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Additional Information

Quasi-metric properties of the dual cone of an asymmetric normed space. *

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Abstract

We obtain some quasi-metric properties of the dual cone of an asymmetric normed space. Thus, we prove that it is balanced, and hence its topology is completely regular. We also prove that it is complete in the sense of D. Doitchinov. These results generalize those obtained in [18] because, in our study, the asymmetric normed space does not necessarily satisfy the T_1 axiom. Moreover, we provide a class of asymmetric normed spaces whose dual cones are right K-sequentially complete. Finally, we represent an arbitrary asymmetric normed space as a function space by using the unit ball of its dual space.

1 Introduction and preliminaries

If (X, q) is an asymmetric normed space, its topological dual X^*, that is, the set of all upper semi-continuous linear functionals on it is a cone and can be regarded as an extended quasi-metric space. In Theorem 3 of [18] the authors proved that if (X, q) is a T_1 asymmetric normed space, the quasi-metric space X^* is balanced and D-complete (complete in the sense of D. Doitchinov). Given that there are important classes of asymmetric normed spaces that are not T_1 (see e.g. [11], [15]), we consider extending these results to any asymmetric normed space.

In the last decades, quasi-metric spaces and asymmetric normed spaces have received considerable attention for their intrinsic interest and applications within and outside mathematics (see e.g. [3], [5], [8], [9], [14], [17]).

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That is why papers in this area continue to be published every year. The topological dual of an asymmetric norm space is a powerful tool in non-symmetric functional analysis, both from the theoretical and the applied point of view (see e.g. [6], [12], [14]). In this paper we prove that the (quasi-metric) dual cone X^* of any asymmetric normed space is balanced and D-complete. We also show that X^* is not always right K-sequentially complete and provide a class of asymmetric normed spaces with right K-sequentially complete dual cones. Finally, we represent an arbitrary asymmetric normed space as a function space by using the unit ball of its dual space.

We start this by recalling several concepts related to quasi-metrics and asymmetric norms that we will need throughout the article. Our basic references in relation to these topics are [8] and [16].

By a *quasi-metric* on a nonempty set X we mean a function $d : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$: (i) $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$, and (ii) $d(x, y) \leq d(x, z) + d(z, x)$.

We will also consider extended quasi-metrics. They satisfy the above two axioms, except that we allow $d(x, y) = +\infty$.

If d is a (n extended) quasi metric, the function d^{-1} defined on $X \times X$ by $d^{-1}(x, y) = d(y, x)$ is a (n extended) quasi-metric on X called the *conjugate* of d and d^s defined on $X \times X$ by $d^s(x, y) = max\{d(x, y), d(y, x)\}$ is a (n extended) metric on X

A *(n extended) quasi-metric* space is a pair (X, d) such that X is a (nonempty) set X and d is a (n extended) quasi-metric on X.

Each (extended) quasi-metric d on X induces a T_0 topology τ_d on X which has as a base the family of the balls $\{B_d(x, r) : x \in X,\ r > 0\}$, where $B_d(x, r) = \{y \in B : d(x, y) < r\}$.

If the quasi-metric satisfies that $d(x, y) = 0 \Leftrightarrow x = y$ then (X, τ_d) is a T_1 space.

A subset Y of a real linear space is a *cone or semilinear space* if for every $x, y \in Y$ and $\alpha \in \mathbb{R}^+$, $x + y \in Y$ and $\alpha x \in Y$. Obviously, every real linear space is a cone.

Let X be a cone. A function $q : X \to \mathbb{R}^+$, is said to be an *asymmetric norm* on X if for all $x, y \in X$ and $\alpha \in \mathbb{R}^+$,

(i) $x = 0$ if and only if $-x \in X$ and $q(x) = q(-x) = 0$;

(ii) $q(\alpha x) = \alpha q(x)$;

(iii) $q(x + y) \leq q(x) + q(y)$.

If we allow that $q(x) = +\infty$, q is called an *extended asymmetric norm.*

If X is a real linear space and q is an asymmetric norm on X, the pair (X, q) is called an *asymmetric normed space.*

If X is a cone and q is an asymmetric norm on X, the pair (X, q) is called an *asymmetric normed cone.*

If q is an asymmetric norm on a real linear space X, then the function q^{-1} defined on X by $q^{-1}(x) = q(-x)$ is also an asymmetric norm on X, called the *conjugate* of q. The function q^s defined on X by $q^s(x) = \max\{q(x), q^{-1}(x)\}$ is a norm on X.

Each asymmetric norm q on a real linear space X induces a quasi-metric d_q on X defined by $d_q(x, y) = q(y - x)$, for all $x, y \in X$. We refer to the topology τ_{d_q} as the topology *induced* by q.

If (X, q) is an asymmetric normed space, by X^{s*} is denoted the topological dual space of the normed space (X, q^s), i.e.,

$$X^{s*} = \{f : (X, q^s) \to (\mathbb{R}, |\cdot|) : f \textit{ is linear and continuous}\}$$

By $(q^s)^*$ is denoted the dual norm on X^{s*}, i.e.,

$$(q^s)^*(f) = sup\{|f(x)| : q^s(x) \leq 1\}.$$

It is well known that $(X^{s*}, (q^s)^*)$ is a Banach space.

If (X, q) is an asymmetric normed space, by X^* is denoted the set

$$X^* = \{f : (X, q) \to (\mathbb{R}, u) : f \textit{ is linear and continuous}\}$$

where $(\mathbb{R}, u)$ is the asymmetric normed linear space given by $u(x) = x^+ = x \vee 0$.

Note that $f \in X^*$ if and only if f is is a linear and upper semicontinuous function from (X, q) into $(\mathbb{R}, |\cdot|)$.

The set X^* is not necessarily a linear space, but it is a convex cone of X^{s*}. In fact, in Corollary 3 of [6] the authors prove that X^* is a linear space if and only if (X, q) is isomorphic to its associated normed space.

X^* is called the *dual cone or dual space* of (X, q).

The function

$$q^*(f) = sup\{f(x)^+ : q(x) \leq 1\} = sup\{f(x) : q(x) \leq 1\}$$

define an asymmetric norm on the cone X^*. Note that if $q^*(f) = 0$ then $f = 0$.

If $f \in X^{s*}$, then $q^*(f)$ can be infinite, that is, q^* is an extended asymmetric norm on X^{s*}.

More information about the dual space of an asymmetric normed space can be found in [1], [8] and [12].

An important class of asymmetric normed spaces is one whose elements are induced by normed lattices. Let us recall that a *normed lattice* $(X, \|\cdot\|, \leq)$ is a real linear lattice $(X, \leq)$ equipped with a norm that satisfies that if $|x| \leq |y|$ then $\|x\| \leq \|y\|$ for all $x, y \in X$, where $|x| = x \vee (-x)$. The set $\{x \in X : x \geq 0\}$ is called the *positive cone* of X.

Our basic references for real normed lattices are [4] and [19].

It was shown in [11] that for any normed lattice $(X, \|\cdot\|, \leq)$, the function q defined on X by $q(x) = \|x^+\|$, with $x^+ = x \vee 0$, is an asymmetric norm on X and the norm q^s is equivalent to the norm $\|\cdot\|$. In this case the asymmetric norm q is called an *asymmetric lattice norm* and the pair (X, q) is an *asymmetric normed lattice.* Note that in this case the topology on X induced by d_q is not T_1 because if $x > y$ then $d_q(x, y) = 0$.

The class of asymmetric normed lattices is of the most interesting ones from the point of view of the applications. Indeed, these spaces have proved to be useful in computer science, mainly in the analysis of complexity of algorithms (see, e.g. [13]). More recently, asymmetric normed lattices have been used in the mathematical development of specific tools for visualization of multi-objective optimization problems ([7]).

2 Quasi-metric properties of X^*

Considering the properties of q^*, the following result is immediate.

Proposition 1. *Let (X, q) an asymmetric normed space. The function d_{q^*} defined on $X^* \times X^*$ by*

$$d_{q^*}(f, g) = \begin{cases} q^*(g - f), & \text{if } g - f \in X^* \\ +\infty, & \text{if } g - f \notin X^* \end{cases}$$

is an extended T_1 quasi-metric on X^*.

It is interesting to note that although $(q^s)^*(f) \leq q^*(f)$, for all $f \in X^{s*}$ (see Corollary 1 of [1]), the topological space $(X^*, \tau_{d_{q^*}})$ is not necessarily metrizable. The following example shows this fact.

Example 1. *Let $X = \mathbb{R}$ and $q(x) = x^+$, then $f \in X^*$ if and only if $f(x) = ax$ with $a \in \mathbb{R}^+$. Moreover $q^*(f) = a$. If $f(x) = ax$ and $g(x) = bx$ with $a, b \in \mathbb{R}^+$, then*

$$d_{q^*}(f, g) = \begin{cases} b - a, & \text{if } b \geq a \\ +\infty, & \text{if } b < a \end{cases}$$

Therefore, $\tau_{d_{q^*}}$ is the Sorgenfrey topology which is not metrizable.

In [10] Doitchinov introduced the notion of balanced quasi-metric space with the aim to give an appropriate theory of completion and proved that every balanced quasi-metric space is Hausdorff and completely regular. In [18] the authors proved that if (X, q) is a T_1 asymmetric normed space, then (X^*, d_{q^*}) is an extended balanced quasi-metric space. We shall now prove that this result remains true even if (X, q) is not T_1.

Recall that an extended quasi-metric space (X, d) is *balanced* if for each pair of sequences $(x_n)_n$, $(y_n)_n$ in X such that $\lim_{n,m\to\infty} d(y_m, x_n) = 0$, and for each $x, y \in X$ and $r, s > 0$ satisfying $d(x, x_n) \leq r$ and $d(y_n, y) \leq s$ for all $n \in \mathbb{N}$, it follows that $d(x, y) \leq r + s$.

The following result is crucial in our study.

Lemma 1. *(Proposition 4 of [12] and Corollary 1 of [1]) Let (X, q) be an asymmetric normed space. Then,*

(a) $X^* = \{f \in X^{s*} : q^*(f) < +\infty\}$

(b) $(q^s)^*(f) \leq q^*(f)$ *for all* $f \in X^{s*}$

(c) $f \in X^*$ *if and only if there exists* $M > 0$ *such that* $f(x) \leq Mq(x)$ *for all* $x \in X$

(d) $f(x) \leq q^*(f)q(x)$ *for all* $f \in X^*$ *and for all* $x \in X$.

Theorem 1. *Let (X, q) an asymmetric normed space. Then (X^*, d_{q^*}) is a balanced extended T_1 quasi-metric space.*

Proof. Let $f, g \in X^*$, $(f_n)_n$ and $(g_n)_n$ be two sequences in X^* and $r, s > 0$ such that

$$\lim_{n,m\to\infty} d_{q^*}(g_m, f_n) = 0, \quad d_{q^*}(f, f_n) \leq r \quad \textit{and} \quad d_{q^*}(g_m, g) \leq s,$$

for all $n, m \in \mathbb{N}$. Then

$$\lim_{n,m\to\infty} q^*(f_n - g_m) = 0, \quad q^*(f_n - f) \leq r \quad and \quad q^*(g - g_m) \leq s,$$

for all $n, m \in \mathbb{N}$. Thus, $f_n - f \in X^*$, $g - g_m \in X^*$, for all $n, m \in \mathbb{N}$ and given $\varepsilon > 0$ there exits $n_0 \in \mathbb{N}$ such that $q^*(f_n - g_m) < \varepsilon$ for all $n, m \geq n_0$, consequently, $f_n - g_m \in X^*$ for all $n, m \geq n_0$.

Let $n, m \geq n_0$. Then

$$g_m(x) - f_n(x) = f_n(-x) - g_m(-x) \leq q^*(f_n - g_m)q(-x) < \varepsilon q(-x)$$

Therefore,

$$g(x) - f(x) = g(x) - g_m(x) + g_m(x) - f_n(x) + f_n(x) - f(x) \leq$$

$$q^*(g - g_m)q(x) + q^*(f_n - g_m)q(-x) + q^*(f_n - f)q(x) <$$

$$\varepsilon q(-x) + (r + s)q(x).$$

Then, $g(x) - f(x) \leq (r + s)q(x)$, for all $x \in X$ and so $q^*(g - f) \leq r + s$, i.e., $d_{q^*}(f, g) \leq r + s$.

□

Corollary 1. *Let (X, q) an asymmetric normed space. Then (X^*, d_{q^*}) is Hausdorff and completely regular.*

There exist several different notions of Cauchy sequence and quasi-metric completeness in the literature (see e.g.[8]). Here we will consider the following ones.

A (n extended) quasi-metric space (X, d) is called *bicomplete* if (X, d^s) is a complete (extended) metric space.

A sequence $(x_n)_n$ in a (n extended) quasi-metric (X, d) is said to be *right (left) $K-$Cauchy* if for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \leq \varepsilon$ whenever $n_0 \leq m \leq n$ ($n_0 \leq n \leq m$).

A (n extended) quasi-metric space (X, d) is *right (left) $K-$sequentially complete* if every right (left) K-Cauchy sequence $(x_n)_n$ in (X, d) converges with respect to the topology τ_d, i.e., there exists $z \in X$ such that $d(z, x_n) \to 0$.

As we have already mentioned, Doitchinov developed in [10] a satisfactory theory of completion for quasi-metric spaces. To this end he introduced a notion of Cauchy sequence, called D-Cauchy sequence in modern terminology.

A sequence $(x_n)_{n\in\mathbb{N}}$ in a (n extended) quasi-metric (X,d) is said to be *D-Cauchy* if there exists a sequence $(y_n)_n$ in X such that $\lim_{n,m\to\infty} d(y_m,x_n)=0$.

A (n extended) quasi-metric space (X,d) is called *D-complete* if every D-Cauchy sequence $(x_n)_n$ in (X,d) converges with respect to the topology τ_d, i.e., there exists $z\in X$ such that $d(z,x_n)\to 0$.

The following result is directly obtained as a corollary of Theorem 1 of [12].

Theorem 2. *Let (X,q) an asymmetric normed space. Then (X^*,d_{q^*}) is bicomplete.*

Theorem 3. *Let (X,q) an asymmetric normed space. Then (X^*,d_{q^*}) is D-complete.*

Proof. Let $(f_n)_n$ be a D-Cauchy sequence in (X^*,d_{q^*}). Then, there exists a sequence $(g_n)_n$ in X^* such that

$$0=\lim_{n,m\to\infty} d_{q^*}(g_m,f_n)=\lim_{n,m\to\infty} q^*(f_n-g_m).$$

Therefore, given $\varepsilon>0$ there exits n_0 such that $q^*(f_n-g_m)<\varepsilon$, for all $n,m\geq n_0$. Thus, since

$$(q^s)^*(g_m-f_n)=(q^s)^*(f_n-g_m)\leq q^*(f_n-g_m)<\varepsilon,$$

we have that

$$(q^s)^*(f_n-f_m)\leq (q^s)^*(f_n-g_m)+q^{s*}(g_m-f_m)<2\varepsilon.$$

Hence, $(f_n)_n$ is a Cauchy sequence in the Banach space (X^{s*},q^{s*}). Then, there exists $f\in X^{s*}$ such that $\lim_{n\to\infty} q^{s*}(f_n-f)=0$. Moreover,

$$(q^s)^*(g_n-f)\leq (q^s)^*(g_n-f_m)+q^{s*}(f_m-f),$$

therefore $\lim_{n\to\infty} q^{s*}(g_n-f)=0$.

Now we show that $\lim_{n\to\infty} q^*(f_n-f)=0$. Let $\varepsilon>0$, then $f_n(x)-g_m(x)<\varepsilon$, for all $n,m\geq n_0$ and for all $x\in X$ such that $q(x)\leq 1$. Since $\lim_{m\to\infty} g_m(x)=f(x)$, we have that , $f_n(x)<f(x)+\varepsilon$ for all $n\geq n_0$ and for all $x\in X$ such that $q(x)\leq 1$. Consequently, $q^*(f_n-f)<\varepsilon$.

Finally, we prove that $f \in X^*$. Since $f_n(x) - g_m(x) < \varepsilon$, for all $n, m \geq n_0$ and for all $x \in X$ such that $q(x) \leq 1$, and $\lim_{n\to\infty} f_n(x) = f(x)$, we have that $f(x) - \varepsilon < g_m(x)$, for all $m \geq n_0$. Hence, $q^*(f - g_m) < \varepsilon$, and then $f - g_m \in X^*$ for all $m \geq n_0$. Therefore, since $f = (f - g_m) + g_m$ and X^* is a cone, it follows that $f \in X^*$.

□

The following is an example of an asymmetric normed space (X, q) such that (X^*, d_{q^*}) is not right $K-$ sequentially complete.

Example 2. *In $\mathbb{R}^2$ we consider the asymmetric norm q given by*

$$q(x, y) = \frac{-y + \sqrt{4x^2 + y^2}}{2}.$$

Next, we show that $(\mathbb{R}^2)^* = \{(a, b) \in \mathbb{R}^2 : b < 0\} \cup \{(0, 0)\}$.

In [2] it is proved that $(\mathbb{R}^2)^* \subseteq \{(a, b) \in \mathbb{R}^2 : b < 0\} \cup \{(0, 0)\}$. Let us now prove the other inclusion.

Let $(a, b) \in \mathbb{R}^2, b < 0$ and let $f(x, y) = ax + by$. If $q(x, y) \leq 1$ then $y \geq x^2 - 1$ so that

$$f(x, y) = ax + |b|(-y) \leq ax + |b|(-x^2 + 1) = -|b|x^2 + ax + |b|.$$

As the parabola $y = -|b|x^2 + ax + |b|$ has its maximum at $x = \frac{a}{2|b|}$ we have that

$$f(x, y) \leq \frac{a^2 + 4|b|^2}{4|b|},$$

Therefore, $f \in X^*$ and $q^*(f) = q^*(a, b) \leq \frac{a^2+4|b|^2}{4|b|}$.

In $(\mathbb{R}^2)^*$ we consider the sequence $\{(1, -1/n)\}$. This sequence is right-K-Cauchy because if $n \geq m$ then

$$q^*((1, -\frac{1}{m}) - (1, -\frac{1}{n})) = q^*(0, \frac{-n + m}{nm}) \leq \frac{n - m}{nm}.$$

Furthermore, since $q^*((1, -1/n) - (1, 0)) \to 0$, the sequence $\{(1, -1/n)\}$ converges to $(1, 0) \notin X^*$. Therefore (X^*, d_{q^*}) is not right-K- sequentially complete.

Theorem 4. *Let (X, q) an asymmetric normed space such that X^* is closed in $(X^{s*}, (q^s)^*)$. Then (X^*, d_{q^*}) is right $K-$ sequentially complete.*

Proof. Let $(f_n)_n$ be a right-K-Cauchy sequence in (X^*, d_{q^*}). Then, given $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $q^*(f_m - f_n) < \varepsilon$, for all $n \geq m \geq n_0$. Since $(q^s)^*(f_m - f_n) \leq q^*(f_m - f_n) < \varepsilon$, we have that $(f_n)_n$ is a Cauchy sequence in the Banach space $(X^{s*}, (q^s)^*)$, then there exists $f \in X^{s*}$ such that $\lim\limits_{n\to\infty} (q^s)^*(f_n - f) = 0$. In addition, $f \in X^*$ because X^* is closed in $(X^{s*}, (q^s)^*)$.

Finally, we prove that $\lim\limits_{m\to\infty} q^*(f_m - f) = 0$. Let $\varepsilon > 0$, and let $x \in X$ such that $q(x) \leq 1$. Then there exists $n_0 \in \mathbb{N}$ such that $f_m(x) - f_{m+k}(x) < \varepsilon$, for all $m \geq n_0$ and for all $k \geq 0$. Since $\lim\limits_{k\to\infty} f_{m+k}(x) = f(x)$, we have that $f_m(x) < f(x) + \varepsilon$ for all $m \geq n_0$ and for all $x \in X$ such that $q(x) \leq 1$. Therefore, $q^*(f_m - f) < \varepsilon$.

□

Corollary 2. *Let $(X, \|\cdot\|, \leq)$ a normed lattice and let $q(x) = \|x^+\|$. Then (X^*, d_{q^*}) is right $K-$ sequentially complete.*

Proof. It is well known that the topological dual of a normed lattice is a Banach normed lattice under its dual norm ($\|f\|^* = sup\{|f(x)| : \|x\| \leq 1\}$) and the induced order ($f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \geq 0$). Since q^s is equivalent to the norm $\|\cdot\|$, the topological dual of $(X, \|\cdot\|)$ is X^{s*} and $(q^s)^*$ is equivalent to $\|\cdot\|^*$.

In Corollary 3 of [1] it is proven that X^* is the positive cone of the lattice normed X^{s*}. Since the positive cone of a normed lattice is closed (5.2 Proposition of [19]), we have that X^* is closed in $(X^{s*}, (q^s)^*)$. Therefore, (X^*, d_{q^*}) is right $K-$ sequentially complete. □

3 The unit ball of X^*

Let (X, q) be an asymmetric normed space and let $B_{q^*} = B_{d_{q*}}(0, 1)$, i.e.,

$$B_{q^*} = \{f \in X^* : d_{q^*}(0, f) \leq 1\} = \{f \in X^* : q^*(f) \leq 1\}$$

The set B_{q^*} is called the unit ball of X^*.

Let $B_{(q^s)^*}$ the unit ball of the normed space $(X^{s*}, (q^s)^*)$, i.e.,

$$B_{(q^s)^*} = \{f \in X^{s*} : (q^s)^*(f) \leq 1\}$$

Since $(q^s)^*(f) \leq q^*(f)$ for all $f \in X^{s*}$, we have that $B_{q^*} \subset B_{(q^s)^*} \cap X^*$. The following example illustrates that this inclusion may be strict.

Example 3. *Let* $X = \{(x_n) \in l_1 : 2x_1 + x_2 = 0\}$ *and let* $q((x_n)) = \sum_{n=1}^{\infty} x_n^+$. *q is an asymmetric norm on X since q is the restriction to X of the asymmetric lattice norm of* l_1.

Let $f : X \to \mathbb{R}$ *given by* $f((x_n)) = \frac{3x_1}{2}$. *Since*

$$f((x_n)) = \frac{3x_1}{2} \leq \frac{3x_1^+}{2} \leq \frac{3}{2} q((x_n)),$$

we have that $f \in X^*$ *and* $q^*(f) = \frac{3}{2}$. *Therefore,* $f \notin B_{q^*}$.

Now, if $(x_n) \in X$ *and* $q^s((x_n)) \leq 1$, *then* $x_1^+ + (-2x_1)^+ \leq 1$ *and* $(-x_1)^+ + (2x_1)^+ \leq 1$. *Therefore,* $3(x_1^+ + (-x_1)^+) \leq 2$ *and so* $\mid x_1 \mid \leq \frac{2}{3}$. *So that* $(q^s)^*(f) \leq 1$ *and then* $f \in B_{(q^s)^*} \cap X^*$.

Proposition 2. *Let* $(X, \|\cdot\|, \leq)$ *be a normed lattice and let* $q(x) = \|x^+\|$ *for all* $x \in X$. *Then* $B_{q^*} = B_{(q^s)^*} \cap X^*$.

Proof. Suppose $f \in B_{(q^s)^*} \cap X^*$. Since X^* is the positive cone of the lattice normed X^{s*} (Corollary 3 of [1]), we have that

$$f(x) \leq f(x^+) \leq (q^s)^*(f)\|x^+\| = (q^s)^*(f)q(x).$$

Hence, $q^*(f) \leq (q^s)^*(f) \leq 1$. □

We conclude this section representing an arbitrary asymmetric normed space as a function space. To this end, we recall that if K is a compact subset of a topological space, then $(C(K), \|f\|_\infty)$ is a Banach lattice, where $C(K) = \{f : K \to \mathbb{R} : f \text{ is continuos}\}$ and $\|f\|_\infty = sup\{|f(x)| : x \in K\}$. If we consider the asymmetric lattice norm $q_\infty(f) = \|f^+\|_\infty$, since q_∞^s is equivalent to $\|\cdot\|_\infty$, it follows that $(C(K), q_\infty)$ is a biBanach space.

Theorem 5. *Let* (X, q) *be an asymmetric normed linear space.* (X, q) *is isometric to a linear subspace of the biBanach space* $(C(B_{q^*}), q_\infty)$.

Proof. By Theorem 4 of [12], B_{q^*} is compact in $(X^*, w(X^{s*}, X)_{|X^*})$, being $w(X^{s*}, X)$ the weak topology on X^{s*} induced by X. Then $(C(B_{q^*}), q_\infty)$ is a biBanach space. Let $i : X \to C(B_{q^*})$ given by $i(x)(f) = f(x)$, with $x \in X$ and $f \in B_{q^*}$. It is obvious that $i(x)$ is $w(X^{s*}, X)$-continuous for all $x \in X$. Furthermore, by Theorem 2 of [12], we have that

$$q_\infty(i(x)) = sup\{(i(x)(f))^+ : f \in B_{q^*}\} = sup\{f(x)^+ : f \in B_{q^*}\} = q(x).$$

□

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