Vector-valued spaces of multiplier statistically convergent series and uniform convergence

Francisco Javier García-Pacheco[1], Ramazan Kama[2] and Marina Murillo-Arcila[3]

[1]Department of Mathematics, College of Engineering, University of Cadiz, 11519 Puerto Real, Spain

[2]Department of Mathematics and Physical Sciences Education, Faculty of Education, Siirt University, The Kezer Campus, Kezer, 56100-Siirt, Turkey

[3]Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, E-46022, València, Spain

garcia.pacheco@uca.es; ra.kama12@gmail.com; mamuar1@upv.es

Abstract

In this paper, we introduce the spaces of vector-valued sequences containing multiplier (weakly) statistically convergent series. The completeness of such spaces is studied as well as some relations between unconditionally convergent and weakly unconditionally Cauchy series of these spaces. We also obtain generalizations of some results regarding uniform convergence of unconditionally convergent series through the concept of statistical convergence. Finally, we provide a version of the Orlicz-Pettis theorem for λ-multiplier convergent operator series by means of the statistical convergence.

.

Keywords and phrases: Statistical convergence, uniform convergence, unconditionally convergent series, weakly unconditionally Cauchy series

AMS Subject Classifications (2010): Primary: 46B15 Secondary: 46B20, 46B25.

1. Introduction

By ω, we denote the space of all real or complex valued sequences. Any vector subspace of ω is called a sequence space. We denote by l_∞ and c_0 the Banach spaces of bounded and null sequences endowed with sup norm, respectively.

Let X be a real Banach space, X^* is the dual space of X and $\sum_k x_k$ is a series in X. A series $\sum_k x_k$ is called weakly unconditionally Cauchy (wuC) if $\left(\sum_{k=1}^n x_{\pi_k}\right)_{n\in\mathbb{N}}$ is weakly Cauchy for every permutation π of $\mathbb{N}$; or equivalently, $\sum_k x_k$ is a wuC series if and only if $\sum_{k=1}^\infty |x^*(x_k)| < \infty$ for every $x^* \in X^*$. A series $\sum_k x_k$ is called unconditionally convergent (uc) if $\sum_{k=1}^\infty x_{\pi_k}$ is convergent for every permutation π of $\mathbb{N}$.

Many results have been obtained on the behaviour of a series of the form $\sum_{k=1}^\infty a_k x_k$, where $(a_k)_{k\in\mathbb{N}}$ is a bounded sequence of real numbers. It is well known that (see [6, 8, 11]):

(1) A series $\sum_k x_k$ is uc if and only if whenever $(a_k)_{k\in\mathbb{N}} \in l_\infty$, the series $\sum_{k=1}^\infty a_k x_k$ is convergent.

(2) A series $\sum_k x_k$ is wuC if and only if whenever $(a_k)_{k\in\mathbb{N}} \in c_0$, the series $\sum_{k=1}^\infty a_k x_k$ is convergent.

(3) If $\sum_k x_k$ is a wuC series, then for every $(a_k)_{k\in\mathbb{N}} \in l_\infty$, the series $\sum_{k=1}^{\infty} a_k x_k$ is weak* convergent in X^{**}, that is, convergent with the topology $\sigma(X^{**}, X^*)$, where X^{**} is the second dual of X.

In [9, 19], the spaces $X(l_\infty)$ and $X(c_0)$, which are also denoted by $BMC(X)$ and $CMC(X)$, respectively, are defined by

$$X(\ell_\infty) = \left\{ (x_k)_{k\in\mathbb{N}} \in X^{\mathbb{N}} : \sum_{k=1}^{\infty} a_k x_k \text{ is convergent for every } (a_k)_{k\in\mathbb{N}} \in \ell_\infty \right\}$$

and

$$X(c_0) = \left\{ (x_k)_{k\in\mathbb{N}} \in X^{\mathbb{N}} : \sum_{k=1}^{\infty} a_k x_k \text{ is convergent for every } (a_k)_{k\in\mathbb{N}} \in c_0 \right\},$$

and it is proved that both spaces are real Banach spaces when endowed with the norm

$$\|(x_k)_{k\in\mathbb{N}}\| = \sup \left\{ \left\| \sum_{k=1}^{n} a_k x_k \right\| : n \in \mathbb{N}, |a_k| \leq 1, k = 1, 2, \ldots, n \right\}. \tag{1.1}$$

In view of $(1, 2)$ above, these spaces can be considered as the spaces of uc and wuC series in X, respectively.

The Hahn-Schur Theorem is one of the most important results in Banach Space Theory. For example, it is used in the original proof of the Orlicz-Pettis Theorem (see [23]). For more information on this theorem, see [28]. In [27], Swartz proves a version of this theorem regarding uniform convergence of uc series in linear metric spaces as follows:

Theorem 1.1. *Let X be a real Banach space and let (x^n) be a sequence in $X(\ell_\infty)$. If for each $(a_k) \in \ell_\infty$, $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_k x_k^n$ exists in X, then there exists $x^0 \in X(\ell_\infty)$ such that $\lim_{n\to\infty} \|x^n - x^0\| = 0$ in $X(\ell_\infty)$.*

Later, in [1–4] the authors obtain generalizations of some results provided in [27] as well as some extensions of Theorem 1.1 for wuC series, with the aid of some convergence methods, such as usual convergence, regular matrix summability and almost summability.

It is well known that statistical convergence is not equivalent to a matrix summability method (see [15]). So, it is a natural to wonder whether it is possible to obtain uniform statistical convergence from any particular situation of pointwise statistical convergence.

The first idea of statistical convergence appeared, under the name of almost convergence, in the first edition (Warsaw, 1935) of the monograph [33] by Zygmund. Since 1951, when Fast [14] (see also [24, 25]) introduced statistical convergence of number sequences in terms of asymptotic density of subsets of $\mathbb{N}$, several applications and generalizations of this notion have been investigated (for references, see [10, 12]). For instance, Maddox [21] and Kolk [17] considered the statistical convergence of sequences taking values in a locally convex space or a normed space, respectively.

Now let us recall some notions about statistical convergence. Let A be a subset of $\mathbb{N}$. We denote by $|A|$ the cardinality of A. For every $n \in \mathbb{N}$, we denote $A(n) := \{k \in$

$A : k \leq n\}$. The density of A is defined by

$$\delta(A) = \lim_{n\to\infty} \frac{1}{n}|A(n)|,$$

in case this limit exists. A sequence (x_k) in X is statistically convergent to $x \in X$, and we write $St - \lim_k x_k = x$, if for every $\varepsilon > 0$

$$\delta(\{k \in \mathbb{N} : \|x_k - x\| < \varepsilon\}) = 1.$$

Also, a sequence (x_k) in X is weakly statistically convergent to $x \in X$, and we write $wSt - \lim_k x_k = x$, if for every $\varepsilon > 0$ and every $f \in X^*$

$$\delta(\{k \in \mathbb{N} : |f(x_k) - f(x)| < \varepsilon\}) = 1.$$

It is simple to observe that every convergent sequence is statistically convergent, and every statistically convergent sequence is weakly statistically convergent.

The paper is organized as follows: in Section 2 we introduce the spaces of statistically and weakly statistically summable vector-valued sequences and show their completeness. Also we prove some auxiliary results in order to state one of the main results of the paper which consists of the existence of uniform statistical convergence from pointwise statistical convergence. In Section 3 after recalling some preliminary results we provide a new version of Orlicz-Pettis Theorem for λ-multiplier convergent operator series by means of the statistical convergence.

2. Vector-valued spaces of multiplier statistical convergent series

Within the following lines we will consider real Banach spaces unless we explicitly say otherwise. The concept of statistical convergence of a series can be given through the sequences of its partial sums (see [13, 30]):

Definition 2.1. Let X be a real normed space. We will say that the series $\sum_k x_k$ is statistically convergent to x_0 if $St - \lim_n \sum_{k=1}^n x_k = x_0$. We will denote it by $St - \sum_k x_k = x_0$.

Next, we will introduce the space of statistically summable vector-valued sequences.

Definition 2.2. Let S be a subspace of l_∞ such that $c_0 \subseteq S$. We define the space

$$X(S, St) = \left\{ (x_k)_{k\in\mathbb{N}} \in X^{\mathbb{N}} : St - \sum_{k=1}^{\infty} a_k x_k \text{ exists for every } (a_k)_{k\in\mathbb{N}} \in S \right\}$$

endowed with the norm given in (1.1).

Remark 2.3. If X is a real Banach space and S is a subspace of ℓ_∞ containing c_0, then one can observe (see also Proposition 2.7 and Remark 2.8) that

$$X(l_\infty) \subseteq X(S, St) \subseteq X(c_0).$$

Our first result shows the completeness of the space $X(S, St)$.

Theorem 2.4. *Let X be a real Banach space and let S be a subspace of ℓ_∞ containing c_0. The space $X(S, St)$ is complete when endowed with the norm given in (1.1).*

Proof. Considering Remark 2.3, it is sufficient prove that $X(S, St)$ is closed in $X(c_0)$. Consider a sequence $(x^n)_{n\in\mathbb{N}} \subseteq X(S, St)$ converging to some $x^0 \in X(c_0)$, that is,

$$\|x^n - x^0\| \to 0 \text{ as } n \to \infty.$$

We will show that $x^0 \in X(S, St)$. Fix $a = (a_k)_{k\in\mathbb{N}} \in S\backslash\{0\}$. There exists $(x_m)_{m\in\mathbb{N}} \subset X$ such that

$$St - \lim_n \sum_{k=1}^{n} a_k x_k^m = x_m \tag{2.1}$$

for each $m \in \mathbb{N}$. Let us see that $(x_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in X. Since $(x^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $X(c_0)$, there exists $m_0 \in \mathbb{N}$ such that for $\varepsilon > 0$ and $p, q \geq m_0$

$$\|x^p - x^q\| \leq \frac{\varepsilon}{3\|a\|}. \tag{2.2}$$

Fixed $p, q \geq m_0$. From (2.1), we can choose $n \in \mathbb{N}$ such that

$$\left\|x_p - \sum_{k=1}^{n} a_k x_k^p\right\| < \frac{\varepsilon}{3} \quad \text{and} \quad \left\|x_q - \sum_{k=1}^{n} a_k x_k^q\right\| < \frac{\varepsilon}{3}. \tag{2.3}$$

Therefore, by (2.2) and (2.3) we have that

$$\begin{aligned}
\|x_p - x_q\| &\leq \left\|x_p - \sum_{k=1}^{n} a_k x_k^p\right\| + \left\|x_q - \sum_{k=1}^{n} a_k x_k^q\right\| + \left\|\sum_{k=1}^{n} a_k \left(x_k^p - x_k^q\right)\right\| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \|a\|\|x^p - x^q\| \\
&= \varepsilon.
\end{aligned}$$

Thus $(x_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in X and hence, by completeness of X, there exists $x_0 \in X$ such that $\lim_m x_m = x_0$. Now, let $\varepsilon > 0$, and we fix $m \in \mathbb{N}$ such that

$$\|x^m - x^0\| \leq \frac{\varepsilon}{3\|a\|} \quad \text{and} \quad \|x_m - x_0\| \leq \frac{\varepsilon}{3}.$$

Also, by (2.1) there is an $A \subset \mathbb{N}$ with $\delta(A) = 1$ such that

$$\left\|\sum_{k=1}^{n} a_k x_k^m - x_m\right\| < \frac{\varepsilon}{3}$$

for every $n \in A$. Consequently,

$$\begin{aligned}
\left\|\sum_{k=1}^{n} a_k x_k^0 - x_0\right\| &\leq \left\|\sum_{k=1}^{n} a_k \left(x_k^0 - x_k^m\right)\right\| + \left\|\sum_{k=1}^{n} a_k x_k^m - x_m\right\| + \|x_m - x_0\| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\end{aligned}$$

for each $n \in A$, and hence we have that $x^0 \in X(S, St)$. $\square$

In a natural way we can consider the space of weakly statistically summable vector-valued sequences, which takes us to the concept of weak statistical convergence of a series.

Definition 2.5. Let X be a real normed space. We will say that the series $\sum_k x_k$ is weakly statistically convergent to x_0 if $St - \lim_n \sum_{k=1}^{n} x^*(x_k) = x^*(x_0)$ for every $x^* \in X^*$. We will denote it by $wSt - \sum_k x_k = x_0$.

Definition 2.6. Let S be a subspace of l_∞ such that $c_0 \subseteq S$. We define the space

$$X_w(S, St) = \left\{ (x_k)_{k\in\mathbb{N}} \in X^{\mathbb{N}} : wSt - \sum_{k=1}^{\infty} a_k x_k \text{ exists for every } (a_k)_{k\in\mathbb{N}} \in S \right\}$$

endowed with the norm given (1.1).

Before stating and proving the completeness of this space, we will give a characterization of the *wuC* series, which is given in terms of statistical convergence, and a remark.

Proposition 2.7. *Let X be a Banach space and $\sum_k x_k$ be a series in X. If $wSt - \sum_k a_k x_k$ is convergent for every $(a_k) \in c_0$, then $\sum_k x_k$ is a wuC series.*

Proof. Let us suppose that $\sum_k x_k$ is not a wuC series. There exists $x^* \in X^*$ such that

$$\sum_k |x^*(x_k)| = +\infty. \tag{2.4}$$

Inductively, we will construct a sequence $(a_k)_{k\in\mathbb{N}}$ in c_0 such that $\sum_k a_k x^*(x_k) = +\infty$ and $a_k x^*(x_k) \geq 0$ for every $k \in \mathbb{N}$, therefore by hypothesis and [30, Proposition 1], we will obtain a contradiction with (2.4). Let $\nu > 1$ and $i_0 = 0$. There exists an increasing sequence (i_n) such that

$$\sum_{k=i_{n-1}+1}^{i_n} |x^*(x_k)| > \nu^{2n\nu}.$$

On the other hand, we define the following sequence:

$$a_k = \begin{cases} \frac{1}{\nu^{n\nu}}, & \text{if } x^*(x_k) \geq 0, \\ -\frac{1}{\nu^{n\nu}}, & \text{if } x^*(x_k) < 0 \end{cases}$$

for $k = i_{n-1}+1, i_{n-1}+2, \ldots, i_n$. After an easy calculation, we obtain that

$$a_k x^*(x_k) \geq 0 \quad \text{for every } k \text{ and } \quad \sum_k a_k x^*(x_k) = +\infty.$$

This completes the proof. ∎

Remark 2.8. We claim that the inclusion $X(l_\infty) \subseteq X(S, St) \subseteq X_w(S, St) \subseteq X(c_0)$ is provided. Indeed, it is sufficient to show $X_w(S, St) \subseteq X(c_0)$. If we take $x = (x_k) \in X_w(S, St)$ then $wSt - \sum_k a_k x_k$ is convergent for every $a = (a_k) \in S$. Also, since $c_0 \subset S$, we have that $wSt - \sum_k a_k x_k$ is convergent for every $(a_k) \in c_0$. By using Proposition 2.7, we obtain that $\sum_k x_k$ is wuC series. This means that $x = (x_k) \in X(c_0)$.

Now, we give the completeness of the space $X_w(S, St)$.

Theorem 2.9. *Let X be a real Banach space and S be a subspace of ℓ_∞ containing c_0. The space $X_w(S, St)$ is complete when endowed with the norm given in (1.1).*

Proof. We will prove that $X_w(S,St)$ is closed in $X(c_0)$. Let $(x^n)_{n\in\mathbb{N}}$ be a sequence in $X_w(S,St)$ and consider $x^0 \in X(c_0)$ such that $\|x^n - x^0\| \to 0$ as $n \to \infty$. We will prove that $x^0 \in X_w(S,St)$. Fix $a = (a_k)_{k\in\mathbb{N}} \in S \setminus \{0\}$. For each $m \in \mathbb{N}$, there exists $x_m \in X$ such that

$$wSt - \lim_n \sum_{k=1}^n a_k x_k^m = x_m.$$

We will first prove that $(x_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in X. Take any $\varepsilon > 0$. An $m_0 \in \mathbb{N}$ can be found so that if $p,q \geq m_0$, then we have that $\|x^p - x^q\| \leq \frac{\varepsilon}{3\|a\|}$. We fix $p,q \geq m_0$ and consider a functional $x^* \in S_{X^*}$ such that $\|x_p - x_q\| = |x^*(x_p - x_q)|$. There exists $n \in \mathbb{N}$ such that

$$\left| x^*(x_p) - \sum_{k=1}^n a_k x^*(x_k^p) \right| < \frac{\varepsilon}{3} \quad \text{and} \quad \left| x^*(x_q) - \sum_{k=1}^n a_k x^*(x_k^q) \right| < \frac{\varepsilon}{3}.$$

It follows that

$$\begin{aligned} \|x_p - x_q\| &\leq \left| x^*(x_p) - \sum_{k=1}^n a_k x^*(x_k^p) \right| + \left| x^*(x_q) - \sum_{k=1}^n a_k x^*(x_k^q) \right| + \left| \sum_{k=1}^n a_k x^* \left(x_k^p - x_k^q \right) \right| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \|a\| \|x^p - x^q\| \\ &= \varepsilon. \end{aligned}$$

Since X is complete, there exists $x_0 \in X$ such that $\lim_m x_m = x_0$. Now, we will prove that $wSt - \lim_n \sum_{k=1}^n a_k x_k^0 = x_0$. If $\varepsilon > 0$ and $x^* \in X^*$, then we can fix $m \in \mathbb{N}$ such that

$$\|x^m - x^0\| \leq \frac{\varepsilon}{3\|a\|\|x^*\|} \quad \text{and} \quad \|x_m - x_0\| \leq \frac{\varepsilon}{3\|x^*\|}.$$

On the other hand, since $wSt - \lim_n \sum_{k=1}^n a_k x_k^m = x_m$, there exists an $A \subset \mathbb{N}$ with $\delta(A) = 1$ such that

$$\left| \sum_{k=1}^n a_k x^*(x_k^m) - x^*(x_m) \right| < \frac{\varepsilon}{3}$$

for every $n \in A$. Therefore,

$$\begin{aligned} \left| \sum_{k=1}^n a_k x^*(x_k^0) - x^*(x_0) \right| &\leq \left| \sum_{k=1}^n a_k x^* \left(x_k^0 - x_k^m \right) \right| \\ &+ \left| \sum_{k=1}^n a_k x^*(x_k^m) - x^*(x_m) \right| + |x^*(x_m) - x^*(x_0)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

for each $n \in A$. This means that $x^0 \in X_w(S,St)$. $\square$

Lemma 2.10. *Let X be a real Banach space and $c_0 \subseteq S \subseteq l_\infty$. If $x = (x_k)_{k\in N} \in X(S, St)$, then the linear mapping*

$$\begin{aligned} T_x : S &\rightarrow X \\ a &\rightarrow T_x(a) = St - \sum_{k=1}^{\infty} a_k x_k \end{aligned}$$

is continuous and $\|T_x\| = \|x\|$.

Proof. Consider $a = (a_k) \in S$ and let $(T_x(a))_n$ denote the sequence of the partial sums of the series $St - \sum_{k=1}^{\infty} a_k x_k$. Since $x = (x_k)_{k\in\mathbb{N}} \in X(S, St)$,

$$\|(T_x(a))_n\| = \left\| St - \sum_{k=1}^{n} a_k x_k \right\| \leq \|a\|\|x\|,$$

and hence we have that $\|T_x(a)\| \leq \|a\|\|x\|$. This shows that T_x is continuous and $\|T_x\| \leq \|x\|$. In order to show that $\|T_x\| = \|x\|$, we fix an arbitrary $\varepsilon > 0$ and choose $n \in \mathbb{N}$ and $a \in c_{00}$ with $\|a\| \leq 1$ such that $\|x\| - \|\sum_{k=1}^{n} a_k x_k\| < \varepsilon$. Then

$$\|x\| \geq \|T_x(a)\| \geq \left\| \sum_{k=1}^{n} a_k x_k \right\| > \|x\| - \varepsilon.$$

The arbitrariness of ε forces the equality $\|T_x\| = \|x\|$. ☐

Remark 2.11. In Lemma 2.10, if we replace the assumption $x = (x_k)_{k\in N} \in X(S, St)$ by $x = (x_k)_{k\in N} \in X_w(S, St)$, then the linear mapping

$$\begin{aligned} T_x : S &\rightarrow X \\ a &\rightarrow T_x(a) = wSt - \sum_{k=1}^{\infty} a_k x_k \end{aligned}$$

is also continuous with $\|T_x\| = \|x\|$.

It is now time to provide our main result in this study: the existence of uniform statistical convergence from pointwise statistical convergence. Observe that this is also a generalization of Theorem 1.1 in terms of statistical convergence method. Before stating and proving this result, we will need some concepts (see [3]).

Definition 2.12. Let X be a real Banach space and M be a vector subspace of X^{**}. The space X is called an M-Grothendieck space if every $\sigma(X^*, X)$-convergent sequence in X^* is also $\sigma(X^*, M)$-convergent. If $M = X^{**}$, then the space X is called a Grothendieck space.

Recall that if F is a set of functions of a given set X into a topological space, then $\sigma(X, F)$ stands for the initial topology on X induced by F, that is, the coarsest topology that makes all functions in F continuous. If $G \subseteq F$, then it is clear that $\sigma(X, G)$ is coarser than $\sigma(X, F)$. Notice that, under the settings of Definition 2.12, if $M \subseteq X$, then X is trivially M-Grothendieck. On the other hand, if X is an M-Grothendieck space and $(x_n^*)_{n\in\mathbb{N}} \subseteq X^*$ is w^*-convergent to some $x^* \in X^*$, then we know that $(x_n^*)_{n\in\mathbb{N}}$ is $\sigma(X^*, M)$-convergent, but it is not necessarily $\sigma(X^*, M)$-convergent to

x^*. The following lemma provides a sufficient condition to assure that if $(x_n^*)_{n\in\mathbb{N}} \subseteq X^*$ is w^*-convergent to some $x^* \in X^*$, then $(x_n^*)_{n\in\mathbb{N}}$ is $\sigma(X^*, M)$-convergent to x^*.

Lemma 2.13. *Let X be a real Banach space and M be a vector subspace of X^{**} containing X. Suppose that X is M-Grothendieck. If $(x_n^*)_{n\in\mathbb{N}} \subseteq X^*$ is w^*-convergent to some $x^* \in X^*$, then $(x_n^*)_{n\in\mathbb{N}}$ is $\sigma(X^*, M)$-convergent to x^*.*

Proof. Since X is M-Grothendieck, there exists $y^* \in X^*$ such that $(x_n^*)_{n\in\mathbb{N}}$ is $\sigma(X^*, M)$-convergent to y^*. All we need to show is that $x^* = y^*$. Fix an arbitrary $x \in X$. Since $X \subseteq M \subseteq X^{**}$ and $(x_n^*)_{n\in\mathbb{N}}$ is w^*-convergent to $x^* \in X^*$ and $\sigma(X^*, M)$-convergent to y^*, we have that $(x_n^*(x))_{n\in\mathbb{N}}$ is both convergent to $x^*(x)$ and $y^*(x)$. As a consequence, $x^*(x) = y^*(x)$. The arbitrariness of $x \in X$ shows that $x^* = y^*$. □

Remark 2.14. Let S be a vector subspace of ℓ_∞ so that $c_0 \subseteq S$. We can (isometrically) identify ℓ_∞ with a subspace of S^{**}. Indeed, consider the map

$$\begin{array}{ccl} \ell_\infty & \to & S^{**} \\ a & \mapsto & \begin{array}{rccl} h_a: & S^* & \to & \mathbb{R} \\ & g & \to & h_a(g) = \sum_{k=1}^\infty a_k g(e^k), \end{array} \end{array} \tag{2.5}$$

where (e^k) is the standard basis of c_0. Notice that, for every $p \in \mathbb{N}$,

$$\sum_{k=1}^p \left|a_k g(e^k)\right| \le \|a\|_\infty \sum_{k=1}^p \left|g(e^k)\right| = \|a\|_\infty \sum_{k=1}^p g\left(\varepsilon_k e^k\right) = \|a\|_\infty g\left(\sum_{k=1}^p \varepsilon_k e^k\right) \le \|a\|_\infty \|g\|,$$

where

$$\varepsilon_k := \begin{cases} +1 & g(e^k) \ge 0, \\ -1 & g(e^k) < 0. \end{cases}$$

This shows that $|h_a(g)| \le \|a\|_\infty \|g\|$, hence $\|h_a\| \le \|a\|_\infty$. In fact, $\|h_a\| = \|a\|_\infty$, since, for every $\varepsilon > 0$, there exists $k_\varepsilon \in \mathbb{N}$ such that $\|a\|_\infty - \varepsilon < |a_{k_\varepsilon}| \le \|a\|_\infty$, thus

$$\|a\|_\infty - \varepsilon < |a_{k_\varepsilon}| = |h_a(g)| \le \|h_a\| \le \|a\|_\infty,$$

where $g \in S^*$ is a Hahn-Banach extension to S of

$$\begin{array}{rccl} e_{k_\varepsilon}: & c_0 & \to & \mathbb{R} \\ & x & \mapsto & x_{k_\varepsilon}. \end{array}$$

As a consequence of Remark 2.14, for every subspace $c_0 \subseteq S \subseteq \ell_\infty$, it makes sense to consider whether S is an ℓ_∞-Grothendieck space.

Lemma 2.15. *Let $c_0 \subseteq S \subseteq \ell_\infty$. Let $a \in S$ Then:*

1. *If $\sum_{k=1}^\infty a_k e^k$ is $\sigma(S, S^*)$-convergent to a, then $h_a(g) = g(a)$ for all $g \in S^*$.*
2. *If $a \in c_0$, then $h_a(g) = g(a)$ for all $g \in S^*$.*
3. *If S is an ℓ_∞-Grothendieck space, then every w^*-convergent sequence $(g_n)_{n\in\mathbb{N}} \subseteq S^*$ to 0 satisfies that*

$$\lim_{n\to\infty} \sum_{k=1}^\infty a_k g_n(e^k) = 0$$

for every $a = (a_k)_{k\in\mathbb{N}} \in \ell_\infty$.

Proof.

(1) Fix an arbitrary $g \in S^*$. Then $\left(\sum_{k=1}^{l} a_k e^k\right)_{l\in\mathbb{N}}$ is $\sigma(S,S^*)$-convergent to a, so $\left(g\left(\sum_{k=1}^{l} a_k e^k\right)\right)_{l\in\mathbb{N}}$ is convergent to $g(a)$, that is, $\left(\sum_{k=1}^{l} a_k g\left(e^k\right)\right)_{l\in\mathbb{N}}$ converges to $g(a)$. However, $\left(\sum_{k=1}^{l} a_k g\left(e^k\right)\right)_{l\in\mathbb{N}}$ converges to $h_a(g)$.
(2) Simply notice that $\sum_{k=1}^{\infty} a_k e^k$ converges to a in the sup norm.
(3) By hypothesis, S is an ℓ_∞-Grothendieck space, therefore there exists $g \in S^*$ such that $h_a(g_n) \to h_a(g)$ as $n \to \infty$ for every $a = (a_k)_{k\in\mathbb{N}} \in \ell_\infty$. We have to show that $h_a(g) = 0$ for all $a = (a_k)_{k\in\mathbb{N}} \in \ell_\infty$. Fix an arbitrary $a = (a_k)_{k\in\mathbb{N}} \in \ell_\infty$. Fix also an arbitrary $K \in \mathbb{N}$. Notice that $a_K := \sum_{k=1}^{K} a_k e^k \in c_{00} \subseteq c_0 \subseteq S$. Therefore, $g_n\left(\sum_{k=1}^{K} a_k e^k\right) \to 0$ as $n \to \infty$ because $(g_n)_{n\in\mathbb{N}}$ is w^*-convergent to 0. On the other hand, $h_{a_K}(g_n) \to h_{a_K}(g)$ as $n \to \infty$. However, $h_{a_K}(g_n) = g_n\left(\sum_{k=1}^{K} a_k e^k\right)$ for every $n \in \mathbb{N}$. This implies that $h_{a_K}(g) = 0$. Finally, it only suffices to realize that $h_{a_K}(g) \to h_a(g)$ as $K \to \infty$.

□

Lemma 2.15 indicates that, if $\sum_{k=1}^{\infty} a_k e^k$ is $\sigma(S,S^*)$-convergent to a for all $a \in S$, then ℓ_∞ contains S in the identification (2.5), hence the hypothesis of Lemma 2.13 are satisfied.

Theorem 2.16. *Let X be a real Banach space and $(x^n)_{n\in\mathbb{N}}$ a sequence in $X(\ell_\infty)$. Assume that the following conditions are satisfied:*

(i) *S is an ℓ_∞-Grothendieck space such that $c_0 \subseteq S \subseteq l_\infty$.*
(ii) *$\lim_{n\to\infty} St - \sum_{i=1}^{\infty} a_i x_i^n$ exists for each $a = (a_i) \in S$.*

Then, there exists $x^0 \in X(\ell_\infty)$ such that $\lim_n \|x^n - x^0\| = 0$ in $X(\ell_\infty)$.

Proof. It suffices to prove that $(x^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $X(\ell_\infty)$. If not, there exist $\varepsilon > 0$ and an increasing sequence $(n_k)_{k\in\mathbb{N}}$ of naturals such that $\|y^k\| > \varepsilon$ for every $k \in \mathbb{N}$, where $y^k = x^{n_k} - x^{n_{k+1}}$. For every $k \in \mathbb{N}$, we can choose $x_k^* \in S_{X^*}$ such that

$$\sum_{i=1}^{\infty} |x_k^*(y_i^k)| > \varepsilon. \tag{2.6}$$

On the other hand, from Lemma 2.10 we can define the following continuous linear mapping:

$$\begin{aligned} T_{y^k} : S &\to X \\ a &\to T_{y^k}(a) = St - \sum_{i=1}^{\infty} a_i y_i^k. \end{aligned}$$

Observe that $\lim_{k\to\infty} T_{y^k}(a) = 0$ for every $a = (a_i) \in S$. Indeed, fix an arbitrary $a = (a_i) \in S$. Note that $(St - \sum_{i=1}^{\infty} a_i x_i^{n_k})_{k\in\mathbb{N}}$ and $\left(St - \sum_{i=1}^{\infty} a_i x_i^{n_{k+1}}\right)_{k\in\mathbb{N}}$ are subsequences of $(St - \sum_{i=1}^{\infty} a_i x_i^n)_{n\in\mathbb{N}}$. Since the limit of $(St - \sum_{i=1}^{\infty} a_i x_i^n)_{n\in\mathbb{N}}$ exists in X by (ii), we conclude that

$$\begin{aligned}\lim_{k\to\infty} T_{y^k}(a) &= \lim_{k\to\infty} St-\sum_{i=1}^{\infty} a_i y_i^k \\ &= \lim_{k\to\infty} St-\sum_{i=1}^{\infty} a_i x_i^{n_k} - \lim_{k\to\infty} St-\sum_{i=1}^{\infty} a_i x_i^{n_{k+1}} \\ &= 0.\end{aligned}$$

Therefore, the sequence $(x_k^*\circ T_{y^k})_k$ is weak* convergent to zero in S^*. From (i) together with Remark 2.14 and (3) in Lemma 2.15, we obtain that

$$\lim_{k\to\infty}\sum_{i=1}^{\infty} a_i(x_k^*\circ T_{y^k})(e^i) = \lim_{k\to\infty}\sum_{i=1}^{\infty} a_i x_k^*(y_i^k) = 0$$

for every $a = (a_i)_{i\in\mathbb{N}} \in \ell_\infty$. Hence, the sequence $\{(x_k^*(y_i^k))_{i\in\mathbb{N}}\}_{k\in\mathbb{N}}$ is weakly convergent to zero in ℓ_1. Since the space ℓ_1 enjoys the Schur property, the sequence $\{(x_k^*(y_i^k))_{i\in\mathbb{N}}\}_{k\in\mathbb{N}}$ is norm convergent to zero in ℓ_1, which contradicts (2.6). $\square$

If we now consider weakly statistical convergence instead of statistical convergence in the above theorem, we obtain the following result:

Corollary 2.17. *Let X be a real Banach space. Suppose that S is a subspace of ℓ_∞ containing c_0 that is an ℓ_∞-Grothendieck space. If $(x^n)_{n\in\mathbb{N}}$ is a sequence in $X(\ell_\infty)$ such that*

$$\lim_{n\to\infty} wSt-\sum_{i=1}^{\infty} a_i x_i^n$$

exists for each $a = (a_i) \in S$, then there exists $x^0 \in X(\ell_\infty)$ such that $\lim_n \|x^n - x^0\| = 0$ in $X(\ell_\infty)$.

3. A version of Orlicz-Pettis Theorem for statistical convergence

The classical Orlicz-Pettis Theorem concerning subseries convergence in the weak and norm topologies of a normed linear space has proven to be a very useful result with applications to many situations in measure and integration theory and the geometric theory of B-spaces [6, 11]. Several versions of the Orlicz-Pettis Theorem have been established for multiplier convergent operator series [7, 18, 26, 28, 29, 31, 32].

Before stating and proving the main result in this final section, we need to introduce and recall several concepts on which we will strongly rely. A sequence space is a vector subspace $\lambda \subseteq \mathbb{R}^{\mathbb{N}}$ endowed with a vector topology finer than the product topology. By an increasing sequence of intervals we mean a sequence (I_m) where, for every $m \in \mathbb{N}$, $I_m := (k_{m-1}, k_m] \cap \mathbb{N}$ with $k_m \in \mathbb{N}$ and $k_0 := 0$. If $x \in \lambda$ and I is an interval of naturals, then $\chi_I x$ stands for the sequence whose k^{th}-term is

$$(\chi_I x)_k := \begin{cases} x_k & \text{if } k \in I, \\ 0 & \text{if } k \notin I. \end{cases}$$

Now, we consider a gliding hump property [28] on a sequence space λ. This property warranties that a series $\sum_k T_k$ which is λ-multiplier statistically convergent for the weak topology of Y is λ-multiplier statistically convergent for the strong topology of Y, where $(T_k) \subseteq \mathcal{L}(X,Y)$ for X, Y Banach spaces.

Definition 3.1. Let λ be a sequence space. We say that λ has the infinite gliding hump property (∞-GHP) if whenever $x \in \lambda$ and (I_m) is an increasing sequence of intervals, there exists a strictly increasing sequence $\{p_m\}$ of naturals with $t_{p_m} > 0$, $t_{p_m} \to \infty$, such that every subsequence of $\{p_m\}$ has a further subsequence $\{q_m\}$ in such a way the coordinatewise sum of the series $\sum_m t_{q_m} \chi_{I_{q_m}} x \in \lambda$.

The following lemma will be used in the proof of Orlicz-Pettis theorem and it can be found in [5]. Before that, we need the following definition.

Definition 3.2. Let $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$. We say that:
- $\mathcal{F}$ is a natural family if $\phi_0(\mathbb{N}) \subseteq \mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$, where $\phi_0(\mathbb{N})$ denotes the family of finite subsets of $\mathbb{N}$.
- $\mathcal{F}$ has the (M)-property if for each sequence $(A_i)_i$ in $\phi_0(\mathbb{N})$ there exists $E \in \mathcal{F}$ and $K \subset \mathbb{N}$ infinite such that $\bigcup_{i\in K} A_i \subseteq E \subseteq \bigcup_{i\in\mathbb{N}} A_i$.

Lemma 3.3. *Let $\mathcal{F}$ be a natural family with the (M) property and let $(x_{ij})_{i,j\in\mathbb{N}}$ be a matrix in a Banach space X. Suppose*

(i) *$St - \lim_i x_{ij} = x_j$ for each $j \in \mathbb{N}$.*

(ii) *For each $B \in \mathcal{F} \setminus \phi_0(\mathbb{N})$, $\{St - \sum_{j\in B} x_{ij}\}_i$ is a Cauchy sequence.*

Then $(x_{ij})_{i,j}$ is strongly uniformly statistically convergent to zero. In particular,

$$St - \lim_i (St - \lim_j x_{ij}) = St - \lim_j (St - \lim_i x_{ij}) = 0 \text{ and } St - \lim_i x_{ii} = 0.$$

We have finally gathered all the necessary tools to prove our version of Orlicz-Pettis Theorem for statistical convergence.

Theorem 3.4. *Let λ be a sequence space having the ∞-GHP, $\{T_k\} \subset L(X,Y)$ for X, Y Banach spaces, and $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ a natural family with the (M)-property. If the series $\sum_{k\in B} T_k$ is λ-multiplier statistically convergent with respect to weak topology of Y for $B \in \mathcal{F} \setminus \phi_0(\mathbb{N})$, then the series $\sum_k T_k$ is $\lambda-$multiplier statistically convergent with respect to strong topology of Y.*

Proof. Let $\sum_{k\in B} T_k$ be λ-multiplier statistically convergent with respect to weak topology of Y for $B \in \mathcal{F} \setminus \phi_0(\mathbb{N})$. Take $\varepsilon > 0$. If $\sum_k T_k$ is not λ-multiplier statistically convergent for the strong topology of Y, there exists $x \in \lambda$, a bounded sequence $(y_n^*) \subset Y^*$ and an increasing sequence of intervals $\{I_n\}$ such that

$$\left| St - \sum_{k\in I_n} y_n^*(T_k x_k) \right| > \varepsilon \tag{3.1}$$

for all $n \in \mathbb{N}$. Since λ has the ∞-GHP, there exists a subsequence $\{p_n\}$ and $t_{p_n} > 0$, $t_{p_n} \to \infty$ such that every subsequence of $\{p_n\}$ has a further subsequence $\{q_n\}$ such that $\sum_n t_{q_n} \chi_{I_{q_n}} x \in \lambda$. Now, we consider the matrix H defined by

$$H = [h_{ij}] = \left[\sum_{m\in I_{p_j}} \frac{y_{p_i}^*}{t_{p_i}} (T_m(t_{p_j} x_m)) \right].$$

We will show the matrix H satisfies the conditions of Lemma 3.3, and hence we will obtain a contradiction with (3.1). Indeed:

(i) The columns of H are statistically convergent to zero because (y_i^*) is bounded and $t_{p_i} \to \infty$.

(ii) Since λ has the ∞-GHP, there is a further subsequence $\{q_j\}$ such that $\sum_j t_{q_j} \chi_{I_{q_j}} x \in \lambda$.

By hypothesis, there exists a sequence y such that

$$wSt - \sum_{j \in B} \sum_{m \in I_{q_j}} T_m(t_{q_j} x_m) = y$$

for $B \in \mathcal{F} \setminus \phi_0(\mathbb{N})$ and hence, from the definition of the matrix H, we obtain

$$St - \lim_i \sum_{j \in B} h_{iq_j} = St - \lim_i \sum_{j \in B} \sum_{m \in I_{q_j}} \frac{y_{p_i}^*}{t_{p_i}} (T_m(t_{q_j} x_m)) = 0.$$

Consequently, the diagonal of H is statistically convergent to zero, and this completes the proof. □

It follows immediately from Theorem 3.4:

Corollary 3.5. *If λ is a sequence space having the ∞-GHP and $\{T_k\} \subset L(X,Y)$ for X, Y Banach spaces, then the following assumptions are equivalent:*

(i) *The series $\sum_k T_k$ is λ-multiplier convergent.*

(ii) *The series $\sum_k T_k x_k$ is statistically convergent for every sequence $\{x_k\} \in \lambda$.*

(iii) *The series $\sum_k T_k x_k$ is weakly statistically convergent for every sequence $\{x_k\} \in \lambda$.*

Acknowledgements. The first author has been supported by Research Grant PGC-101514-B-I00 awarded by the Ministry of Science, Innovation and Universities of Spain. The third author has been supported by MEC, grant PID2019-105011GB-I00.

References

[1] A. Aizpuru, F.J. García-Pacheco, C. Pérez-Eslava, Matrix summability and uniform convergence of series, Proc. Amer. Math. Soc., 135(11) (2007), 3571-3579.

[2] A. Aizpuru, A. Gutiérrez-Dávila, F.J. Pérez-Fernández, Boolean algebras and uniform convergence of series, J. Math. Anal. Appl., 284 (2003), 89-96.

[3] A. Aizpuru, F.J. Pérez-Fernández, Spaces of S-bounded multiplier convergent series, Acta Math. Hungar., 87 (2000), 103-114.

[4] A. Aizpuru, R. Armario, F.J. García-Pacheco, F.J. Pérez-Fernández, Banach limits and uniform almost summability, J. Math. Anal. Appl., 379(1) (2011), 82-90.

[5] A. Aizpuru, M. Nicasio-Llach, F. Rambla-Barreno, A remark about the Orlicz-Pettis theorem and the statistical convergence. Acta Mathematica Sinica, English Series, 2010, 26:305-310.

[6] F.Albiac, N.J. Kalton, Topics in Banach Spaces Theory, Springer Inc., 2006.

[7] B. Altay , R. Kama, On Cesàro summability of vector valued multiplier spaces and operator valued series, Positivity, 2018, 22:575-586.

[8] C. Bessaga, A. Pelczynski, On bases and unconditional convergence of series in Banach spaces, Stud. Math., 17 (1958), 151-164.

[9] Q. Bu, C. Wu, Unconditionally convergent series of operators on Banach spaces, J. Math. Anal. Appl., 207 (1997), 291-299.

[10] J. Connor, A topological and functional analytic approach to statistical convergence, Applied Numer. Harmon. Anal., Birkhuser Boston, Boston, MA 1999, 403-413.

[11] J. Diestel, Sequences and Series in Banach Spaces, Springer-Verlag, New York, 1984.

[12] G. Di Maio, L.D. Kočinac, Statistical convergence in topology, Topology and its Applications, 156 (2008), 28-45.

[13] M. Dindoš, T. Salat, V. Toma, Statistical convergence of infinite series, Czechoslovak Math. J., 53 (128)(4) (2003), 989-1000.

[14] H. Fast, Sur la convergence statistique, Colloq. Math., 2 (1951), 241-244.

[15] J.A. Fridy, On statistical convergence, Analysis, 5 (1985), 301-313.

[16] V.M. Kadets, M.I. Kadets, Rearrangements of Series in Banach Spaces, Translations of Mathematical Monographs. 86. Amer. Math. Soc., Providence, 1991.

[17] E. Kolk, The statistical convergence in Banach spaces, Acta et Comment Univ Tartu, 928 (1991), 41-52.

[18] F. León-Saavedra, M. Romero de la Rosa, A. Sala, OrliczPettis theorem through summability methods. Mathematics 2019, 7, 895.

[19] R. Li, Q. Bu, Locally convex spaces containing no copy of c_0, J. Math. Anal. Appl., 172 (1993), 205-211.

[20] G.G. Lorentz, A contribution to the theory of divergent sequences, Acta Math., 80 (1948), 167-190.

[21] I.J. Maddox, Statistical convergence in a locally convex space, Math. Proc. Camb. Philos Soc., 104 (1988), 141-145.

[22] C.W. McArthur, On relationships amongst certain spaces of sequences in an arbitrary Banach space, Canad. J. Math., 8 (1956), 192-197.

[23] B.J. Pettis, On integration in vector spaces, Trans. Am. Math. Soc. 44 (1938), 277-304.

[24] I.J. Schoenberg, The integrability of certain functions and related summability methods, Amer. Math. Monthly, 66 (1959), 361-375.

[25] H. Steinhaus, Sur la convergence ordinarie et la convergence asymptotique, Colloq. Math., 2 (1951), 73-74.

[26] C. Stuart, C. Swartz, Generalization of the Orlicz-Pettis theorems, Proyecciones, 2005, 24:37-48.

[27] C. Swartz, The Schur lemma for bounded multiplier convergent series, Math. Ann. 263 (1983), 283-288.

[28] C. Swartz, Multiplier Convergent Series, World Scientific, Singapore, 2009.

[29] C. Swartz , Orlicz-Pettis theorems for multiplier convergent operator valued series, Proyecciones, 2003, 22:135-144.

[30] B.C. Tripathy, On statistically convergent series, Punjab. Univ. J. Math., 32 (1999), 1-8.

[31] J. Wu, S. Lu, A general Orlicz-Pettis theorem, Taiwanese J Math, 2002, 6:433-440.

[32] T. Yuanhong, L. Ronglu, Orlicz-Pettis theorem for $\lambda-$multiplier convergent operator series, Bulletin of the Australian Mathematical Society, 2007, 75:247-252.

[33] A. Zygmund, Trigonometric Series, Cambridge University Press, Cambridge, 1979.