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On Principal Fuzzy Metric Spaces

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Abstract: In this paper, we deal with the notion of fuzzy metric space $(\mathcal{X}, \mathcal{M}, *)$, or simply $\mathcal{X}$, due to George and Veeramani. It is well known that such fuzzy metric spaces, in general, are not completable and also that there exist p-Cauchy sequences which are not Cauchy. We prove that if every p-Cauchy sequence in $\mathcal{X}$ is Cauchy, then $\mathcal{X}$ is principal, and we observe that the converse is false, in general. Hence, we introduce and study a stronger concept than principal, called strongly principal. Moreover, $\mathcal{X}$ is called weak p-complete if every p-Cauchy sequence is p-convergent. We prove that if $\mathcal{X}$ is strongly principal (or weak p-complete principal), then the family of p-Cauchy sequences agrees with the family of Cauchy sequences. Among other results related to completeness, we prove that every strongly principal fuzzy metric space where $\mathcal{M}$ is strong with respect to an integral (positive) t-norm $*$ admits completion.

Keywords: fuzzy metric; Cauchy sequence; principal fuzzy metric; p-Cauchy sequence; completeness; completion

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1. Introduction

In this paper, we deal with the concept of fuzzy metric space $(\mathcal{X}, \mathcal{M}, *)$ due to George and Veeramani [1,2]. These types of fuzzy metric spaces are close to probabilistic metric spaces (PM-spaces) [3] and fuzzy metric spaces in the sense of Kramosil and Michalek [4]. In the same way as the mentioned spaces, a topology $\tau_{\mathcal{M}}$, deduced from $\mathcal{M}$, is defined on $\mathcal{X}$. The family of open balls $\{B(x,r,t) : r \in]0,1[, t \in \mathbb{R}^+\}$ is a base for $\tau_{\mathcal{M}}$. In [5,6], the authors proved that $\tau_{\mathcal{M}}$ is metrizable, and so many metric concepts were extended to the fuzzy context, some of them inherited from PM-spaces. Now, a significant difference between fuzzy metric spaces and PM-spaces (or classical metric spaces) is that, in general, fuzzy metric spaces do not admit completion [7,8]. For this reason, an interesting problem in this fuzzy context is to find large classes of completable fuzzy metric spaces.

Several concepts of convergent sequences have been introduced in our fuzzy context (see [9–11] and references therein). In particular, a weaker concept than convergence called p-convergence (Definition 2) was introduced by D. Mihet in [12] devoted to fixed point theory, which is currently a topic of high activity in this context (see, for instance, [13–19]).

In [20], the authors called *principal* those fuzzy metric spaces in which every p-convergent sequence is convergent and gave the following characterization: $\mathcal{X}$ is *principal* if and only if for each $x \in \mathcal{X}$ the family $\{B(x,r,t) : r \in]0,1[\}$ is a local base at x for $\tau_{\mathcal{M}}$, for each $t \in \mathbb{R}^+$. In addition, according to the concept of p-convergence, the authors introduced the concepts of a p-Cauchy sequence (Definition 7) and p-complete fuzzy metric space which here will be called weak p-complete (Definition 10). We notice that in a principal fuzzy metric space one can find p-Cauchy sequences that are not Cauchy.

In this paper, we approach two questions: the first one is related to those fuzzy metric spaces in which every p-Cauchy sequence is Cauchy; the second one is to provide a wide family of fuzzy metric spaces that admit completion. Both approaches are carried out by introducing a stronger concept than the principal fuzzy metric, named the strongly principal fuzzy metric. This concept arises from the observation of the characterization of principal fuzzy metrics, above shown, but applied to a particular base of the uniformity $\mathcal{U}_{\mathcal{M}}$ on $\mathcal{X}$, compatible with $\tau_{\mathcal{M}}$, deduced from $\mathcal{M}$, instead of the family of open balls (Definition 9). The class of strongly principal fuzzy metric spaces is a wide family that contains stationary fuzzy metrics (Definition 3) and a well-known fuzzy metric as the standard one.

Concerning the first question, we prove that if every p-Cauchy sequence in $\mathcal{X}$ is Cauchy then $\mathcal{X}$ is principal (Theorem 1). In addition, we prove that if $\mathcal{X}$ is strongly principal then every p-Cauchy sequence is Cauchy (Proposition 3) and we ignore if the converse of this proposition is true (Problem 1). On the other hand, if $\mathcal{X}$ is principal and weakly p-complete then the family of p-Cauchy sequences in $\mathcal{X}$ agrees with the family of Cauchy sequences in $\mathcal{X}$ (Proposition 5). Then, there arises a question related to Problem 1: Does a weak p-complete principal fuzzy metric space which is not strongly principal exist (Problem 2)?

With respect to the second question, using Proposition 6, we prove that every strongly principal fuzzy metric space $(\mathcal{X}, \mathcal{M}, *)$, where $\mathcal{M}$ is strong for an integral (positive) t-norm $*$, is completable (Corollary 3). Several examples throughout the paper illustrate the theory.

The structure of the paper is as follows. After preliminaries (Section 2), we introduce and study, in Section 3, the concept of a strongly principal fuzzy metric. Section 4 is devoted to completeness and weak p-completeness in (principal) fuzzy metric spaces and the completion of strongly principal fuzzy metric spaces.

2. Preliminaries

In the following, $\mathbb{R}^+$ will denote the set of positive real numbers, i.e., $\mathbb{R}^+$ is the interval $]0, \infty[$.

Definition 1 (Ref. [1]). *A fuzzy metric space is an ordered triple $(\mathcal{X}, \mathcal{M}, *)$ such that $\mathcal{X}$ is a (non-empty) set, $*$ is a continuous t-norm and $\mathcal{M}$ is a fuzzy set on $\mathcal{X} \times \mathcal{X} \times \mathbb{R}^+$ satisfying the following conditions, for all $x, y, z \in \mathcal{X}$ and $t, s \in \mathbb{R}^+$:*

(GV1) $\mathcal{M}(x, y, t) > 0$;
(GV2) $\mathcal{M}(x, y, t) = 1$ *if and only if* $x = y$;
(GV3) $\mathcal{M}(x, y, t) = \mathcal{M}(y, x, t)$;
(GV4) $\mathcal{M}(x, y, t) * \mathcal{M}(y, z, s) \leq \mathcal{M}(x, z, t + s)$;
(GV5) *The function* $\mathcal{M}_{xy} : \mathbb{R}^+ \to]0, 1]$ *is continuous, where* $\mathcal{M}_{xy}(t) = \mathcal{M}(x, y, t)$ *for each* $t \in \mathbb{R}^+$.

If $(\mathcal{X}, \mathcal{M}, *)$ is a fuzzy metric space, we say that $(\mathcal{M}, *)$, or simply $\mathcal{M}$, is a fuzzy metric on $\mathcal{X}$. In addition, we say that $(\mathcal{X}, \mathcal{M})$, or simply $\mathcal{X}$, is a fuzzy metric space.

Let $(\mathcal{X}, d)$ be a metric space. Denote by $\cdot$ the usual product on $[0, 1]$, and let $\mathcal{M}_d$ be the fuzzy set defined on $\mathcal{X} \times \mathcal{X} \times \mathbb{R}^+$ by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}. \tag{1}$$

Then, $(\mathcal{M}_d, \cdot)$ is a fuzzy metric on $\mathcal{X}$ called the *standard fuzzy metric* induced by d [1].

George and Veeramani proved in [1] that every fuzzy metric $\mathcal{M}$ on $\mathcal{X}$ generates a topology $\tau_{\mathcal{M}}$ on $\mathcal{X}$ which has as a base the family of open sets of the form $\{\mathcal{B}_{\mathcal{M}}(x, r, t) : x \in \mathcal{X}, r \in]0, 1[, t \in \mathbb{R}^+\}$, where $\mathcal{B}_{\mathcal{M}}(x, r, t) = \{y \in \mathcal{X} : \mathcal{M}(x, y, t) > 1 - r\}$ for all $x \in \mathcal{X}$, $r \in]0, 1[$ and $t \in \mathbb{R}^+$. In the case of the standard fuzzy metric $\mathcal{M}_d$, it is well known that the topology $\tau(d)$ on $\mathcal{X}$ deduced from d satisfies $\tau(d) = \tau_{\mathcal{M}_d}$. If confusion is not possible, we write $\mathcal{B}$ instead of $\mathcal{B}_{\mathcal{M}}$. From now on, we suppose $\mathcal{X}$ is endowed with the topology $\tau_{\mathcal{M}}$.

A t-norm $*$ is called integral (positive) if $x * y > 0$ for all $x, y \in]0,1]$.

Definition 2 (Ref. [21]). *A fuzzy metric space $(\mathcal{X}, \mathcal{M}, *)$ is said to be strong (non-Archimedean) if for all $x, y, z \in \mathcal{X}$ and all $t \in \mathbb{R}^+$ it satisfies*

$$M(x,z,t) \geq \mathcal{M}(x,y,t) * \mathcal{M}(y,z,t). \quad (2)$$

Definition 3 (Ref. [8]). *A fuzzy metric $\mathcal{M}$ on $\mathcal{X}$ is said to be stationary if $\mathcal{M}$ does not depend on t, i.e., if for each $x, y \in \mathcal{X}$ the function $\mathcal{M}_{xy}$ defined in axiom (GV5) is constant, for all $x, y \in \mathcal{X}$.*

Proposition 1 (Ref. [1]). *A sequence $\{x_n\}$ in a fuzzy metric space $(\mathcal{X}, \mathcal{M}, *)$ converges to x_0 if and only if $\lim_n \mathcal{M}(x_0, x_n, t) = 1$, for all $t \in \mathbb{R}^+$.*

Definition 4 (Ref. [20]). *A fuzzy metric space $(\mathcal{X}, \mathcal{M}, *)$ is said to be* principal *(or simply, $\mathcal{M}$ is principal) if $\{\mathcal{B}(x,r,t) : r \in]0,1[\}$ is a local base at $x \in X$, for each $x \in X$ and each $t \in \mathbb{R}^+$.*

If confusion is not possible, we also say, simply, that $\mathcal{X}$, or $\mathcal{M}$, is principal. This terminology is applied, without mention, as usual, to other concepts.

Definition 5 (Ref. [2]). *A sequence $\{x_n\}$ in a fuzzy metric space $(\mathcal{X}, \mathcal{M}, *)$ is called Cauchy if for each $\varepsilon \in]0,1[$ and each $t \in \mathbb{R}^+$ there exists $n_0 \in \mathbb{N}$ such that $\mathcal{M}(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$, or equivalently, $\lim_{m,n} \mathcal{M}(x_n, x_m, t) = 1$ for all $t \in \mathbb{R}^+$.*

$(\mathcal{X}, \mathcal{M}, *)$ is called complete if every Cauchy sequence in $\mathcal{X}$ is convergent with respect to $\tau_{\mathcal{M}}$.

Definition 6 (Ref. [12]). *A sequence $\{x_n\}$ in a fuzzy metric space $(\mathcal{X}, \mathcal{M}, *)$ is called p-convergent to x_0, or simply p-convergent, if $\lim_n \mathcal{M}(x_n, x_0, t_0) = 1$ for some $t_0 \in \mathbb{R}^+$.*

Definition 7. *A sequence $\{x_n\}$ in a fuzzy metric space $(\mathcal{X}, \mathcal{M}, *)$ is called p-Cauchy if there exists $t_0 \in \mathbb{R}^+$ such that for each $\epsilon \in]0,1[$ there exists $n_0 \in \mathbb{N}$ such that $\mathcal{M}(x_n, x_m, t_0) > 1 - \epsilon$ for all $n, m \geq n_0$, or equivalently, $\lim_{n,m} \mathcal{M}(x_n, x_m, t_0) = 1$ for some $t_0 \in \mathbb{R}^+$.*

Observe that every p-convergent sequence is p-Cauchy.

Definition 8 (Ref. [7]). *Let $(\mathcal{X}, \mathcal{M}, *)$ and $(\mathcal{Y}, \mathcal{N}, \diamond)$ be two fuzzy metric spaces. A mapping f from $\mathcal{X}$ to $\mathcal{Y}$ is called an isometry if, for each $x, y \in \mathcal{X}$ and each $t \in \mathbb{R}^+$, $\mathcal{M}(x,y,t) = \mathcal{N}(f(x), f(y), t)$ and, in this case, if f is a bijection, $\mathcal{X}$ and $\mathcal{Y}$ are called isometric. A fuzzy metric completion of $(\mathcal{X}, \mathcal{M})$ is a complete fuzzy metric space $(\mathcal{X}^*, \mathcal{M}^*)$ such that $(\mathcal{X}, \mathcal{M})$ is isometric to a dense subspace of $\mathcal{X}^*$. $\mathcal{X}$ is said to be completable if it admits a fuzzy metric completion.*

3. Strongly Principal Fuzzy Metric Spaces

Let $(\mathcal{X}, \mathcal{M}, *)$ be a fuzzy metric space. Similarly to the case of metric spaces, the family of bands $\{U_{r,t} : r \in]0,1[, t \in \mathbb{R}^+\}$ is a natural base for a uniformity $\mathcal{U}_M$ on $\mathcal{X}$, induced by $\mathcal{M}$, which is compatible with $\tau_{\mathcal{M}}$ [6], where $U_{r,t} = \{(x,y) \in X \times X : M(x,y,t) > 1 - r)\}$.

Now, according to the concept of principal fuzzy metric spaces ([20], Definition 9), we introduce the following concept.

Definition 9. *We say that the fuzzy metric space $(\mathcal{X}, \mathcal{M}, *)$ is strongly principal (s-principal, for short) if $\{U_{r,t} : r \in]0,1[\}$ is a base for $\mathcal{U}_{\mathcal{M}}$, for each $t \in \mathbb{R}^+$.*

Observe that if $\mathcal{X}$ is s-principal then, for each $t \in \mathbb{R}^+$, $\{U_{r,t}(x) : r \in]0,1[\}$ is a local base at x, for all $x \in \mathcal{X}$, where $U_{r,t}(x) = \{y \in X : M(x,y,t) > 1 - r\} = B(x,r,t)$. Then, we have the following proposition.

Proposition 2. *Every s-principal fuzzy metric space is principal.*

Next, we show some examples of s-principal fuzzy metric spaces.

Example 1.

(a) *Stationary fuzzy metric spaces are, obviously, s-principal.*

(b) *The standard fuzzy metric space is s-principal. Indeed, let d be a metric on $\mathcal{X}$ and consider $\mathcal{M}_d$. Fix $t_0 \in \mathbb{R}^+$ and let $\varepsilon \in]0,1[$ and $t \in \mathbb{R}^+$ be arbitrary. The basic band $U_{\varepsilon,t}$ of $\mathcal{U}_{\mathcal{M}_d}$ is $U_{\varepsilon,t} = \{(x,y) \in \mathcal{X}^2 : \frac{t}{t+d(x,y)} > 1-\varepsilon\}$. Choose $\delta \in]0,1[$ such that $\frac{t_0 \cdot \delta}{1-\delta} < \frac{t \cdot \varepsilon}{1-\varepsilon}$. We claim that $U_{\delta,t_0} \subset U_{\varepsilon,t}$. Indeed, let $(x,y) \in U_{\delta,t_0}$. We have that $\mathcal{M}_d(x,y,t_0) = \frac{t_0}{t_0+d(x,y)} > 1-\delta$, and hence $d(x,y) < \frac{\delta}{1-\delta} \cdot t_0$. Then, for $(x,y) \in U_{\delta,t_0}$ we have that $\mathcal{M}_d(x,y,t) = \frac{t}{t+d(x,y)} > \frac{t}{t+\frac{\delta}{1-\delta}\cdot t_0} > \frac{t}{t+\frac{t\cdot\varepsilon}{1-\varepsilon}} = 1-\varepsilon$, and thus $(x,y) \in U_{\varepsilon,t}$.*

(c) *It is well known [22] that $\mathcal{M}(x,y,t) = \frac{\min\{x,y\}+t}{\max\{x,y\}+t}$ is a fuzzy metric on $\mathcal{X} = \mathbb{R}^+$ for the product t-norm. We will see that $\mathcal{M}$ is s-principal.*
Fix $t_0 \in \mathbb{R}^+$, and let $\varepsilon \in]0,1[$ and $t \in \mathbb{R}^+$ be arbitrary. If $t_0 \le t$, it is obvious that $U_{\varepsilon,t_0} \subset U_{\varepsilon,t}$. Suppose $t_0 > t$ and choose $\delta = \frac{t\cdot\varepsilon}{t_0} < \varepsilon$. Let $(x,y) \in U_{\delta,t_0}$. Then, $\frac{\min\{x,y\}+t_0}{\max\{x,y\}+t_0} > 1 - \frac{t\cdot\varepsilon}{t_0}$. An easy computation shows that from the previous inequality we obtain the following one:

$$\min\{x,y\} + t > \max\{x,y\} + t - \varepsilon \cdot \left(\max\{x,y\} \cdot \frac{t}{t_0} + t\right).$$

Then, $\frac{\min\{x,y\}+t}{\max\{x,y\}+t} > 1 - \varepsilon \cdot \left(\frac{\max\{x,y\}\cdot\frac{t}{t_0}+t}{\max\{x,y\}+t}\right) > 1-\varepsilon$. Hence, $U_{\delta,t_0} \subset U_{\varepsilon,t}$.

The converse of Proposition 2 is false, as is shown in the following example.

Example 2. *(A principal non-s-principal fuzzy metric space)*

(a) *Let $\mathcal{X} =]0,1[$ and define the fuzzy set $\mathcal{M}$ on $\mathcal{X}^2 \times \mathbb{R}^+$ by*

$$\mathcal{M}(x,y,t) = \begin{cases} 1 & x = y \\ xyt & x \neq y, t \le 1 \\ xy & x \neq y, t > 1 \end{cases}. \tag{3}$$

In [20], Example 19, it is proven that $(\mathcal{M},\cdot)$ is a principal fuzzy metric on $\mathcal{X}$. We will see that $\mathcal{M}$ is not s-principal.
Notice that $\mathcal{U}_{\mathcal{M}}$ is the discrete uniformity, since $U_{\frac{1}{2},\frac{1}{2}}$ is the diagonal Δ of $\mathcal{X} \times \mathcal{X}$.
For $t = 1$, the family $\{U_{r,1} : r \in]0,1[\}$ is constituted by the bands $U_{r,1} = \{(x,y) \in \mathcal{X}^2 : xy > 1-r\} \cup \Delta$, where $r \in]0,1[$. Obviously, $U_{r,1} \neq \Delta$ for all $r \in]0,1[$, and then $U_{r,1} \not\subset U_{\frac{1}{2},\frac{1}{2}}$ for all $r \in]0,1[$, and so $\{U_{r,1} : r \in]0,1[\}$ is not a base for $\mathcal{U}_{\mathcal{M}}$, and hence $\mathcal{M}$ is not s-principal.

(b) *Let $\{x_n\}$ and $\{y_n\}$ be two strictly increasing sequences of positive real numbers, which converge to 1, with respect to the Euclidean metric, with $A \cap B = \emptyset$, where $A = \{x_n : n \in \mathbb{N}\}$ and $B = \{y_n : n \in \mathbb{N}\}$. Put $\mathcal{X} = A \cup B$ and define a fuzzy set $\mathcal{M}$ on $\mathcal{X}^2 \times \mathbb{R}^+$ as follows:*

$$\mathcal{M}(x,y,t) = \begin{cases} 1 & \text{if } x = y \\ x \wedge y \wedge t & \text{if } x \in A, y \in B \text{ or } x \in B, y \in A, \text{ and } t < 1 \\ x \wedge y & \text{elsewhere} \end{cases}. \tag{4}$$

Then, $(\mathcal{X},\mathcal{M},\wedge)$ is a strong fuzzy metric space ([21], Example 41).
Let $x \in \mathcal{X}$ and, without loss of generality, suppose $x = x_n$ for some $n \in \mathbb{N}$. Let $t \in \mathbb{R}^+$. For $y \in \mathcal{X}$ and $y \neq x$, we have that $\mathcal{M}(x,y,t) \le x$, and so if we choose $r \in]0,1[$ such that $x < 1-r$ we have $B(x,r,t) = \{x\}$, and thus $\tau_{\mathcal{M}}$ is the discrete topology on $\mathcal{X}$.

Clearly, $\{B(x,r,t) : r \in]0,1[\}$ is a local base at x, for all $x \in \mathcal{X}$ and $t \in \mathbb{R}^+$, and then $\mathcal{M}$ is principal. We will see that $\mathcal{M}$ is not s-principal.

Suppose that x_{n_0+1} is the least element of A satisfying $\frac{1}{2} < x_{n_0+1}$ and y_{n_1+1} is the least element of B satisfying $\frac{1}{2} < y_{n_1+1}$. We have that

$$\begin{aligned} U_{\frac{1}{2},\frac{1}{2}} = {} & (\{(x,x) : x \in \{x_1,\ldots,x_{n_0},y_1,\ldots,y_{n_1}\}) \bigcup (\{x_{n_0+1},x_{n_0+2},\ldots\} \times \{x_{n_0+1},x_{n_0+2},\ldots\}) \bigcup \\ & (\{y_{n_1+1},y_{n_1+2},\ldots\} \times \{y_{n_1+1},y_{n_1+2},\ldots\}) \end{aligned}$$

Now, let $r \in]0,1[$. Suppose that x_m and y_n are the least elements of A and B, respectively, such that $x_m > 1-r$ and $y_n > 1-r$. We have that

$$\begin{aligned} U_{r,1} = {} & (\{(x,x) : x \in \{x_1,\ldots,x_{m-1},y_1,\ldots,y_{n-1}\}) \bigcup \\ & (\{x_m,x_{m+1},\ldots,y_n,y_{n+1},\ldots\} \times \{x_m,x_{m+1},\ldots,y_n,y_{n+1},\ldots\}) \end{aligned}$$

and then $U_{r,1} \not\subset U_{\frac{1}{2},\frac{1}{2}}$ for all $r \in]0,1[$. Then, $\{U_{r,1} : r \in]0,1[\}$ is not a base for $\mathcal{U}_{\mathcal{M}}$ and hence $\mathcal{M}$ is not s-principal.

Proposition 3. *In an s-principal fuzzy metric space, every p-Cauchy sequence is Cauchy.*

Proof. Suppose $\mathcal{M}$ is an s-principal fuzzy metric on $\mathcal{X}$ and let $\{x_n\}$ be a p-Cauchy sequence in X. Let $\varepsilon \in]0,1[$, $t \in \mathbb{R}^+$ and suppose that $\lim_{m,n} \mathcal{M}(x_m,x_n,t_0) = 1$ for some $t_0 \in \mathbb{R}^+$. Now, since $\mathcal{M}$ is s-principal there exists $\delta \in]0,1[$ such that $U_{\delta,t_0} \subset U_{\varepsilon,t}$. On the other hand, since $\lim_{m,n} \mathcal{M}(x_m,x_n,t_0) = 1$, we can find $n_0 \in \mathbb{N}$ such that $\mathcal{M}(x_m,x_n,t_0) > 1-\delta$ for $m,n \geq n_0$, i.e., $(x_m,x_n) \in U_{\delta,t_0}$ for all $m,n \geq n_0$, and so $\{x_n\}$ is Cauchy. □

The following corollary is obvious.

Corollary 1. *In an s-principal fuzzy metric space, a sequence is Cauchy if and only if it is p-Cauchy.*

Theorem 1. *Let $(\mathcal{X},\mathcal{M},*)$ be a fuzzy metric space. If every p-Cauchy sequence in $\mathcal{X}$ is Cauchy, then $\mathcal{M}$ is principal.*

Proof. Suppose $\mathcal{M}$ is not principal. Then, there exist $x_0 \in \mathcal{X}$ and $t_0 \in \mathbb{R}^+$ such that $\{B(x_0,r,t_0) : r \in]0,1[\}$ is not a local base at x_0 for $\tau_{\mathcal{M}}$. Consequently, we can find $t \in \mathbb{R}^+$ and $\varepsilon \in]0,1[$ such that $B(x_0,r,t_0) \not\subset B(x_0,\varepsilon,t)$ for all $r \in]0,1[$.

We construct, by induction, a sequence $\{x_n\}$ in $\mathcal{X}$ such that $x_n \in B(x_0,\frac{1}{n},t_0) \setminus B(x_0,\varepsilon,t)$, for $n = 2,3,\ldots$, and consider the sequence $\{y_n\}$ defined by $y_{2n} = x_0$ and $y_{2n-1} = x_n$, for $n \geq 2$.

By construction, $\mathcal{M}(x_n,x_0,t_0) > 1-\frac{1}{n}$, so $\lim_n \mathcal{M}(x_n,x_0,t_0) = 1$ and, in consequence, $\lim_n \mathcal{M}(y_n,x_0,t_0) = 1$, and then $\lim_{m,n} \mathcal{M}(y_n,y_m,2t_0) \geq \lim_{m,n}(\mathcal{M}(y_n,x_0,t_0) * \mathcal{M}(x_0,y_m,t_0)) = 1$, i.e., $\{y_n\}$ is p-Cauchy.

Now, $\{y_n\}$ is not Cauchy since for the above values $\varepsilon \in]0,1[$ and $t \in \mathbb{R}^+$ we have that for each $n_0 \in \mathbb{N}$ we can find $2n-1 > n_0$ such that $\mathcal{M}(y_{2n-1},y_{2n},t) = \mathcal{M}(x_n,x_0,t) \leq 1-\varepsilon$, and hence $\{y_n\}$ is not Cauchy. □

The converse of the last theorem is false, as is shown in the following example.

Example 3. *(A p-Cauchy non-Cauchy sequence in a principal fuzzy metric space)*

Consider the principal fuzzy metric space $(\mathcal{X},\mathcal{M},*)$ of (b) in Example 2. We define the sequence $\{z_n\}$ in $\mathcal{X}$ as follows: $z_n = x_n$ if n is even and $z_n = y_n$ if n is odd. Then, $\{z_n\}$ is a p-Cauchy sequence since, for $t_0 = 1$, we have $\lim_{n,m} \mathcal{M}(z_n,z_m,t_0) = \lim_{n,m}(z_n \wedge z_m) = 1$ due to $\lim_n z_n = \lim_n x_n = \lim_n y_n = 1$. Now, $\{z_n\}$ is not Cauchy, since $\lim_{n,m} \mathcal{M}(z_n,z_m,t)$ does not exist *for all $t \in]0,1[$. Indeed, given $t \in]0,1[$, there exists n_0 such that $z_n \wedge z_m \wedge t = t$ for all $n, m \geq n_0$, due to $\lim_n z_n = 1$, and so $\lim_{n,m} \mathcal{M}(z_n, z_m, t) = t < 1$.*

Notation 1. *Denote by $\mathcal{P}$ and $s\mathcal{P}$ the families of principal and s-principal fuzzy metric spaces, respectively, and denote by $\mathcal{H}$ the family of fuzzy metric spaces in which all p-Cauchy sequences are Cauchy. By Proposition 3 and Theorem 1, we have the chain of inclusions*

$$s\mathcal{P} \subset \mathcal{H} \subset \mathcal{P}.$$

By Example 3, we know that the inclusion $\mathcal{H} \subset \mathcal{P}$ is strict but we do not know if the inclusion $s\mathcal{P} \subset \mathcal{H}$ is strict. Thus, the following is an open question.

Problem 1. *How can we find a non-s-principal fuzzy metric space in which all p-Cauchy sequences are Cauchy?*

4. Completeness and Weak p-Completenes (w-p-Completeness)

Definition 10. *A fuzzy metric space $(\mathcal{X}, \mathcal{M}, *)$ is called weak p-complete (w-p-complete, for short) if every p-Cauchy sequence in $\mathcal{X}$ is p-convergent in $\mathcal{X}$.*

Remark 1. *Notice that in [20] w-p-completeness is called p-completeness.*

There is not any relationship between completeness and w-p-completeness, as is shown in the following example.

Example 4.

(a) *The principal fuzzy metric space of (a) in Example 2 is complete and it is not w-p-complete ([20], Example 19).*

(b) *The non-principal fuzzy metric space of [20], Example 18, is w-p-complete and non-complete.*

(c) *(A non-principal complete w-p-complete fuzzy metric space) Let $\mathcal{X} = \mathbb{R}^+$ and let $\varphi : \mathbb{R}^+ \to]0,1]$ be a function given by $\varphi(t) = t$ if $t \leq 1$ and $\varphi(t) = 1$ elsewhere. Define the fuzzy set $\mathcal{M}$ on $\mathcal{X}^2 \times \mathbb{R}^+$ by*

$$\mathcal{M}(x,y,t) = \begin{cases} 1 & x = y \\ \frac{\min\{x,y\}}{\max\{x,y\}} \cdot \varphi(t) & x \neq y \end{cases}. \tag{5}$$

In [20], Example 13, it is proven that $(\mathcal{X}, \mathcal{M}, \cdot)$ is a non-principal fuzzy metric space. Now, the only Cauchy sequences are the constant sequences, and then it is complete.
We will see that $(\mathcal{X}, \mathcal{M}, \cdot)$ is w-p-complete. Suppose that $\{x_n\}$ is a p-Cauchy sequence in $\mathcal{X}$. Then, $\lim_{m,n} \mathcal{M}(x_m, x_n, 1) = 1$ and in such a case we have $\mathcal{M}(x_m, x_n, 1) = \frac{\min\{x_m, x_n\}}{\max\{x_m, x_n\}}$. In consequence, $\{x_n\}$ is a Cauchy sequence in the stationary fuzzy metric space $(\mathcal{X}, \mathcal{M}_0, \cdot)$ where $\mathcal{M}_0(x,y) = \frac{\min\{x,y\}}{\max\{x,y\}}$, and by [23], Theorem 16, $\{x_n\}$ is a convergent sequence in $\tau_{\mathcal{M}_0}$, and so there exists $x_0 \in \mathbb{R}^+$ such that $\lim_n \mathcal{M}_0(x_0, x_n) = 1$, i.e., $\lim_n \mathcal{M}(x_0, x_n, 1) = 1$, and hence $\{x_n\}$ is p-convergent.

(d) *(A non-principal non-complete non-w-p-complete fuzzy metric space)*
Let $\mathcal{X} =]0,1[$, $A = \mathcal{X} \cap \mathbb{Q}$ and $B = \mathcal{X} - A$. Define the fuzzy set $\mathcal{M}$ on $\mathcal{X}^2 \times \mathbb{R}^+$ by

$$\mathcal{M}(x,y,t) = \begin{cases} \frac{\min\{x,y\}+1}{\max\{x,y\}+1} \cdot t & \text{if } x \in A, y \in B \text{ or } x \in B, y \in A, \text{ and } t < 1 \\ \frac{\min\{x,y\}+1}{\max\{x,y\}+1} & \text{elsewhere} \end{cases}. \tag{6}$$

It is easy to verify that $\mathcal{M}$ is a fuzzy metric on $\mathcal{X}$ for the product t-norm (compare with [22], Example 18).

We describe the topology $\tau_{\mathcal{M}}$ on $\mathcal{X}$. Notice that if $x \in A$ then $\{y \in B : \mathcal{M}(x, y, \frac{1}{2}) > 1 - \frac{1}{2}\} = \emptyset$ and if $x \in B$ then $\{y \in A : \mathcal{M}(x, y, \frac{1}{2}) > 1 - \frac{1}{2}\} = \emptyset$. Now, it is easy to verify that the open balls of the local base at $x \in \mathcal{X}$, for $\tau_{\mathcal{M}}$, given by $\{B\left(x, \frac{1}{n}, \frac{1}{n}\right) : n > 2\}$, are

$$B(x, \frac{1}{n}, \frac{1}{n}) = \begin{cases}]\frac{n-1}{n} \cdot x - \frac{1}{n}, \frac{n-1}{n} \cdot x + \frac{1}{n}[\cap A & \text{if } x \in A \\ ]\frac{n-1}{n} \cdot x - \frac{1}{n}, \frac{n-1}{n} \cdot x + \frac{1}{n}[\cap B & \text{if } x \in B \end{cases}. \tag{7}$$

Thus, $\tau_{\mathcal{M}}$ is finer than the usual topology of $\mathbb{R}$, restricted to $\mathcal{X}$, but $\tau_{\mathcal{M}}$ is not the discrete topology.
Now, $B(1, r, 1) =]1 - 2r, 1]$ for each $r \in]0, 1[$. On the other hand, $B(1, \frac{1}{2}, \frac{1}{2}) = A$, and then $\{B(1, r, 1) : r \in]0, 1[\}$ is not a local base at $x = 1$, since $B(1, r, 1) \not\subset A$ for all $r \in]0, 1[$, and hence $\mathcal{M}$ is not principal.
Now, consider a strictly increasing sequence $\{b_n\}$, contained in B, converging to 1, in the usual topology of $\mathbb{R}$. It is easy to verify that $\{b_n\}$ is Cauchy. In addition, $\{b_n\}$ is p-convergent since $\lim_n \mathcal{M}(b_n, 1, 1) = 1$. Nevertheless, by [20], Corollary 6, $\{b_n\}$ does not converge in $\mathcal{X}$, due to $\lim_n \mathcal{M}(b_n, 1, \frac{1}{2}) = \frac{1}{2}$. Hence, $\mathcal{X}$ is not complete.
Consider now two strictly decreasing sequences $\{q_n\} \subset A$ and $\{i_n\} \subset B$ converging to 0, in the usual topology of $\mathbb{R}$, such that $q_{2n+2} < i_{2n+1} < q_{2n}$ for each $n \in \mathbb{N}$. Let $\{x_n\}$ be the sequence defined by

$$x_n = \begin{cases} q_n & \text{if } n = 2p \\ i_n & \text{if } n = 2p + 1 \end{cases}, p = 0, 1, 2, \ldots \tag{8}$$

We have that $\lim_{m,n} \mathcal{M}(x_m, x_n, 1) = 1$ and so $\{x_n\}$ is p-Cauchy. However, $\{x_n\}$ is not Cauchy, since $M\left(x_n, x_{x+1}, \frac{1}{2}\right) < \frac{1}{2}$ for each $n \in \mathbb{N}$. Now, if $q \in \mathcal{X}$, then for $t \geq 1$ we have that $\lim_n \mathcal{M}(x_n, q, t) = \lim_n \frac{x_{n+1}}{q+1} \leq \frac{1}{q+1} < 1$, and for $t < 1$ we have that $\lim_n \mathcal{M}(x_n, q, t)$ does not exist, and so $\{x_n\}$ is not p-convergent in $\mathcal{X}$, and then $\mathcal{X}$ is not w-p-complete.

In the case that $\mathcal{X}$ is principal, the situation is distinct since it is easy to prove the following proposition.

Proposition 4. *If $\mathcal{X}$ is principal and w-p-complete, then $\mathcal{X}$ is complete.*

We prove in the following proposition that the class of principal w-p-complete spaces is contained in $\mathcal{H}$.

Proposition 5. *In a principal w-p-complete fuzzy metric space, the family of Cauchy sequences agrees with the family of p-Cauchy sequences.*

Proof. Let $\{x_n\}$ be a p-Cauchy sequence in the principal fuzzy metric space $\mathcal{X}$. Since $\mathcal{X}$ is w-p-complete, then $\{x_n\}$ is p-convergent in $\mathcal{X}$, and thus $\{x_n\}$ is convergent in $\mathcal{X}$, since $\mathcal{X}$ is principal, and then $\{x_n\}$ is Cauchy.
Obviously, Cauchy sequences are p-Cauchy. □

The following is a natural question.

Problem 2. *Does a principal w-p-complete fuzzy metric space which is not strongly principal exist?*

If the answer to this problem is affirmative, then Problem 1 is also answered and $s\mathcal{P} \neq \mathcal{P}$.
In order to obtain a nice result on completion in fuzzy metric spaces, we need the following proposition.

Proposition 6. *Let* $(\mathcal{X}, \mathcal{M}, *)$ *be an s-principal fuzzy metric space and let* $\{a_n\}$ *and* $\{b_n\}$ *be two sequences in* $\mathcal{X}$. *If* $\lim_n \mathcal{M}(a_n, b_n, t_0) = 1$ *for some* $t_0 \in \mathbb{R}^+$, *then* $\lim_n \mathcal{M}(a_n, b_n, t) = 1$ *for all* $t \in \mathbb{R}^+$.

Proof. Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences satisfying $\lim_n \mathcal{M}(a_n, b_n, t_0) = 1$ for some $t_0 \in \mathbb{R}^+$. Let $t \in \mathbb{R}^+$. Since $\mathcal{M}$ is s-principal, for $\varepsilon \in]0,1[$ we can find $\delta \in]0,1[$ such that $U_{\delta,t_0} \subset U_{\varepsilon,t}$. Since $\lim_n \mathcal{M}(a_n, b_n, t_0) = 1$, then there exists $n_0 \in \mathbb{N}$ such that $\mathcal{M}(a_n, b_n, t_0) > 1 - \delta$ for all $n \geq n_0$, i.e., $(a_n, b_n) \in U_{\delta,t_0}$ for $n \geq n_0$, and hence $(a_n, b_n) \in U_{\varepsilon,t}$ for $n \geq n_0$, and so $\mathcal{M}(a_n, b_n, t) > 1 - \varepsilon$ for $n \geq n_0$, and then $\lim_n \mathcal{M}(a_n, b_n, t) = 1$, since ε is arbitrary. □

Theorem 2 (Ref. [8]). *A fuzzy metric space* $(\mathcal{X}, \mathcal{M}, *)$ *admits completion if and only if, for each pair of Cauchy sequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *in* $\mathcal{X}$, *the following conditions are satisfied:*

(c1) *The assignment* $t \to \lim_n \mathcal{M}(\alpha_n, \beta_n, t)$ *for each* $t \in \mathbb{R}^+$ *is a continuous function on* $\mathbb{R}^+$, *provided with the usual topology of* $\mathbb{R}$.
(c2) $\lim_n \mathcal{M}(\alpha_n, \beta_n, t_0) = 1$ *for some* $t_0 \in \mathbb{R}^+$ *implies* $\lim_n \mathcal{M}(\alpha_n, \beta_n, t) = 1$ *for all* $t \in \mathbb{R}^+$.
(c3) $\lim_n \mathcal{M}(\alpha_n, \beta_n, t) > 0$ *for all* $t \in \mathbb{R}^+$.

Attending to Proposition 6 and Theorem 2, we obtain the following theorem.

Theorem 3. *An s-principal fuzzy metric space* $(\mathcal{X}, \mathcal{M}, *)$ *is completable if and only if* $\mathcal{X}$ *satisfies (c1) and (c3).*

In Theorem 4.6 of [24], it is proven that condition (c1) is satisfied for strong fuzzy metrics, and thus we obtain the following corollary.

Corollary 2. *A strong s-principal fuzzy metric space* $(\mathcal{X}, \mathcal{M}, *)$ *is completable if and only if* $\mathcal{X}$ *satisfies condition (c3).*

On the other hand, in Theorem 35 of [21], the following result is proven.

Theorem 4. *Let* $(\mathcal{X}, \mathcal{M}, *)$ *be a strong fuzzy metric space and suppose that* $*$ *is integral (positive). If* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are a pair of Cauchy sequences in* $\mathcal{X}$ *and* $t \in \mathbb{R}^+$, *then* $\{\mathcal{M}(\alpha_n, \beta_n, t)\}_n$ *converges in* $]0,1]$.

Therefore, an immediate consequence is the following corollary.

Corollary 3. *If* $(\mathcal{X}, \mathcal{M}, *)$ *is a strong s-principal fuzzy metric space and* $*$ *is integral, then* $\mathcal{X}$ *is completable.*

We cannot replace s-principal by principal in the last corollary. Indeed, the principal fuzzy metric space of (b) Example 2 is strong for the t-norm minimum ([21], Example 41) and it is not completable ([8], Example 2).

The next example shows an application of Corollary 3.

Example 5. *Consider the fuzzy metric space* $(\mathcal{X}, \mathcal{M}, *)$ *where* $\mathcal{X} =]0,1]$, $\mathcal{M}(x, y, t) = \frac{\min\{x,y\}+t}{\max\{x,y\}+t}$ *and* $*$ *is the product t-norm.*

It is well known [23] that $\tau_{\mathcal{M}}$ *is the usual topology of* $\mathbb{R}$ *restricted to* $]0,1]$.

The sequence $\{x_n\}$ *in* $\mathcal{X}$ *defined by* $x_n = \frac{1}{n}$ *for all* $n \in \mathbb{N}$ *is p-Cauchy. Indeed, given* $t_0 = 1$ *we have* $\lim_{n,m} \mathcal{M}(x_n, x_m, t_0) = \frac{\min\{x_n,x_m\}+1}{\max\{x_n,x_m\}+1} = 1$. *Consequently,* $\{x_n\}$ *is Cauchy due to* $\mathcal{M}$ *being strongly principal (see (c) in Example 1). Then,* $\mathcal{X}$ *is not complete since* $\{x_n\}$ *does not converge, obviously, in* $\mathcal{X}$.

Now, $\mathcal{M}$ is strong for the (integral) t-norm product (see [21]) and so, by Corollary 3, we conclude that $(\mathcal{X}, \mathcal{M}, *)$ admits completion. It is left to the reader to verify that $(\tilde{\mathcal{X}}, \tilde{\mathcal{M}}, *)$ is the completion of $(\mathcal{X}, \mathcal{M}, *)$, where $\tilde{\mathcal{X}} = [0,1]$ and $\tilde{\mathcal{M}}$ is the above expression of $\mathcal{M}$ extended to $[0,1]$.

As an application of our main results, we can prove the classical one, which claims that every metric admits completion. Below, we show such an affirmation.

Let d be a metric on $\mathcal{X}$ and consider the corresponding standard fuzzy metric space $(\mathcal{X}, \mathcal{M}_d, \cdot)$. It is well known that the class of $\mathcal{M}_d$-Cauchy ($\mathcal{M}_d$-convergent) sequences agrees with the class of d-Cauchy (d-convergent) sequences. For this reason, it is easy to conclude that $(\mathcal{X}, \mathcal{M}_d, \cdot)$ is complete if and only if $(\mathcal{X}, d)$ is complete.

Suppose that $(\mathcal{X}, d)$ is a non-complete metric space. Since $(\mathcal{X}, \mathcal{M}_d, \cdot)$ is strongly principal (see (b) in Example 1) and strong (see [21]) and $\cdot$ is integral, we can apply Corollary 3 to conclude that $(\mathcal{X}, \mathcal{M}_d, \cdot)$ admits completion. On account of [8], $(\tilde{\mathcal{X}}, \tilde{\mathcal{M}}_d, \cdot)$ is a completion of $(\mathcal{X}, \mathcal{M}_d, \cdot)$, where $\tilde{\mathcal{X}}$ is the set of all equivalence classes of Cauchy sequences in $\mathcal{X}$, under the equivalence relation $\{\alpha_n\} \sim \{\beta_n\} \Leftrightarrow \lim_n \mathcal{M}(\alpha_n, \beta_n, t) = 1$ for all $t > 0$, and $\tilde{\mathcal{M}}_d$ is given by $\tilde{\mathcal{M}}_d(\alpha, \beta, t) = \frac{t}{t + \lim_n d(\alpha_n, \beta_n)}$, whenever $\{\alpha_n\}$ and $\{\beta_n\}$ are Cauchy sequences of the classes $\alpha, \beta \in \tilde{\mathcal{X}}$, respectively. It means that there exists the metric $\tilde{d}$ on $\tilde{\mathcal{X}}$ given by $\tilde{d}(\alpha, \beta) = \lim_n d(\alpha_n, \beta_n)$. In other words, $\tilde{\mathcal{M}}_d$ is actually the standard fuzzy metric $\mathcal{M}_{\tilde{d}}$ on $\tilde{\mathcal{X}}$, and since $(\tilde{\mathcal{X}}, \mathcal{M}_{\tilde{d}}, \cdot)$ is complete, it is easy to conclude that $(\tilde{\mathcal{X}}, \tilde{d})$ is the metric completion of $(\mathcal{X}, d)$.

Explanatory 1. *The problem of finding fuzzy metrics satisfying condition (c2) was approached in [25]. There, a family of fuzzy metrics was found, called stratified, that satisfy (c2). Both families, stratified and s-principal fuzzy metrics, are two wide classes of fuzzy metrics that include stationary fuzzy metrics and the standard fuzzy metric. Now, they are different. Indeed, the fuzzy metric of (c) in Example 1 is s-principal and not stratified, and the fuzzy metric of (c) in Example 4 is stratified and it is not s-principal.*

5. Conclusions

In this paper, we have approached the problem of finding the class $\mathcal{H}$ of fuzzy metric spaces characterized as follows: $\mathcal{X} \in \mathcal{H}$ if and only if every p-Cauchy sequence in $\mathcal{X}$ is Cauchy. We have proven that if every p-Cauchy sequence in $\mathcal{X}$ is Cauchy then $\mathcal{X}$ is principal and that the converse is false. Nevertheless, if, in addition to being principal, $\mathcal{X}$ is weak p-complete then the converse is true, as it has been demonstrated. Thus, we have introduced the class of strongly principal fuzzy metrics and we have proven that strongly principal fuzzy metric spaces are completable, whenever the fuzzy metric is strong with respect to an integral t-norm. In addition, we have shown that if $\mathcal{X}$ is strongly principal then the class of p-Cauchy sequences in $\mathcal{X}$ agrees with the class of Cauchy sequences, and we ignore if the converse is true (Problem 1). As future line of research, we propose to the reader to answer Problem 1 and also Problem 2, related to Problem 1, involving a condition of completeness.

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