-1},$$

so we have to determine X_2 and X_3.
If we have $[w^{(k)}, x^{(k)}; F] = F'(\alpha)\left(I + Y_2e_k + Y_3e_k^2 + O\left(e_k^3\right)\right)$, then

$$\begin{aligned}
[w^{(k)}, x^{(k)}; F]^{-1}[w^{(k)}, x^{(k)}; F] &= \left(I + X_2e_k + X_3e_k^2\right)\left(I + Y_2e_k + Y_3e_k^2\right) + O\left(e_k^3\right) \\
&= I + (X_2 + Y_2)e_k + (X_3 + Y_3 + X_2Y_2)e_k^2 + O\left(e_k^3\right).
\end{aligned}$$

Since one also has $[w^{(k)}, x^{(k)}; F]^{-1}[w^{(k)}, x^{(k)}; F] = I$, it follows that the terms X_2 and X_3 must be

$$\begin{aligned}
X_2 &= -Y_2, \\
X_3 &= -Y_3 - X_2Y_2.
\end{aligned}$$

replacing then the values of Y_2 and Y_3, one has

$$\begin{aligned}
X_2 &= -C_2(2I + \gamma F'(\alpha)), \\
X_3 &= 4C_2^2 + \gamma C_2F'(\alpha)C_2 + 2\gamma C_2^2F'(\alpha) + \gamma^2(C_2F'(\alpha))^2 - C_3\left(3I + 3\gamma F'(\alpha) + \gamma^2F'(\alpha)F'(\alpha)\right).
\end{aligned}$$

Let us calculate $y^{(k)}$. Starting from the above relations, one has

$$\begin{aligned}
y^{(k)} - \alpha =& e_k - [w^{(k)}, x^{(k)}; F]^{-1} F(x^{(k)}) \\
=& e_k - \left(I + X_2 e_k + X_3 e_k^2\right) F'(\alpha)^{-1} F'(\alpha) \left(e_k + C_2 e_k^2 + C_3 e_k^3\right) + O\left(e_k^4\right) \\
=& e_k - e_k - C_2 e_k^2 - C_3 e_k^3 - X_2 e_k^2 - X_2 C_2 e_k^3 - X_3 e_k^3 + O\left(e_k^4\right) \\
=& -(C_2 + X_2) e_k^2 - (C_3 + X_2 C_2 + X_3) e_k^3 + O\left(e_k^4\right) \\
=& \; C_2 (I + \gamma F'(\alpha)) e_k^2 \\
& - \left(2C_2^2 + 2\gamma C_2^2 F'(\alpha) + \gamma^2 (C_2 F'(\alpha))^2 - C_3 \left(2I + 3\gamma F'(\alpha) + \gamma^2 F'(\alpha) F'(\alpha)\right)\right) e_k^3 \\
& + O\left(e_k^4\right).
\end{aligned}$$

Let us calculate e_{k+1}. Firstly, we must calculate $\mu^{(k)} = I - [w^{(k)}, x^{(k)}; F]^{-1} [y^{(k)}, w^{(k)}; F]$. One has

$$\begin{aligned}
[y^{(k)}, w^{(k)}; F] =& F'(w^{(k)}) + \frac{1}{2} F''(w^{(k)})(y^{(k)} - w^{(k)}) + \frac{1}{6} F'''(w^{(k)})(y^{(k)} - w^{(k)})^2 \\
& + O\left((y^{(k)} - w^{(k)})^3\right) \\
=& F'(\alpha) \big(I + C_2 (I + \gamma F'(\alpha)) e_k \\
& + \left(\gamma C_2^2 F'(\alpha) + \gamma C_2 F'(\alpha) C_2 + C_2^2 + C_3 \left(I + 2\gamma F'(\alpha) + \gamma^2 F'(\alpha)^2\right)\right) e_k^2 \big) \\
& + O\left(e_k^3\right).
\end{aligned}$$

Thus

$$\begin{aligned}
\mu^{(k)} &= I - [w^{(k)}, x^{(k)}; F]^{-1} [y^{(k)}, w^{(k)}; F] \\
&= C_2 e_k + \left(-C_2 \left(C_2 \left(3 + \gamma F'(\alpha)\right) + \gamma F'(\alpha) C_2\right) + C_3 \left(2 + \gamma F'(\alpha)\right)\right) e_k^2 + O\left(e_k^3\right).
\end{aligned}$$

We denote $M_3 = -C_2 \left(C_2 \left(3 + \gamma F'(\alpha)\right) + \gamma F'(\alpha) C_2\right) + C_3 \left(2 + \gamma F'(\alpha)\right)$, therefore $\mu^{(k)} = C_2 e_k + M_3 e_k^2 + O\left(e_k^3\right)$, and one has

$$\begin{aligned}
H(\mu^{(k)}) &= H_0 + H_1 \mu^{(k)} + \frac{1}{2} H_2 (\mu^{(k)})^2 + O\left(\mu^3\right) = I + \mu^{(k)} + \frac{H_2}{2} (\mu^{(k)})^2 + O\left((\mu^{(k)})^3\right) \\
&= I + C_2 e_k + M_3 e_k^2 + \frac{H_2}{2} C_2^2 e_k^2 + O\left(e_k^3\right) \\
&= I + C_2 e_k + \left(M_3 + \frac{H_2}{2} C_2^2\right) e_k^2 + O\left(e_k^3\right).
\end{aligned}$$

We calculate $[y^{(k)}, x^{(k)}; F]$ using the Genochi-Hermite formula we have

$$[y^{(k)}, x^{(k)}; F] = F'(x^{(k)}) + \frac{1}{2} F''(x^{(k)}) h_1 + \frac{1}{6} F'''(x^{(k)}) h_1^2 + O\left(h_1^3\right).$$

where $h_1 = y^{(k)} - x^{(k)} = -[w^{(k)}, x^{(k)}; F]^{-1}F(x^{(k)})$.
Replacing appropriately, one has

$$\begin{aligned}
[y^{(k)}, x^{(k)}; F] =& F'(x^{(k)}) + \frac{1}{2}F''(x^{(k)})h_1 + \frac{1}{6}F'''(x^{(k)})h_1^2 + O\left(h_1^3\right) \\
&= F'(\alpha)\left(I + C_2e_k + \left(C_3 + C_2^2\left(I + \gamma F'(\alpha)\right)e_k^2\right)\right) + O\left(e_k^3\right).
\end{aligned}$$

Calculating the inverse of this divided difference operator as we have done above, one has

$$[y^{(k)}, x^{(k)}; F]^{-1} = \left(I - C_2e_k - \left(C_3 - C_2\left(I + \gamma F'(\alpha)C_2 - \gamma C_2F'(\alpha)\right)\right)e_k^2 + O\left(e_k^3\right)\right)[F'(\alpha)]^{-1}.$$

Denoting by $R_2 = -C_2$ and $R_3 = -\left(C_3 - C_2\left(I + \gamma F'(\alpha)C_2 - \gamma C_2F'(\alpha)\right)\right)$ we have $[y^{(k)}, x^{(k)}; F]^{-1} = \left(I + R_2e_k + R_3e_k^2 + O\left(e_k^3\right)\right)[F'(\alpha)]^{-1}$.

Then, we calculate $e_{k+1} = e_y - H(\mu^{(k)})[y^{(k)}, x^{(k)}; F]^{-1}F(y^{(k)})$, where $e_y = y^{(k)} - \alpha$

$$e_{k+1} = e_y - \left(I + C_2e_k + \left(M_3 + \frac{H_2}{2}C_2^2\right)e_k^2\right)\left(I + R_2e_k + R_3e_k^2\right)\left(e_y + C_2e_y^2\right) + O_3\left(e_k, e_y\right).$$

As $e_y = -(C_2 + X_2)e_k^2 - (C_3 + X_2C_2 + X_3)e_k^3 + O\left(e_k^4\right)$, then we obtain

$$\begin{aligned}
e_{k+1} &= e_y - \left(I + C_2e_k + \left(M_3 + \frac{H_2}{2}C_2^2\right)e_k^2\right)\left(I + R_2e_k + R_3e_k^2\right)\left(e_y + C_2e_y^2\right) + O_5\left(e_k\right) \\
&= e_y - \left(I + \left(C_2 + R_2\right)e_k + \left(M_3 + \frac{H_2}{2}C_2^2 + R_3 + C_2R_2\right)e_k^2\right)\left(e_y + C_2e_y^2\right) + O_5\left(e_k\right) \\
&= -\left(\left(C_2 + R_2\right)e_k + \left(M_3 + \frac{H_2}{2}C_2^2 + R_3 + C_2R_2\right)e_k^2\right)e_y - C_2e_y^2 + O_5\left(e_k\right) \\
&= -\left(\left(C_2 - C_2\right)e_k + \left(M_3 + \frac{H_2}{2}C_2^2 + R_3 - C_2C_2\right)e_k^2\right)e_y - C_2\left(C_2 + X_2\right)^2e_k^4 + O_5\left(e_k\right) \\
&= -C_2\left(C_2 + X_2\right)^2e_k^4 - \left(M_3 + \frac{H_2}{2}C_2^2 + R_3 - C_2C_2\right)e_k^2e_y + O_5\left(e_k\right) \\
&= \left(-C_3\left(I + \gamma F'\left(\alpha\right)\right) + C_2\gamma F'\left(\alpha\right)C_2 + C_2^2\left(\frac{6 - H_2}{2}I + \gamma F'\left(\alpha\right)\right)\right)C_2\left(I + \gamma F'\left(\alpha\right)\right)e_k^4 \\
&+ O\left(e_k^5\right).
\end{aligned}$$

Thus it is proven that family (8.4) has order of convergence 4.
If we assume that $H_2 = 2$, then the error equation is obtained as follows

$$\begin{aligned}
e_{k+1} =& \left(-C_3(I + \gamma F'(\alpha)) + C_2\left(\left(I + \gamma F'(\alpha)\right)C_2 + C_2(I + \gamma F'(\alpha))\right)\right)C_2\left(I + \gamma F'(\alpha)\right)e_k^4) \\
&+ O\left(e_k^5\right).
\end{aligned}$$

□

Introducing memory to M_4

As it was done in Chapter 4, we are going to introduce memory to the multidimensional parametric family M_4.

As we can see in the error equation (8.6), if it were satisfied that $I + \gamma F'(\alpha) = 0$, then we would increase the order of convergence to at least 5. But as in the unidimensional case, we do not know α or $F'(\alpha)$, so we cannot define $\gamma = -[F'(\alpha)]^{-1}$. What we do, then, is to obtain an approximation of $F'(\alpha)$ based on the functional evaluations already performed.

In the first case, we use the inverse of the divided difference operator at nodes $x^{(k)}$ and $x^{(k-1)}$, that is, we choose $\gamma_k = -[x^{(k)}, x^{(k-1)}, F]^{-1}$. If we replace the parameter of family M_4 by the above approximation, we obtain a method with memory, which we denote by M_4D.

Theorem 8.2.2. *Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a sufficiently differentiable function in a convex neighbourhood of α which we denote by $D \subset \mathbb{R}^n$ such that $F(\alpha) = 0$. We assume that $F'(\alpha) \neq 0$. Let $H(t)$ be a real matrix function satisfying that $H_0 = 1$, $H_1 = 1$, $H_2 = 2$ and $\|H_3\| < \infty$, where I is the identity matrix of size $n \times n$. Then, taking an estimation $x^{(0)}$ close enough to α, the sequence of iterates $\{x^{(k)}\}_{k\geq 0}$ generated by method M_4D converges to α with order $p = 2 + \sqrt{6} \approx 4.449$.*

Proof. The error equation of M_4 under the above conditions, proven in Theorem 8.2.1, is

$$e_{k+1} = \left(-C_3(I + \gamma F'(\alpha)) + C_2\left(\left(I + \gamma F'(\alpha)\right)C_2 + C_2(I + \gamma F'(\alpha))\right)\right)C_2\left(I + \gamma F'(\alpha)\right)e_k^4 + O\left(e_k^5\right). \tag{8.9}$$

Let us consider now the Taylor expansion of $F(x^{(k-1)})$, $F'(x^{(k-1)})$ and $F''(x^{(k-1)})$ around α

$$\begin{aligned}
F(x^{(k-1)}) &= F'(\alpha)\left(e_{k-1} + C_2e_{k-1}^2 + C_3e_{k-1}^3 + C_4e_{k-1}^4 + C_5e_{k-1}^5 + O\left(e_{k-1}^6\right)\right),\\
F'(x^{(k-1)}) &= F'(\alpha)\left(I + 2C_2e_{k-1} + 3C_3e_{k-1}^2 + 4C_4e_{k-1}^3 + 5C_5e_{k-1}^4 + O\left(e_{k-1}^5\right)\right),\\
F''(x^{(k-1)}) &= F'(\alpha)\left(2C_2I + 6C_3e_{k-1} + 12C_4e_{k-1}^2 + 20C_5e_{k-1}^3 + O\left(e_{k-1}^4\right)\right).
\end{aligned}$$

Let us calculate $[x^{(k)}, x^{(k-1)}; F]$ using the Genochi-Hermite formula.

$$[x^{(k)}, x^{(k-1)}; F] = F'(x^{(k-1)}) + \frac{1}{2}F''(x^{(k-1)})h_2 + O\left(h_2^2\right),$$

being $h_2 = e_k - e_{k-1}$. Then,

$$[x^{(k)}, x^{(k-1)}; F] = F'(\alpha)\left(I + C_2(e_k + e_{k-1})\right) + O_2\left(e_k, e_{k-1}\right).$$

We then calculate the inverse of this divided difference operator

$$[x^{(k)}, x^{(k-1)}; F]^{-1} = (I - C_2(e_k + e_{k-1}))F'(\alpha)^{-1} + O_2\left(e_k, e_{k-1}\right).$$

Then, $\gamma_k = -(I - C_2(e_k + e_{k-1}))[F'(\alpha)]^{-1} + O_2\left(e_k, e_{k-1}\right)$.
Therefore,

$$\begin{aligned} I + \gamma_k F'(\alpha) &= I - (I - C_2(e_k + e_{k-1})) + O_2(e_{k-1}, e_k) \\ &= C_2(e_k + e_{k-1})) + O_2(e_{k-1}, e_k). \end{aligned}$$

Thus $I + \gamma_k F'(\alpha) \sim e_{k-1}$.
By error equation (8.9) and the above relation, one has

$$e_{k+1} \sim e_{k-1}^2 e_k^4. \tag{8.10}$$

We assume that the R-order of the method is at least p. Therefore, it is satisfied that

$$e_{k+1} \sim D_{k,p} e_k^p,$$

where $D_{k,p}$ tends to the asymptotic error constant, D_p, when $k \to \infty$. Analogously,

$$e_k \sim D_{k-1,p} e_{k-1}^p.$$

Then, we have

$$e_{k+1} \sim D_{k,p}(D_{k-1,p} e_{k-1}^p)^p = D_{k,p} D_{k-1,p}^p e_{k-1}^{p^2}. \tag{8.11}$$

In the same way as relation (8.10) is obtained, one has

$$e_{k+1} \sim e_{k-1}^2 (D_{k-1,p} e_{k-1}^p)^4 = D_{k-1,p}^4 e_{k-1}^{4p+2}. \tag{8.12}$$

Then, by equating the exponents of e_{k-1} of (8.11) and (8.12), one has

$$p^2 = 4p + 2,$$

whose only positive solution is the order of convergence of method M_4D, that is $p \approx 4.449$, according to Theorem 2.1.1. □

We have previously seen in Chapter 4 that the Kurchatov divided difference operator obtains better approximations to the Jacobian than the usual divided difference operator at same nodes. For that reason in this case we also use the Kurchatov operator at nodes $x^{(k)}$ and $x^{(k-1)}$, that is, we choose $\gamma_K = -[2x^{(k)} - x^{(k-1)}, x^{(k-1)}, F]^{-1}$. If we substitute the parameter of family M_4 by the above approximation, we obtain a method with memory, which we denote by M_4K.

Theorem 8.2.3. *Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a sufficiently differentiable function in a convex neighbourhood of α which we denote by $D \subset \mathbb{R}^n$ such that $F(\alpha) = 0$. We assume that $F'(\alpha) \neq 0$. Let $H(t)$ be a real matrix function satisfying that $H_0 = 1$, $H_1 = 1$, $H_2 = 2$ and $\|H_3\| < \infty$, where I is the identity matrix of size $n \times n$. Then, taking an estimation $x^{(0)}$ close enough to α, the sequence of iterates $\{x^{(k)}\}_{k\geq 0}$ generated by method M_4K converges to α with order $p = 2 + 2\sqrt{2} \approx 4.8284$.*

Proof. We consider the Taylor development of $F(x^{(k-1)})$, $F'(x^{(k-1)})$, $F''(x^{(k-1)})$ and $F'''(x^{(k-1)})$ around α.

Let us calculate $[2x^{(k)} - x^{(k-1)}, x^{(k-1)}; F]$ using the Genochi-Hermite formula.

$$[2x^{(k)} - x^{(k-1)}, x^{(k-1)}; F] = F'(x^{(k-1)}) + \frac{1}{2}F''(x^{(k-1)})h_3 + \frac{1}{6}F'''(x^{(k-1)})h_3^2 + O\left(h_3^3\right),$$

being $h_3 = 2(e_k - e_{k-1})$. Then

$$[2x^{(k)} - x^{(k-1)}, x^{(k-1)}; F] \sim F'(\alpha)\left(I + 2C_2e_k - 2C_3e_{k-1}e_k + C_3e_{k-1}^2 + 4C_3e_k^2\right).$$

We then calculate the inverse of this divided difference operator

$$[2x^{(k)} - x^{(k-1)}, x^{(k-1)}; F]^{-1} \sim (I - 2C_2e_k - C_3e_{k-1}^2 + 2C_3e_{k-1}e_k + e_k^2)F'(\alpha)^{-1}.$$

Therefore,

$$\begin{aligned} I + \gamma_k F'(\alpha) &= I - (I - 2C_2e_k - C_3e_{k-1}^2 + 2C_3e_{k-1}e_k + 4(C_2^2 - C_3)e_k^2) + O_3\left(e_k, e_{k-1}\right) \\ &= 2C_2e_k + C_3e_{k-1}^2 - 2C_3e_{k-1}e_k - 4(C_2^2 - C_3)e_k^2)F'(\alpha)^{-1} \\ &+ O_3\left(e_k, e_{k-1}\right). \end{aligned}$$

Thus $I + \gamma_k F'(\alpha)$ can behave as e_k, as e_ke_{k-1}, as e_k^2 or as e_{k-1}^2.
Obviously the factors e_ke_{k-1} and e_k^2 tend faster to zero than e_k. Then we have to see if the behaviour is like e_k or like e_{k-1}^2.

On the other hand, we assume that the R-order of the method is at least p. Therefore, it is satisfied that

$$e_{k+1} \sim D_{k,p}e_k^p,$$

where $D_{k,p}$ tends to the asymptotic error constant, D_p, when $k \to \infty$. Analogously,

$$e_k \sim D_{k-1,p}e_{k-1}^p.$$

Then, we have

$$\frac{e_k}{e_{k-1}^2} \sim \frac{D_{k-1,p}e_{k-1}^p}{e_{k-1}^2} \sim e_{k-1}^{p-2}.$$

Then if $p \geq 2$, we will have that the behaviour is like e_{k-1}^2. Thus $I + \gamma_k F'(\alpha) \sim e_{k-1}^2$.
By error equation (8.9) and the above relation, one has

$$e_{k+1} \sim e_{k-1}^4 e_k^4. \tag{8.13}$$

On the other hand, by assuming that the R-order of the method is at least p we have

$$e_{k+1} \sim D_{k,p}(D_{k-1,p}e_{k-1}^p)^p = D_{k,p}D_{k-1,p}^p e_{k-1}^{p^2}. \tag{8.14}$$

In the same way as relation (8.13) is obtained, one has

$$e_{k+1} \sim e_{k-1}^4 (D_{k-1,p} e_{k-1}^p)^4 = D_{k-1,p}^4 e_{k-1}^{4p+4}. \tag{8.15}$$

Then, by equating the exponents of e_{k-1} of (8.14) and (8.15), one has

$$p^2 = 4p + 4,$$

whose only positive solution is the order of convergence of method M_4K, that is $p \approx 4.8284$, according to Theorem 2.1.1. □

In the previous cases we used nodes $x^{(k)}$ and $x^{(k-1)}$, but we have also carried out the functional evaluation of $y^{(k-1)}$, so what we do next is to use the same divided difference operators replacing the node $x^{(k-1)}$ by the node $y^{(k-1)}$.

That is, we choose $\gamma_k = -[x^{(k)}, y^{(k-1)}, F]^{-1}$. If we substitute the parameter of family M_4 by the above approximation, we obtain a method with memory, which we denote by M_4D_Y.

Theorem 8.2.4. *Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a sufficiently differentiable function in a convex neighbourhood of α which we denote by $D \subset \mathbb{R}^n$ such that $F(\alpha) = 0$. We assume that $F'(\alpha) \neq 0$. Let $H(t)$ be a real matrix function satisfying that $H_0 = 1$, $H_1 = 1$, $H_2 = 2$ and $\|H_3\| < \infty$, where I is the identity matrix of size $n \times n$. Then, taking an estimation $x^{(0)}$ close enough to α, the sequence of iterates $\{x^{(k)}\}_{k\geq 0}$ generated by method M_4D_Y converges to α with order $p = 5$.*

Proof. Let us consider the Taylor development of $F(y^{(k-1)})$, $F'(y^{(k-1)})$ and $F''(y^{(k-1)})$ around α

$$F(y^{(k-1)}) = F'(\alpha)\left(e_{y,k-1} + C_2 e_{y,k-1}^2 + C_3 e_{y,k-1}^3 + C_4 e_{y,k-1}^4 + C_5 e_{y,k-1}^5 + O\left(e_{y,k-1}^6\right)\right),$$
$$F'(y^{(k-1)}) = F'(\alpha)\left(I + 2C_2 e_{y,k-1} + 3C_3 e_{y,k-1}^2 + 4C_4 e_{y,k-1}^3 + 5C_5 e_{y,k-1}^4 + O\left(e_{y,k-1}^5\right)\right),$$
$$F''(y^{(k-1)}) = F'(\alpha)\left(2C_2 I + 6C_3 e_{y,k-1} + 12C_4 e_{y,k-1}^2 + 20C_5 e_{y,k-1}^3 + O\left(e_{y,k-1}^4\right)\right).$$

Let us calculate $[x^{(k)}, y^{(k-1)}; F]$ using the Genochi-Hermite formula.

$$[x^{(k)}, y^{(k-1)}; F] = F'(y^{(k-1)}) + \frac{1}{2} F''(y^{(k-1)}) h_4 + O\left(h_4^2\right),$$

where $h_4 = e_k - e_{y,k-1}$. Then

$$[x^{(k)}, y^{(k-1)}; F] = F'(\alpha)\left(I + C_2(e_k + e_{y,k-1})\right) + O_2\left(e_k, e_{y,k-1}\right).$$

We then calculate the inverse of this divided difference operator

$$[x^{(k)}, y^{(k-1)}; F]^{-1} = (I - C_2(e_k + e_{y,k-1}))F'(\alpha)^{-1} + O_2\left(e_k, e_{y,k-1}\right).$$

Therefore, $\gamma_k = -(I - C_2(e_k + e_{y,k-1}))F'(\alpha)^{-1} + O_2\left(e_k, e_{y,k-1}\right)$. Thus,

$$\begin{aligned} I + \gamma_k F'(\alpha) &= I - (I - C_2(e_k + e_{y,k-1})) + O_2(e_{y,k-1}, e_k) \\ &= C_2(e_k + e_{y,k-1})) + O_2(e_{y,k-1}, e_k). \end{aligned}$$

Therefore $I + \gamma_k F'(\alpha) \sim e_{y,k-1}$.
By error equation (8.9) and the above relation, one has

$$e_{k+1} \sim e_{y,k-1}^2 e_k^4.$$

We assume that the R-order of the method is at least p. Therefore, it is satisfied that

$$e_{k+1} \sim D_{k,p} e_k^p,$$

where $D_{k,p}$ tends to the asymptotic error constant, D_p, when $k \to \infty$. On the other hand, we assume that sequence $\{y^{(k)}\}_{k \geq 0}$ has R-order, at least, p_1. Therefore, it is satisfied

$$e_{y,k} \sim D_{k,p_1} e_k^{p_1},$$

where D_{k,p_1} tends to the asymptotic error constant, D_{p_1}, when $k \to \infty$. Then, we have

$$\frac{e_k}{e_{y,k-1}} = \frac{e_{k-1}^p}{e_{k-1}^{p_1}} = e_{k-1}^{p-p_1}.$$

Thus $I + \gamma_k F'(\alpha) \sim e_{y,k-1}$, if it is satisfied that $p \geq p_1$. By error equation (8.9) and the above relation, one has

$$e_{k+1} \sim e_{y,k-1}^2 e_k^4. \tag{8.16}$$

On the other hand, by assuming that the R-order of the method is at least p we have

$$e_{k+1} \sim D_{k,p}(D_{k-1,p} e_{k-1}^p)^p = D_{k,p} D_{k-1,p}^p e_{k-1}^{p^2}. \tag{8.17}$$

In the same way as relation (8.16) is obtained, and assuming that sequence $\{y^{(k)}\}_{k \geq 0}$ has R-order at least p_1, one has

$$e_{k+1} \sim e_{y,k-1}^2 e_k^4 \sim (e_{k-1}^{p_1})^2 (e_{k-1}^p)^4 \sim e_{k-1}^{4p+2p_1}. \tag{8.18}$$

On the other hand, by error equation of $e_{y,k}$ one has

$$e_{y,k} \sim (I + \gamma F'(\alpha)) e_k^2 \sim e_{y,k-1} e_k^2. \tag{8.19}$$

By assuming that sequence $\{y^{(k)}\}_{k \geq 0}$ has R-order at least p_1, one has

$$e_{y,k} \sim e_k^{p_1} \sim e_{k-1}^{pp_1}. \tag{8.20}$$

Then, by equating the exponents of e_{k-1} of (8.17) and (8.18), and by equating the exponents of e_{k-1} of (8.19) and (8.20), one has

$$\begin{aligned} p^2 &= 4p + 2p_1, \\ pp_1 &= 2p + p_1 \end{aligned}$$

whose only positive solution is the order of convergence of method M_4D_Y, that is $p = 5$ and $p_1 = 2.5$, according to Theorem 2.1.1. □

In the next case, we choose $\gamma = -[2x^{(k)} - y^{(k-1)}, y^{(k-1)}, F]^{-1}$, that is, the Kurchatov divided difference operator at nodes $x^{(k)}$ and $y^{(k-1)}$. If we substitute the parameter of family M_4 by the above approximation, we obtain a method with memory, which we denote by M_4K_Y.

Theorem 8.2.5. *Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a sufficiently differentiable function in a convex neighbourhood of α which we denote by $D \subset \mathbb{R}^n$ such that $F(\alpha) = 0$. We assume that $F'(\alpha) \neq 0$. Let $H(t)$ be a real matrix function satisfying that $H_0 = 1$, $H_1 = 1$, $H_2 = 2$ and $\|H_3\| < \infty$, where I is the identity matrix of size $n \times n$. Then, taking an estimation $x^{(0)}$ close enough to α, the sequence of iterates $\{x^{(k)}\}_{k\geq 0}$ generated by method M_4K_Y converges to α with order $p = 6$.*

Proof. Let us consider the Taylor development of $F(y^{(k-1)})$, $F'(y^{(k-1)})$,$F''(y^{(k-1)})$ and $F'''(y^{(k-1)})$ around α as we have done before in Theorem 8.2.4. Let us calculate $[2x^{(k)} - y^{(k-1)}, y^{(k-1)}; F]$ using the Genochi-Hermite formula.

$$[2x^{(k)} - y^{(k-1)}, y^{(k-1)}; F] = F'(y^{(k-1)}) + \frac{1}{2}F''(y^{(k-1)})h_5 + \frac{1}{6}F'''(y^{(k-1)})h_5^2 + O\left(h_5^3\right),$$

being $h_5 = 2(e_k - e_{y,k-1})$. Then

$$[2x^{(k)} - y^{(k-1)}, y^{(k-1)}; F] \sim F'(\alpha)\left(I + 2C_2e_k - 2C_3e_{y,k-1}e_k + C_3e_{y,k-1}^2 + 4C_3e_k^2\right).$$

We then calculate the inverse of this divided difference operator

$$[2x^{(k)} - y^{(k-1)}, y^{(k-1)}; F]^{-1} \sim (I - 2C_2e_k - C_3e_{y,k-1}^2 + 2C_3e_{y,k-1}e_k + 4(C_2^2 - C_3)e_k^2)F'(\alpha)^{-1}.$$

Then,

$$\begin{aligned} I + \gamma_k F'(\alpha) &= I - (I - 2C_2e_k - C_3e_{y,k-1}^2 + 2C_3e_{y,k-1}e_k + 4(C_2^2 - C_3)e_k^2) + O_3\left(e_k, e_{y,k-1}\right) \\ &= 2C_2e_k + C_3e_{y,k-1}^2 - 2C_3e_{y,k-1}e_k - 4(C_2^2 - C_3)e_k^2 + O_3\left(e_k, e_{y,k-1}\right). \end{aligned}$$

Thus $I + \gamma_k F'(\alpha)$ can behave as e_k, as $e_ke_{y,k-1}$, as e_k^2 or as $e_{y,k-1}^2$.
Obviously the factors $e_ke_{y,k-1}$ and e_k^2 tend faster to zero than e_k. Then we have to see if the behaviour is like e_k or like $e_{y,k-1}^2$.

On the other hand, we assume that the R-order of the method is at least p. Therefore, it is satisfied that

$$e_{k+1} \sim D_{k,p} e_k^p,$$

where $D_{k,p}$ tends to the asymptotic error constant, D_p, when $k \to \infty$. On the other hand, we assume that sequence $\{y^{(k)}\}_{k\geq 0}$ has R-order, at least, p_1. Therefore, it is satisfied

$$e_{y,k} \sim D_{k,p_1} e_k^{p_1},$$

where D_{k,p_1} tends to the asymptotic error constant, D_{p_1}, when $k \to \infty$. Then, we have

$$\frac{e_k}{e_{y,k-1}^2} \sim \frac{D_{k-1,p} e_{k-1}^p}{e_{k-1}^{2p_1}} \sim e_{k-1}^{p-2p_1}.$$

If $p \geq 2p_1$, we will have that the behaviour is like the behaviour of $e_{y,k-1}^2$, otherwise the behaviour will be like the behaviour of e_k.
Thus, if we assume that $p \geq 2p_1$, one has $I + \gamma_k F'(\alpha) \sim e_{k-1}^2$.
By error equation (8.9) and the above relation, one has

$$e_{k+1} \sim e_{y,k-1}^4 e_k^4. \tag{8.21}$$

On the other hand, by assuming that the R-order of the method is at least p we have

$$e_{k+1} \sim D_{k,p}(D_{k-1,p} e_{k-1}^p)^p = D_{k,p} D_{k-1,p}^p e_{k-1}^{p^2}. \tag{8.22}$$

In the same way as relation (8.21), is obtained, and assuming that sequence $\{y^{(k)}\}_{k\geq 0}$ has R-order at least p_1, one has

$$e_{k+1} \sim e_{y,k-1}^4 (e_{k-1}^p)^4 \sim e_{k-1}^{4p1} + e_{k-1}^{4p} \sim e_{k-1}^{4p+4p_1}. \tag{8.23}$$

On the other hand, by error equation of $e_{y,k}$ one has

$$e_{y,k} \sim (I + \gamma F'(\alpha)) e_k^2 \sim e_{y,k-1}^2 e_k^2. \tag{8.24}$$

By assuming that sequence $\{y^{(k)}\}_{k\geq 0}$ has R-order at least p_1, one has

$$e_{y,k} \sim e_k^{p_1} \sim e_{k-1}^{pp_1}. \tag{8.25}$$

Then, by equating the exponents of e_{k-1} of (8.22) and (8.23), and by equating the exponents of e_{k-1} of (8.24) and (8.25), one has

$$\begin{aligned} p^2 &= 4p + 4p_1, \\ pp_1 &= 2p + 2p_1 \end{aligned}$$

whose only positive solution is $p = 6$ and $p_1 = 3$, which coincides with the R-order of convergence, thus proving that the order of method M_4K_Y is 6, according to Theorem 2.1.1. $\square$

In this case, by extending the iterative methods proposed in Chapter 4 to systems, we obtain the same order of convergence as in the unidimensional case, both with the parametric family and with the different schemes with memory obtained.

Convergence analysis of M_7

In the following result, we establish the order of convergence of parametric family M_7, which is independent of the value of parameter γ, $\gamma \neq 0$.
In the unidimensional case, we obtained that the three-step family had order 6. In this case, since we have defined the function G differently to make sense in the multidimensional case, we have that this three-step family has order 7, as we are going to see below.

Theorem 8.2.6. *Let $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a sufficiently differentiable function in a convex neighbourhood D of α, which is a root of $F(\alpha) = 0$. We assume that $F'(\alpha)$ is non singular. Let $H(t)$ be a real matrix function that satisfies $H_0 = 1$, $H_1 = 1$ and $\|H_2\| < \infty$, where I is the identity matrix of size $n \times n$. Let us also consider a multivariate matrix function $G(p, q)$ such that $G_0 = 1$, $G_{11} = G_{12} = 0$, $G_{2,1} = 0$, $G_{2,2} = 2$, $G_{2,3} = 0$ and $\|G_{3,i}\| < \infty$ for $i = 1, \ldots, 4$. Then, taking an estimate $x^{(0)}$ close enough to α, the sequence of iterates $\{x^{(k)}\}_{k\geq 0}$ generated by family M_7 converges to α with order 7.*

Proof. We have already proven

$$\begin{aligned} y^{(k)} - \alpha \sim\ & C_2(I + \gamma F'(\alpha))e_k^2 \\ & - \left(2C_2^2 + 2\gamma C_2^2 F'(\alpha) + \gamma^2 (C_2 F'(\alpha))^2 - C_3(2I + 3\gamma F'(\alpha) + \gamma^2 F'(\alpha)F'(\alpha))\right) e_k^3. \end{aligned}$$

We denote by

$$Z_1 = \left(-C_3(I + \gamma F'(\alpha)) + C_2\left(\left(3I - \frac{H_2}{2}I + \gamma F'(\alpha)\right) C_2 + \gamma C_2 F'(\alpha)\right)\right) C_2\left(I + \gamma F'(\alpha)\right).$$

Then, we have already proven

$$z^{(k)} - \alpha = Z_1 e_k^4 + O\left(e_k^5\right).$$

Applying the Genochi-Hermite formula, we obtain

$$[z^{(k)}, y^{(k)}; F] = F'(\alpha)\left(I + C_2^2(I + \gamma F'(\alpha))e_k^2 + D_3 e_k^3\right) + O\left(e_k^4\right),$$

being

$$D_3 = -(2C_2^2 + 2\gamma C_2^2 F'(\alpha) + \gamma^2((C_2 F'(\alpha))^2 - C_3(2I + 3\gamma F'(\alpha) + \gamma^2 F'(\alpha))^2) + C_2^4(I + \gamma F'(\alpha)).$$

Calculating the inverse of this divided difference operator as we have done above in Theorem 8.2.4, we obtain

$$[z^{(k)}, y^{(k)}; F]^{-1} = \left(I - C_2^2(I + \gamma F'(\alpha))e_k^2 + J_3 e_k^3 + O\left(e_k^4\right)\right)[F'(\alpha)]^{-1},$$

being $J_3 = ((C_2^2(I+\gamma F'(\alpha))^2 - D_3)$. Now, we calculate $\nu^{(k)}$ and obtain

$$
\begin{aligned}
\nu^{(k)} &= I - [w^{(k)}, x^{(k)}; F]^{-1}[z^{(k)}, y^{(k)}; F]H(\mu^{(k)}) \\
&= I - \left(I + X_2 e_k + X_3 e_k^2\right)\left(I + C_2^2\left(I+\gamma F'(\alpha)\right)e_k^2\right)\left(I + C_2 e_k + \left(M_3 + \frac{H_2}{2}\right)e_k^2\right) \\
&= -(X_2 + C_2)e_k - \left(X_3 - C_2\left(I+\gamma F'(\alpha)\right) + X_2 C_2 + \left(M_3 + \frac{H_2}{2}\right)\right)e_k^2 + O\left(e_k^3\right) \\
&= I - (X_2 + C_2)e_k - V_2 e_k^2 + O\left(e_k^3\right),
\end{aligned}
$$

being $V_2 = X_3 - C_2(I+\gamma F'(\alpha)) + X_2 C_2 + \left(M_3 + \frac{H_2}{2}\right)$.
By denoting

$$
\begin{aligned}
R = &-M_3(X_2 + C_2) - C_2 V_2 \\
&+ \frac{1}{6}\left(G_{31}C_2^3 - G_{32}C_2^2(X_2 + C_2) + G_{33}C_2(X_2+C_2)^2 - G_{34}(X_2+C_2)^3\right).
\end{aligned}
$$

Then,

$$
\begin{aligned}
G(\mu^{(k)}, \nu^{(k)}) &\sim I + \mu^{(k)}\nu^{(k)} + \frac{\left(G_{31}(\mu^{(k)})^3 + G_{32}(\mu^{(k)})^2\nu^{(k)} + G_{33}\mu^{(k)}(\nu^{(k)})^2 + G_{34}(\nu^{(k)})^3\right)}{6} \\
&= I - C_2(X_2 + C_2)e_k^2 + R e_k^3 + O\left(e_k^4\right).
\end{aligned}
$$

From that, the error equation can be expressed as

$$
\begin{aligned}
x^{(k+1)} - \alpha &= e_z - G(\mu^{(k)}, \nu^{(k)})[z^{(k)}, y^{(k)}; F]^{-1}F(z^{(k)}) \\
&= e_z - \left(I + (-C_2(X_2 + C_2) - C_2^2(I+\gamma F'(\alpha)))e_k^2 + (R + J_3)e_k^3\right)(e_z + C_2 e_z^2) \\
&\quad + O\left(e_k^8\right).
\end{aligned}
$$

As $X_2 = -C_2(2I + \gamma F'(\alpha))$, then $X_2 + C_2 = -C_2(I + \gamma F'(\alpha))$. So, $-C_2(X_2+C_2) - C_2^2(I + \gamma F'(\alpha)) = 0$. From this,

$$
\begin{aligned}
e_{k+1} &= e_z - G(\mu^{(k)}, \nu^{(k)})[z^{(k)}, y^{(k)}; F]^{-1}F(z^{(k)}) \\
&= e_z - \left(I + (R + J_3)e_k^3\right)(e_z + C_2 e_z^2) + O\left(e_k^8\right) \\
&= -(R + J_3)e_k^3 e_z + O\left(e_k^8\right) \\
&= -(R + J_3)Z_1 e_k^7 + O\left(e_k^8\right).
\end{aligned}
$$

Thus, it is proven that parametric family M_7 has order of convergence 7. In particular, if $G_{31} = G_{32} = G_{34} = 0$ and $G_{33} = 13$, then $e_{k+1} \sim (I + \gamma F'(\alpha))^4 e_k^7$. $\square$

Introducing memory to M_7

As we did with class M_4, in this section we introduce memory, in different ways, to family M_7.

- If we choose $\gamma_k = -[x^{(k)}, x^{(k-1)}, F]^{-1}$, then replacing the parameter of family M_7 by this value, we obtain a method with memory, denoted by M_7D.
- Choosing $\gamma = -[2x^{(k)} - x^{(k-1)}, x^{(k-1)}, F]^{-1}$ and replacing it in family M_7, yields a method with memory, denoted by M_7K.
- If we choose $\gamma_k = -[x^{(k)}, y^{(k-1)}, F]^{-1}$ and replacing it in M_7, a new scheme with memory, M_7D_Y, is obtained.
- Finally, choosing $\gamma = -[2x^{(k)} - y^{(k-1)}, y^{(k-1)}, F]^{-1}$ and replacing it in M_7, a new scheme with memory, M_7K_Y, is obtained.

The order of convergence of all these methods with memory is established in the next result, whose proof is similar to that of the previous results.

Theorem 8.2.7. *Let $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a sufficiently differentiable function in a convex neighbourhood D of α such that $F(\alpha) = 0$. We assume that $F'(\alpha)$ is non singular. Let H and G be real matrix functions that satisfy $H_0 = 1$, $H_1 = 1$ and $H_2 = 2$, and $G_0 = 1$, $G_{11} = G_{12} = 0$, $G_{2,1} = 0$, $G_{2,3} = 0$, $G_{2,2} = 2$, $G_{3,3} = 13$ and $G_{3,i} = 0$ for $i = 1, 2, 3, 4$, where I is the identity matrix of size $n \times n$. Then, taking an estimate $x^{(0)}$ close enough to α, we have*

- *the sequence of iterates $\{x^{(k)}\}_{k \geq 0}$ generated by method M_7D converges to α with order* $p = \dfrac{7 + \sqrt{65}}{2} \approx 7.5311.$
- *the sequence of iterates $\{x^{(k)}\}_{k \geq 0}$ generated by scheme M_7K converges to α with order* $p = \dfrac{7 + \sqrt{78}}{2} \approx 7.9159.$
- *the sequence of iterates $\{x^{(k)}\}_{k \geq 0}$ generated by method M_7D_Y converges to α with order* $p = 4 + \sqrt{17} \approx 8.1231.$
- *the sequence of iterates $\{x^{(k)}\}_{k \geq 0}$ generated by scheme M_7K_Y converges to α with order* $p = \dfrac{9 + \sqrt{89}}{2} \approx 9.21699.$

In these methods with memory, we could also use variable $z^{(k-1)}$ in order to obtain a better approximation of the parameter. Thus, if we choose $\gamma_k = -[x^{(k)}, z^{(k-1)}, F]^{-1}$, and replace the parameter of family M_7 by this expression, we obtain a new method with memory, denoted by M_7D_Z.

In the same way, the approximation by the Kurchatov divided difference at nodes $x^{(k)}$ and

$z^{(k-1)}$, $\gamma = -[2x^{(k)} - z^{(k-1)}, z^{(k-1)}, F]^{-1}$, gives us a scheme with memory, M_7K_Z, whose convergence we are going to establish. In the following, we prove the order of convergence using these parameter approximations because for the previous cases, it was carried out in a similar way to what was done with family M_4.

Theorem 8.2.8. *Let $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a sufficiently differentiable function in a convex neighbourhood D of α such that $F(\alpha) = 0$. We assume that $F'(\alpha)$ is non singular. Let $H(t)$ and $G(t)$ be real matrix functions that satisfy $H_0 = 1$, $H_1 = 1$ and $H_2 = 2$, and that $G_0 = 1$, $G_{11} = G_{12} = 0$, $G_{2,1} = 0$, $G2, 3 = 0$, $G_{2,2} = 2$, $G_{3,3} = 13$ and $G_{3,i} = 0$ for $i = 1, 2, 3, 4$, where I is the identity matrix of size $n \times n$. Then, taking an estimate $x^{(0)}$ close enough to α, the sequence of iterates $\{x^{(k)}\}_{k\geq 0}$ generated by method M_7D_Z converges to α with order $\dfrac{9+\sqrt{89}}{2} \approx 9.21699$, and the sequence of iterates $\{x^{(k)}\}_{k\geq 0}$ generated by method M_7K_Z converges to α with order 11.*

Proof. Let us consider the Taylor development of $F(z^{(k-1)})$, $F'(z^{(k-1)})$ and $F''(z^{(k-1)})$ around α as was done in theorem 8.2.2. Applying the Genochi-Hermite formula we obtain

$$[x^{(k)}, z^{(k-1)}; F] = F'(\alpha)\left(I + C_2(e_k + e_{z,k-1})\right) + O_2\left(e_k, e_{z,k-1}\right).$$

Then, we calculate the inverse of this divided difference operator.

$$[x^{(k)}, z^{(k-1)}; F]^{-1} = (I - C_2(e_k + e_{z,k-1}))F'(\alpha)^{-1} + O_2\left(e_k, e_{z,k-1}\right).$$

Therefore

$$I + \gamma_k F'(\alpha) = C_2(e_k + e_{z,k-1})) + O_2(e_{z,k-1}, e_k).$$

Let us suppose that the R-order of the method is at least p and sequence $\{z^{(k)}\}_{k\geq 0}$ has R-order p_1. Then, it follows

$$\frac{e_k}{e_{z,k-1}} \sim \frac{e_{k-1}^{p}}{e_{k-1}^{p_1}} \sim e_{k-1}^{p-p_1}.$$

Thus $I + \gamma_k F'(\alpha) \sim e_{z,k-1}$ if it is satisfied that $p \geq p_1$. By error equation (8.6) and the above relation we have

$$e_{k+1} \sim e_{z,k-1}^4 e_k^7. \tag{8.26}$$

Assuming that the R-order of the method is at least p, we have (8.11). In the same way as relation (8.26) is obtained, and supposing that sequence $\{z^{(k)}\}_{k\geq 0}$ has R-order at least p_1 we obtain

$$e_{k+1} \sim e_{z,k-1}^4 e_k^7 \sim (e_{k-1}^{p_1})^4 (e_{k-1}^{p})^7 \sim e_{k-1}^{7p+4p_1}. \tag{8.27}$$

By other way, from the error equation of $e_{z,k}$, we have

$$e_{z,k} \sim \left(I + \gamma F'(\alpha)\right) e_k^4 \sim e_{z,k-1}^2 e_k^4 \sim e_{k-1}^{4p+2p_1}. \tag{8.28}$$

Assuming that sequence $\{z^{(k)}\}_{k\geq 0}$ has R-order at least p_1, we assure

$$e_{z,k} \sim e_k^{p_1} \sim e_{k-1}^{pp_1}. \tag{8.29}$$

Then, by equaling the exponents of e_{k-1} of (8.11) and (8.27), and by equaling the exponents of e_{k-1} of (8.28) and (8.29), it follows

$$\begin{aligned} p^2 &= 7p + 4p_1, \\ pp_1 &= 4p + 2p_1, \end{aligned}$$

whose only positive solution is $p = \dfrac{9+\sqrt{89}}{2} \approx 9.21699$ and $p_1 \approx 5.1085$, that is the order of convergence of method M_7D_Z, according to Theorem 2.1.1.

Now, we calculate $[2x^{(k)} - z^{(k-1)}, z^{(k-1)}; F]$ by using the Genochi-Hermite formula

$$[2x^{(k)} - z^{(k-1)}, z^{(k-1)}; F] \sim F'(\alpha)\left(I + 2C_2e_k - 2C_3e_{z,k-1}e_k + C_3e_{z,k-1}^2 + 4C_3e_k^2\right).$$

Then, the inverse of this divided difference operator is

$$[2x^{(k)} - z^{(k-1)}, z^{(k-1)}; F]^{-1} \sim (I - 2C_2e_k - C_3e_{z,k-1}^2 + 2C_3e_{z,k-1}e_k + e_k^2)F'(\alpha)^{-1}.$$

Therefore,

$$I + \gamma_k F'(\alpha) \sim 2C_2e_k + C_3e_{z,k-1}^2 - 2C_3e_{z,k-1}e_k - 4(C_2^2 - C_3)e_k^2)F'(\alpha)^{-1}.$$

Thus, $I + \gamma_k F'(\alpha)$ can have the behaviour of e_k or $e_{z,k-1}^2$, since the factors $e_k e_{z,k-1}$ and e_k^2 tend to have higher speed at 0 than e_k, so we have to see whether e_k or $e_{z,k-1}^2$ converges faster. Suppose the R-order of the method is at least p. As sequence $z^{(k)}$ has R-order p_1, we have

$$\frac{e_k}{e_{z,k-1}^2} \sim \frac{D_{k-1,p}e_{k-1}^p}{e_{k-1}^{2p_1}} \sim e_{k-1}^{p-2p_1}.$$

Then, if we assume that $p \geq 2p_1$, we have that the behaviour will be like that of $e_{z,k-1}^2$, that is, $I + \gamma_k F'(\alpha) \sim e_{z,k-1}^2$.

From the error equation and the above relation the following relation is obtained

$$e_{k+1} \sim e_{z,k-1}^8 e_k^7. \tag{8.30}$$

In addition, relation (8.11) holds since the R-order of the method is at least p.

In the same way as relation (8.30) is obtained, and taking into account that sequence $z^{(k)}$ has R-order p_1, we obtain

$$e_{k+1} \sim e_{z,k-1}^8 (e_{k-1}^p)^7 \sim e_{k-1}^{8p_1} e_{k-1}^{7p} \sim e_{k-1}^{7p+8p_1}. \tag{8.31}$$

On the other hand, by error equation of $e_{z,k}$ it is obtained

$$e_{z,k} \sim e_{z,k-1}^4 e_k^4. \tag{8.32}$$

Then by equaling the exponents of e_{k-1} of (8.11) and (8.31), and by equaling the exponents of e_{k-1} of (8.32) and (8.29) it is obtained

$$\begin{aligned} p^2 &= 7p + 8p_1, \\ pp_1 &= 4p + 4p_1, \end{aligned}$$

whose only positive solution is $p \approx 11.3523$ and $p_1 \approx 6.17$, therefore it does not satisfy the property for which $I + \gamma_k F'(\alpha) \sim e_{z,k-1}^2$, thus $I + \gamma_k F'(\alpha) \sim e_k$, and therefore

$$e_{k+1} \sim e_k^4 e_k^7 \sim e_k^{11}. \tag{8.33}$$

Thus, we conclude that the order of method M_7K_Z is $p = 11$, according to Theorem 2.1.1. □

As we can see, by introducing memory to families M_4 and M_7 we have managed to increase the order up to 2 and 4 units, thus obtaining methods with memory up to order 6 and 11, respectively.

Next, we show Table 8.1 where we have a collection of the different convergence orders obtained by introducing memory to families M_4 and M_7.

Table 8.1: Collection of the different orders of convergence

Parameter approximation using	Method Name	Order
	M_4	4
$[x^{(k)}, x^{(k-1)}; F]$	M_4D	$2+\sqrt{6} \approx 4.4495$
$[2x^{(k)} - x^{(k-1)}, x^{(k-1)}; F]$	M_4K	$2+2\sqrt{2} \approx 4.8284$
$[x^{(k)}, y^{(k-1)}; F]$	M_4D_Y	5
$f[2x^{(k)} - y^{(k-1)}, y^{(k-1)}]$	M_4K_Y	6
	M_7	7
$[x^{(k)}, x^{(k-1)}; F]$	M_7D	$\frac{1}{2}(7+\sqrt{65}) \approx 7.5311$
$[2x^{(k)} - x^{(k-1)}, x^{(k-1)}; F]$	M_7K	$\frac{1}{2}(7+\sqrt{65}) \approx 7.9159$
$[x^{(k)}, y^{(k-1)}; F]$	M_7D_Y	$4+\sqrt{17} \approx 8.1231$
$[2x^{(k)} - y^{(k-1)}, y^{(k-1)}; F]$	M_7K_Y	$\frac{1}{2}(9+\sqrt{89}) \approx 9.21699$
$[x^{(k)}, z^{(k-1)}; F]$	M_7D_z	$\frac{1}{2}(9+\sqrt{89}) \approx 9.21699$
$[2x^{(k)} - z^{(k-1)}, z^{(k-1)}; F]$	M_7K_z	11

8.3 Numerical experiments

In this section, we perform several numerical experiments in order to see the behaviour of families M_4, M_7 and the partners derived from them with memory. We present two numerical experiments, one of them is to solve the Hammerstein equation and other is an academical nonlinear system, in which we also make a comparison with two known methods with order of convergence 8. These schemes are method $CCGT1$, which can be found in [65], and method $NM8$, which can be found in [66]

We would like to point out that in this case Matlab 2020b has been used to carry out the numerical experiments, with an arithmetical precision of 1000 digits. As stopping criterion we choose that

$$\left\|x^{(k+1)} - x^{(k)}\right\|_2 + \left\|F(x^{(k)})\right\|_2 < 10^{-50}.$$

We use also a maximum of 100 iterations.

For all methods and all numerical experiments the following matrix functions have been selected as weight functions

- $H(t) = t^2 + t + I$,
- $G(t, r) = I + t\ r + \dfrac{13}{6}\ t\ r^2$,

where I is the identity matrix.

In the different tables we show the following data

- the norm of the function evaluated in the last approximation, $\left\|F(x^{(k+1)})\right\|_2$,
- the norm of the distance between the last two approximations, $\left\|x^{(k+1)} - x^{(k)}\right\|_2$,
- the number of iterations necessary to satisfy the required tolerance,
- and the approximated computational order of convergence (ACOC).

In this example, we consider the well-known Hammerstein integral equation (see [3]), which is given as follows

$$x(s) = 1 + \frac{1}{5}\int_0^1 F(s,t)x(t)^3 dt, \tag{8.34}$$

where $x \in \mathbb{C}[0, 1]$, $s, t \in [0, 1]$ and the kernel F is

$$F(s,t) = \begin{cases} (1-s)t, & t \le s, \\ s(1-t), & s \le t. \end{cases}$$

We transform the above equation into a finite-dimensional problem by using Gauss–Legendre quadrature formula given as $\int_0^1 f(t)dt \approx \sum_{i=1}^{7} \omega_i f(t_i)$, where the abscissas t_i and the weights ω_i are determined for $n = 7$ (see Table 7.1)

By denoting the approximations of $x(t_i)$ by x_i, $i = 1, \ldots, 7$, one gets the system of nonlinear equations

$$5x_i - 5 - \sum_{j=1}^{7} a_{ij} x_j^3 = 0,$$

where $i = 1, \ldots, 7$ and

$$a_{ij} = \begin{cases} w_j t_j (1 - t_i) & j \leq i, \\ w_j t_i (1 - t_j) & i < j. \end{cases}$$

We start from the initial approximation $x^{(0)} = (0.5, \ldots, 0.5)^T$, we choose as initial approximations for $x^{(-1)}$, $y^{(-1)}$ and $z^{(-1)}$ vector $(0.4, \ldots, 0.4)^T$. In Table 8.2, the results obtained by each method for the Hammerstein's equation are shown.

Table 8.2: Numerical results of Hammerstein's equation

Method	$\left\|x^{(k+1)} - x^{(k)}\right\|_2$	$\left\|F(x^{(k+1)})\right\|_2$	Iteration	ACOC
$M_{4,-1}$	1.01573×10^{-166}	2.41351×10^{-666}	5	3.99986
M_4D	2.05575×10^{-154}	4.58506×10^{-688}	4	4.4952
M_4K	7.31331×10^{-177}	2.37182×10^{-853}	4	4.9600
M_4D_Y	7.12038×10^{-204}	1.62192×10^{-1021}	4	4.9971
M_4K_Y	1.31159×10^{-295}	3.12493×10^{-1776}	4	5.9975
$M_{7,-1}$	4.53896×10^{-171}	9.75609×10^{-1027}	4	6.9996
M_7D	2.42252×10^{-79}	3.32054×10^{-516}	3	7.5291
M_7K	2.45812×10^{-82}	3.43156×10^{-571}	3	7.8613
M_7D_Y	2.35271×10^{-88}	4.7518×10^{-622}	3	8.1898
M_7K_Y	3.16092×10^{-101}	7.79194×10^{-813}	3	9.1692
M_7D_Z	3.17456×10^{-99}	2.21039×10^{-796}	3	9.2162
M_7K_Z	2.85847×10^{-114}	1.65181×10^{-1032}	3	10.9981

We can see that in all cases the ACOC is close to the theoretical convergence order demonstrated in Section 2 and that the number of iterations required is similar for the methods of the same family, being one unit higher in the case of the methods without memory.

It can be seen that the best results for these numerical experiments are given by methods with memory that use the Kurchatov divided difference operator to approximate the parameter of family. These methods give the closest approximations to the solution and the biggest ACOC.

We also approximate the solution of the following academic system of nonlinear equations. In this case, we compare the results obtained with the different methods proposed with those provided by two known schemes without memory, both of order 8. These schemes are method $CCGT1$ and method $NM8$.

The system that we use in our experiment, denoted by System F, is

$$\begin{cases} F_i(x) = x_i^2 x_{i+1} - 1 = 0, \\ F_{200}(x) = x_{200}^2 x_1 - 1 = 0, \end{cases}$$

a system with 200 unknowns and 200 equations.

For this example we use an initial estimation $x^{(0)} = (0.9, \ldots, 0.9)^T$, and as initial approximations for $x^{(-1)}$, $y^{(-1)}$ and $z^{(-1)}$ vector $(0.7, \ldots, 0.7)^T$.

Table 8.3: Numerical results for System F

Method	$\left\|x^{(k+1)} - x^{(k)}\right\|_2$	$\left\|F(x^{(k+1)})\right\|_2$	Iteration	ACOC
$M_{4,-1}$	2.61975×10^{-71}	2.275×10^{-102}	5	3.92262
M_4D	1.84221×10^{-51}	6.63527×10^{-83}	4	4.36222
M_4K	5.58564×10^{-57}	5.32079×10^{-89}	4	4.71234
M_4D_Y	3.72943×10^{-54}	1.45955×10^{-85}	4	5.1739
M_4K_Y	1.51937×10^{-63}	1.46077×10^{-95}	4	5.9701
$M_{7,-1}$	4.32096×10^{-77}	1.40986×10^{-123}	4	6.93731
M_7D	1.10549×10^{-89}	2.70395×10^{-137}	4	7.53147
M_7K	8.76112×10^{-51}	6.9915×10^{-401}	3	7.85679
M_7D_Y	1.52107×10^{-94}	1.2392×10^{-139}	4	8.19609
M_7K_Y	8.76112×10^{-51}	6.9915×10^{-401}	3	9.18679
M_7D_Z	5.21322×10^{-97}	8.1106×10^{-144}	4	9.22566
M_7K_Z	8.76112×10^{-51}	6.9915×10^{-401}	3	10.9754
$CCGT1$	2.64372×10^{-64}	1.59087×10^{-516}	3	8.09479
$NM8$	2.81063×10^{-292}	3.0869×10^{-2337}	4	8.0

The results obtained for system F and for each method are shown in Table 8.3. We can see that the number of iterations change for the family M_7 and their partners with memory. In this case the iterations are between 3 and 4, making the methods that perform 4 iterations have the ACOC closer to the theoretical convergence order.

It can be seen that the best results for these numerical experiments are given by the partners with memory that use Kurchatov's divided difference operator, although these are also the ones that perform the fewest iterations, which means that they are still closer to the solution than the rest.

As we can see in the tables, our methods M_7K, M_7K_Y and M_7K_Z are quite similar to the results obtained by method $CCGT1$, and that method $NM8$ performs one more iteration than them to satisfy the tolerance, so it would be more advisable in practice to use the methods derived from the parametric family.

Real dynamics on an uncoupled polynomial system

In the previous sections we have introduced memory to two parametric families and studied the order of convergence of the proposed methods. These are important concepts of iterative methods, but not the only ones. Another important concept is the behaviour of the method according to the initial estimate chosen, since we would like to know in advance if the method converge to any of the solutions according to the estimate taken. This study let us know the stability of the method. In this case, the method is analyzed for a given function and the behaviour shown graphically with dynamical planes. This procedure is explained in many papers, for example, see [19].

The system of nonlinear equations of which we analyse the behaviour is the following

$$\begin{cases} x_1^2 - 1 = 0, \\ x_2^2 - 1 = 0, \end{cases}$$

where $(x_1, x_2)^T \in \mathbb{R}^2$.
We now that the roots of this system are $(-1,-1)^T$, $(-1,1)^T$, $(1,-1)^T$ and $(1,1)^T$.

For all methods the following matrix functions have been selected as weight functions because they are the easiest polynomial functions that satisfy the conditions:

- $H(\mu^{(k)}) = (\mu^{(k)})^2 + \mu^{(k)} + I$,
- $G(\mu^{(k)}, \nu^{(k)}) = I$,

where I is the identity matrix of size 2×2.
To generate the dynamical planes, we have chosen a mesh of 400×400 points, and what we do is apply our methods to each of these points, taking the point as the initial estimate. We have also defined that the maximum number of iterations that each initial estimate must do is 80, and that we determine that the starting point converges to one of the solutions if the distance to that solution is less than 10^{-3}. We represented in orange the initial points that converge to the root $(1,1)^T$, in green the initial points that converge to the root $(1,-1)^T$, in blue the initial points that converge to the root $(-1,1)^T$, in red the initial points that converge to the root $(-1,-1)^T$ and in black the initial points that do not converge to any root.

Figure 8.1: Dynamical planes of M_4 and their partners with memory

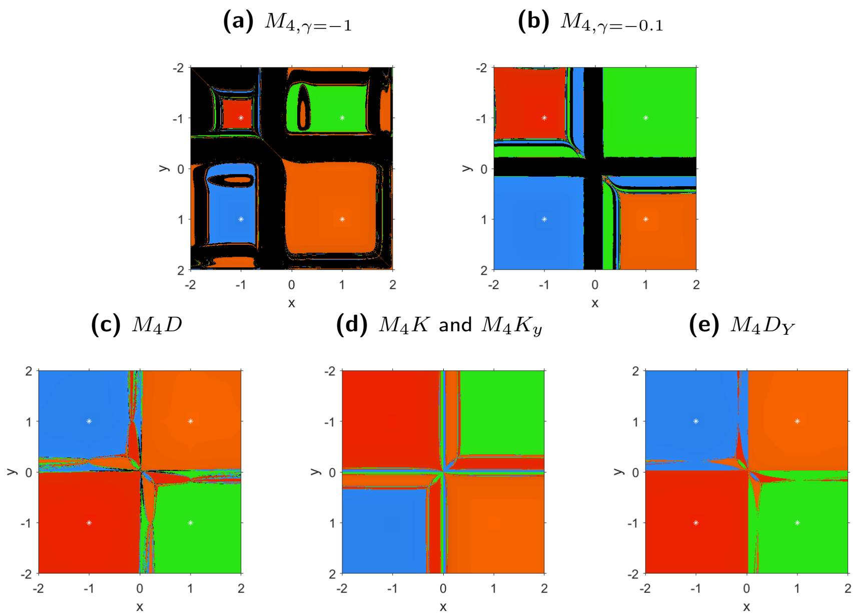
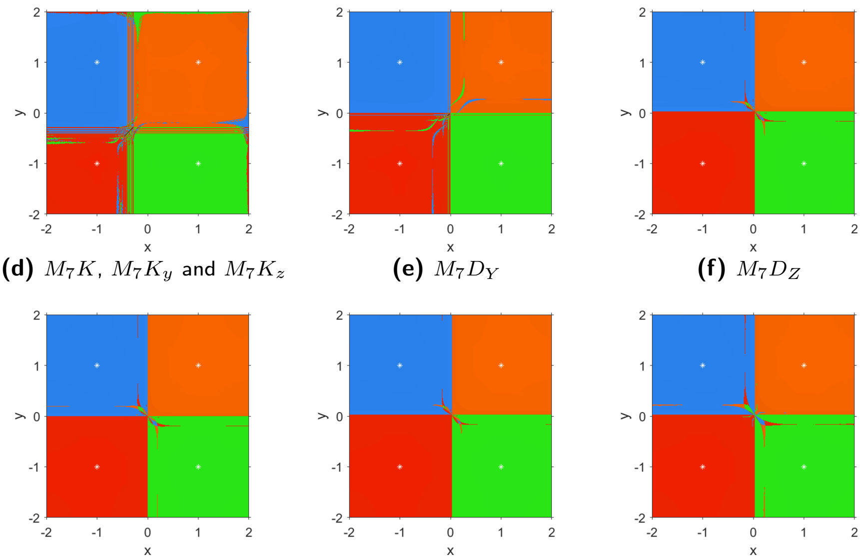


Figure 8.2: Dynamical planes of M_7 and their partners with memory



(a) $M_{7,\gamma=-1}$ **(b)** $M_{7,\gamma=-0.1}$ **(c)** M_7D

(d) M_7K, M_7K_y and M_7K_z **(e)** M_7D_Y **(f)** M_7D_Z

In addition to having increased the convergence order by introducing memory, it can be noticed in Figures 8.1 and 8.2, that for the selected system, the introduction of memory has also helped to obtain a more stable behaviour and wider sets of converging initial guesses to the roots.

8.4 Conclusions

In this chapter, two parametric families of iterative methods with orders of convergence 4 and 7, respectively, for solving systems of nonlinear equations, have been designed.

Memory has been introduced, in different ways, to these two families in order to obtain iterative methods with higher order of convergence without the need to increase the number of functional evaluations per iteration. These methods with memory have managed to increase the order by up to 2 units for the family of order 4 and up to 4 units for the family of order 7.

But not only does the introduction of memory improves the order of convergence, but as we have seen in the dynamical planes that have been carried out, it has also improved the behaviour of the method, since we obtain that more points converge when it comes to the methods with memory, or else the attraction zones of the roots are simpler.

In the numerical experiments, the theoretical results are confirmed, and when comparing our methods with other known ones of high order (order of convergence 8) it can be seen that most of the proposed methods obtain a closer approximation to the solution than known methods, and it also can see that the partners with memory obtain better results in these cases.

Chapter 9

Iterative methods for simultaneous solutions of nonlinear systems

Based on [Chinesta, F.; Cordero, A.; Garrido, N.; Torregrosa, J.R.; Triguero-Navarro, P. (2023). Simultaneous roots for vectorial problems. Computational and Applied Mathematics. Submitted]

9.1 Introduction

In a large number of problems in applied mathematics, we need to solve a system of equations, and in many cases these systems are nonlinear. We cannot always solve these systems exactly, due to the complexity of the problem. For this reason, we obtain an approximation to the solution of the problem.

One way to obtain these approximations is by using iterative methods. What iterative schemes do is that, starting from an initial approximation, they generate a succession of approximations that, under certain conditions, converge to that solution. Some known iterative methods to solve nonlinear systems are designed in: [53] by Cordero et al, [62] by Chicharro et al and [25] by Neta and Johnson.

But what if instead of wanting only one of the solutions, we want to obtain more than one of them simultaneously? One option would be, by using two different initial approximations, to obtain the approximate solutions. But with this we have a problem, what happens if both estimates converge to the same root?

In what follows, we generate several dynamical planes for Newton' and Steffensen's methods, [2] and [11], in order to illustrate this problem. What we draw in these cases is if the initial points converge or not to the roots of our problems in the same way as we have done in Chapter 5.

We show the dynamical planes associated with each of the procedures when applied to a simple quadratic polynomial, $p(x) = x^2 - 1$, whose roots are 1 and -1.

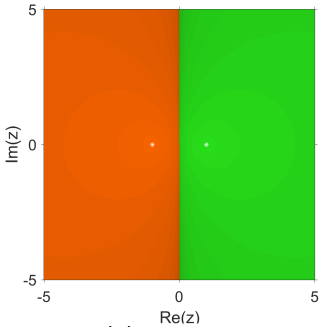
To generate the dynamical planes (see [14]), we have chosen a mesh of 400×400 points, and what we do is to apply our methods to each of these points, taking the point as the initial estimate. Each of the axes corresponds to the real and the imaginary part of the initial guess, respectively.

We have also defined that the maximum number of iterations for each initial estimate is 80, and we determine that the initial point converges to one of the solutions if the distance to that solution is less than 10^{-3}.

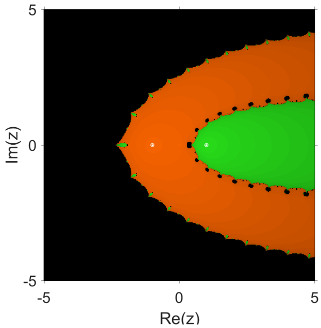
We represent in orange the initial points that converge to root -1, in green the initial points converging to root 1, and in black the initial points that do not converge to any root.

Figure 9.1: Dynamical planes of Newton' and Steffensen's scheme

(a) Newton



(b) Steffensen



In this case, if we take two different initial approximations and both are in the same basin of attraction, in the end we will obtain that the sequences converge to the same solution.

However, if we design a scheme that calculates both sequences of approximations at the same time, taking into account who is the other iterate, we will avoid this problem.

In Chapter 5, we have designed an iterative step that can be added to any iterative method that solves nonlinear equations, so if the initial iterative scheme has order of convergence p, the new scheme will have order of convergence $2p$ ($3p$ if the nonlinear function is polynomical), but in addition, the new iterative procedure will obtain the solutions of the equation simultaneously.

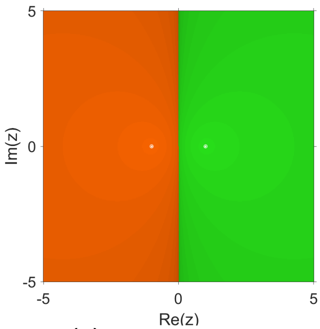
In Figures 9.2 and 9.3, we generate the dynamical planes of the Modified Newton and Steffensen schemes, applied to the same equation and with the same convergence criterion.

In this case, one of the axes is the initial estimate $x_1^{(0)}$ and the other is $x_2^{(0)}$. We represent the initial point in purple if component of the iterate of the point on the $x_1^{(0)}$ converges to the root -1 and the component in the axis $x_2^{(0)}$ converges to the root 1. We represent the initial point in yellow if component of the iterate of the point on the $x_1^{(0)}$ converges to the root 1 and the component in the axis $x_2^{(0)}$ converges to the root -1. In case of non convergence, we represent the starting point blue.

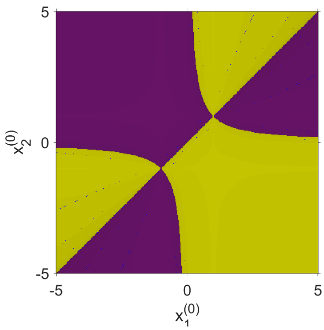
In Figure 9.2, we show the dynamical planes obtained for Newton's and Modified Newton's methods applied to the quadratic polynomial. As we can see, the basins of attraction occupy the whole space in Newton's scheme, as well as for its variant for finding roots simultaneously.

Figure 9.2: Dynamical planes of Newton and Modified Newton

(a) Newton



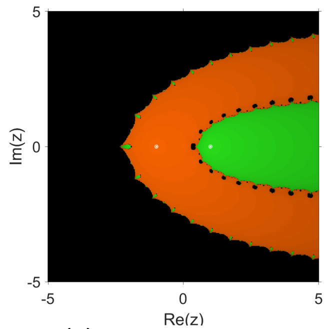
(b) Modified Newton



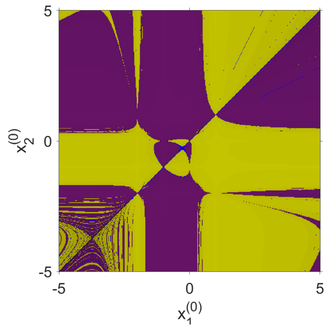
In Figure 9.3, we show the dynamical planes obtained for Steffensen's and Modified Steffensen's methods applied to the quadratic polynomial. As we can observe in this case, Steffensen's scheme does not converge in some areas, for example at the point z = -5, although we can observe that its variant does converge to the roots at any point of this mesh, except a small area where $x_1^{(0)} = x_2^{(0)} = 0$.

Figure 9.3: Dynamical planes of Steffensen and Modified Steffensen

(a) Steffensen



(b) Modified Steffensen



As we can see in Figure 9.1, now taking initial estimations $x_1^{(0)} = 2$ and $x_2^{(0)} = 5$ we converge to both roots, something that iterating both non simultaneous methods in parallel we did not achieve, since both were in the basin of attraction of the root 1.

What we do in this chapter is modify the step presented in Chapter 5 so it can be applicable to systems, thus designing an iterative step that can be added to any iterative procedure that solves systems of nonlinear equations, so that we duplicate the order, and we can apply this method to obtain solutions simultaneously.

This chapter is structured as follows. In Section 9.2, we modify the step and study the order of convergence of this step. In Section 9.3, we carry out several numerical experiments with known iterative schemes to which we add the simultaneous step, to see their behaviour, and we finish the work in Section 9.4 with conclusions derived from the study.

9.2 Convergence analysis

Let $F(x) = 0$, $F : \mathbb{R}^m \to \mathbb{R}^r$, be a system of nonlinear equations, where the number of unknowns is m and the number of equations is r. Let us notice that the system is written as a column vector of size $r \times 1$.

Suppose that this system has n solutions, which we denote by $\alpha_i = (\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_m})$ for $i = 1, \ldots, n$.

We are going to design an iterative step to obtain all the solutions simultaneously. Therefore, we take a set of n initial approximations which we denote by $x_i^{(0)} = \left(x_{i_1}^{(0)}, x_{i_2}^{(0)}, \ldots, x_{i_m}^{(0)}\right)$, $i = 1, .., n$.

If we define $\dfrac{1}{x_i^{(k)} - x_j^{(k)}} := \left(\dfrac{1}{x_{i_1}^{(k)} - x_{j_1}^{(k)}}, \dfrac{1}{x_{i_2}^{(k)} - x_{j_2}^{(k)}}, \ldots, \dfrac{1}{x_{i_m}^{(k)} - x_{j_m}^{(k)}}\right)$.
We then design the iterative step in the following way:

$$x_i^{(k+1)} = x_i^{(k)} - \left(F'\left(x_i^{(k)}\right) - F\left(x_i^{(k)}\right) \sum_{j \neq i} \frac{1}{x_i^{(k)} - x_j^{(k)}}\right)^{-1} F\left(x_i^{(k)}\right). \tag{9.1}$$

As we can see, the size of matrix $F'\left(x_i^{(k)}\right)$ is $r \times m$ which matches the size of the product of $F\left(x_i^{(k)}\right)$ and $\dfrac{1}{x_i^{(k)} - x_j^{(k)}}$, $j \neq i$, as the above vectors are a column vector of size $r \times 1$ and a row vector of size $1 \times m$, respectively.
We denote this point-to-point method by PS. We are going to prove that its order of convergence is 2.

Theorem 9.2.1. *Let $F : \mathbb{R}^m \longrightarrow \mathbb{R}^r$ be a sufficiently differentiable function in a convex neighbourhood of α_i which we denote by $D_i \subset \mathbb{R}^m$ such that $F(\alpha_i) = 0$ for $i = 1, ..., n$. We assume that $F'(\alpha_i)$ is non singular for $i = 1, ..., n$. Then, taking an estimate $x_i^{(0)} \in \mathbb{R}^m$ close enough to α_i for $i = 1, ..., n$, the sequences of iterates $\{x_i^{(k)}\}_{k \geq 0}$ generated by method PS converges to α_i with order 2.*

Proof. Let us denote $F = (F_1, F_2, \ldots, F_r)$ where $F_p : \mathbb{R}^m \to \mathbb{R}^q$ are the coordinate functions of F for $p = 1, 2, \ldots, r$.
Consider now the Taylor development of $F_p\left(x_i^{(k)}\right)$ around α for $p = 1, 2, \ldots, r$:

$$F_p\left(x_i^{(k)}\right) = \sum_{j_1=1}^{m} \frac{\partial F_p(\alpha)}{\partial x_{j_1}} e_{i,k_{j_1}} + \sum_{j_1=1}^{m} \sum_{j_2=1}^{m} \frac{\partial^2 F_p(\alpha)}{\partial x_{j_1} \partial x_{j_2}} e_{i,k_{j_1}} e_{i,k_{j_2}} + O_3(e_{i,k}), \tag{9.2}$$

where $e_{i,k_{j_1}} = x^{(k)}_{i_{j_1}} - \alpha_{i_{j_1}}$ for $j_1 \in \{1,2,\ldots,m\}$ and $i \in \{1,\ldots,n\}$, and where $O_3\left(e_{i,k}\right)$ the elements where the sums of the exponents of $e_{i,k_{j_1}}$ is greater than or equal to 3 with $j_1 \in \{1,2,\ldots,m\}$.

If we derive this with respect to variable x_{i_q}, with $q = 1,2,\ldots,m$, we get

$$\frac{\partial F_p\left(x_i^{(k)}\right)}{\partial x_{i_q}} = \frac{\partial F_p\left(\alpha\right)}{\partial x_{i_q}} + \sum_{j_1=1}^{m} \frac{\partial^2 F_p\left(\alpha\right)}{\partial x_{i_q}\partial x_{j_1}} e_{i,k_{j_1}} + O_2\left(e_{i,k}\right). \tag{9.3}$$

To simplify notation, we denote by E_{i,k_q} the following

$$E_{i,k_q} = \sum_{j\neq i}^{m} \frac{1}{e_{i,k_q} - e_{j,k_q} + \alpha_{i_q} - \alpha_{j_q}}.$$

Then,

$$\begin{aligned}\frac{\partial F_p\left(x_i^{(k)}\right)}{\partial x_{i_q}} - F_p\left(x_i^{(k)}\right)\sum_{j\neq i}^{m}\frac{1}{x_{i_q}^{(k)} - x_{j_q}^{(k)}} =& \frac{\partial F_p\left(\alpha\right)}{\partial x_{i_q}} + \sum_{j_1=1}^{m} \frac{\partial^2 F_p\left(\alpha\right)}{\partial x_{i_q}\partial x_{j_1}} e_{i,k_{j_1}} \\ & - \sum_{j_1=1}^{m}\frac{\partial F_p\left(\alpha\right)}{\partial x_{j_1}} e_{i,k_{j_1}} E_{i,k_q} + O_2\left(e_{i,k}\right).\end{aligned} \tag{9.4}$$

We simplify as follows

$$\sum_{j_1=1}^{m} \frac{\partial^2 F_p\left(\alpha\right)}{\partial x_{i_q}\partial x_{j_1}} e_{i,k_{j_1}} - \sum_{j_1=1}^{m}\frac{\partial F_p\left(\alpha\right)}{\partial x_{j_1}} e_{i,k_{j_1}} E_{i,k_q} = \sum_{j_1=1}^{m}\left(\frac{\partial^2 F_p\left(\alpha\right)}{\partial x_{i_q}\partial x_{j_1}} - \frac{\partial F_p\left(\alpha\right)}{\partial x_{j_1}}E_{i,k_q}\right) e_{i,k_{j_1}}.$$

Then, we can rewrite (9.4) as

$$\frac{\partial F_p\left(x_i^{(k)}\right)}{\partial x_{i_q}} - F_p\left(x_i^{(k)}\right)\sum_{j\neq i}^{m}\frac{1}{x_{i_q}^{(k)} - x_{j_q}^{(k)}} = \frac{\partial F_p\left(\alpha\right)}{\partial x_{i_q}} + \sum_{j_1=1}^{m} A_{i_q,j_1} e_{i,k_{j_1}} + O_2\left(e_{i,k}\right),$$

being $A_{i_q,j_1} := \left(\frac{\partial^2 F_p\left(\alpha\right)}{\partial x_{i_q}\partial x_{j_1}} - \frac{\partial F_p\left(\alpha\right)}{\partial x_{j_1}}E_{i,k_q}\right)$ for $j_1 \in \{1,2,\ldots,m\}$.

We denote $A_{i_q} := \left(A_{i_q,1}, A_{i_q,2}, \ldots, A_{i_q,m}\right)$, and then define matrix A that is composed by rows A_{i_1}, A_{i_2}, …, A_{i_m} we obtain

$$F'\left(x_i^{(k)}\right) - F\left(x_i^{(k)}\right)\sum_{j\neq i}\frac{1}{x_i^{(k)} - x_j^{(k)}} = F'\left(\alpha\right)\left(I + F'\left(\alpha\right)^{-1} A e_{i,k}\right) + O_2\left(e_{i,k}\right). \tag{9.5}$$

From (9.5), it follows

$$\left(F'\left(x_i^{(k)}\right) - F\left(x_i^{(k)}\right)\sum_{j\neq i}\frac{1}{x_i^{(k)}-x_j^{(k)}}\right)^{-1} = \left(I - F'(\alpha)^{-1} Ae_{i,k}\right)^{-1} F'(\alpha)^{-1} + O_2\left(e_{i,k}\right). \tag{9.6}$$

Then, by using (9.6), the error equation is

$$\begin{aligned} e_{i,k+1} &= e_{i,k} - \left(I - F'(\alpha)^{-1} Ae_{i,k}\right)\left(e_{i,k} + C_2 e_{i,k}^2\right) + O_3\left(e_{i,k}\right) \\ &= \left(F'(\alpha)^{-1} A - C_2\right) e_{i,k}^2 + O_3\left(e_{i,k}\right). \end{aligned}$$

It is therefore proven that method PS has convergence order 2. □

Let ϕ be the non-simultanous fixed point function of a known iterative method, we define PS_ϕ as follows

$$\begin{aligned} y_i^{(k)} &= \phi\left(x_i^{(k)}\right), \\ x_i^{(k+1)} &= PS\left(y_1^{(k)}, \ldots, y_n^{(k)}\right), \end{aligned}$$

in other words, an iterative method in which we use ϕ as a predictor and then scheme PS as a simultaneous corrector.

Theorem 9.2.2. *Let $F : \mathbb{R}^m \longrightarrow \mathbb{R}^r$ be a sufficiently differentiable function in a convex neighbourhood of α_i which we denote by $D_i \subset \mathbb{R}^m$ such that $F(\alpha_i) = 0$ for $i = 1, ..., n$. We assume that $F'(\alpha_i)$ is non singular for $i = 1, ..., n$. Then, taking an estimate $x_i^{(0)} \in \mathbb{R}^m$ close enough to α_i for $i = 1, ..., n$, the sequences of iterates $\{x_i^{(k)}\}_{k\geq 0}$ generated by method PS_ϕ converges to α_i with order $2p$, where p is the order of convergence of ϕ.*

Proof. By Theorem 9.2.1,

$$e_{i,k+1} = \left(F'(\alpha)^{-1} A - C_2\right) e_{i,y,k}^2 + O_3\left(e_{i,y,k}\right), \tag{9.7}$$

where $e_{i,y,k} = y_i^{(k)} - \alpha_i$ and $e_{i,k} = x_i^{(k)} - \alpha_i$. Since we have that ϕ has order of convergence p, this means that $e_{i,y,k} \sim e_{i,k}^p$.
Substituting the last relation into equation (9.7) we obtain that

$$e_{i,k+1} \sim e_{i,y,k}^2 \sim \left(e_{i,k}^p\right)^2 \sim e_{i,k}^{2p}.$$

Thus it is proven that iterative method PS_ϕ duplicates the order of convergence of the predictor non-simultaneous scheme ϕ. □

9.3 Numerical experiments

We use Matlab R2020b with variable precision arithmetics of 1000 digits for the computational calculations. As a stopping criterion we use that the mean of the norm of function F evaluated at the last iterations is less than a tolerance of 10^{-50}, that is, if we try to find n solutions,

$$\frac{1}{n}\sum_{i=1}^{n}\left\|F\left(x_i^{(k+1)}\right)\right\| < 10^{-50}.$$

We denote this mean by $\left\|F\left(x^{(k+1)}\right)\right\| := \frac{1}{n}\sum_{i=1}^{n}\left\|F\left(x_i^{(k+1)}\right)\right\|$. We also denote by $x^{(k+1)} = \left(x_1^{(k+1)}, x_2^{(k+1)}, \ldots, x_n^{(k+1)}\right)$. We also use a maximum of 100 iterations as a stopping criterion.

For the numerical experiments we use three different methods. The first one is procedure PS, that we have defined in (9.1). The others two schemes are PS_N and PS_{N2} that are the composition of PS with N and $N2$, where N denote Newton's method and $N2$ denotes double Newton, that is

$$\begin{aligned} y_i^{(k)} &= x_i^{(k)} - F'\left(x_i^{(k)}\right)^{-1} F\left(x_i^{(k)}\right), \text{ for } i = 1, \ldots, n \\ x_i^{(k+1)} &= y_i^{(k)} - F'\left(y_i^{(k)}\right)^{-1} F\left(y_i^{(k)}\right), \text{ for } i = 1, \ldots, n. \end{aligned}$$

The numerical results we are going to compare the methods in the different examples are:

- the approximation obtained,
- the mean of the norm of the system evaluated in that set of approximations,
- the norm of the distance between the last two sets of approximations,
- the necessary number of iterations to satisfy the required tolerance,
- the computational time and the approximate computational convergence order (ACOC).

We are going to solve two nonlinear problems.

The first one is to find the points of intersection of the circle centred at the $(0,0)$ with radius $\sqrt{2}$ and the ellipse $\{(x,y) \in \mathbb{R}^2 : 3x^2 + 2xy + 3y^2 = 5\}$, that is, find the points (x,y) that are solutions of the following system:

$$\begin{cases} x^2 + y^2 = 2, \\ 3x^2 + 2xy + 3y^2 = 5. \end{cases} \tag{9.8}$$

The exact solutions of this problem are

$$\left\{\pm\left(\sqrt{1+\frac{\sqrt{3}}{2}}, \frac{-1}{\sqrt{4+2\sqrt{3}}}\right), \pm\left(\sqrt{1-\frac{\sqrt{3}}{2}}, \frac{-1}{\sqrt{4-2\sqrt{3}}}\right)\right\}.$$

Then, we take as initial estimations $x_1^{(0)} = (1, -0.5)$, $x_2^{(0)} = (-1, 0.5)$, $x_3^{(0)} = (0.5, -1)$ and $x_4^{(0)} = (-0.5, 1)$.

Table 9.1: Results for intersection of ellipse and circle

Method	$\left\|x^{(k+1)} - x^{(k)}\right\|$	$\left\|F\left(x^{(k+1)}\right)\right\|$	Iteration	ACOC
PS	2.1809×10^{-39}	1.5874×10^{-77}	8	1.9993
PS_N	1.6021×10^{-94}	5.635×10^{-188}	4	3.5975
PS_{N2}	2.5749×10^{-45}	2.6135×10^{-362}	3	9.2015

As we can see in Table 9.1, all the methods converge to all the roots and obtain good results for the chosen initial points. The approximate computational convergence order coincides with the theoretical one or is greater than that one. The methods using predictor obtain better results in terms of the number of iterations needed to satisfy the stopping criterion, but this is expected given that the order of convergence is higher.

The second academical problem to be solved is obtaining all the critical points of function $g\left(x, y\right) = \frac{1}{3}x^3 + y^2 + 2xy - 6x - 3y + 4$.

To calculate the critical points we calculate the gradient of the function, and solve approximately when this gradient is 0, that is, $\nabla g\left(x, y\right) = 0$.

As initial estimations we choose $x_1^{(0)} = (0, 1)$ y $x_2^{(0)} = (2, -1)$.

Table 9.2: Results for equation $\nabla g\left(x, y\right) = 0$

Method	$\left\|x^{(k+1)} - x^{(k)}\right\|$	$\left\|F\left(x^{(k+1)}\right)\right\|$	Iteration	ACOC
PS	3.7423×10^{-40}	5.5804×10^{-80}	8	2.0003
PS_N	6.7853×10^{-61}	1.151×10^{-121}	4	4.0
PS_{N2}	2.3299×10^{-30}	8.2801×10^{-244}	3	8.2457

In this case, for all methods, vectors $\left(-1, \frac{5}{2}\right)$ and $\left(3, -\frac{3}{2}\right)$ are obtained as approximations to the solutions.

As we can see in Table 9.2, the approximate computational convergence order also coincides with the theoretical one in this numerical experiment. The methods obtain solutions very close to the solution of this problem as can be seen in the third column of Table 9.2.

Now, we solve the following nonlinear system with size 200×200

$$\begin{cases} F_i(x) = (x_i^2 - 1)(x_{i+1}^2 - 1) = 0, \quad i = 1, \ldots, 199, \\ F_{200}(x) = (x_1^2 - 1)(x_{200}^2 - 1) = 0. \end{cases} \tag{9.9}$$

where $x = (x_1, x_2 \ldots, x_{199}, x_{200}) \in \mathbb{R}^{200}$.

As initial points we choose $x_1^{(0)} = 0.8(1, 1, \ldots, 1)$ and $x_2^{(0)} = -0.8(1, 1, \ldots, 1)$. In this case, we use Matlab2020 with variable precision arithmetics of 10 digits and 10^{-5} as the tolerance. The tolerance has been lowered as the system was larger than the previous ones.

The results obtained for the proposed methods and the system F are shown in Table 9.3.

Table 9.3: Results for system $F(x) = 0$

Method	$\left\|x^{(k+1)} - x^{(k)}\right\|$	$\left\|F(x^{(k+1)})\right\|$	Iteration	ACOC
PS	3.637×10^{-4}	3.3385×10^{-7}	17	2.0777
PS_N	4.2398×10^{-4}	4.5234×10^{-7}	3	3.2312
PS_{N2}	6.605×10^{-5}	0	2	—

In this case, the importance of using a predictor method is shown since the number of iterations needed to satisfy the stopping criterion has been considerably reduced compared to the number of iterations needed by method PS, but all of them converge to two solutions of the problem.

9.4 Conclusions

In this chapter, we have defined an iterative step that can be added to any iterative method for systems of nonlinear equations in such a way that a new iterative scheme for finding the roots simultaneously is obtained, and this new obtained procedure has double the order of convergence of the original iterative method.

We have selected different known iterative schemes to which we have added this step, and we have carried out different numerical experiments to see the behaviour of these new iterative methods which coincides with the results obtained theoretically.

Chapter 10

Dynamical analysis of multidimensional iterative methods with memory

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10.1 Introduction

As was discussed in Chapters 7 and 8, in applied mathematics, iterative methods for solving systems of nonlinear equations are an essential instrument for solving problems, as in most of these problems it is often complicated or impossible to solve these systems. This is why iterative schemes are employed since, by giving an initial approximation which is close enough to the solution, an approximation of a solution is obtained with the required precision.

As we have already mentioned, the initial point needs to be close to the solution to guarantee convergence, which is why dynamical analysis is increasingly important, as it allows us to see how the initial approximations behave.

The stability of iterative fixed point methods can be studied by using real or complex dynamical tools applied to a rational operator resulting from the application of the iterative scheme to low-degree polynomials. Such dynamical techniques allow to compare or deepen in known iterative schemes, as can be seen in [67, 68, 69], and to analyse the qualitative properties of new iterative procedures without memory (see [15, 70, 71, 72]) or with memory (see, for example, [73, 74]). It also changes if the scheme is multidimensional, as we can see in [73, 75, 76, 18, 77, 78, 79].

Before, it was only possible to study the stability of vectorial schemes without memory or scalar methods with memory. In both cases using multidimensional discrete dynamical systems. That is the reason in this chapter, we are going to lay the foundations for our future work in the study of the dynamics of iterative methods with memory to approximate the solutions of nonlinear systems.

This chapter is structured as follows. In Section 10.2, we present the necessary theoretical concepts and some results obtained. In Section 10.3, we apply the theoretical results obtained in Section 10.2 to some well-known multidimensional iterative methods with memory. We choose different types of systems to see the behaviour of these iterative schemes. Finally, in Section 10.4 we draw some conclusions about this chapter.

10.2 Theoretical concepts

We begin with $F : \mathbb{R}^n \to \mathbb{R}^n$ that defines a system of nonlinear equations, for which we want to obtain an approximation to the solution.

It has been already stated that the standard form of an iterative method with memory that uses only two previous iterations to calculate the next one is:

$$x^{(k+1)} = \phi\left(x^{(k-1)}, x^{(k)}\right),\ k \geq 1,$$

where $x^{(0)}$ and $x^{(1)}$ are the initial approximations.

From here on, we assume that when introducing memory to the iterative method, operator ϕ depends not only on $x^{(k)}$ but also depends on the previous iterate, $x^{(k-1)}$, since otherwise a study of the dynamics of a iterative method without memory for systems would be carried out.

A function defined from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^n$ cannot have fixed points since to be a fixed point of a operator, the point and its image by the function must coincide. Therefore, an auxiliary function O is defined as follows:

$$O\left(x^{(k-1)}, x^{(k)}\right) = \left(x^{(k)}, x^{(k+1)}\right) = \left(x^{(k)}, \phi\left(x^{(k-1)}, x^{(k)}\right)\right), \ k = 1, 2, \ldots$$

If $\left(x^{(k-1)}, x^{(k)}\right)$ is a fixed point of O, then

$$O\left(x^{(k-1)}, x^{(k)}\right) = \left(x^{(k-1)}, x^{(k)}\right),$$

and by the definition of O, one has

$$\left(x^{(k-1)}, x^{(k)}\right) = \left(x^{(k)}, x^{(k+1)}\right),$$

thus, the discrete dynamical system $O\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ is defined as

$$O(z, x) = (x, \phi(z, x)),$$

where ϕ is the operator associated with the vectorial iterative method with memory.

To simplify the notation, we denote by $x = x^{(k)}$ and $z = x^{(k-1)}$.

Then, a point (z, x) is a fixed point of O if $z = x$ and $x = \phi(z, x)$. If (z, x) is a fixed point of operator O that does not satisfy $F(x) = 0$, it is called a strange fixed point.

The basin of attraction of a fixed point (z^*, x^*) is defined as the set of pre-images of any order such that

$$\mathcal{A}(z^*, x^*) = \{(w, y) \in \mathbb{R}^{n\times n} \colon O^m(w, y) \to (z^*, x^*), \ m \to \infty\}.$$

To study the character of the fixed points, we use Theorem 2.2.4.

If one eigenvalue λ of $G'(x)$ satisfies $|\lambda| = 1$, then x is not hyperbolic and we cannot conclude anything about the character of this fixed point.

We want to deduce a more specific result for determining the character of the fixed points (z, x) of operator O. To do this, we calculate the Jacobian matrix of O, denoted by O' which has size $2n \times 2n$. The result is matrix

$$O'(z, x) = \begin{pmatrix} & 0_{n\times n} & & & I_{n\times n} & \\ \dfrac{\partial \phi_1}{\partial z_1} & \cdots & \dfrac{\partial \phi_1}{\partial z_n} & \dfrac{\partial \phi_1}{\partial x_1} & \cdots & \dfrac{\partial \phi_1}{\partial x_n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \dfrac{\partial \phi_n}{\partial z_1} & \cdots & \dfrac{\partial \phi_n}{\partial z_n} & \dfrac{\partial \phi_n}{\partial x_1} & \cdots & \dfrac{\partial \phi_n}{\partial x_n} \end{pmatrix}.$$

We denote by $\dfrac{\partial \phi}{\partial z}$ and $\dfrac{\partial \phi}{\partial x}$ matrices

$$\frac{\partial \phi}{\partial z} = \begin{pmatrix} \dfrac{\partial \phi_1}{\partial z_1} & \dots & \dfrac{\partial \phi_1}{\partial z_n} \\ \vdots & & \vdots \\ \dfrac{\partial \phi_n}{\partial z_1} & \dots & \dfrac{\partial \phi_n}{\partial z_n} \end{pmatrix},$$

and

$$\frac{\partial \phi}{\partial x} = \begin{pmatrix} \dfrac{\partial \phi_1}{\partial x_1} & \dots & \dfrac{\partial \phi_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial \phi_n}{\partial x_1} & \dots & \dfrac{\partial \phi_n}{\partial x_n} \end{pmatrix}.$$

Therefore, the Jacobian matrix $O'(z,x)$ is defined as a block matrix

$$O'(z,x) = \begin{pmatrix} 0_{n\times n} & I_{n\times n} \\ \dfrac{\partial \phi}{\partial z} & \dfrac{\partial \phi}{\partial x} \end{pmatrix}.$$

We need to calculate the eigenvalues of the Jacobian matrix $O'(z,x)$ evaluated at the fixed points for determining their character. To do this we need to use the following result that can be found in [80].

Theorem 10.2.1. *If $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A, B, C, D are matrices of size $n \times n$ and $AB = BA$, then ,*

$$\textit{det}(M) = \textit{det}(DA - CB).$$

Theorem 10.2.2. *The eigenvalues of $O'(z,x)$ are those satisfying:*

$$\det\left(\lambda^2 I_{n\times n} - \lambda \frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial z}\right) = 0.$$

Proof. It is easy to see that

$$\lambda I_{2n\times 2n} - O'(z,x) = \begin{pmatrix} \lambda I_{n\times n} & -I_{n\times n} \\ -\dfrac{\partial \phi}{\partial z} & \lambda I_{n\times n} - \dfrac{\partial \phi}{\partial x} \end{pmatrix}.$$

By applying Theorem 10.2.1 for calculating the determinant of a block matrix, we obtain

$$\det\left(\lambda I_{2n\times 2n} - O'(z,x)\right) = \det\left(\lambda\left(\lambda I_{n\times n} - \frac{\partial \phi}{\partial x}\right) - \frac{\partial \phi}{\partial z}\right).$$

Then, λ is an eigenvalue of $O'(z,x)$ if

$$\det\left(\lambda\left(\lambda I_{n\times n}-\frac{\partial\phi}{\partial x}\right)-\frac{\partial\phi}{\partial z}\right)=0,$$

which is the same as

$$\det\left(\lambda^2 I_{n\times n}-\lambda\frac{\partial\phi}{\partial x}-\frac{\partial\phi}{\partial z}\right)=0.$$

□

In particular, $\lambda=0$ is an eigenvalue of $O'(z,x)$ if 0 is an eigenvalue of $\frac{\partial\phi}{\partial z}$ since

$$0=\det\left(-O'(z,x)\right)=\det\left(-\frac{\partial\phi}{\partial z}\right).$$

To study the eigenvalues of $O'(z,x)$, we use Theorem 10.2.2. In particular, for at least one of the eigenvalues to be 0, it must be satisfied that det $\left(O'(z,x)\right)=0$. For this reason, we present the following result.

Theorem 10.2.3. *The determinant of $O'(z,x)$ is zero if, and only if, it satisfies*

$$\det\left(\frac{\partial\phi}{\partial z}(z,x)\right)=0.$$

Proof. The determinant of O' can be calculated as follows

$$\det(O'(z,x))=\det\begin{pmatrix}0_{n\times n} & I_{n\times n}\\ \dfrac{\partial\phi}{\partial z} & \dfrac{\partial\phi}{\partial x}\end{pmatrix}=\det\left(-\frac{\partial\phi}{\partial z}\right)=(-1)^n\det\left(\frac{\partial\phi}{\partial z}\right).$$

Therefore, $\det(O'(z,x))=0$ if, and only if, $\det\left(\dfrac{\partial\phi}{\partial z}(z,x)\right)=0.$ □

Another relevant concept in a dynamical study is the critical point. In this case, we use definition 10 as definition of critical point.

This is a restrictive definition of a critical point since it is usually sufficient that the determinant of the Jacobian matrix cancels out, but in this case, if we do not use the above definition, we obtain critical point surfaces because of the form of the operator.

10.3 On the qualitative analysis of some vectorial iterative schemes with memory

In this section, we present the dynamical study of two simple vectorial methods with memory: Kurchatov's scheme, see [12], whose expression is as follows

$$x^{(k+1)} = x^{(k)} - [2x^{(k)} - x^{(k-1)}, x^{(k-1)}; F]^{-1} F\left(x^{(k)}\right), \quad k = 1, 2, \dots, \tag{10.1}$$

and Steffensen's method with memory, see [2], whose expression is

$$x^{(k+1)} = x^{(k)} - [x^{(k)} + \gamma^{(k)} F\left(x^{(k)}\right), x^{(k)}; F]^{-1} F\left(x^{(k)}\right), \quad k = 1, 2, \dots, \tag{10.2}$$

where $\gamma^{(k)} = -[x^{(k)}, x^{(k-1)}; F]^{-1}$.

This study is performed on two polynomial systems of different degrees. To see a range of behaviours, we choose an uncoupled system of degree 3 and a coupled system of degree 2. The reason of choosing different degrees is because for the uncoupled system if we use polynomials of degree 2, the operator does not depend on the previous iteration.

10.3.1 Uncoupled Third Order System

We perform this dynamical study on a system with size 2×2 in order to use graphical tools. However, these results can be easily extended to higher-dimension systems. The system, denoted by $p(x) = 0$, is as follows:

$$\begin{cases} x_1^3 - 1 = 0, \\ x_2^3 - 1 = 0, \end{cases} \tag{10.3}$$

where $(x_1, x_2) \in \mathbb{R}^2$. The only root with real components of this system is $(1, 1)$.

To make the study simpler, we denote $x^{(k-1)}$ by z and $x^{(k)}$ by x as was done in the theoretical study.

10.3.1.1 Kurchatov's Method

Operator ϕ_K of Kurchatov's method on the cubical system is

$$\phi_K(z, x) = \begin{pmatrix} x_1 + \frac{1 - x_1^3}{4x_1^2 - 2x_1 z_1 + z_1^2} \\ x_2 + \frac{1 - x_2^3}{4x_2^2 - 2x_2 z_2 + z_2^2} \end{pmatrix}. \tag{10.4}$$

Theorem 10.3.1. *The only fixed point of operator $O_K(z, x) = (x, \phi_K(z, x))$ has equal components $z = x = (1, 1)$ and has superattracting character.*

Proof. We calculate matrices $\frac{\partial \phi_K}{\partial z}$ and $\frac{\partial \phi_K}{\partial x}$ appearing in the dynamical study,

$$\frac{\partial \phi_K}{\partial z}(z,x) = \begin{pmatrix} -\frac{2(x_1^3-1)(x_1-z_1)}{(4x_1^2-2x_1z_1+z_1^2)^2} & 0 \\ 0 & -\frac{2(x_2^3-1)(x_2-z_2)}{(4x_2^2-2x_2z_2+z_2^2)^2} \end{pmatrix},$$

and

$$\frac{\partial \phi_K}{\partial x}(z,x) = \begin{pmatrix} \frac{9x_1^2z_1^2-12x_1^3z_1+12x_1^4-4x_1(z_1^3+2)+z_1(z_1^3+2)}{(4x_1^2-2x_1z_1+z_1^2)^2} & 0 \\ 0 & \frac{9x_2^2z_2^2-12x_2^3z_2+12x_2^4-4x_2(z_2^3+2)+z_2(z_2^3+2)}{(4x_2^2-2x_2z_2+z_2^2)^2} \end{pmatrix}.$$

If we evaluate these matrices at the fixed point, we get

$$\frac{\partial \phi_K}{\partial z}((1,1),(1,1)) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$\frac{\partial \phi_K}{\partial x}((1,1),(1,1)) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

By Theorem 10.2.2, it follows that $\det(\lambda I - O'_K((1,1),(1,1))) = \det(\lambda^2 I)$. From the above relation it follows that the only eigenvalue associated with the fixed point is $\lambda = 0$. So, fixed point $x = z = (1,1)$ is a superattracting point. □

Regarding the critical points of operator $O_K(z,x)$, we have

Theorem 10.3.2. *Operator $O_K(z,x)$ has four categories of critical points, denoted by $C_i(z,x)$ $i = 1,2,3,4$, defined in Table 10.1. The notation of Table 10.1 is understood in such a way that, for example, the points $C_2(z,x)$ are those that satisfy that $x_1 = 1$ and $z_2 = x_2$, and the other components are arbitrary.*

Table 10.1: Categories of critical points of operator O_K

	$x_1 = 1$	$z_1 = x_1$
$x_2 = 1$	$C_1(z,x)$	$C_3(z,x)$
$z_2 = x_2$	$C_2(z,x)$	$C_4(z,x)$

Proof. To do this, we calculate the eigenvalues of $O'_K(z,x)$ for any point (z,x) and obtain those satisfying the condition that all their eigenvalues are 0.

It is obtained that the critical points are those $z = (z_1, z_2)$ and $x = (x_1, x_2)$ that satisfy these two expressions

$$\left(x_1^3 - 1\right)(x_1 - z_1) = 0 \quad \text{and} \quad \left(x_2^3 - 1\right)(x_2 - z_2) = 0. \tag{10.5}$$

It follows that the four categories of points defined in Table 10.1 are critical points of operator $O_K(z,x)$. $\square$

It can be checked that the points of category $C_1(z,x)$ are a preimage of the only fixed point since $\phi_K(z,(1,1)) = (1,1)$. We are only going to study the orbit of the points of category $C_2(z,x)$ and $C_4(z,x)$ since the points $C_1(z,x)$ converge to the fixed point for any value of z, and the fixed points of category $C_3(z,x)$ have a symmetrical study to that of the fixed points of category $C_2(z,x)$.

- Operator ϕ_K evaluated at the critical points of category $C_2(z,x)$ has the following form

$$\phi_K(C_2(z,x)) = \begin{pmatrix} 1 \\ x_2 + \frac{1-x_2^3}{3x_2^2} \end{pmatrix}.$$

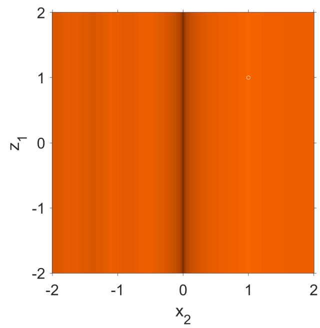
The convergence of these points only depends on x_2 and z_1, as we can see in the expression of ϕ_K. For this reason, we draw planes to see the convergence of these points with these two variables.

To draw these planes of the points of the category $C_2(z,x)$, what we are going to do is see which of these points belong to the basins of attraction of the attracting fixed points, that is, which of these points converge to the attractor.

We make a mesh of 400×400 points of the set $[-2,2] \times [-2.2]$. On one of the axes, we have x_2, and on the other, z_1, and with them we construct our points of category $C_2(z,x)$. We take each of these points $C_2(z,x)$ and apply operator ϕ_K on it. If it converges to the only attracting fixed point, which is $(1,1)$, then we represent it in orange. As convergence criterion, we have used that the distance from the iteration to the fixed point is less than 10^{-3} in less than 40 iterations. If this is not satisfied, the mesh point is represented black.

As can be seen in Figure 10.1, we have a slower convergence when x_2 approaches the value 0 because of the shape of the operator, but we still have convergence. In the rest of the cases, the convergence to the point $(1,1)$ is clear.

Figure 10.1: Behaviour of critical points $C_2(z,x)$



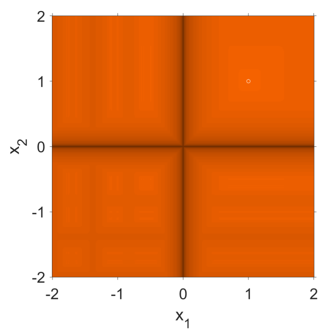
- Next, we evaluate operator ϕ_K at the critical points of category $C_4(z,x)$. In this case, the operator is

$$\phi_K(C_4(z,x)) = \begin{pmatrix} x_1 + \frac{1-x_1^3}{3x_1^2} \\ x_2 + \frac{1-x_2^3}{3x_2^2} \end{pmatrix}.$$

  In this case, the critical points of category $C_4(z,x)$ depend on variables x_1 and x_2. For this reason, we draw the convergence plane of the critical points depending on these variables.

  As in the previous case, it is shown in Figure 10.2 that if any of the variables approach the value 0 we have slow convergence but that in the rest of the points the convergence to the point $(1,1)$ is clear.

Figure 10.2: Behaviour of critical points $C_4(z,x)$



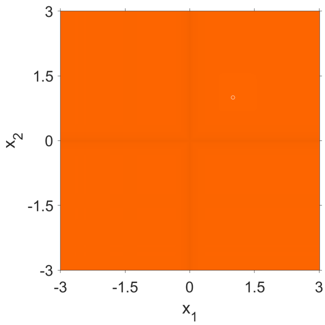
To conclude the dynamical study of Kurchatov's method for this system, let us draw some dynamical planes in order to see the behaviour of the points in general (to see how we generate the planes of convergence see [15]).

To draw these planes, given that we have an operator with 4 variables, what we have done is to select a parameter a, so that $z = x - (a, a)$. We try different values of a to see which one gives the best results. Usually, testing with small values of a gives good results. Thus, our variables would be x_1 and x_2, and the variables z are a variation of these.

To make the dynamical planes, we have chosen a mesh of 400×400 points, where the chosen point of the mesh is the starting point. We study the orbit of the initial point. If the seed converges to $(1, 1)$, it is represented orange, and if it does not converge, it is represented black. We define convergence to the point $(1, 1)$ because the distance of the iteration is less than 10^{-3}, and this convergence is realised in, at most, 40 iterations.

We have tested with different values of a over a wide range and obtained that there are the same dynamical plane for different values of a. As we can see in Figure 10.3, all initial points converge to the root $(1, 1)$ showing the good stability properties of this iterative scheme with memory, even in this multidimensional case.

Figure 10.3: Dynamical plane of Kurchatov's scheme for $a = -0.1$



10.3.1.2 Steffensen's Scheme

In this section, we perform the dynamical study of Steffensen's scheme with memory for system $p(x) = 0$. Operator ϕ_S obtained by Steffensen's method is

$$\phi_S(z,x) = \begin{pmatrix} x_1 + \dfrac{(x_1^3-1)^2}{(x_1^2+x_1z_1+z_1^2)\left(\left(\frac{1-x_1^3}{x_1^2+x_1z_1+z_1^2}+x_1\right)^3-x_1^3\right)} \\ x_2 + \dfrac{(x_2^3-1)^2}{(x_2^2+x_2z_2+z_2^2)\left(\left(\frac{1-x_2^3}{x_2^2+x_2z_2+z_2^2}+x_2\right)^3-x_2^3\right)} \end{pmatrix}. \tag{10.6}$$

Theorem 10.3.3. *Operator $O_S(z, x) = (x, \phi_S(z, x))$ has four fixed points, which are*

- *fixed point $(z, x) = (S_1, S_1)$, with $S_1 = (1, 1)$,*

- *strange fixed point* $(z, x) = (S_2, S_2)$, *being* $S_2 = (0, 0)$,
- *strange fixed point* $(z, x) = (S_3, S_3)$, *being* $S_3 = (1, 0)$,
- *strange fixed point* $(z, x) = (S_4, S_4)$, *with* $S_4 = (0, 1)$.

The strange fixed points are not hyperbolic and the fixed point (S_1, S_1) *is a superattracting fixed point.*

Proof. In order to study the character of these fixed points, we need to obtain matrices $\frac{\partial \phi_S}{\partial z}$ and $\frac{\partial \phi_S}{\partial x}$. We denote by $GZ_i(z, x)$ for $i = 1, 2$ the following expression:

$$GZ_i(z,x) = \frac{\left(x_i^3 - 1\right)^2 (x_i + 2z_i)\left(x_i^2 + x_i z_i + z_i^2\right)\left(3x_i^2 z_i + x_i^3 + 3x_i z_i^2 + 2\right)}{\left(6x_i^4 z_i^2 + x_i^3\left(6z_i^3 + 1\right) + 3x_i^2\left(z_i^4 + z_i\right) + 3x_i^5 z_i + x_i^6 + 3x_i z_i^2 + 1\right)^2}. \tag{10.7}$$

Thus,

$$\frac{\partial \phi_S}{\partial z}(z,x) = \begin{pmatrix} GZ_1(z,x) & 0 \\ 0 & GZ_2(z,x) \end{pmatrix}.$$

We denote by $GX_i(z, x)$ for $i = 1, 2$ expression:

$$\begin{aligned} GX_i(z,x) = {} & \frac{36x_i^4 z_i^2 + x_i^3\left(39z_i^3 + 7\right) + 12x_i^2\left(2z_i^4 + z_i\right) + 18x_i^5 z_i}{\left(6x_i^4 z_i^2 + x_i^3\left(6z_i^3 + 1\right) + 3x_i^2\left(z_i^4 + z_i\right) + 3x_i^5 z_i + x_i^6 + 3x_i z_i^2 + 1\right)^2} \\ & + \frac{4x_i^6 + 6x_i z_i^2\left(z_i^3 + 2\right) + 3z_i^3 + 1}{\left(6x_i^4 z_i^2 + x_i^3\left(6z_i^3 + 1\right) + 3x_i^2\left(z_i^4 + z_i\right) + 3x_i^5 z_i + x_i^6 + 3x_i z_i^2 + 1\right)^2}. \end{aligned} \tag{10.8}$$

Thus,

$$\frac{\partial \phi_S}{\partial x}(z,x) = \begin{pmatrix} \left(x_1^3 - 1\right)\left(z_1^3 - 1\right) GX_1(z,x) & 0 \\ 0 & \left(x_2^3 - 1\right)\left(z_2^3 - 1\right) GX_2(z,x) \end{pmatrix}.$$

Let us now deduce the character of the fixed points (S_i, S_i), for $i = 1, \ldots, 4$.

- For the point associated to S_1, the related matrices are

  $$\frac{\partial \phi_S}{\partial z}(S_1, S_1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

  and

  $$\frac{\partial \phi_S}{\partial x}(S_1, S_1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

  We can see that both eigenvalues are 0 and, from Theorem 10.2.2, we conclude that the fixed point is superattracting point.

- For the point associated to S_2, we obtain

$$\frac{\partial \phi_S}{\partial z}(S_2, S_2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$\frac{\partial \phi_S}{\partial x}(S_2, S_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

By applying Theorem 10.2.2, the eigenvalues of this point are the values λ satisfying

$$\begin{aligned} 0 = \det(\lambda I - O'_S(S_2, S_2)) \quad &= \det\left(\lambda^2 I - \lambda \tfrac{\partial \phi_S}{\partial x}(S_2, S_2) - \tfrac{\partial \phi_S}{\partial z}(S_2, S_2)\right) \\ &= \lambda^2(\lambda - 1)^2. \end{aligned} \tag{10.9}$$

It follows that the eigenvalues are 0 and 1, so we cannot conclude anything about the character of this strange fixed point as it is not hyperbolic.

- For the fixed point associated to S_3, the matrices are

$$\frac{\partial \phi_S}{\partial z}(S_3, S_3) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$\frac{\partial \phi_S}{\partial x}(S_3, S_3) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The eigenvalues associated with this fixed point are those λ values that satisfy

$$\begin{aligned} 0 = \det(\lambda I - O'_S(S_3, S_3)) \quad &= \det\left(\lambda^2 I - \lambda \tfrac{\partial \phi_S}{\partial x}(S_3, S_3) - \tfrac{\partial \phi_S}{\partial z}(S_3, S_3)\right) \\ &= \lambda^3(\lambda - 1). \end{aligned} \tag{10.10}$$

It follows that the eigenvalues are 0 and 1, so again the point is not hyperbolic.

- Finally, let us study the character of the fixed point associated with S_4. The matrices for this fixed point are

$$\frac{\partial \phi_S}{\partial z}(S_4, S_4) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$\frac{\partial \phi_S}{\partial x}(S_4, S_4) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

So the eigenvalues of the fixed point associated with S_4 are the values that satisfy

$$\begin{aligned} 0 = \det(\lambda I - O'_S(S_4, S_4)) \quad &= \det\left(\lambda^2 I - \lambda \tfrac{\partial \phi_S}{\partial x}(S_4, S_4) - \tfrac{\partial \phi_S}{\partial z}(S_4, S_4)\right) \\ &= \lambda^3(\lambda - 1). \end{aligned} \tag{10.11}$$

It follows that the eigenvalues are 0 and 1, so again the point is not hyperbolic.

□

Now, let us calculate the critical points.

Theorem 10.3.4. *The critical points of operator $O_S(z,x)$ are vectors $z=(z_1,z_2)$ and $x=(x_1,x_2)$, which satisfy that they are of one of the following 16 categories, which we denote by $CS_i(z,x)$ for $i=1,\dots,16$. Table 10.2 is a summary of the different categories of critical points we obtain.*

Table 10.2: Categories of critical points of operator O_S

	$x_1=1$	$z_1=-\frac{1}{2}x_1$	$z_1=\dfrac{\pm\sqrt{-3x_1^4-24x_1}-3x_1^2}{6x_1}$	
$x_2=1$	CS_1	CS_5	CS_9	CS_{13}
$z_2=-\frac{1}{2}x_2$	CS_2	CS_6	CS_{10}	CS_{14}
$z_2=\dfrac{\sqrt{-3x_2^4-24x_2}-3x_2^2}{6x_2}$	CS_3	CS_7	CS_{11}	CS_{15}
$z_2=\dfrac{-\sqrt{-3x_2^4-24x_2}-3x_2^2}{6x_2}$	CS_4	CS_8	CS_{12}	CS_{16}

Proof. We are working in real multidimensional dynamics, so it is assumed that the critical points have real numbers as their components.

If we define $DGX_i(z,x)$, $i=1,2$, as

$$DGX_i(z,x)=\lambda^2-\lambda(x_1^3-1)(z_1^3-1)GX_i(z,x)-GX_i(z,x),$$

then, we can check that

$$\begin{aligned}\det(\lambda I-O_S'(x,z)) &= \det\left(\lambda^2 I-\lambda\frac{\partial\phi_S}{\partial x}-\frac{\partial\phi_S}{\partial z}\right)\\ &= \det\begin{pmatrix} DGX_1(z,x) & 0\\ 0 & DGX_2(z,x)\end{pmatrix}.\end{aligned} \tag{10.12}$$

Additionally, it follows that all eigenvalues are zero if the point (z,x) can be expressed as one of the forms given in Table 10.2. □

As we can see on Table 10.2, there exists a certain symmetry relation in the components of the following $CS_i(z,x)$.

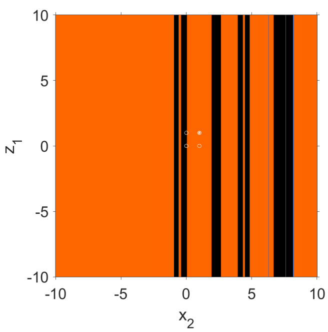
Table 10.3: Symmetry relation between $CS_i(z,x)$

$CS_2(z,x)$	$CS_5(z,x)$
$CS_3(z,x)$	$CS_9(z,x)$
$CS_4(z,x)$	$CS_{13}(z,x)$
$CS_7(z,x)$	$CS_{10}(z,x)$
$CS_8(z,x)$	$CS_{14}(z,x)$
$CS_{12}(z,x)$	$CS_{15}(z,x)$

For that reason, and because the operator also satisfies certain symmetry with the components, we only study the behaviour of certain categories of critical points.

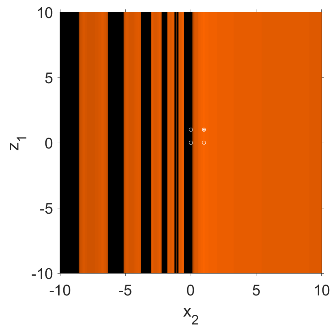
- The behaviour of the critical points $CS_2(z,x)$ is analysed with the following plane where the convergence to the fixed points is shown in different colours. In this case, if the distance from the iteration to the fixed point is less than 10^{-3}, we say that the iteration is in the basin of attraction of the fixed point. In this case, it is represented orange if the critical point converges to (S_1,S_1), blue if it converges to the strange fixed point (S_3,S_3), red if it converges to the strange fixed point (S_4,S_4) and green if it converges to the point (S_2,S_2). If the points are represented black, they have not converged to any of the fixed points in less than 40 iterations. In this case, we have that $x_1 = 1$, and the value z_2 depends on x_2, so the variables of the axes are x_2 and z_1 as shown in Figure 10.4.

Figure 10.4: Convergence of the critical points of category $CS_2(z,x)$



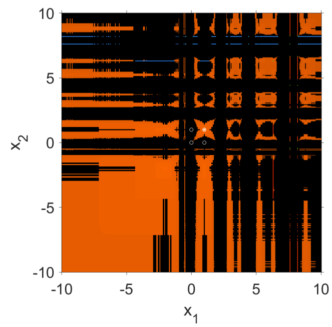
- In a similar way to the previous case, we study the convergence of the critical points of category $CS_3(z,x)$ and of category $CS_4(z,x)$. In these cases, the value x_1 is also fixed as 1 and the value z_2 depends on x_2; for this reason, the variables of the axes are x_2 and z_1 as in the previous cases and as can be seen in Figure 10.5. In this case, we have that the behaviour of both categories of critical points is the same; for that reason, we only show one dynamical plane.

Figure 10.5: Convergence of the critical points of category $CS_3(z,x)$ and $CS_4(z,x)$



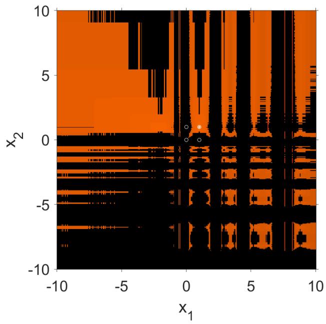
- For the critical points of category $CS_6(z,x)$, the convergence study is similar to the previous ones, but in this case none of the variables are fixed, and it is z_1 and z_2 that depend on x_1 and x_2, respectively; for this reason, the dynamical plane has as axis variables the values of x_1 and x_2, as shown in Figure 10.6.

Figure 10.6: Convergence of the critical points of category $CS_6(z,x)$



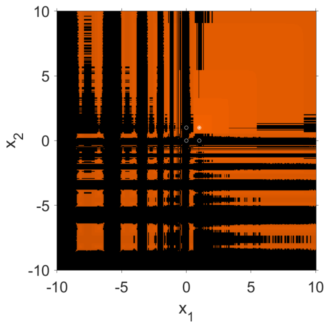
- For the critical points of category $CS_7(z,x)$ and $CS_8(z,x)$, we also have as variables on the axes the values of x_1 and x_2, as shown in Figure 10.7. In this case, we have decided to show only one dynamical plane because the behaviour of both categories of critical points is the same.

Figure 10.7: Convergence of the critical points of category $CS_7(z, x)$ and $CS_8(z, x)$



- For the critical points of category $CS_{11}(z, x)$, $CS_{12}(z, x)$ and $CS_{16}(z, x)$, we also have as variables on the axes the values of x_1 and x_2. In this case, we have that the behaviour of these 3 categories of critical points is the same; for that reason, we only show one dynamical plane (Figure 10.8).

Figure 10.8: Convergence of the critical points of category $CS_{11}(z, x)$, $CS_{12}(z, x)$ and $CS_{16}(z, x)$



10.3.2 Coupled Second-Order System

Now, we are going to perform the dynamical analysis of these method applied to other system that has a more complicated aspect since the variables cannot be separated, that is to say, we do not have that the first component of the operator only depends on the first components of the variables of x and z and the same with the second component; instead, in this case, we have that both components of the operator depend on both components of the vectors. The next system we solve, denoted by $q(x) = 0$, is

$$\begin{cases} x_1x_2 + x_1 - x_2 - 1 = 0, \\ x_1x_2 - x_1 + x_2 - 1 = 0, \end{cases} \tag{10.13}$$

where $(x_1, x_2) \in \mathbb{R}^2$. The real roots of this system are $(-1, -1)$ and $(1, 1)$.

10.3.2.1 Kurchatov's Scheme

We then study the stability of vectorial Kurchatov's scheme applied to this system. If we apply Kurchatov's scheme to the proposed system, we obtain the following operator:

$$\varphi_K(z, x) = \begin{pmatrix} \frac{1 - x_2z_1 + x_1(x_2+z_2)}{2x_1 - z_1 + z_2} \\ \frac{1 - x_2z_1 + x_1(x_2+z_2}{2x_1 - z_1 + z_2} \end{pmatrix}. \tag{10.14}$$

Theorem 10.3.5. *The only fixed points of operator $O_K(z, x) = (z, \varphi_K(z, x))$ are $z = x = (-1, -1)$ and $z = x = (1, 1)$, and both have superattracting character.*

Proof. Now, we calculate the matrices $\frac{\partial \varphi_K}{\partial z}$ and $\frac{\partial \varphi_K}{\partial x}$ to obtain the character of these fixed points.

$$\frac{\partial \varphi_K}{\partial z}(z, x) = \begin{pmatrix} \frac{x_1(z_2 - x_2) - x_2z_2 + 1}{(2x_1 - z_1 + z_2)^2} & \frac{2x_1^2 - x_1(x_2+z_1) + x_2z_1 - 1}{(2x_1 - z_1 + z_2)^2} \\ \frac{x_1(z_2 - x_2) - x_2z_2 + 1}{(2x_1 - z_1 + z_2)^2} & \frac{2x_1^2 - x_1(x_2+z_1) + x_2z_1 - 1}{(2x_1 - z_1 + z_2)^2} \end{pmatrix},$$

and

$$\frac{\partial \varphi_K}{\partial x}(z, x) = \begin{pmatrix} \frac{x_2(z_1+z_2) - z_1z_2 + z_2^2 - 2}{(2x_1 - z_1 + z_2)^2} & \frac{x_1 - z_1}{2x_1 - z_1 + z_2} \\ \frac{x_2(z_1+z_2) - z_1z_2 + z_2^2 - 2}{(2x_1 - z_1 + z_2)^2} & \frac{x_1 - z_1}{2x_1 - z_1 + z_2} \end{pmatrix}.$$

If we evaluate the previous matrices in the fixed points, we obtain in both cases that

$$\frac{\partial \phi}{\partial z}(\pm((1, 1), (1, 1))) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$\frac{\partial \phi}{\partial x}(\pm((1, 1), (1, 1))) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, by Theorem 10.2.2, both eigenvalues are 0 for all the fixed points. For that reason, both fixed points are superattracting points. □

Theorem 10.3.6. *Operator $O_K(z, x) = (z, \varphi_K(z, x))$ has two categories of critical points. They have one of the following two structures:*

- $C_+(z, x) = (z_1, z_2, x_1, x_1)$ *where* $z_2 = -x_1 + \sqrt{2 - x_1^2} + z_1$.

- $C_{-}(z,x) = (z_1, z_2, x_1, x_1)$ *where* $z_2 = -x_1 - \sqrt{2 - x_1^2} + z_1$.

Proof. From Theorem 10.2.2 and the form of $\frac{\partial \varphi_K}{\partial x}$ and $\frac{\partial \varphi_K}{\partial x}$, we can see that the eigenvalues are all of them zero if

$$x_1 - x_2 = 0 \quad \text{and} \quad (z_2 - z_1 + x_1)^2 = 2 - x_1^2.$$

□

Let us draw the orbit of these critical points. In this case, we draw on the abscissa axis the values of x_1, which is the same value as x_2, and we draw on the other axis the value of z_1 since z_2 is obtained from x_1 and z_2.

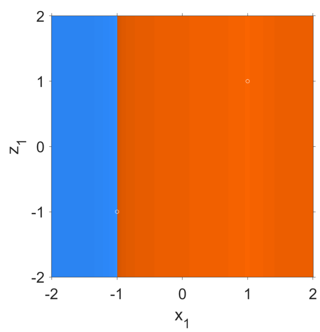
To generate these convergence planes of the points of category $C_{+}(z,x)$, we are going to see which of these points belong to the basins of attraction of the attracting fixed points, that is, which them converge to the attracting fixed points.

To do this, we make a mesh of 400×400 points in the set $[-2,2] \times [-2,2]$. We made sure that increasing the set did not alter the behaviour. On one of the axes, we have the variable x_1, and on the other, the variable z_1, and with these variables we construct our points of category $C_{+}(z,x)$. We take each of these points of category $C_{+}(z,x)$, and we apply our operator φ_K on them.

If this initial point converges to $(1,1)$, we represent it in orange, and if converges to $(-1,-1)$, we represent it in blue. As convergence criteria, we have that the distance from the iteration to the fixed point is less than 10^{-3} in less than 40 iterations. If this is not satisfied, we represent it in black.

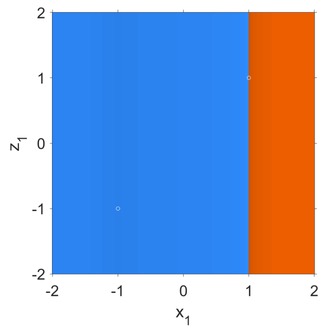
Figure 10.9 shows the plane of convergence for the points of category $C_{+}(z,x)$.

Figure 10.9: Convergence of the critical points of category $C_{+}(z,x)$



In the same way that the plane of convergence of the points of category $C_+(z,x)$ is generated, we generate the plane of convergence of the points of category $C_-(z,x)$, which is shown in Figure 10.10.

Figure 10.10: Convergence of the critical points of category $C_-(z,x)$



We observe in this planes of convergence, Figures 10.9 and 10.10, that global convergence to the roots of the system exists.

To conclude the dynamical study of the Kurchatov method for this system, we draw some dynamical planes in order to see the behaviour of the points in general. To draw these planes, given that we have an operator with 4 variables, what we have done is to select a parameter a, so that $z = x - (a,a)$. Thus, our variables would be x_1 and x_2, and the variables z are a variation of these.

To make the dynamical planes, we have chosen a mesh of 400×400 points. If the initial point converges to the point $(1,1)$, it is represented orange; if it converges to point $(-1,-1)$, it is represented blue; and if it does not converge to any point, is represented black.

We have tested with different values of a over a wide range and obtained that there are similar dynamical planes for different values of a; for that reason, in Figures 10.11, 10.12 and 10.13 we show how they behave differently from each other. As we can see on this figure, we have that all initial approximation converge to the roots of the polynomial.

Figure 10.11: Dynamical plane of Kurchatov's scheme with $a = 0.1$

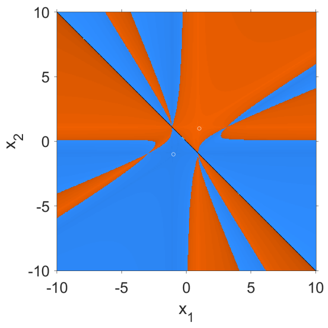


Figure 10.12: Dynamical plane of Kurchatov's scheme with $a = -1$

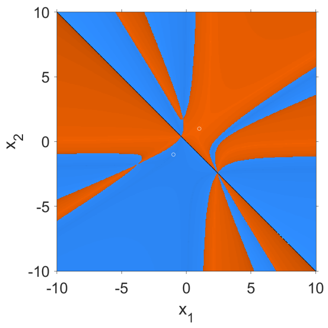
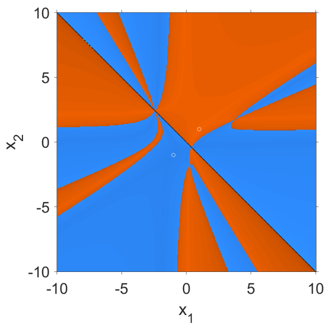


Figure 10.13: Dynamical plane of Kurchatov's scheme with $a = 1$



10.3.2.2 Steffensen's Scheme

We now continue with the study of the stability of Steffensen's vectorial method on the coupled system presented. If we apply Steffensen's scheme with memory to system $q(x) = 0$, we obtain the following operator

$$\theta_S(z,x) = \begin{pmatrix} \frac{2x_1^2x_2+x_1x_2z_2+x_1-x_2+z_2}{2x_1^2+x_1(x_2+z_2)+x_2z_2-1} \\ \frac{2x_1^2x_2+x_1x_2z_2+x_1-x_2+z_2}{2x_1^2+x_1(x_2+z_2)+x_2z_2-1} \end{pmatrix}. \tag{10.15}$$

Theorem 10.3.7. *Operator $O(z,x) = (x, \theta_S(z,x))$ has three fixed points, that is,*

- $z = x = (-1,-1)$, *which is a superattracting fixed point.*
- $z = x = (1,1)$, *which is a superattracting fixed point.*
- $z = x = (0,0)$, *which is a non-hyperbolic strange fixed point.*

Proof. Let us calculate the matrices $\frac{\partial \theta_S}{\partial z}$ and $\frac{\partial \theta_S}{\partial x}$ to obtain the character of the fixed points.

$$\frac{\partial \theta_S}{\partial z}(z,x) = \begin{pmatrix} 0 & -\frac{(x_1^2-1)(x_2^2-1)}{(2x_1^2+x_1(x_2+z_2)+x_2z_2-1)^2} \\ 0 & -\frac{(x_1^2-1)(x_2^2-1)}{(2x_1^2+x_1(x_2+z_2)+x_2z_2-1)^2} \end{pmatrix}, \tag{10.16}$$

$$\frac{\partial \theta_S}{\partial x}(z,x) = \begin{pmatrix} \frac{(x_2^2-1)(2x_1^2+4x_1z_2+z_2^2+1)}{(2x_1^2+x_1(x_2+z_2)+x_2z_2-1)^2} & \frac{(x_1^2-1)(4x_1^2+4x_1z_2+z_2^2-1)}{(2x_1^2+x_1(x_2+z_2)+x_2z_2-1)^2} \\ \frac{(x_2^2-1)(2x_1^2+4x_1z_2+z_2^2+1)}{(2x_1^2+x_1(x_2+z_2)+x_2z_2-1)^2} & \frac{(x_1^2-1)(4x_1^2+4x_1z_2+z_2^2-1)}{(2x_1^2+x_1(x_2+z_2)+x_2z_2-1)^2} \end{pmatrix}. \tag{10.17}$$

For the fixed points associated with the roots, both matrices are the zero matrix. So, by Theorem 10.2.2, both eigenvalues are 0. Then, the fixed points associated with the roots are superattracting points. Let see what happens to the strange fixed point. The matrices are

$$\frac{\partial \theta_S}{\partial z}((0,0),(0,0)) = \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix}, \tag{10.18}$$

$$\frac{\partial \theta_S}{\partial x}((0,0),(0,0)) = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}. \tag{10.19}$$

By Theorem 10.2.2, the eigenvalues for that strange fixed point are the values λ that satisfy the following equation:

$$\det(\lambda I - O_S'((0,0),(0,0)) = \det\left(\lambda^2 I - \lambda \frac{\partial \theta_S}{\partial x} - \frac{\partial \theta_S}{\partial z}\right)((0,0),(0,0)) = \lambda^2(\lambda^2+1) = 0. \tag{10.20}$$

So, the eigenvalues are $\pm i$ and 0. We cannot determine the character of that non-hyperbolic fixed point. $\square$

Theorem 10.3.8. *Operator $O_S(z,x) = (x, \theta_S(z,x))$ has six categories of critical points. These categories of points are preimages of one of the fixed points and are described in Table 10.4.*

Table 10.4: Categories of critical points of operator O_S

	$x_1 = 1$	$x_1 = -1$	$z_2 = -x_2 - 2x_1$
$x_2 = 1$	$CP_1(z,x)$ if $z_2 \neq -3$		$CP_5(z,x)$ if $\lvert x_1\rvert \neq 1$
$x_2 = -1$		$CP_2(z,x)$ if $z_2 \neq 3$	$CP_6(z,x)$ if $\lvert x_1\rvert \neq 1$
$z_2 = -1 - 2x_1$	$CP_3(z,x)$ if $x_2 \neq -1$		
$z_2 = 1 - 2x_1$		$CP_4(z,x)$ if $x_2 \neq 1$	

Proof. Since

$$\begin{aligned}\det(\lambda I - O'_S(z,x)) &= \lambda^4 + \lambda^2 \frac{(-1+x_1^2)(-1+x_2^2) - \lambda x_2^2}{(-1+2x_1^2+x_2z_2+x_1(x_2+z_2))^2} \\ &+ \lambda^3 \frac{x_1^2(7-4x_1^2-2x_2^2) - 4x_1(-2+x_1^2+x_2^2)z_2 - (-2+x_1^2+x_2^2)z_2^2}{(-1+2x_1^2+x_2z_2+x_1(x_2+z_2))^2},\end{aligned}$$

then, all the eigenvalues are zero, if (z,x) has one of the categories shown in Table 10.4.

1. Since θ_S evaluated at the $CP_2(z,x)$, $CP_4(z,x)$ and $CP_6(z,x)$ points is $(-1,-1)$, then those categories of critical points belong to the basin of attraction of $(-1,-1)$.

2. Since θ_S evaluated at the $CP_1(z,x)$, $CP_3(z,x)$ and $CP_5(z,x)$ points is $(1,1)$, then those categories of critical points belong to the basin of attraction of $(1,1)$.

□

Below, we draw a dynamical plane of Steffensen's scheme with memory in the same way as was done for the Kurchatov's scheme. In Figure 10.14, we can see a black region; that is because we have slow convergence to the roots in that region since there are no critical points outside the basins of attraction of the roots, so there cannot be convergence to any point other than the roots. Here, we also tried different values for the parameter a, and similar results were obtained, although the larger the parameter was, the slow convergence zone was increased.

Figure 10.14: Dynamical plane of Steffensen's method with $a = -0.1$

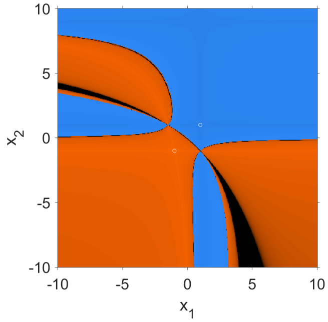


Figure 10.15: Dynamical plane of Steffensen's method with $a = -1$

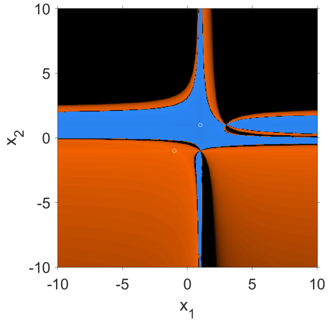
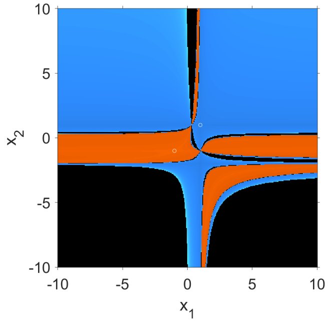


Figure 10.16: Dynamical plane of Steffensen's method with $a = 1$



10.4 Conclusions

In recent years, the design of new vectorial iterative schemes with memory for solving nonlinear problems is an expanding area in Numerical Analysis. Although it is possible to test these methods numerically, there was no possibility to analyse their performance qualitatively so far, since all existing techniques, including both complex and real discrete dynamical ones, were not designed to handle the high dimension of the rational functions involved.

We have tested this proposed procedure by analysing the behaviour of Kurchatov' and Steffensen's multidimensional schemes on coupled and uncoupled nonlinear polynomial systems. Results shown the applicability of our procedure and offer many opportunities for research in the future.

Chapter 11

Conclusions and future work

> *"No human enquiry can be a science unless it pursues its path through mathematical exposition and demonstration."*
> *Leonardo da Vinci*

11.1 Conclusions

The results obtained during the course of this Doctoral Thesis are summarised below.

In Chapter 3, a family of optimal multi-step iterative methods for solving nonlinear equations is designed as a variant of n-times compose Newton's scheme. The order of convergence of the n-steps element of the class is 2^n, only performing $n+1$ functional evaluations per iteration, therefore, that method is an optimal procedure. We perform the dynamical analysis for the elements of order 2, 4 and 8 and compare the dynamical behaviour of these schemes and other known procedures of similar order.

In Chapter 4, based on Traub's method, two parametric families of derivative-free iterative schemes with weight function for nonlinear equations are designed, denoted by M_4 and M_6. Under certain conditions, this families have order 4 and 6, respectively and the family M_4 is a class of optimal iterative procedures. Memory is introduced to both families in order to increase the order of convergence without performing more functional evaluations per iteration, increasing the order by up to two units for the family of order 4, and increasing it by up to three units for the family of order 6. This introduction of memory is performed by using Newton interpolating polynomial of different degrees. A complex dynamical analysis is performed for class M_4, obtaining for which parameter values, the class of iterative schemes show more stable performance, making parameter planes as a graphical representation. At the same time that this analysis is performed, a real multidimensional dynamical analysis is also performed for certain memory variants of this family, in order to make comparisons between the iterative class and its memory variants, beyond the order of convergence. We observe that procedures with memory show a more stable and predictable behaviour.

In Chapter 5, we have designed of an iterative step for obtaining simple roots of a nonlinear equation simultaneously. It is obtained that the order of convergence of this step is 2, and it is also analysed that it can be added to any other method ϕ, thus generating a predictor-corrector scheme, denoted by ϕ_S, that approximates roots simultaneously with twice the order of convergence of the predictor procedure used for arbitrary equations and three times the order of convergence in the case of polynomial equations. How the behaviour of the schemes is modified by adding this step of simultaneity is graphically represented in this chapter for Newton's method, Steffensen's scheme and others, illustrating that non-convergence points of the original procedure becomes convergence in the simultaneous case.

In Chapter 6, iterative methods, based on Kurchatov's scheme, for obtaining roots of equations with multiplicity greater than 1 are presented, denoted by KM, KMD and KMS. This schemes do not require the value of this multiplicity in their iterative expression, because to know this value, it is necessary to know the solutions of the problem, and if we want to obtain all the roots, we must change the value of the multiplicity depending on which root we want to converge to. The proposed procedures KM and KMD have second-order of convergence. Method KM has derivatives in its iterative expression, but it can be seen through the dynamical analysis that has wider zones of convergence for roots with different multiplicities. Method KMD is a variant derivative-free of KM and scheme KMS is procedure KM combined with the iterative step defined in Chapter 5, thus obtaining an iterative scheme that converges simultaneously to several roots without the need to take into account whether they are single or multiple or whether they have different multiplicities. This method has convergence order four for arbitrary equations and order 6 in the case of polynomial equations.

In Chapter 7, based on two known iterative methods for nonlinear equations, a parametric class of iterative schemes for the approximation of nonlinear systems of equations is designed. This class has derivatives in its iterative expression. The schemes of this class has convergence order 3, and increases to order 4 when the parameter has null value. By performing a unidimensional complex dynamical study for this family, we find out for which parameter values the most stable procedures are obtained.

In Chapter 8, the iterative classes proposed in Chapter 4 are extended to the resolution of nonlinear systems. In this case, the family M_4 of order 4 maintains the order, but the family M_6 of order 6 manages to increase the order of convergence by one unit, thus obtaining a parametric class of iterative schemes of order 7 for nonlinear systems, denoted by M_7. As in Chapter 4, memory is introduced to these families, increasing the order by two units and four units, respectively, that is, schemes of up to order 6 are obtained for the case of the iterative class of order 4 and methods of up to order 11 for the case of the iterative class of order 6. This introduction of memory is made by using divided difference operators. In this chapter, in one of the numerical experiments there are dynamical planes to obtain a graphical representation of the behaviour of the iterative procedures presented in this chapter.

Chapter 9 focuses on the modification of the iterative step proposed in Chapter 5 in order to adapt it to the resolution of nonlinear systems. The obtained step maintains the order of convergence that we had for nonlinear equations, and it is also proved that it can be added to any iterative method for systems obtaining a predictor-corrector procedure that doubles the order of the predictor scheme.

In Chapter 10, some theoretical results are obtained to carry out the dynamical study for iterative schemes with memory that solve systems of nonlinear equations. Once these theoretical concepts have been defined, the dynamical analysis of Steffensen' and Kurchatov's method for two different systems of nonlinear equations, is carried out. On the one hand, we study what happens in the case where the system is uncoupled, that is, the components do not interact with each other, while on the other hand, we study what happens in the case of a coupled system, where the behaviour of each component involves both components. It is shown in the dynamical analysis, that the theoretical results obtained are useful for the realisation of the analysis.

11.2 Future work

In the following, we describe the future lines of research that emerge from the research carried out and the results obtained.

- In this Thesis, we have proposed how to study the dynamics of vectorial methods with memory and we have developed the study for vectorial Kurchatov's and Steffensen's schemes applied to several nonlinear problems. Thus, one of the future lines is to apply this tool to other vectorial iterative procedures.
- An iterative method has been proposed that simultaneously obtains several roots with different multiplicities for nonlinear equations, therefore one of the future lines will be to extend this to the case of nonlinear systems. It is also planned to modify the schemes that obtain simultaneous roots or multiple roots in order to increase the order of convergence by use different techniques such as the introduction of weight functions, the introduction of memory, etc., and to study the dynamical analysis of the iterative procedures.
- The q-calculus (quantum calculus) area has been an important research interest in mathematics for the last few decades. The q-analogue of the ordinary derivative has wide applications so that some of the recently developed iterative methods use q-derivatives instead of the usual derivatives. Thus, designing and analysing iterative methods using this technique is one of the lines of research that we intend to pursue in the coming years.

- Throughout the Thesis, it has been distinguished if it was a nonlinear equation or a nonlinear system, but not the case of matrix equations. Therefore, another of the future lines of research will be the study and development of iterative methods that allow the approximation of the solution of matrix equations.

Appendix A

Merits

A.1 Publications

[53] A. Cordero, E. G. Villalba, J. R. Torregrosa, and P Triguero-Navarro. "Convergence and Stability of a Parametric Class of Iterative Schemes for Solving Nonlinear Systems". Mathematics 9, (2021).

[28] A. Cordero, J. R. Torregrosa, and P. Triguero-Navarro. "A General Optimal Iterative Scheme with Arbitrary Order of Convergence". Symmetry 5, 13, (2021).

[81] A. Cordero, N. Garrido, J. R. Torregrosa, and P. Triguero-Navarro. "Symmetry in the Multidimensional Dynamical Analysis of Iterative Methods with Memory". Symmetry 3, 14, (2022).

[47] A. Cordero, N. Garrido, J. R. Torregrosa, and P. Triguero-Navarro. "Iterative schemes for finding all roots simultaneously of nonlinear equations". Applied Mathematics Letters 134, (2022).

[38] A. Cordero, N. Garrido, J. R. Torregrosa, and P. Triguero-Navarro. "Memory in the iterative processes for nonlinear problems". Mathematical Methods in the Applied Sciences 46 (4), 4145- 4158, (2023).

[82] E. Martinez, E. G. Villalba, and P. Triguero-Navarro. "Semilocal Convergence of a Multi-Step Parametric Family of Iterative Methods". Symmetry 15, 536, (2023).

[83] A. Cordero, N. Garrido, J. R. Torregrosa, and P. Triguero-Navarro. "Improving the order of a fifth-order family of vectorial fixed point schemes by introducing memory". Fixed Point Theory Journal 24 (1), 155-172, (2023).

[84] A. Cordero, N. Garrido, J. R. Torregrosa, and P. Triguero-Navarro. "Modifying Kurchatov's method to find multiple roots". Applied Numerical Mathematics, *submitted*, (2022).

[85] A. Cordero, N. Garrido, J. R. Torregrosa, and P. Triguero-Navarro. "Three-step iterative weight function scheme with memory for solving nonlinear problems". Mathematical Methods in Applied Science, *accepted*, (2022).

[86] A. Cordero, N. Garrido, J. R. Torregrosa, and P. Triguero-Navarro. "Design of iterative methods with memory for solving nonlinear systems". Mathematical Methods in Applied Science, (2023).

[87] A. Cordero, N. Garrido, J. R. Torregrosa, and P. Triguero-Navarro. "An iterative scheme to obtain multiple solutions simultaneously". Applied Mathematical Letters, *submitted*, (2023).

[88] F. Chinesta, A. Cordero, N. Garrido, J. R. Torregrosa, and P. Triguero-Navarro. "Simultaneous roots for vectorial problems". Computational and Applied Mathematics, *submitted*, (2023).

[89] A. Cordero, E. G. Villalba, J. R. Torregrosa, and P. Triguero-Navarro. "Introducing memory to a family of multi-step multidimensional iterative methods with weight function". Expositiones Mathematicae, *submitted*, (2023).

A.2 Conferences

- Mathematical Modelling in Engineering & Human Behaviour (IMM 2021). ISBN: 978-84-09-36287-5. June, 2021. Universitat Politècnica de València, Spain.
- 8th International Conference on Recent Advances in Pure and Applied Mathematics (ICRAPAM 2021), September, 2021. Turkey.
- Jornada Resolución de ecuaciones: Métodos iterativos y Aplicaciones. June, 2022. Granada, Spain.
- XXVII Congreso de Ecuaciones Diferenciales y Aplicaciones XVII Congreso de Matemática Aplicada (CEDYA/CMA). July, 2022. Zaragoza, Spain.
- Mathematical Modelling in Engineering & Human Behaviour (IMM 2022). ISBN:978-84-09-47037-2. July, 2022. Universitat Politècnica de València, Spain.
- XV Jornadas de Analisis Numérico y Aplicaciones (XV JANA). November, 2022. Universidad de la Rioja, Spain.

A.3 Others

A.3.1 Related to the area of Mathematics

- Degree in Mathematics (2019), by Universitat de València.
- University Master's Degree in Mathematical Research (2020), by Universitat de València and Universitat Politècnica de València.

A.3.2 Teaching merits

- Type of teaching: Official teaching
  - Name of the subject/course: Fundamentos matemáticos II
  - University degree: Doble Titulación. Grado en Ingeniería Forestal y del Medio Natural y Grado en Ciencias Ambientales (itinerario Valencia-Gandía); Grado en Ingeniería Forestal y del Medio Natural
  - Year: 2022-2023
  - Type of hours/credits ECTS: Hours
  - No. of hours/credits ECTS: 26
  - Entity: Universitat Politècnica de València
- Type of teaching: Official teaching
  - Name of the subject/course: Matemáticas
  - University degree: Grado en Biotecnología
  - Year: 2022-2023
  - Type of hours/credits ECTS: Hours
  - No. of hours/credits ECTS: 34
  - Entity: Universitat Politècnica de València
- Type of teaching: Official teaching
  - Name of the subject/course: Matemáticas
  - University degree: Grado en Biotecnología

  - Year: 2021-2022
  - Type of hours/credits ECTS: Hours
  - No. of hours/credits ECTS: 60
  - Entity: Universitat Politècnica de València

A.3.3 Contracts and grants

- Superior Research Technician at Instituto de Matemática Multidisciplinar, Universitat Politècnica de València (BECA FPI-UPV 2020). Period: 01/03/2021 - up to date.
- Aid for mobility within the Programme for the Training of Research Staff (FPI) of the UPV 2021.

A.3.4 Research stay

- Research stay at Arts et Metiers Institute of Technology-ENSAM.
  - Supervisor of the stay: Francisco Chinesta Soria.
  - Period: 1 March 2022–6 June 2022.

A.3.5 Related to the area of languages

- English: Aptis General (C), British Council, 2022.

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On numerous occasions, when an applied mathematics problem is being solved, it is necessary to solve a nonlinear problem. It is not always possible to solve these nonlinear problems analytically, so iterative methods are used in order to obtain an approximation to the solution of the problem.

The work developed in this doctoral thesis is based on the study and design of iterative methods to obtain approximations to the solution of nonlinear equations and systems of nonlinear equations.

In this dissertation, the composition of iterative schemes, the introduction of weight functions or the introduction of memory are used. These techniques are used to design methods with a higher order of convergence or to modify existing methods in order to be applied to problems that cannot be solved by the original methods, such as obtaining solutions with a multiplicity greater than one, obtaining solutions simultaneously or the applicability to non differentiable problems. Dynamical analysis is performed to obtain the behaviour of the initial estimations by real, complex or multidimensional dynamical techniques, focusing one of the chapters of the memory on how to perform the dynamical analysis of multidimensional iterative methods with memory.

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