A Montel-type theorem for Hardy spaces of holomorphic functions

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Abstract

We give a version of the Montel theorem for Hardy spaces of holomorphic functions on an infinite dimensional space. Precisely, we show that any bounded sequence of holomorphic functions in some Hardy space, has a subsequence that converges uniformly over compact subsets to a function that also belongs to the same Hardy space. As a by-product of our results for spaces of functions on infinitely many variables, we also provide an elementary proof of a Montel-type theorem for the Hardy space of Dirichlet series.

1 Introduction

Montel's theorem is one of the basic results in the classical function theory of one complex variable (see e.g. [5, Theorem 2.9]): every sequence of holomorphic functions on some open set Ω of the complex plane that is uniformly bounded on the compact subsets of Ω has a subsequence that converges uniformly (on the compact subsets) to some holomorphic function defined on Ω. It is also classical and well knownt that this result extends to holomophic functions on infinitely dimensional Banach spaces (precise definitions are given below). It should be noted that, if the sequence belongs to a certain class of functions (e.g. a Hardy space of holomorphic functions) and one applies Montel's theorem, there is no guarantee that the limit function also belongs to the same class. We show here that if we start with a sequence in some Hardy space of holomorphic functions H_p (again, see below for definitions), that is (norm) bounded, then the limit function also belongs to H_p (this is Theorem 1).

Given a complex Banach space X and an open set $U \subseteq X$, a function $f : U \to \mathbb{C}$ is said to be holomorphic (see [8, Chapter 15]) if it is Fréchet differentiable at every point of U, that is if for every $x \in U$ there exists a functional $x^* \in X^*$ so that

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - x^*(h)}{\|h\|} = 0 \,.$$

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On ℓ_2 (the space of square-summable complex sequences) we consider the open set

$$\ell_2 \cap \mathbb{D}^{\mathbb{N}} = \{z = (z_n)_n \in \ell_2 \colon |z_n| < 1 \text{ for all } n\}$$

and, for each $1 \leq p < \infty$, the Hardy space $H_p(\ell_2 \cap \mathbb{D}^{\mathbb{N}})$ consisting of all holomorphic functions $f : \ell_2 \cap \mathbb{D}^{\mathbb{N}} \to \mathbb{C}$ for which

$$\|f\|_{H_p(\ell_2 \cap \mathbb{D}^{\mathbb{N}})} = \sup_{n\in\mathbb{N}} \sup_{0<r<1} \left(\int_{\mathbb{T}^n} |f(rw_1, \cdots, rw_n, 0, \cdots)|^p \mathrm{d}w_1 \cdots \mathrm{d}w_n \right)^{\frac{1}{p}} < \infty \tag{1}$$

(here and everywhere else in this note the n-dimensional torus $\mathbb{T}^n = \{(z_1, \ldots, z_n) \colon |z_j| = 1 \text{ for all } j\}$ is considered with the normalised Lebesgue measure).

Our aim in this note is to prove the following Montel-type result.

Theorem 1. *Let $(f_n) \subseteq H_p(\ell_2 \cap \mathbb{D}^{\mathbb{N}})$ be a bounded sequence. Then there exist $f \in H_p(\ell_2 \cap \mathbb{D}^{\mathbb{N}})$ and a subsequence (f_{n_k}) that converges to f uniformly on the compact subsets of $\ell_2 \cap \mathbb{D}^{\mathbb{N}}$.*

These Hardy spaces are natural extensions of the classical $H_p(\mathbb{D})$ spaces of functions of one variable (see, for example, [6, Chapter 20]), consisting of those holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ so that

$$\|f\|_{H_p(\mathbb{D})} = \sup_{0<r<1} \left(\int_{\mathbb{T}} |f(rw)|^p \mathrm{d}w \right)^{\frac{1}{p}} < \infty\,. \tag{2}$$

Observe that the one-variable version of Theorem 1 is the following.

Theorem 2. *Let $(f_n) \subseteq H_p(\mathbb{D})$ be a bounded sequence. Then there exist $f \in H_p(\mathbb{D})$ and a subsequence (f_{n_k}) that converges to f uniformly on the compact subsets of $\mathbb{D}$.*

This seems to be well known within the area, although we could not find it explicitly in the literature. It follows essentially from the Banach-Alaoglu theorem (with which we can extract a weakly*-convergent subsequence) and the fact that weak*-convergence on $H_p(\mathbb{D})$ is equivalent to being bounded and uniformly convergent on the compact sets of $\mathbb{D}$ [6, Chapter 20, Proposition 3.15].
Another perspective is to use to the classical Montel's theorem for holomorphic functions. Indeed, let (f_n) be a bounded sequence in $H_p(\mathbb{D})$. Using [8, Corollary 13.13]

$$|f_n(z)| \leq \|f_n\|_{H_p(\mathbb{D})} (1 - |z|^2)^{-\frac{1}{p}},$$

we know that the sequence is uniformly bounded on the compact subsets of $\mathbb{D}$. Then, by Montel's theorem there is a subsequence (f_{n_k}) which converges uniformly over compact subsets to a holomorphic function f. It remains to show that f actually belongs to $H_p(\mathbb{D})$. To do this fix $r < 1$ and by passing to the (uniform) limit in the inequality

$$\left(\int_{\mathbb{T}} |f_{n_k}(rz)|^p \mathrm{d}z \right)^{\frac{1}{p}} \leq C\,,$$

we have $\left(\int_{\mathbb{T}} |f(rz)|^p \mathrm{d}z \right)^{\frac{1}{p}} \leq C$. Taking the sup over all $0 < r < 1$ we conclude that $f \in H_p(\mathbb{D})$ and $\|f\|_{H_p(\mathbb{D})} \leq C$.

We will later show that our approach in infinite-dimensional spaces, when looked in one variable, becomes absolutely elementary and provides a self-contained proof of Theorem 2, which uses only basic tools on the theory of one complex variable and do not even appeal to Montel's classical theorem. Obviously, the same argument would give the analogous result for Hardy spaces on the n-th dimensional polydisc.

Let us denote by $H_p(\mathbb{T})$ the closed subspace of $L_p(\mathbb{T})$ consisting of those functions f for which the Fourier coefficient $\hat{f}(n)$ is 0 whenever $n < 0$. A classical, well known result states that $H_p(\mathbb{T}) = H_p(\mathbb{D})$ as Banach spaces (see again [6, Chapter 20]), establishing a sort of bridge connecting function theory and harmonic analysis. The spaces $H_p(\ell_2 \cap \mathbb{D}^{\mathbb{N}})$ that we have just considered are also closely related with harmonic analysis. Let us explain briefly how. On the infinite polytorus $\mathbb{T}^{\mathbb{N}}$ (which is a compact abelian group) we consider the product of the normalised Lebesgue measure (which is the Haar measure) and (see [12, Chapter 8]) $H_p(\mathbb{T}^{\mathbb{N}})$ the class of functions in $L_p(\mathbb{T}^{\mathbb{N}})$ whose Fourier coefficients $\hat{f}(\alpha)$ vanish unless $\alpha_k \geq 0$ for every k (which corresponds to the classical Hardy space on $\mathbb{T}$). It is then well known (see [4] or [8, Theorem 13.2]) that $H_p(\ell_2 \cap \mathbb{D}^{\mathbb{N}}) = H_p(\mathbb{T}^{\mathbb{N}})$.

Ordinary Dirichlet series are formal series of the form $\sum a_n n^{-s}$, where $a_n \in \mathbb{C}$ and s is a complex variable. It should be noted that the Hardy spaces of holomorphic functions on infinite dimensional spaces mentioned above are also closely related with the Hardy spaces of Dirichlet series, introduced by Bayart in [2]. There he gave a Montel-type theorem for Dirichlet series in the space $\mathcal{H}_\infty$ (precise definitions are given below). Recently, Defant and Schoolmann obtained in [10] and in collaboration with the first and third authors in [7] versions for Hardy spaces of general Dirichlet spaces of the form $\sum a_n e^{-\lambda_n s}$ (a much more general setting), based on harmonic analysis on compact groups. Here, as a consequence of Theorem 1, we provide an elementary approach to Montel-type theorems for ordinary Dirichlet series. More details on this are given in Section 3. We would like to emphazise that we do not know of any way to deduce Theorem 1 (which is novel, and we believe is interesting within the field of holomorphic functions on infinite dimensional spaces) from the result for Dirichlet series and hence from any of the results in [10] or [7].

2 The proof of the result

We prove Theorem 1 in several steps. First we use a diagonal procedure to find a subsequence that converges pointwise on some dense subset of $\ell_2 \cap \mathbb{D}^{\mathbb{N}}$. In a second step we see that this sequence is uniformly Cauchy on the compact subsets of $\ell_2 \cap \mathbb{D}^{\mathbb{N}}$ and, hence, converges. Finally we show that the limit function belongs to the Hardy space. We need two results that we state now without proof. The first one is [8, Lemma 2.16], and provides a locally Lipschitz condition on compact subsets.

Lemma 3. *Let X be a Banach space, $U \subseteq X$ an open set, and $K \subseteq U$ a compact set. If $f : U \to \mathbb{C}$ is holomorphic and bounded, then for every $0 < s < r = \operatorname{dist}(X \setminus U, K)$, all $x \in K$ and $y \in \overline{B(x,s)}$*

$$|f(x) - f(y)| \leq \frac{1}{r-s} \|x - y\| \sup_{z \in U} |f(z)|.$$

The second lemma that we need (the proof of which can be found in [8, Corollary 13.20

and (13.25)]), allows us to estimate $|f(z)|$ for $z \in \ell_2 \cap \mathbb{D}^{\mathbb{N}}$ in terms of $\|z\|_2 = (\sum_{j=1}^{\infty} |z_j|^2)^{1/2}$ and $\|z\|_\infty := \sup_{j\in\mathbb{N}} |z_j|$.

Lemma 4. *If $1 \le p < \infty$, then*

$$|f(z)| \le e^{\frac{\|z\|_2^2}{1-\|z\|_\infty^2}} \|f\|_{H_p(\ell_2\cap\mathbb{D}^{\mathbb{N}})} .$$

for every $f \in H_p(\ell_2 \cap \mathbb{D}^{\mathbb{N}})$ and $z \in \ell_2 \cap \mathbb{D}^{\mathbb{N}}$.

We are now ready to prove Theorem 1.

Proof of Theorem 1. We begin by finding a subsequence (f_{n_k}) of f_n that converges pointwise on some dense subset of $\ell_2 \cap \mathbb{D}^{\mathbb{N}}$. Fix, then, some $\{x_m : m \in \mathbb{N}\}$ dense in $\ell_2 \cap \mathbb{D}^{\mathbb{N}}$. By Lemma 4, $(f_n(x_1))_n$ is bounded in $\mathbb{C}$ and we can find a subsequence $f_{n(k,1)}$ and $c_1 \in \mathbb{C}$, so that

$$c_1 = \lim_{k\to 1} f_{n(k,1)}(x_1) \text{ and } |f_{n(k,1)}(x_1) - c_1| < \frac{1}{k} \text{ for all } k \in \mathbb{N} .$$

Suppose that we have found subsequences $\big(n(k,1)\big)_{k=1}^\infty, \dots, \big(n(k,m)\big)_{k=1}^\infty$ in such a way that, for each $1 \le j \le m$, $\big(n(k,j)\big)_{k=1}^\infty$ is a subsequence of $\big(n(k,j-1)\big)_{k=1}^\infty$ and

$$c_j = \lim_{k\to\infty} f_{n(k,j)}(x_j) \text{ and } |f_{n(k,j)}(x_j) - c_j| < \frac{1}{k} \text{ for all } k \in \mathbb{N} . \tag{3}$$

Again, $f_{n(k,m)}(x_{m+1})$ is bounded in $\mathbb{C}$ and then there is a subsequence $f_{n(k,m+1)}$ that converges to a certain $c_{m+1} \in \mathbb{C}$, with $|f_{n(k,m+1)}(x_{m+1}) - c_{m+1}| < \frac{1}{k}$ for all $k \in \mathbb{N}$. In this way we define $(c_j)_j \subseteq \mathbb{C}$ and $(n(k,j))_{k,j} \subseteq \mathbb{N}$ in such a way that $\big(n(k,j)\big)_{k=1}^\infty$ is a subsequence of $\big(n(k,j-1)\big)_{k=1}^\infty$ and (3) holds for every j. We define now $n_k = n(k,k)$ and observe that for each fixed m, if $k \ge m$ then

$$|f_{n_k}(x_m) - c_m| = |f_{n(k,k)}(x_m) - c_m| < \frac{1}{k} . \tag{4}$$

Thus $\big(f_{n_k}(x_m)\big)_{k=1}^\infty$ converges to c_m for every $m \in \mathbb{N}$, and we have found the subsequence that we were looking for.
The second step of the proof is to see that $(f_{n_k})_k$ converges to some $f \in H_p(\ell_2 \cap \mathbb{D}^{\mathbb{N}})$ uniformly over the compact sets. To do this take some compact $K \subseteq \ell_2 \cap \mathbb{D}^{\mathbb{N}}$ and set $r = \operatorname{dist}\big(\ell_2 \setminus (\ell_2 \cap \mathbb{D}^{\mathbb{N}}), K\big)$, since $\ell_2 \cap \mathbb{D}^{\mathbb{N}}$ is an open set in ℓ_2 then $r > 0$. Fix now $0 \ne z \in K$ and fix j so that $z_j \ne 0$. Define $y_i = z_i$ if $i \ne j$ and $y_j = \frac{z_j}{|z_j|}$; then clearly $y = (y_i)_i \in \ell_2 \setminus (\ell_2 \cap \mathbb{D}^{\mathbb{N}})$ and, therefore,

$$r \le \|y - z\|_2 = \left| z_j - \frac{z_j}{|z_j|} \right| = \left| \frac{z_j|z_j| - z_j}{|z_j|} \right| = \big| |z_j| - 1 \big| = 1 - |z_j| < 1 .$$

This shows that $\|z\|_\infty \le 1 - r$ for every $z \in K$ (since the inequality obviously holds also for $z = 0$). Consider now $B = \bigcup_{z\in K} B(z, \frac{r}{2})$. Given $x \in B$ we can find $z \in K$ so that $\|x - z\|_\infty \le \|x - z\|_2 < \frac{r}{2}$ and, then,

$$\|x\|_\infty < \|z\|_\infty + \frac{r}{2} \le 1 - \frac{r}{2} < 1 .$$

With this, we can find a constant $\lambda_B > 0$ so that $\|x\|_2 \leq \lambda_B$ for every $x \in B$ and, denoting $M = \sup_n \|f_n\|_{H_p(\ell_2 \cap \mathbb{D}^{\mathbb{N}})}$, this and Lemma 4 yield

$$|f_{n_k}(x)| \leq e^{\frac{\|x\|_2^2}{1-\|x\|_\infty^2}} \|f_{n_k}\|_{H^p(\ell_2 \cap \mathbb{D}^{\mathbb{N}})} \leq e^{\frac{\lambda_B^2}{r - \frac{r^2}{4}}} M$$

for every $x \in B$. This shows that each f_{n_k} is bounded on B. But B is open and contains K so, if we take $r_B = \operatorname{dist}(\ell_2 \setminus B, K) > 0$ and fix $0 < \varepsilon < \frac{r_B}{2}$, from Lemma 3 we know that

$$|f_{n_k}(z) - f_{n_k}(y)| \leq \frac{1}{r_B - \varepsilon} \|z - y\| \sup_{x \in B} |f_{n_k}(x)| \leq \frac{1}{r_B - \varepsilon} \|z - y\| e^{\frac{\lambda_B^2}{r - \frac{r^2}{4}}} M$$

for every $z \in K$ and all $y \in \overline{B(z, \varepsilon)}$. Since $\{B(z,\varepsilon) : z \in K\}$ is an open cover of K, there exist $z_1, \cdots, z_N$ in K such that

$$K \subseteq B(z_1, \varepsilon) \cup \cdots \cup B(z_N, \varepsilon)$$

and, by the denseness of $\{x_m : m \in \mathbb{N}\}$, for each $i = 1, \ldots, N$ we can find $m_i \in \mathbb{N}$ so that $x_{m_i} \in B(z_i, \varepsilon)$. In this way, for every $z \in K$ there are z_j and x_{m_j}, such that $z, x_{m_j} \in \overline{B(z_j, \varepsilon)}$ and, then,

$$\begin{aligned}
|f_{n_k}(z) - f_{n_k}(x_{m_j})| &\leq |f_{n_k}(z) - f_{n_k}(z_j)| + |f_{n_k}(z_j) - f_{n_k}(x_{m_j})| \\
&\leq \frac{M e^{\frac{\lambda_B^2}{r - \frac{r^2}{4}}}}{r_B - \varepsilon} \Big(\|z - z_j\| + \|x_{m_j} - z_j\| \Big) \leq \frac{4 e^{\frac{\lambda_B^2}{r - \frac{r^2}{4}}} M}{r_B} \varepsilon \,.
\end{aligned}$$

Now, if $k \geq l \geq \max\{m_1, \cdots, m_N, \frac{1}{\varepsilon}\}$, using (4) we get

$$\begin{aligned}
&|f_{n_k}(z) - f_{n_l}(z)| \\
&\quad \leq |f_{n_k}(z) - f_{n_k}(x_{m_j})| + |f_{n_k}(x_{m_j}) - c_{m_j}| + |f_{n_l}(x_{m_j}) - c_{m_j}| + |f_{n_l}(z) - f_{n_l}(x_{m_j})| \\
&\quad < \frac{2}{l} + \frac{4 e^{\frac{\lambda_B^2}{r - \frac{r^2}{4}}} M}{r_B} \varepsilon
\end{aligned}$$

This shows that the sequence f_{n_k} is uniformly Cauchy on K and, since K was arbitrary the sequence is uniformly Cauchy on the compact subsets of $\ell_2 \cap \mathbb{D}^{\mathbb{N}}$. Then, since the space of holomorphic functions endowed with the topology of uniform convergence on compact sets is complete (see e.g. [8, Theorem 15.48]), it converges to some holomorphic $f : \ell_2 \cap \mathbb{D}^{\mathbb{N}} \to \mathbb{C}$.

To complete the proof it is only left to check that in fact $f \in H_p(\ell_2 \cap \mathbb{D}^{\mathbb{N}})$. Let $M = \sup_n \|f_n\|_{H_p(\ell_2 \cap \mathbb{D}^{\mathbb{N}})} > 0$ and fix $n \in \mathbb{N}$ and $0 < r < 1$. Since $r\mathbb{T}^n \times \{0\} \subseteq \ell_2 \cap \mathbb{D}^{\mathbb{N}}$ is compact, there is $k = k(n,r) \in \mathbb{N}$ such that $|f(z) - f_{n_k}(z)| \leq \frac{1}{2}$ for every $z \in r\mathbb{T}^n \times \{0\}$ and, therefore

$$\begin{aligned}
&\left(\int_{\mathbb{T}^n} |f(rw_1, \ldots, rw_n, 0, \ldots)|^p \mathrm{d}w_1 \cdots \mathrm{d}w_n \right)^{\frac{1}{p}} \\
&\qquad \leq \left(\int_{\mathbb{T}^n} |f(rw_1, \ldots, rw_n, 0, \ldots) - f_{n_k}(rw_1, \ldots, rw_n, 0, \ldots)|^p \mathrm{d}w_1 \cdots \mathrm{d}w_n \right)^{\frac{1}{p}}
\end{aligned}$$

$$+\left(\int_{\mathbb{T}^n}|f_{n_k}(rw_1,\cdots,rw_n,0,\cdots)|^p\mathrm{d}w_1\cdots\mathrm{d}w_n\right)^{\frac{1}{p}}$$

$$\leq\left(\int_{\mathbb{T}^n}\frac{1}{2^p}\mathrm{d}w\right)^{\frac{1}{p}}+\|f_{n_k}\|_{H_p}\leq\frac{1}{2}+M.$$

Then we have that

$$\sup_{n\in\mathbb{N}}\sup_{0<r<1}\left(\int_{\mathbb{T}^n}|f(rw_1,\cdots,rw_n,0,\cdots)|^p\mathrm{d}w_1\cdots\mathrm{d}w_n\right)^{\frac{1}{p}}\leq\frac{1}{2}+M,$$

and shows that, in fact, $f\in H_p(\ell_2\cap\mathbb{D}^{\mathbb{N}})$. □

The one-variable version (Theorem 2) can be deduced from Theorem 1. Let us briefly explain how. First of all, it is pretty straightforward to see that $H_p(\mathbb{D})$ can be isometrically embedded in $H_p(\ell_2\cap\mathbb{D}^{\mathbb{N}})$. On the other hand, every holomorphic function $f:\ell_2\cap\mathbb{D}^{\mathbb{N}}\to\mathbb{C}$ defines a family of coefficients in the following way (see [9] or [8, Chapter 15]). For each n and $\alpha\in\mathbb{N}_0^n$ (where $\mathbb{N}_0=\mathbb{N}\cup\{0\}$) the α-th coefficient of f is defined as

$$c_\alpha(f):=\frac{1}{(2\pi i)^n}\int\limits_{|\lambda_1|=r}\cdots\int\limits_{|\lambda_n|=r}\frac{f(\lambda_1,\cdots,\lambda_N,0,\dots)}{\lambda_1^{\alpha_1+1}\cdots\lambda_n^{\alpha_n+1}}\mathrm{d}\lambda_n\cdots\mathrm{d}\lambda_1\,. \tag{5}$$

Then, denoting $\mathbb{N}_0^{(\mathbb{N})}=\bigcup_{n\in\mathbb{N}}\mathbb{N}_0^n\times\{0\}$ we have a unique family of coefficients $\big(c_\alpha(f)\big)_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}}$ associated to f.

If we start with a bounded sequence $(f_n)_n$ in $H_p(\mathbb{D})$ we may look at it as belonging to $H_p(\ell_2\cap\mathbb{D}^{\mathbb{N}})$, and Theorem 1 gives us a subsequence $(f_{n_k})_k$ converging uniformly on the compacts (of $\ell_2\cap\mathbb{D}^{\mathbb{N}}$) to some $f\in H_p(\ell_2\cap\mathbb{D}^{\mathbb{N}})$. But (5) and the uniform convergence on compacts show that $c_\alpha(f_k)$ converges to $c_\alpha(f)$. Hence, these coefficients are 0 for every α for which there is some $N\geq 2$ with $\alpha_N\neq 0$. Then it is also easy to see (using e.g. [8, Theorem 13.2]) that in fact $f\in H_p(\mathbb{D})$ and, since every compact set in $\mathbb{D}$ is compact in $\ell_2\cap\mathbb{D}^{\mathbb{N}}$ the argument is completed.

This is, however, too long a way to prove Theorem 2 (go to an infinite dimensional space in order to come back to dimension 1). As a matter of fact the strategy to prove Proposition 1 can be adapted (and simplified) to give a direct proof of the one-dimensional result. The replacements for Lemmas 3 and 4 are direct consequences of Cauchy's Integral Formula (as in, e.g. [5, Theorem 5.4]). If $f\in H_1(\mathbb{D})$ and $z\in\mathbb{D}$ we may take any $|z|<r<1$ to have

$$|f(z)|=\left|\frac{1}{2\pi i}\int_{|\zeta|=r}\frac{f(\zeta)}{\zeta-z}\mathrm{d}\zeta\right|\leq\frac{r}{r-|z|}\int_{\mathbb{T}}|f(rw)|\mathrm{d}w\leq\frac{r}{r-|z|}\|f\|_{H_1(\mathbb{D})}\,.$$

Since this holds for every r we have

$$|f(z)|\leq\frac{1}{1-|z|}\|f\|_{H_1(\mathbb{D})} \tag{6}$$

for every $f\in H_1(\mathbb{D})$ and $z\in\mathbb{D}$.

Suppose now that $f:\mathbb{D}\to\mathbb{C}$ is holomorphic and $z_1,z_2\in s\mathbb{D}$ for some $0<s<1$. Using again Cauchy's Integral Formula we have

$$|f(z_1)-f(z_2)| = \frac{1}{2\pi}\left|\int_{|\zeta|=s} f(\zeta)\frac{z_1-z_2}{(\zeta-z_1)(\zeta-z_2)}\mathrm{d}\zeta\right| \leq |z_1-z_2|\frac{s}{(s-|z_1|)(s-|z_2|)}\sup_{|\zeta|=s}|f(\zeta)|\,. \tag{7}$$

We now present an alternative and elementary proof of Theorem 2 that only uses basic facts of function theory of one compex variables.

Alternative proof of Theorem 2. We begin by taking $\{x_m \colon m \in \mathbb{N}\}$ a countable dense subset of $\mathbb{D}$. By (6), for each fixed m, the sequence $(f_n(x_m))_n$ is bounded in $\mathbb{C}$ (recall that $\|\cdot\|_{H_1} \leq \|\cdot\|_{H_p}$). With exactly the same diagonal procedure as in the proof of Proposition 1 we can define a subsequence $(f_{n_k})_k$ and $\{c_m \colon m \in \mathbb{N}\}$ so that $f_{n_k}(x_m) \to c_m$ as $k \to \infty$ for every m.The key point now is to see that the sequence $(f_{n_k})_k$ is uniformly Cauchy on every compact set $K \subseteq \mathbb{D}$. It suffices to take $K = \overline{r\mathbb{D}}$ for $0 < r < 1$. Fix, then, such an r and $0 < \varepsilon < 1-r$. By the density of the set $\{x_m\}_m$ and compactness we may find $x_1, \dots, x_N \in \overline{r\mathbb{D}}$ (belonging to the dense set, these may not be the corresponding to $m = 1, \dots, N$ but we prefer to keep the notation as neat as possible) so that $\overline{r\mathbb{D}} \subseteq \mathbb{D}(x_1, \varepsilon) \cup \cdots \cup \mathbb{D}(x_N, \varepsilon)$ (where $\mathbb{D}(x, \varepsilon)$ denotes the open disc in $\mathbb{C}$ centred in x and with radius ε). For each $j = 1, \dots, N$ the sequence $(f_{n_k}(x_j))_k$ is Cauchy, so we can find k_0 so that $|f_{n_{k_1}}(x_j) - f_{n_{k_2}}(x_j)| < \varepsilon$ for every $k_1, k_2 \geq k_0$ and all $j = 1, \dots, N$. Given $z \in \overline{r\mathbb{D}}$, there is some j so that $|z - x_j| < \varepsilon$. Then, taking any $r < s < 1$, (7) and (6) give

$$|f_{n_k}(z) - f_{n_k}(x_j)| \leq \varepsilon \frac{s}{(s-r)^2}\frac{1}{1-r}\|f_{n_k}\|_{H_p(\mathbb{D})}\,.$$

We may choose $s = r + \frac{1-r}{2}$ and, then $\frac{s}{(s-r)^2}\frac{1}{1-r} = \frac{2(r+1)}{(1-r)^3} = K_r$. Denoting $M = \sup_k \|f_{n_k}\|_{H_p(\mathbb{D})}$ we have $|f_{n_{k_1}}(z) - f_{n_{k_2}}(z)| \leq \varepsilon(1 + 2K_r M)$ for every $z \in \overline{r\mathbb{D}}$ and $k_1, k_2 \geq k_0$. This shows that the sequence $(f_{n_k})_k$ is uniformly Cauchy on every compact subset of $\mathbb{D}$. Since the space of holomorphic functions on $\mathbb{D}$ (with the topology of uniform convergence on compact sets) is complete [5, Corollary 2.3], it converges to some holomorphic function $f : \mathbb{D} \to \mathbb{C}$. Exactly the same arguments mentioned after the statement of Theorem 2 show that f actually belongs to $H_p(\mathbb{D})$. □

Let us finish this section by noting that as a matter of fact, Theorem 1 holds in a more general setting. Let X be a Banach sequence space (i.e. a subspace of $\mathbb{C}^{\mathbb{N}}$ that contains all the canonical vectors $(e_n)_n$, endowed with a complete norm such that $\|e_n\| = 1$ for all n and, if $x = (x_n)_n \in \mathbb{C}^{\mathbb{N}}$ and $y = (y_n)_n \in X$ are sequences so that $|x_n| \leq |y_n|$ for every n, then $x \in X$ and $\|x\| \leq \|y\|$) and consider the open set

$$X \cap \mathbb{D}^{\mathbb{N}} = \{z = (z_n)_n \in X \colon |z_n| < 1 \text{ for all } n\}\,.$$

For each $1 \leq p < \infty$, the Hardy space $H_p(X \cap \mathbb{D}^{\mathbb{N}})$ is defined as consisting of all holomorphic functions $f : X \cap \mathbb{D}^{\mathbb{N}} \to \mathbb{C}$ for which

$$\|f\|_{H_p(X\cap\mathbb{D}^{\mathbb{N}})} = \sup_{n\in\mathbb{N}}\sup_{0<r<1}\left(\int_{\mathbb{T}^n} |f(rw_1, \cdots, rw_n, 0, \cdots)|^p \mathrm{d}w_1\cdots\mathrm{d}w_n\right)^{\frac{1}{p}} < \infty$$

(here again we take on the n-dimensional torus $\mathbb{T}^n$ the normalised Lebesgue measure). If $X \hookrightarrow \ell_2$ then [8, Remark 13.22] shows that $H_p(X \cap \mathbb{D}^{\mathbb{N}}) = H_p(\ell_2 \cap \mathbb{D}^{\mathbb{N}})$ for every $1 \leq p < \infty$, and every compact set in $X \cap \mathbb{D}^{\mathbb{N}}$ is also compact in $\ell_2 \cap \mathbb{D}^N$. Then as an immediate consequence of Theorem 1 we have

Corollary 5. *Let X be a Banach sequence space with $X \hookrightarrow \ell_2$ (with continuous inclusion) and $1 \leq p < \infty$. If $(f_n) \subseteq H_p(X \cap \mathbb{D}^{\mathbb{N}})$ is a bounded sequence, then there exist $f \in H_p(X \cap \mathbb{D}^{\mathbb{N}})$ and a subsequence (f_{n_k}) that converges to f uniformly on the compact subsets of $X \cap \mathbb{D}^{\mathbb{N}}$.*

3 An application to Dirichlet series

Dirichlet series are formal series of the form $\sum a_n n^{-s}$, where $a_n \in \mathbb{C}$ and s is a complex variable. It is well known that Dirichlet series converge on half-planes and there define holomorphic functions. The space $\mathcal{H}_\infty$ consists of all Dirichlet series that converge on $[\operatorname{Re} s > 0]$ and there define a bounded holomorphic function, which the norm defined as the supremum on $[\operatorname{Re} s > 0]$ is a Banach space. The reader is referred to [8, 11] for a complete account on this theory.
Given a sequence of Dirichlet series in $\mathcal{H}_\infty$ one can look at them as functions on the right half-plane and try to apply Montel's theorem. This would have two problems: one would get a subsequence that converges uniformly on the compacts (and not on half-planes, which are the natural setting for Dirichlet series), and, moreover, it would converge to some holomorphic function on $[\operatorname{Re} s > 0]$ which might or might not be represented by a Dirichlet series. So one could say that the classical Montel's theorem is useless within the context of Dirichlet series. Bayart overcame this problem in [2, Lemma 18], proving a Montel-type theorem for Dirichlet series: *let $\{\sum a_n^{(N)} n^{-s}\}_N$ be a bounded sequence in $\mathcal{H}_\infty$; then it has a subsequence that converges to a Dirichlet series $\sum a_n n^{-s}$ in $\mathcal{H}_\infty$ uniformly on $[\operatorname{Re} s > \sigma]$ for every $\sigma > 0$.* This has several interesting applications within the functional-analytic theory of Dirichlet series.
Let us look at this result from a slightly different point of view. If $\sum a_n n^{-s}$ is a Dirichlet series in $\mathcal{H}_\infty$ and $D(s)$ is the holomorphic function that it defines on $[\operatorname{Re} s > 0]$, then for $\varepsilon > 0$ we have

$$\sup_{\operatorname{Re} s > \varepsilon} \Big| \sum_{n=1}^{\infty} a_n \frac{1}{n^s} \Big| = \sup_{\operatorname{Re} s > \varepsilon} |D(s)| = \sup_{\operatorname{Re} s > 0} |D(s+\varepsilon)| = \sup_{\operatorname{Re} s > 0} \Big| \sum_{n=1}^{\infty} a_n \frac{1}{n^{s+\varepsilon}} \Big| = \sup_{\operatorname{Re} s > 0} \Big| \sum_{n=1}^{\infty} \frac{a_n}{n^\varepsilon} \frac{1}{n^s} \Big| .$$

So Bayart's Montel theorem for Dirichlet series tells us that, if D_N is the sequence of functions associated to the Dirichlet series, then there is a subsequence and a Dirichlet series with limit function D so that the translated Dirichlet series $D_{N_k}(s+\varepsilon)$ converges to $D(s+\varepsilon)$ uniformly on $[\operatorname{Re} s > 0]$ for every $\varepsilon > 0$. Or, to put it in other terms: *let $\{\sum a_n^{(N)} n^{-s}\}_N$ be a bounded sequence in $\mathcal{H}_\infty$; then there is a subsequence $\{\sum a_n^{(N_k)} n^{-s}\}_k$ and a Dirichlet series $\sum a_n n^s$ in $\mathcal{H}_\infty$ so that $\{\sum \frac{a_n^{(N_k)}}{n^\varepsilon} n^{-s}\}_k$ converges in $\mathcal{H}_\infty$ to $\sum \frac{a_n}{n^\varepsilon} n^{-s}$ for every $\varepsilon > 0$.*

Hardy spaces of Dirichlet series (denoted $\mathcal{H}_p$, for $1 \leq p < \infty$) were introduced by Bayart in [2] in the following way: given $1 \leq p < \infty$, the expresion

$$\Big\| \sum_{n=1}^{N} a_n n^{-s} \Big\|_p = \lim_{R \to \infty} \left(\frac{1}{2R} \int_{-R}^{R} \Big| \sum_{n=1}^{N} a_n n^{-it} \Big|^p dt \right)^{\frac{1}{p}} .$$

defines a norm on the space of Dirichlet polynomials (i.e. finite Dirichlet series). Then the space $\mathcal{H}^p$ is defined as the completion of the Dirichlet polynomials under this norm. It is well known [2, Theorem 3] (see also [8, Remark 12.13]) that the Dirichlet series that lie on $\mathcal{H}^p$ define holomorphic functions on the half-plane $[\operatorname{Re} s > 1/2]$. These spaces are also closely related with the Hardy spaces of holomorphic functions that we considered above. A Dirichlet series $\sum a_n n^{-s}$ belongs to $\mathcal{H}_p$ if and only if there exists $f \in H_p(\ell_2 \cap \mathbb{D}^{\mathbb{N}})$ so that $a_n = c_\alpha(f)$ (recall (5)) whenever $n = \mathfrak{p}^\alpha$ (where $\mathfrak{p} = (\mathfrak{p}_k)_k$ is the sequence of prime numbers), and both have the same norm (see [4, Theorem 3.9] or [8, Corollary 13.3]). In other words,

$$\mathcal{H}_p = H_p(\ell_2 \cap \mathbb{D}^{\mathbb{N}}) \tag{8}$$

as Banach spaces.

Our aim now is to show the following version of Bayart's Montel-type theorem for Hardy spaces.

Theorem 6. *Let $\{\sum a_n^{(N)} n^{-s}\}_N$ be a bounded sequence in $\mathcal{H}^p$. Then there exist $\sum a_n n^{-s} \in \mathcal{H}^p$, and a subsequence $\{\sum a_n^{(N_k)} n^{-s}\}_k$ such that $\{\sum \frac{a_n^{(N_k)}}{n^\varepsilon} n^{-s}\}_k$ converges to $\sum \frac{a_n}{n^\varepsilon} n^{-s}$ in $\mathcal{H}^p$ for every $\varepsilon > 0$.*

Our approach is to switch from uniform convergence on compact subsets (see Theorem 1) to the convergence of translates of the series in $\mathcal{H}^p$. Note that $\varepsilon = 0$ cannot be taken in Theorem 6. For example, given $1 \le p < \infty$ the sequence of monomials $(n^{-s})_n$ is bounded ($\|n^{-s}\|_{\mathcal{H}_p} = 1$) but it does not have a convergent subsequence.

Recently and independently in [10, Theorem 5.8], using techniques of harmonic analysis on compact abelian groups, a Montel-type theorem has been obtained in the more general setting of general Dirichlet series with frequencies satisfying Bohr's theorem. Although there is no statement written explicitly for the classical Hardy spaces of Dirichlet series. Also, Bayart [3] has drawn our attention to a direct proof of Theorem 6 based on the $p = 2$ case (somewhat easier to handle) and an argument through translation of Dirichlet series. We go here a different way, showing how it can be obtained also from Theorem 1.

Let us make first a short comment that we will use within the proof. First of all, let us recall that there is a constant $C > 0$ so that, for every $1 \le p < \infty$, all $x \ge 2$ and $\sum a_n n^{-s} \in \mathcal{H}^p$,

$$\Big\| \sum_{n \le x} a_n n^{-s} \Big\|_{\mathcal{H}^p} \le C \log x \Big\| \sum a_n n^{-s} \Big\|_{\mathcal{H}^p} \tag{9}$$

(see [8, Theorem 12.5]). If we translate the series a little bit we can say more.
If a Dirichlet series $\sum a_n n^{-s}$ belongs to $\mathcal{H}^p$ for some $1 \le p < \infty$, then the translated series $\sum \frac{a_n}{n^\varepsilon} n^{-s}$ belongs to $\mathcal{H}^q$ for all $\varepsilon > 0$ and every $p < q < \infty$ (see [2, Section 3] or [8, Theorem 12.9]). This, combined with the fact that the monomials $\{n^{-s}\}_n$ form a Schauder basis of $\mathcal{H}^q$ (see [1]) for $1 < q < \infty$ and the monotonicity of the $\mathcal{H}_p$-norms immediately imply

$$\lim_{l \to \infty} \Big\| \sum \frac{a_n}{n^\varepsilon} n^{-s} - \sum_{n=1}^{l} \frac{a_n}{n^\varepsilon} n^{-s} \Big\|_{\mathcal{H}^p} = 0 \tag{10}$$

for every $\varepsilon > 0$.

Proof of Theorem 6. By (8), we have a bounded sequence $(f_N)_N$ in $H_p(\ell_2 \cap \mathbb{D}^{\mathbb{N}})$ and, by Theorem 1, a subsequence $(f_{N_k})_k$ that converges uniformly on the compact sets to some

$f \in H_p(\ell_2 \cap \mathbb{D}^{\mathbb{N}})$. Moreover, if $\sum a_n n^{-s} \in \mathcal{H}^p$ is the Dirichlet series associated to f by (8), then (5) (and the uniform convergence on compact sets) yields $a_n^{(N_k)} = c_\alpha(f_{N_k}) \to c_\alpha(f) = a_n$ as $k \to \infty$.

Fix $\varepsilon > 0$ and note that, from (10) we have that the partial sums of $\sum \frac{a_n}{n^\varepsilon} n^{-s}$ and $\sum \frac{a_n^{(N_k)}}{n^\varepsilon} n^{-s}$ converge to the corresponding series. Our first aim is to show that we can control the convergence of all these partial sums uniformly in some sense. To be more precise, our goal now is to show that for every $\eta > 0$ there is some $l_0 > 0$ such that

$$\Big\| \sum \frac{a_n}{n^\varepsilon} n^{-s} - \sum_{n=1}^{l} \frac{a_n}{n^\varepsilon} n^{-s} \Big\|_{\mathcal{H}^p} < \eta \quad \text{and} \quad \Big\| \sum \frac{a_n^{(N_k)}}{n^\varepsilon} n^{-s} - \sum_{n=1}^{l} \frac{a_n^{(N_k)}}{n^\varepsilon} n^{-s} \Big\|_{\mathcal{H}^p} < \eta \tag{11}$$

for every $l \geq l_0$. To begin with we fix $l \geq 2$ and (10) gives

$$\Big\| \sum \frac{a_n}{n^\varepsilon} n^{-s} - \sum_{n=1}^{l} \frac{a_n}{n^\varepsilon} n^{-s} \Big\|_{\mathcal{H}^p} = \lim_{j \to \infty} \Big\| \sum_{n=l+1}^{j} \frac{a_n}{n^\varepsilon} n^{-s} \Big\|_{\mathcal{H}^p}.$$

We denote $M = \max \big\{ \| \sum a_n n^{-s} \|_{\mathcal{H}^p}, \sup_N \| \sum a_n^{(N)} n^{-s} \|_{\mathcal{H}^p} \big\}$ and, for each $j \geq l+1$ Abel summation and (9) give

$$\begin{aligned}
&\Big\| \Big(\sum_{n=l+1}^{j} a_n n^{-s} \Big) \frac{1}{j^\varepsilon} + \sum_{n=l+1}^{j-1} \Big(\sum_{m=l+1}^{n} a_m m^{-s} \Big) \Big(\frac{1}{n^\varepsilon} - \frac{1}{(n+1)^\varepsilon} \Big) \Big\|_{\mathcal{H}^p} \\
&\leq \Big\| \sum_{n=l+1}^{j} a_n n^{-s} \Big\|_{\mathcal{H}^p} \frac{1}{j^\varepsilon} + \sum_{n=l+1}^{j-1} \Big\| \sum_{m=l+1}^{n} a_m m^{-s} \Big\|_{\mathcal{H}^p} \Big(\frac{1}{n^\varepsilon} - \frac{1}{(n+1)^\varepsilon} \Big) \\
&\leq \Big(\Big\| \sum_{n=1}^{j} a_n n^{-s} \Big\|_{\mathcal{H}^p} + \Big\| \sum_{n=1}^{l} a_n n^{-s} \Big\|_{\mathcal{H}^p} \Big) \frac{1}{j^\varepsilon} \\
&\quad + \sum_{n=l+1}^{j-1} \Big(\Big\| \sum_{m=1}^{n} a_m m^{-s} \Big\|_{\mathcal{H}^p} + \Big\| \sum_{m=1}^{l} a_m m^{-s} \Big\|_{\mathcal{H}^p} \Big) \Big(\frac{1}{n^\varepsilon} - \frac{1}{(n+1)^\varepsilon} \Big) \\
&\leq \big(C \log(j) M + C \log(l) M \big) \frac{1}{j^\varepsilon} \\
&\quad + \sum_{n=l+1}^{j-1} \big(C \log(n) M + C \log(l) M \big) \Big(\frac{1}{n^\varepsilon} - \frac{1}{(n+1)^\varepsilon} \Big) \\
&\leq 2C \log(j) M \frac{1}{j^\varepsilon} + \sum_{n=l+1}^{j-1} 2C \log(n) M \frac{\varepsilon}{n^{\varepsilon+1}} \\
&\leq 2C \log(j) M \frac{1}{j^\varepsilon} + \sum_{n=l+1}^{j-1} 2C \log(n) M \frac{\varepsilon}{n^{\varepsilon+1}}.
\end{aligned}$$

Hence

$$\Big\| \sum \frac{a_n}{n^\varepsilon} n^{-s} - \sum_{n=1}^{l} \frac{a_n}{n^\varepsilon} n^{-s} \Big\|_{\mathcal{H}^p} \leq \varepsilon C M \sum_{n=l+1}^{\infty} \frac{\log(n)}{n^{\varepsilon+1}}.$$

For each fixed k, exactly the same computations give the same inequality for $\sum a_n^{(N_k)} n^{-s}$. Since the term on the right-hand-side tends to 0 as $l \to \infty$ we can find l_0 satisfying (11).

Set $L = \max_{1\le n\le l_0}\{\|n^{-s}\|_{\mathcal{H}^p}\}$ and pick $k_0 \in \mathbb{N}$, such that if $k \ge k_0$ then $|a_n^{(N_k)} - a_n| < \frac{\eta}{l_0 L}$ for all $1 \le n \le l_0$. With all this we finally have, for $k \ge k_0$

$$\begin{aligned}\Big\|\sum \frac{a_n^{(N_k)}}{n^\varepsilon}n^{-s} - \sum \frac{a_n}{n^\varepsilon}n^{-s}\Big\|_{\mathcal{H}^p} \le& \Big\|\sum \frac{a_n^{(N_k)}}{n^\varepsilon}n^{-s} - \sum_{n=1}^{l_0} \frac{a_n^{(N_k)}}{n^\varepsilon}n^{-s}\Big\|_{\mathcal{H}^p} \\ &+ \Big\|\sum \frac{a_n}{n^\varepsilon}n^{-s} - \sum_{n=1}^{l_0} \frac{a_n}{n^\varepsilon}n^{-s}\Big\|_{\mathcal{H}^p} + \Big\|\sum_{n=1}^{l_0} \frac{a_n^{(N_k)} - a_n}{n^\varepsilon}n^{-s}\Big\|_{\mathcal{H}^p} \\ \le& 2\eta + \sum_{n=1}^{l_0} \frac{|a_n^{(N_k)} - a_n|}{n^\varepsilon}\|n^{-s}\|_{\mathcal{H}^p} \\ \le& 2\eta + \sum_{n=1}^{l_0} |a_n^{(N_k)} - a_n|\,\|n^{-s}\|_{\mathcal{H}^p} < 3\eta\,. \qquad \square\end{aligned}$$

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