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On the Correlation between Banach Contraction Principle and Caristi's Fixed Point Theorem in *b*-Metric Spaces

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Abstract: We solve a question posed by E. Karapinar, F. Khojasteh and Z.D. Mitrović in their paper "A Proposal for Revisiting Banach and Caristi Type Theorems in *b*-Metric Spaces". We also characterize the completeness of *b*-metric spaces with the help of a variant of the contractivity condition introduced by the authors in the aforementioned article.

Keywords: *b*-metric space; complete; fixed point; the Banach contraction principle; Caristi's fixed point theorem

1. Introduction

In order to investigate correlations between the Banach contraction principle and results of Caristi type in the realm of *b*-metric spaces, Karapinar, Khojasteh and Mitrović proved in [1] (Theorem 1) the following interesting result by using a new type of contractions.

Theorem 1 ([1]). *Let $\mathcal{T}$ be a self mapping of a complete b-metric space $(\mathcal{X}, \mathfrak{b}, s)$ such that there is a function $F : \mathcal{X} \to \mathbb{R}$ (the set of real numbers) satisfying the following two conditions:*

(c1) F is bounded from below, i.e., there is an $a \in \mathbb{R}$ such that $\inf F(\mathcal{X}) > a$;

(c2) for every $u, v \in \mathcal{X}$:

$$\mathfrak{b}(u, \mathcal{T}u) > 0 \Rightarrow \mathfrak{b}(\mathcal{T}u, \mathcal{T}v) \leq (F(u) - F(\mathcal{T}u))\mathfrak{b}(u, v).$$

Then $\mathcal{T}$ has a fixed point.

They also gave an example of a complete metric space where we can apply Theorem 1 above but not the Banach contraction principle, and raised the following question [1] (Remark 1): "It is natural to ask if the Banach contraction principle is a consequence of Theorem 1 (over metric spaces)".

In this note we solve that question in the negative. With the help of a variant of Theorem 1 we also obtain a characterization of complete *b*-metric spaces which should be compared with the classical result given by Hu in [2], that a necessary and sufficient condition for a metric space to be complete is that every Banach contraction on each of its closed subsets has a fixed point.

Let us recall that many authors have contributed to the development of a consistent theory of fixed point for *b*-metric spaces (the bibliographies of [1], and [3–5] contain a high account of references to this respect). In particular, the Banach contraction principle [6] admits, *mutatis mutandis*, a full extension to *b*-metric spaces [7] (Theorem 2.1) (see also [3,8,9]), and regarding the extension of Caristi's fixed point theorem [10] to *b*-metric spaces, significant contributions are given, among others, in [11] (Theorem 2.4), as well as in [3] (Corollary 12.1), [7] (Example 2.8) and [12] (Theorem 3.1).

2. Background

In this section we remind some definitions and properties which will be of help to the reader.

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The set of non-negative real numbers and the set of natural numbers will be represented by $\mathbb{R}^+$ and $\mathbb{N}$, respectively.

The notion of a b-metric space has been considered by several authors under different names (see e.g., [13] and [3] (Chapter 12) for details). In our context we adapt that notion as given by Czerwik in [14].

A b-metric space is a triple $(\mathcal{X}, \mathfrak{b}, s)$, where $\mathcal{X}$ is a set, s is a real number with $s \geq 1$, and $\mathfrak{b} : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ is a function satisfying, for every $u, v, w \in \mathcal{X}$, the following conditions:

(b1) $\mathfrak{b}(u, v) = 0$ if and only if $u = v$;

(b2) $\mathfrak{b}(u, v) = \mathfrak{b}(v, u)$;

(b3) $\mathfrak{b}(u, v) \leq s(\mathfrak{b}(u, w) + \mathfrak{b}(w, v))$.

If $(\mathcal{X}, \mathfrak{b}, s)$ is a b-metric space the function $\mathfrak{b}$ is said to be a b-metric on $\mathcal{X}$. Of course, every metric space is a b-metric space where $s = 1$.

It is well-known (see e.g., [13,15,16]) that, as in the metric case, each b-metric $\mathfrak{b}$ on a set $\mathcal{X}$ induces a metrizable topology $\mathfrak{T}_\mathfrak{b}$ for which a subset $\mathcal{A}$ of $\mathcal{X}$ is declared open provided that for each $u \in \mathcal{A}$ there is an $r > 0$ such that $\mathcal{B}(u, r) \subseteq \mathcal{A}$, where $\mathcal{B}(u, r)= \{v \in \mathcal{X} : \mathfrak{b}(u, v) < r\}$.

An important consequence is that a sequence $(u_n)_{n\in\mathbb{N}}$ in a b-metric space $(\mathcal{X}, \mathfrak{b}, s)$ is $\mathfrak{T}_\mathfrak{b}$-convergent to an $w \in \mathcal{X}$ if and only if $\lim_n \mathfrak{b}(w, u_n) = 0$.

In the sequel all topological properties corresponding to a b-metric space $(\mathcal{X}, \mathfrak{b}, s)$ will refer to the topology $\mathfrak{T}_\mathfrak{b}$.

It is appropriate to point out that, unlike the metric case, the set $\mathcal{B}(u, r)$ is not necessarily $\mathfrak{T}_\mathfrak{b}$-open (see [16] (Example on pages 4310–4311), [17] (Example 3.9)).

Moreover, it is well known that, contrarily to the classical metric case, there exist b-metrics that are not continuous functions (see e.g., [17] (Examples 3.9 and 3.10)).

Finally, we recall that the notions of Cauchy sequence and of complete b-metric space are defined exactly as the corresponding ones that for metric spaces.

3. Results and Examples

We begin this section giving an example that solves the question raised in [1] (Remark 1).

Example 1. *Let $(\mathcal{X}, \mathfrak{b}, 1)$ be the metric space where $\mathcal{X} := \mathbb{R}^+$ and $\mathfrak{b}$ is the metric on $\mathcal{X}$ given by $\mathfrak{b}(u, u) = 0$ for all $u \in \mathcal{X}$, and $\mathfrak{b}(u, v) = \max\{u, v\}$ whenever $u \neq v$.*

It is clear that $(\mathcal{X}, \mathfrak{b}, 1)$ is complete because the only non-eventually constant Cauchy sequences are those that converge to 0.

Let $\mathcal{T}$ be the self mapping of $\mathcal{X}$ given by $\mathcal{T}u = u/2$ for all $u \in \mathcal{X}$.

Since $\mathfrak{b}(\mathcal{T}u, \mathcal{T}v) = \mathfrak{b}(u, v)/2$ for all $u, v \in \mathcal{X}$, all conditions of the Banach contraction principle are satisfied.

Next we show that, however, the condition (c2) of Theorem 1 is not fulfilled.

Indeed, let $F : \mathcal{X} \to \mathbb{R}$ be any bounded from below function.

Take $u_0 \in \mathcal{X}\backslash\{0\}$. Then $\mathfrak{b}(\mathcal{T}^n u_0, \mathcal{T}^{n+1}u_0) = 2^{-n}u_0 > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Suppose that the condition (c2) holds. Thus, we have

$$\begin{aligned} 2^{-(n+1)}u_0 &= \mathfrak{b}(\mathcal{T}^{n+1}u_0, \mathcal{T}^{n+2}u_0) \\ &\leq (F(\mathcal{T}^n u_0) - F(\mathcal{T}^{n+1}u_0))\mathfrak{b}(\mathcal{T}^n u_0, \mathcal{T}^{n+1}u_0) \\ &= (F(\mathcal{T}^n u_0) - F(\mathcal{T}^{n+1}u_0))2^{-n}u_0, \end{aligned}$$

and, hence,

$$F(\mathcal{T}^n u_0) \geq 2^{-1} + F(\mathcal{T}^{n+1}u_0),$$

for all $n \in \mathbb{N} \cup \{0\}$. Therefore,

$$\begin{aligned} F(u_0) &\geq \frac{1}{2} + F(\mathcal{T}u_0) \geq \frac{1}{2} + \frac{1}{2} + F(\mathcal{T}^2u_0) \geq ... \geq \frac{n+1}{2} + F(\mathcal{T}^{n+1}u_0) \\ &\geq \frac{n+1}{2} + \inf F(\mathcal{X}), \end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$, a contradiction.

Remark 1. *Since we are working in the more general context of b-metric spaces, it would be interesting to give an example of a Banach contraction on a non-metric complete b-metric space that does not satisfy condition (c2) of Theorem 1. For it, we proceed to modify Example 1 in the following fashion: Fix $p \in \mathbb{N} \backslash \{1\}$. Let $\mathcal{X} := \mathbb{R}^+$ and $\mathfrak{b}_p : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ defined by $\mathfrak{b}_p(u,u) = 0$ for all $u \in \mathcal{X}$, and $\mathfrak{b}_p(u,v) = (\max\{u,v\})^p$ whenever $u \neq v$. Then $(\mathcal{X}, \mathfrak{b}_p, 2^{p-1})$ is a non-metric complete b-metric space (see e.g., [18] (Example 2.2) or [3] (Example 12.2)). Let $\mathcal{T}$ be the self mapping of $\mathcal{X}$ given in Example 1. Then, it fulfills the conditions of the Banach contraction principle for b-metric spaces ([7] (Theorem 2.1)) with constant of contraction 2^{-p}. Analogously to Example 1 we can check that it does not satisfy condition (c2) for any bounded from below function $F : \mathcal{X} \to \mathbb{R}$ because, otherwise, for any $u_0 \in \mathcal{X} \backslash \{0\}$ we should $F(u_0) \geq 2^{-p}(n+1) + \inf F(\mathcal{X})$ for all $n \in \mathbb{N} \cup \{0\}$, a contradiction.*

In the sequel, a self mapping $\mathcal{T}$ of a b-metric space $(\mathcal{X}, \mathfrak{b}, s)$ such that there is a function $F : \mathcal{X} \to \mathbb{R}$ for which conditions (c1) and (c2) are satisfied will said to be a correlation contraction (on $(\mathcal{X}, \mathfrak{b}, s)$).

We wonder if Theorem 1 allows us to obtain a characterization of complete b-metric spaces in the style of Hu's characterization of metric completeness mentioned in Section 1. In this direction, the next is an example of a non-complete metric space such that every correlation contraction on any of its (non necessarily closed) subsets has a fixed point.

Example 2. *Let $\mathfrak{b}$ be the metric on $\mathbb{N}$ defined by $\mathfrak{b}(n,n) = 0$ for all $n \in \mathbb{N}$, and $\mathfrak{b}(n,m) = \max\{1/n, 1/m\}$ whenever $n \neq m$.*

Then $(\mathbb{N}, \mathfrak{b}, 1)$ is not complete because $(n)_{n\in\mathbb{N}}$ is a non-convergent Cauchy sequence.

Now let $\mathcal{T}$ be a correlation contraction on a (non-empty) subset $\mathcal{A}$ of $(\mathbb{N}, \mathfrak{b}, 1)$. Then, there is a function $F : \mathcal{A} \to \mathbb{R}$ for which conditions (c1) and (c2) are satisfied.

Suppose that $\mathcal{T}$ has no fixed points. Then $|\mathcal{A}| \geq 2$, and $\mathfrak{b}(n, \mathcal{T}n) > 0$ for all $n \in \mathcal{A}$.

Choose an $m_0 \in \mathcal{A}$. Since $\mathcal{T}$ has no fixed points we get $\mathcal{T}^n m_0 \neq \mathcal{T}^{n+1} m_0$, for all $n \in \mathbb{N} \cup \{0\}$.

Hence, by condition (c2),

$$\mathfrak{b}(\mathcal{T}^{n+1}m_0, \mathcal{T}m_0) \leq (F(\mathcal{T}^n m_0) - F(\mathcal{T}^{n+1}m_0))\mathfrak{b}(\mathcal{T}^n m_0, m_0)$$

for all $n \in \mathbb{N} \cup \{0\}$.

Since $1/\mathcal{T}m_0 \leq \mathfrak{b}(\mathcal{T}^{n+1}m_0, \mathcal{T}m_0)$, and $\mathfrak{b}(\mathcal{T}^n m_0, m_0) \leq 1$, we deduce that

$$\frac{1}{\mathcal{T}m_0} + F(\mathcal{T}^{n+1}m_0) \leq F(\mathcal{T}^n m_0),$$

for all $n \in \mathbb{N} \cup \{0\}$.

Therefore,

$$\begin{aligned} F(m_0) \quad &\geq \quad \frac{1}{\mathcal{T}m_0} + F(\mathcal{T}m_0) \geq \frac{2}{\mathcal{T}m_0} + F(\mathcal{T}^2 m_0) \\ &\geq \quad ... \geq \frac{n}{\mathcal{T}m_0} + F(\mathcal{T}^n m_0) \geq \frac{n}{\mathcal{T}m_0} + \inf F(\mathcal{A}), \end{aligned}$$

for all $n \in \mathbb{N}$, which yields a contradiction.

Motivated by the preceding example, in Definition 1 below we present a modification of the notion of a correlation contraction from which a characterization of b-metric completeness will be obtained via a fixed point result.

To this end, we first recall that a partial order on a set $\mathcal{X}$ is a reflexive, antisymmetric, and transitive binary relation on $\mathcal{X}$. If $\preceq$ is a partial order on $\mathcal{X}$, for each $u \in \mathcal{X}$ we denote by $u \uparrow$ the set $\{v \in \mathcal{X} : u \preceq v\}$.

On the other hand, given a b-metric space $(\mathcal{X}, \mathfrak{b}, s)$ we shall denote by $acc((\mathcal{X}, \mathfrak{T}_{\mathfrak{b}}))$ the set of all accumulation points of the metrizable topological space $(\mathcal{X}, \mathfrak{T}_{\mathfrak{b}})$. Hence $acc((\mathcal{X}, \mathfrak{T}_{\mathfrak{b}}))$ consists of all points $w \in \mathcal{X}$ for which there is a sequence of distinct points in $\mathcal{X}$ that $\mathfrak{T}_{\mathfrak{b}}$-converges to w.

Definition 1. *Let $(\mathcal{X}, \mathfrak{b}, s)$ be a b-metric space. We say that a self mapping $\mathcal{T}$ of $\mathcal{X}$ is a $\preceq$-correlation contraction (on $(\mathcal{X}, \mathfrak{b}, s)$) if there is a partial order $\preceq$ on $\mathcal{X}$ such that the following conditions hold:*

(c3) $\mathcal{T}$ is non-decreasing, i.e., $u \preceq v \Rightarrow \mathcal{T}u \preceq \mathcal{T}v$ for all $u, v \in \mathcal{X}$;

(c4) there is $u_0 \in \mathcal{X}$ such that $u_0 \preceq \mathcal{T}u_0$;

(c5) there is a bounded from below function $F : \mathcal{X} \to \mathbb{R}$ such that for every $u \in \mathcal{X}$ and $v \in (u \uparrow) \cup acc((\mathcal{X}, \mathfrak{T}_{\mathfrak{b}}))$,

$$\mathfrak{b}(u, \mathcal{T}u) > 0 \Rightarrow \mathfrak{b}(\mathcal{T}u, \mathcal{T}v) \leq (F(u) - F(\mathcal{T}u))\mathfrak{b}(u, v).$$

Remark 2. *The existence of $\preceq$-correlation contractions on a given b-metric space $(\mathcal{X}, \mathfrak{b}, s)$ is always guaranteed. Indeed, let i be the identity mapping on $\mathcal{X}$, and $\preceq_D$ the discrete partial order on $\mathcal{X}$, i.e., $u \preceq_D v \Leftrightarrow u = v$. It is obvious that conditions (c3)–(c5) are fulfilled for any bounded from below function $F : \mathcal{X} \to \mathbb{R}$ (in particular, (c5) directly follows from the fact that $\mathfrak{b}(u, iu) = 0$ for all $u \in \mathcal{X}$).*

Furthermore, it follows from Theorem 1 that every correlation contraction $\mathcal{T}$ on a complete b-metric space $(\mathcal{X}, \mathfrak{b}, s)$ is a $\preceq_D$-correlation contraction on it: It suffices to observe that conditions (c3) and (c5) are trivially satisfied and, for (c4), notice that every fixed point w of $\mathcal{T}$ obviously verifies $w \preceq_D \mathcal{T}w$.

We now establish the following variant of Theorem 1.

Theorem 2. *Let $(\mathcal{X}, \mathfrak{b}, s)$ be a complete* b*-metric space. Then, every $\preceq$-correlation contraction on it has a fixed point.*

Proof. Let $\mathcal{T}$ be a $\preceq$-correlation contraction on $(\mathcal{X}, \mathfrak{b}, s)$. Then, there is a partial order $\preceq$ on $\mathcal{X}$ and a bounded from below function $F : \mathcal{X} \to \mathbb{R}$ for which conditions (c3)–(c5) are fulfilled.

Let $u_0 \in \mathcal{X}$ be such that $u_0 \preceq \mathcal{T}u_0$. Therefore, by (c3), $\mathcal{T}^n u_0 \preceq \mathcal{T}^{n+1} u_0$ for all $n \in \mathbb{N} \cup \{0\}$.

If $\mathcal{T}^n u_0 = \mathcal{T}^{n+1} u_0$ for some n, $\mathcal{T}^n u_0$ is a fixed point of $\mathcal{T}$.

So, we assume that $\mathcal{T}^n u_0 \neq \mathcal{T}^{n+1} u_0$ for all $n \in \mathbb{N} \cup \{0\}$. Thus $\mathfrak{b}(\mathcal{T}^n u_0, \mathcal{T}^{n+1} u_0) > 0$ for all $n \in \mathbb{N} \cup \{0\}$, and we can apply condition (c5), which implies that

$$\mathfrak{b}(\mathcal{T}^{n+1} u_0, \mathcal{T}^{n+2} u_0) \leq (F(\mathcal{T}^n u_0) - F(\mathcal{T}^{n+1} u_0))\mathfrak{b}(\mathcal{T}^n u_0, \mathcal{T}^{n+1} u_0),$$

for all $n \in \mathbb{N} \cup \{0\}$.

Thus $F(\mathcal{T}^{n+1} u_0) < F(\mathcal{T}^n u_0)$ for all $n \in \mathbb{N} \cup \{0\}$, so $(F(\mathcal{T}^n u_0)_{n \in \mathbb{N}}$ is a strictly decreasing sequence in $\mathbb{R}$. Hence it converges to the real number $\inf_n \mathcal{T}^n(u_0)$ (recall that F is bounded from below), and consequently it is a Cauchy sequence in $\mathbb{R}$.

Now, by repeating the argument given by the authors in their proof of Theorem 1 ([1], lines 12–22 of page 2 and line 1 of page 3), we deduce that $(\mathcal{T}^n u_0)_{n \in \mathbb{N}}$ is a Cauchy sequence in $(\mathcal{X}, \mathfrak{b}, s)$. Therefore, there exists $w \in \mathcal{X}$ such that $(\mathcal{T}^n u_0)_{n \in \mathbb{N}}$ $\mathfrak{T}_{\mathfrak{b}}$-converges to w. Thus $w \in acc((\mathcal{X}, \mathfrak{T}_{\mathfrak{b}}))$, and again we can apply (c5) to deduce that

$$\mathfrak{b}(\mathcal{T}^{n+1} u_0, \mathcal{T}w) \leq (F(\mathcal{T}^n u_0) - F(\mathcal{T}^{n+1} u_0))\mathfrak{b}(\mathcal{T}^n u_0, w),$$

for all $n \in \mathbb{N} \cup \{0\}$.

Since $(F(\mathcal{T}^n u_0)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ and $\lim_n \mathfrak{b}(\mathcal{T}^n u_0, w) = 0$, we conclude that $\lim_n \mathfrak{b}(\mathcal{T}^{n+1} u_0, \mathcal{T}w) = 0$, so $w = \mathcal{T}w$. This completes the proof. □

We turn our attention to the relationship between Theorems 1 and 2. In connection with this, and as we have point out above, in Example 1 of [1], the authors presented an instance of a complete metric space where we can apply Theorem 1 but not the Banach contraction principle. By the second part of Remark 2, we also can apply Theorem 2 to every correlation contraction in [1] (Example 1). The following is an example where we can apply Theorem 2 but not Theorem 1.

Example 3. *Let* $(\mathcal{X}, \mathfrak{b}, 1)$ *be the complete metric space where* $\mathcal{X} := \mathbb{N} \cup \{0\}$ *and* $\mathfrak{b}$ *is the metric on* $\mathcal{X}$ *given by* $\mathfrak{b}(u,u) = 0$ *for all* $u \in \mathcal{X}$, $\mathfrak{b}(0,n) = \mathfrak{b}(n,0) = 1/n$ *for all* $n \in \mathbb{N}$, *and* $\mathfrak{b}(n,m) = \max\{1/n, 1/m\}$ *whenever* $n, m \in \mathbb{N}$ *with* $n \neq m$.

Clearly $acc((\mathcal{X}, \mathfrak{T}_{\mathfrak{b}})) = \{0\}$.

Let $\mathcal{T}$ *be the self mapping of* $\mathcal{X}$ *defined by* $\mathcal{T}0 = 0$ *and* $\mathcal{T}n = n(n+1)$ *for all* $n \in \mathbb{N}$, *and let* $\leq$ *be the usual order on* $\mathcal{X}$. *Then* $\mathcal{T}u \leq \mathcal{T}v$ *whenever* $u \leq v$, *and* $u \leq \mathcal{T}u$ *for all* $u, v \in \mathcal{X}$, *so conditions (c3) and (c4) hold.*

Now take $F : \mathcal{X} \to \mathbb{R}$ *defined by* $F(0) = 0$ *and* $F(n) = 1/n$ *for all* $n \in \mathbb{N}$. *We have* $\inf F(\mathcal{X}) = 0$. *Furthermore, for each* $u, v \in \mathcal{X}$, *with* $u \neq v$, *such that* $\mathfrak{b}(u, \mathcal{T}u) > 0$ *and* $v \in (u \uparrow) \cup acc((\mathcal{X}, \mathfrak{T}_{\mathfrak{b}}))$, *we get*

$$\begin{aligned}\mathfrak{b}(\mathcal{T}u, \mathcal{T}v) &= \frac{1}{\mathcal{T}u} = \frac{1}{u(u+1)} = \left(\frac{1}{u} - \frac{1}{u(u+1)}\right)\frac{1}{u} \\ &= (F(u) - F(\mathcal{T}u))\mathfrak{b}(u,v).\end{aligned}$$

We have shown that $\mathcal{T}$ *is a* $\leq$*-correlation contraction on* $(\mathcal{X}, \mathfrak{b}, 1)$.

However $\mathcal{T}$ *is not a correlation contraction on* $(\mathcal{X}, \mathfrak{b}, 1)$: *Otherwise, its restriction to* $\mathbb{N}$ *would also be a correlation contraction on* $(\mathbb{N}, \mathfrak{b}, 1)$ *and, by Example 2 it would have, at least, a fixed point belonging to* $\mathbb{N}$.

We finish the paper with our promised characterization of b-metric completeness and with two observations related to it.

The following lemma, that provides a full b-metric generalization of the corresponding result for metric spaces, will be useful in the proof of the 'only if' part of our characterization.

Lemma 1. *If* $\mathcal{C}$ *is a closed subset of a complete b-metric space* $(\mathcal{X}, \mathfrak{b}, s)$, *then* $(\mathcal{C}, \mathfrak{b}\,|_{\mathcal{C}}, s)$ *is also a complete* b*-metric space.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(\mathcal{C}, \mathfrak{b}\,|_{\mathcal{C}}, s)$. Then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{X}, \mathfrak{b}, s)$. Therefore there exists $x \in \mathcal{X}$ such that $(x_n)_{n\in\mathbb{N}}$ $\mathfrak{T}_{\mathfrak{b}}$-converges to x. Since $\mathcal{C}$ is closed we get that $x \in \mathcal{C}$. Hence $(\mathcal{C}, \mathfrak{b}\,|_{\mathcal{C}}, s)$ is complete. □

Theorem 3. *A* b*-metric space is complete if and only if every* $\preceq$*-correlation contraction on any of its closed subsets has a fixed point.*

Proof. Let $\mathcal{C}$ be a closed subset of a complete b-metric space $(\mathcal{X}, \mathfrak{b}, s)$ and let $\mathcal{T}$ be a $\preceq$-correlation contraction on $\mathcal{C}$ endowed with the restriction of $\mathfrak{b}$. By Lemma 1, $(\mathcal{C}, \mathfrak{b}\,|_{\mathcal{C}}, s)$ is complete. We deduce from Theorem 2 that $\mathcal{T}$ has a fixed point (in $\mathcal{C}$).

For the converse, suppose that $(\mathcal{X}, \mathfrak{b}, s)$ is a non-complete b-metric space for which every $\preceq$-correlation contraction on any of its closed subsets has a fixed point. Then, there exists a non-convergent Cauchy sequence $(u_n)_{n\in\mathbb{N}}$ in $(\mathcal{X}, \mathfrak{b}, s)$, with $u_n \neq u_m$ whenever $n \neq m$.

From standard arguments we can find a sequence $(j(n))_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that the following properties are fulfilled:

(P1) $j(1) > 1$, $j(n+1) > \max\{n+1, j(n)\}$ for all $n \in \mathbb{N}$;

and

(P2) for each $n \in \mathbb{N}$, $\mathfrak{b}(u_{j(n)}, u_k) < 2^{-(n+1)} b(u_n, u_m)$ whenever $k \geq j(n)$ and $m \neq n$.

Put $\mathcal{C} = \{u_n : n \in \mathbb{N}\}$ and define a self mapping $\mathcal{T}$ on $\mathcal{C}$ by $\mathcal{T}u_n = u_{j(n)}$ for all $n \in \mathbb{N}$. Of course $\mathcal{T}$ has no fixed points because $j(n) > n$ and thus $u_n \neq u_{j(n)}$ for all $n \in \mathbb{N}$.

Finally, we going to check that $\mathcal{T}$ is a $\preceq$-correlation contraction on the closed subset $\mathcal{C}$ of $(\mathcal{X}, \mathfrak{b}, s)$.

Let $\preceq$ be the partial order on $\mathcal{C}$ defined by

$$u_n \preceq u_m \Leftrightarrow n \leq m.$$

Condition (c3) is clearly verified: Indeed, if $u_n \preceq u_m$ we deduce that $n \leq m$, so $j(n) \leq j(m)$ and consequently $\mathcal{T}u_n = u_{j(n)} \preceq u_{j(m)} = \mathcal{T}u_m$.

Moreover $u_n \preceq \mathcal{T}u_n$ for all $n \in \mathbb{N}$ because, by (P1), $n < j(n)$. so condition (c4) is also fulfilled.

Let now $F : \mathcal{C} \to \mathbb{R}$ defined by $F(u_n) = 2^{-n}$ for all $n \in \mathbb{N}$. Thus $\inf F(\mathcal{C}) = 0$.

Pick $u_n \in \mathcal{C}$. Then $\mathfrak{b}(u_n, \mathcal{T}u_n) > 0$. For each $u_m \in \mathcal{C} \backslash \{u_n\}$ such that $u_m \in u_n \uparrow$, we have $n < m$, and hence, $j(n) < j(m)$, by (P1). Then, from (P2) we deduce that

$$\begin{aligned} \mathfrak{b}(\mathcal{T}u_n, \mathcal{T}u_m) &= \mathfrak{b}(u_{j(n)}, u_{j(m)}) < 2^{-(n+1)}\mathfrak{b}(u_n, u_m) \\ &\leq (2^{-n} - 2^{-j(n)})\mathfrak{b}(u_n, u_m) \\ &= (F(u_n) - F(\mathcal{T}u_n))\mathfrak{b}(u_n, u_m). \end{aligned}$$

Hence, condition (c5) is also satisfied (note that $acc((\mathcal{C}, \mathfrak{T}_{\mathfrak{b}} |_{\mathcal{C}}))$ is the empty set).

We conclude that $(\mathcal{X}, \mathfrak{b}, s)$ is complete. □

Remark 3. *Although the function F constructed in the proof of Theorem 3 satisfies $F(u_n) > 0$ for all $n \in \mathbb{N}$, we could have selected it to fulfill $F(u_n) < 0$ for all $n \in \mathbb{N}$. For instance, by defining $F(u_n) = 2^{-n} - 1$ for all $n \in \mathbb{N}$.*

Remark 4. *The metric space $(\mathbb{N}, \mathfrak{b}, 1)$ constructed in Example 2 is not complete. Hence, by Theorem 3, it has closed subsets endowed with $\preceq$-correlation contractions that are free of fixed points. In fact, the restriction to $\mathbb{N}$ of the $\preceq$-correlation contraction $\mathcal{T}$ constructed in Example 3 provides an instance of this situation.*

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