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On a connection between the N-dimensional fractional Laplacian and 1-D operators on lattices [☆]

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ABSTRACT

We show the remarkable fact that the nonlocal property of the discrete N-dimensional fractional Laplacian acting in the second variable of the lattice $\mathbb{N} \times \mathbb{Z}^N$ can be exchanged with an equivalent memory corresponding to a power of a one-dimensional operator that acts only on the first variable of the complete lattice $\mathbb{Z} \times \mathbb{Z}^N$. This property allows to reduce the number of calculations and leads to more complete analytical solutions of mathematical models on lattices. The connection is established by showing that a first order equation in the first variable, and of fractional order $\alpha > 0$ in the second, has the same solution as another of order $1/\alpha$ in the first variable and integer order in the second. As a result, we provide for the first time the fundamental solution for the N-dimensional heat equation discrete in time and space.

1. Introduction

In recent years, much attention has been paid to the discrete fractional Laplacian operator as the natural counterpart of the continuous one [1,6–8,23,18,24,25]. One of the most natural definitions in the N-dimensional case can be found in [18, Section 6] where it is defined as

$$(-\Delta_{d,N})^{\alpha} f(\boldsymbol{n}) = \frac{1}{\Gamma(-\alpha)} \int_0^{\infty} (e^{t\Delta_{d,N}} f(\boldsymbol{n}) - f(\boldsymbol{n})) \frac{dt}{t^{1+\alpha}}, \quad \boldsymbol{n} = (n_1, ..., n_N) \in \mathbb{Z}^N, \tag{1}$$

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whenever $0 < \alpha < 1$ and $f \in \ell^\infty(\mathbb{Z}^N)$. Here $e^{t\Delta_{d,N}}$ is the semidiscrete heat semigroup generated by the N-dimensional discrete Laplacian $\Delta_{d,N}$ defined as

$$\Delta_{d,N} f(\boldsymbol{n}) = \sum_{j=1}^{N} \left(f(\boldsymbol{n}+e_j) - 2f(\boldsymbol{n}) + f(\boldsymbol{n}-e_j) \right),$$

where e_j denotes the unit vector in the positive direction of the j-th coordinate. It is known [18, Section 6] that the semidiscrete heat semigroup is represented by

$$e^{t\Delta_{d,N}}\varphi(\mathbf{n}) = \sum_{\mathbf{k}\in\mathbb{Z}^N} \mathbf{T}_t(\mathbf{n}-\mathbf{k})\varphi(\mathbf{k}), \quad t \geq 0, \quad \mathbf{n} \in \mathbb{Z}^N,$$

where the kernel is given by

$$\mathbf{T}_t(\mathbf{n}) = \prod_{j=1}^{N} e^{-2t} I_{n_j}(2t), \quad t \geq 0, \quad \mathbf{n} \in \mathbb{Z}^N,$$

and I_n denotes the modified Bessel function.

The 1-dimensional case has been extensively studied in recent articles by Ciaurri et al. [6,8,18]. The operator $(-\Delta_{d,1})^\alpha$ can be used to describe the non-local motion of a particle (electron) in a one-dimensional chain with atoms located at all integer lattice points in $\mathbb{Z}$, see [22]. Tarasov [27] provides a formulation of fractional calculus for N-dimensional lattices. See also [28] for the exact discretization of the fractional Laplacian for N-dimensional spaces.

On the other side, when we consider anomalous diffusion processes, several classes of fractional *in time* operators have been proposed in the literature, the most popular being the Riemann-Liouville or Caputo type. In the discrete context, there are several approaches that might be appropriate from either an applied or analytic perspective [11,17,20].

One of the most important facts why this type of fractional operators (in space and time) is relevant in the current literature, is due to their ability to capture memory effects in the mathematical modeling, which are absent in the integer case. This type of phenomenology has been shown widely. However, the existence of a probable relationship between memory in time and memory in space for fractional operators, as well as a plausible explanation for this kind of interaction, has been an open problem for some time.

Probably the first insight about this kind of relationship was given in 2002 by Kulish and Lage in [16], where, in the context of fluid mechanics, they establish the existence of a relationship between the operators Δ (the Laplacian) and $D^{1/2}$ (half order Riemann-Liouville) proving that a PDE of first order in time and second order in space has the same solution as a PDE of half order in time and first order in space. One of the main advantages of this conversion is the fact it can significantly reduce the number of computations as well as lead to more comprehensive analytical solutions [13, Section 6.1.2.4], [16, Section 4].

This problem was later considered in [14, Theorem 1.1], where the authors proved a link between integer powers of operators acting in space, A^n, and the fractional powers, $D^{1/n}$, of the Riemann-Liouville operator acting in time, under the condition that A is the generator of a C_0-semigroup (in case $n = 2$) as well as generalized families of operators related to the abstract Cauchy problem of fractional order (in case $n \neq 2$). This result explains previous studies by Baeumer, Meerschaert and Nane [5] among others. See also [4] for further developments in this research line in the context of stochastic processes.

The connection between the discrete fractional Laplacian $(-\Delta_d)^\alpha$, and the *continuous in time* fractional order operator of Liouville type (left side) ${}_L D^{1/\alpha}$, was made in [7]. This result has subsequently been useful for discussions on generalized diffusion of graphs by Estrada et al. in [9], and by Padgett et al. in [22, Section 2.3] in the context of anomalous diffusion in one-dimensional disordered systems. This result closes the problem in the case of operators acting on a semi-lattice.

However, the connection between the discrete fractional Laplacian with some *discrete time* fractional order operators, that is, operators acting on a complete lattice, remains open.

In this paper, we solve this problem, by showing a relationship between the discrete fractional Laplacian and the following discrete in time fractional order operator defined in [19, formula (28)] by Ortigueira et al., in the context of signal analysis:

$$D_{\nabla}^{\beta} f(n) := \sum_{j=-\infty}^{n} \frac{\Gamma(-\beta+n-j)}{\Gamma(-\beta)(n-j)!} f(j), \quad n \in \mathbb{Z}, \tag{2}$$

which is initially defined for all $\beta > 0$ except positive integer values (see (7) below for an extension). It is worth mentioning this operator approximates the forward Liouville derivative [19]. We also note that similar definitions have appeared in relation to fractional partial difference-differential equations in articles by Abadias et al. [2,3].

Our main results in this article can be summarized as follows: We first show that (1) is equivalent to convolving f with a distinguished kernel. In particular, this allows us to extend the definition given in (1) for all $\alpha > 0$ and to conclude that there is a connection between the fractional Laplacian as defined in (1) and the Riesz derivative (see [21, Section 5.2]). Then, we find by the first time the fundamental solution for the following N-dimensional heat equation discrete in time, and equipped with a discrete fractional N-dimensional Laplacian

$$\begin{cases} v(m+1,\boldsymbol{n}) - v(m,\boldsymbol{n}) = -(-\Delta_{d,N})^{\alpha} v(m+1,\boldsymbol{n}), \quad \boldsymbol{n} \in \mathbb{Z}^N, \quad m \in \mathbb{N}, \\ v(0,\boldsymbol{n}) = \phi(\boldsymbol{n}) \end{cases} \tag{3}$$

for any $\alpha > 0$. We note that fundamental solutions for the heat and wave equations, but only on a semi-lattice, have been already shown by other authors. See e.g. [10,15] and references therein.

Our main result shows that in case $\alpha = \frac{1}{p}$, $p \in \mathbb{N}$, the solution of (3) coincides with the solution of the following equation, equipped with the fractional order operator defined in (2) acting in discrete time ($m \in \mathbb{N}$), and the N-dimensional Laplacian acting in discrete space ($\boldsymbol{n} \in \mathbb{Z}^N$)

$$\begin{cases} D_{\nabla}^{p} v(m,\boldsymbol{n}) = (-1)^{p+1} \Delta_{d,N} v(m,\boldsymbol{n}), \quad \boldsymbol{n} \in \mathbb{Z}^N, \quad m \in \mathbb{N}, \\ v(0,\boldsymbol{n}) = \phi(\boldsymbol{n}) \\ v(-j,\boldsymbol{n}) = (Id + (-\Delta_{d,N})^{1/p})^j \phi(\boldsymbol{n}), \quad j \in \mathbb{N}. \end{cases} \tag{4}$$

This fact reveals the significant property that the spatial memory of the N-dimensional fractional Laplacian $-(-\Delta_{d,N})^{1/p}$ can be exchanged with the (one-dimensional) temporal integer derivative D_{∇}^{p}, being the spatial memory converted into temporal memory that is hosted in the past, or history, of the model.

To exemplify how this result helps to reduce the number of computations, we illustrate the case $p = 2$, $N = 1$ where the model (3) admits a complicated structure while the model (4) supports the simple form:

$$v(m,n) = v(m,n+1) + v(m,n-1) - 2v(m-1,n) + v(m-2,n), \quad m \in \mathbb{N}, \ n \in \mathbb{Z},$$

with given initial condition $v(0,n)$ and the only knowledge of $v(-1,n)$ in the past. Finally, we show that the lattice equations $v(m+1,\boldsymbol{n}) - v(m,\boldsymbol{n}) = (-\Delta_{d,N})^{\alpha} v(m+1,\boldsymbol{n})$ and $D_{\nabla}^{1/\alpha} v(m,\boldsymbol{n}) = \Delta_d v(m,\boldsymbol{n})$ with initial condition $v(0,\boldsymbol{n}) = \varphi(\boldsymbol{n})$, defined on the lattice $\mathbb{Z} \times \mathbb{Z}^N$, have the same solution for any $0 < \alpha \leq \frac{1}{2+\log_2 N}$ and the amount of memory that depends on α in the discrete fractional Laplacian, appears in the initial data for negative values of m in the second named equation. In other words, we show that the spatial memory of the discrete fractional Laplacian for an equation defined on the half lattice $\mathbb{N}\times\mathbb{Z}^N$ appears in the past history of a discrete in time equation on the entire lattice $\mathbb{Z}\times\mathbb{Z}^N$.

2. Preliminaries

In what follows we denote $\mathbb{N}=\{0,1,2,3,\ldots\}$. The discrete time Fourier transform for a sequence ϕ is defined by

$$\mathcal{F}_{\mathbb{Z}^N}(\phi)(\boldsymbol{\theta})\equiv\widehat{\phi}(\boldsymbol{\theta}):=\sum_{\boldsymbol{j}\in\mathbb{Z}^N}e^{i\boldsymbol{j}\cdot\boldsymbol{\theta}}\phi(\boldsymbol{j}),\qquad \boldsymbol{\theta}=(\theta_1,\theta_2,\ldots,\theta_N)\in[-\pi,\pi]^N. \tag{5}$$

The inverse discrete time Fourier transform is stated as follows:

$$\check{\phi}(\boldsymbol{n}):=\frac{1}{(2\pi)^N}\int\limits_{[-\pi,\pi]^N}\phi(\boldsymbol{\theta})e^{-i\boldsymbol{n}\cdot\boldsymbol{\theta}}d\boldsymbol{\theta},\quad \boldsymbol{n}\in\mathbb{Z}^N. \tag{6}$$

In what follows $\delta_i(j)$ denotes the Kronecker delta. Given $\beta\in\mathbb{R}$, we consider the sequence:

$$k^\beta(n)=\begin{cases}\dfrac{\Gamma(\beta+n)}{\Gamma(\beta)\Gamma(n+1)} & n\in\mathbb{N},\ \beta\in\mathbb{R}\setminus\{-1,-2,..\},\\[2ex] (\delta_0-\delta_1)^{*(-\beta)}(n) & n\in\mathbb{N},\ \beta\in\{-1,-2,...\},\end{cases} \tag{7}$$

where Γ is the Euler gamma function and $p^{*n}=\underbrace{p*p*\ldots*p}_{n\text{-times}}$ where $*$ denotes the convolution of sequences given by $(u*v)(n)=\sum_{j=0}^{n}u(n-j)v(j)$. In [17], it was proven the generating function of the sequence $(k^\beta(j))_{j=0}^\infty$:

$$\sum_{j=0}^{\infty}k^\beta(j)z^j=\frac{1}{(1-z)^\beta},\quad \beta\in\mathbb{R},\quad |z|<1 \tag{8}$$

and some other properties concerning the sequence k^β.

It should be noted that defining k^β instead of using binomial coefficients or using the Pochhammer symbol has several advantages, as it has been demonstrated in recent articles, see e.g. [11] and its references. Observe that the (negative) integer case of β, i.e. the second part of (7), is motivated by the formula (8) and the property $\delta_i*\delta_j=\delta_{i+j}$.

Remark 2.1. Concerning convergence of the series (8), we note that if $-\beta$ is neither a natural number nor zero, the series converges absolutely for $|z|<1$ and diverges for $|z|>1$. For $z=-1$, the series converges for $\beta<1$ and diverges for $\beta\geq1$. For $z=1$, it converges absolutely for $\beta<0$ and diverges for $\beta>0$. If $-\beta=n$ is a natural number, the series (8) is reduced to a finite sum (binomial formula), see [12, Formula 1.110].

We recall that the forward Euler operator of a given sequence f is defined by

$$D^1_\Delta f(n):=f(n+1)-f(n),\ n\in\mathbb{N}.$$

The following definition can be found in [19, formula (27) with $h=1$].

Definition 2.2. Given $\beta \in \mathbb{R}_+$, the fractional difference of order β of a given bounded sequence f is defined by

$$D_{\nabla}^{\beta} f(n) := \sum_{j=-\infty}^{n} k^{-\beta}(n-j)f(j) = \sum_{j=0}^{\infty} k^{-\beta}(j)f(n-j), \quad n \in \mathbb{Z}. \tag{9}$$

Observe that the series converges because $k^{-\beta}$ has order $O(1/n^{\beta+1})$, see [11, Proposition 3.1]. As an illustrative example, we note that formula (9) when $\beta = 2$ reads as follows:

$$\begin{aligned} D_{\nabla}^{2} f(n) &= \sum_{j=-\infty}^{n} k^{-2}(n-j)f(j) = \sum_{j=-\infty}^{n} (\delta_0 - \delta_1)^{*2}(n-j)f(j) \\ &= \sum_{j=-\infty}^{n} (\delta_0 - 2\delta_1 + \delta_2)(n-j)f(j) = f(n) - 2f(n-1) + f(n-2), \quad n \in \mathbb{Z}. \end{aligned} \tag{10}$$

Remark 2.3. In [19] it is shown that the fractional operator $D_{\nabla}^{\alpha} f(n)$ approximates the forward Liouville derivative of order $\alpha > 0$ given by $D_t^{\alpha} f(t) = \frac{\partial^m}{\partial t^m} \int_{-\infty}^{t} g_{n-\alpha}(t-s)f(s)$ where $m = \lfloor \alpha \rfloor + 1, t \in \mathbb{R}$ and, for every $t > 0$, $g_{\beta}(t) := \frac{t^{\beta}}{\Gamma(\beta)}$.

3. Main results

We begin with the following result, that generalizes [7, Theorem 2] to the N-dimensional case.

Theorem 3.1. *For all $0 < \alpha < 1$ and $f \in \ell^{\infty}(\mathbb{Z}^N)$ the following holds:*

$$(-\Delta_{d,N})^{\alpha} f(\boldsymbol{n}) = \sum_{\boldsymbol{j} \in \mathbb{Z}^N} K^{\alpha}(\boldsymbol{n} - \boldsymbol{j}) f(\boldsymbol{j}), \quad \boldsymbol{n} \in \mathbb{Z}^N,$$

where

$$K^{\alpha}(\boldsymbol{n}) := \frac{1}{(2\pi)^N} \int_{-[\pi,\pi]^N} \left(\sum_{j=0}^{N} 4\sin^2(\theta_j/2) \right)^{\alpha} e^{-i\boldsymbol{n}\cdot\boldsymbol{\theta}} d\boldsymbol{\theta}, \quad \boldsymbol{n} \in \mathbb{Z}^N, \quad \alpha > 0. \tag{11}$$

Proof. By [18, Lemma 6.5] we have

$$\mathcal{F}_{\mathbb{Z}^N}((-\Delta_{d,N})^{\alpha} f)(\boldsymbol{\theta}) = \left(\sum_{j=1}^{N} 4\sin^2(\theta_j/2) \right)^{\alpha} \mathcal{F}_{\mathbb{Z}^N}(f)(\boldsymbol{\theta}).$$

On the other hand, we have

$$\mathcal{F}_{\mathbb{Z}^N}(K^{\alpha})(\boldsymbol{\theta}) = \left(\sum_{j=1}^{N} 4\sin^2(\theta_j/2) \right)^{\alpha}, \tag{12}$$

and the claim follows from the convolution and uniqueness properties of the Fourier transform. □

Since the formula for K^{α} holds for any $\alpha > 0$, we could extend the definition of the fractional Laplacian to the case $\alpha \geq 1$ by means of the right-hand side of the above theorem. As a further consequence, the following property of associativity holds:

$$(-\Delta_{d,N})^{\alpha}(-\Delta_{d,N})^{\beta} = (-\Delta_{d,N})^{\alpha+\beta} \quad \text{whenever} \quad \alpha+\beta > -1. \tag{13}$$

For the 1-dimensional case, see [7,18] and the references therein.

In light of the above result, it is worth comparing the definition of fractional Laplacian given here with the two-sided fractional derivatives introduced by Ortigueira [21]. According to [21, Definition 2.1] a two-sided Grünwald-Letnikov type fractional derivative of a real function f is defined by

$$D_{\theta}^{\beta} f(x) = \lim_{h\to 0} \frac{1}{h^{\beta}} \sum_{n\in\mathbb{Z}} \frac{(-1)^n \Gamma(\beta+1)}{\Gamma(\frac{\beta+\theta}{2}-n+1)\Gamma(\frac{\beta-\theta}{2}+n+1)} f(x-nh), \tag{14}$$

where $\beta > -1$ is the derivative order and $\theta \in \mathbb{R}$ is an asymmetry parameter. Define:

$$K_{\theta}^{\beta}(n) = \frac{(-1)^n \Gamma(\beta+1)}{\Gamma(\frac{\beta+\theta}{2}-n+1)\Gamma(\frac{\beta-\theta}{2}+n+1)}, \quad n \in \mathbb{Z}. \tag{15}$$

Choosing $\theta = 0$ and $\beta = 2\alpha$ we obtain

$$K_0^{2\alpha}(n) = \frac{(-1)^n \Gamma(2\alpha+1)}{\Gamma(\alpha-n+1)\Gamma(\alpha+n+1)}, \quad n \in \mathbb{Z}, \tag{16}$$

which matches (11) in the 1-dimensional case (see [7, Remark 1]). The above observation shows that there is an interesting connection between the fractional Laplacian as defined in (1) and the Riesz derivative (see [21, Section 5.2]).

The first result of this article is the following theorem.

Theorem 3.2. *For any $\alpha > 0$, and $\phi \in \ell^{\infty}(\mathbb{Z}^N)$, the N-dimensional heat equation with discrete time and discrete space*

$$\begin{cases} D_{\Delta}^1 v(m,\boldsymbol{n}) = -(-\Delta_{d,N})^{\alpha} v(m+1,\boldsymbol{n}), & \boldsymbol{n} \in \mathbb{Z}^N, \quad m \in \mathbb{N}, \\ v(0,\boldsymbol{n}) = \phi(\boldsymbol{n}) \end{cases} \tag{17}$$

admits as unique solution the formula

$$v(m,\boldsymbol{n}) = \sum_{\boldsymbol{j}\in\mathbb{Z}^N} G_m^{\alpha}(\boldsymbol{n}-\boldsymbol{j})\phi(\boldsymbol{j}) \quad \boldsymbol{n} \in \mathbb{Z}^N, \quad m \in \mathbb{Z}, \tag{18}$$

where

$$G_m^{\alpha}(\boldsymbol{n}) := \frac{1}{(2\pi)^N} \int_{[-\pi,\pi]^N} e^{-i\boldsymbol{n}\cdot\boldsymbol{\theta}} (1 + (\sum_{j=1}^N 4\sin^2(\theta_j/2))^{\alpha})^{-m} d\boldsymbol{\theta}. \tag{19}$$

Proof. Let check that $v(m,\boldsymbol{n}) = \sum_{\boldsymbol{j}\in\mathbb{Z}^N} G_m^{\alpha}(\boldsymbol{n}-\boldsymbol{j})\phi(\boldsymbol{j})$ is a solution of (17). Indeed, let denote $a_{\theta} := \sum_{j=1}^N (4\sin^2(\theta_j/2))$, then for every $\boldsymbol{n} \in \mathbb{Z}^N$ and $m \in \mathbb{N}$ we have:

$$\begin{aligned} D_{\Delta}^1 G_m^{\alpha}(\boldsymbol{n}) &= \frac{1}{(2\pi)^N} \int_{[-\pi,\pi]^N} e^{-i\boldsymbol{n}\cdot\boldsymbol{\theta}} [(1+a_{\theta}^{\alpha})^{-(m+1)} - (1+a_{\theta}^{\alpha})^{-m}] d\boldsymbol{\theta} \\ &= \frac{-1}{(2\pi)^N} \int_{[-\pi,\pi]^N} e^{-i\boldsymbol{n}\cdot\boldsymbol{\theta}} a_{\theta}^{\alpha} (1+a_{\theta}^{\alpha})^{-(m+1)} d\boldsymbol{\theta}. \end{aligned} \tag{20}$$

As a result,

$$\begin{aligned} D^1_\Delta v(m,\boldsymbol{n}) &= \sum_{\boldsymbol{j}\in\mathbb{Z}^N} D^1_\Delta G^\alpha_m(\boldsymbol{n}-\boldsymbol{j})\phi(\boldsymbol{j}) \\ &= -\sum_{\boldsymbol{j}\in\mathbb{Z}^N} \frac{1}{(2\pi)^N} \int\limits_{[-\pi,\pi]^N} e^{-i(\boldsymbol{n}-\boldsymbol{j})\cdot\boldsymbol{\theta}} a^\alpha_\theta (1+a^\alpha_\theta)^{-(m+1)}\phi(\boldsymbol{j})d\boldsymbol{\theta} \quad \boldsymbol{n}\in\mathbb{Z}^N, \quad m\in\mathbb{N}. \end{aligned} \tag{21}$$

On the other hand, from Theorem 3.1 and Fubini's theorem, we get for each $\boldsymbol{n}\in\mathbb{Z}^N$ and $m\in\mathbb{N}$ that:

$$\begin{aligned} (-\Delta_{d,N})^\alpha G^\alpha_m(\boldsymbol{n}) &= \frac{1}{(2\pi)^N} \int\limits_{[-\pi,\pi]^N} \left(\sum_{\boldsymbol{j}\in\mathbb{Z}^N} K^\alpha(\boldsymbol{j}) e^{-i(\boldsymbol{n}-\boldsymbol{j})\cdot\boldsymbol{\theta}} \right) (1+a^\alpha_\theta)^{-m} d\boldsymbol{\theta} \\ &= \frac{1}{(2\pi)^N} \int\limits_{[-\pi,\pi]^N} e^{-i\boldsymbol{n}\cdot\boldsymbol{\theta}} a^\alpha_\theta (1+a^\alpha_\theta)^{-m} d\boldsymbol{\theta}, \end{aligned} \tag{22}$$

where we used the identity (12) in the last equality. Consequently, we get

$$\begin{aligned} (-\Delta_{d,N})^\alpha v(m+1,\boldsymbol{n}) &= \sum_{\boldsymbol{j}\in\mathbb{Z}^N} (-\Delta_{d,N})^\alpha G^\alpha_{m+1}(\boldsymbol{n}-\boldsymbol{j})\phi(\boldsymbol{j}) \\ &= \sum_{\boldsymbol{j}\in\mathbb{Z}^N} \frac{1}{(2\pi)^N} \int\limits_{[-\pi,\pi]^N} e^{i(\boldsymbol{n}-\boldsymbol{j})\cdot\boldsymbol{\theta}} a^\alpha_\theta (1+a^\alpha_\theta)^{-(m+1)} \phi(\boldsymbol{j}) d\boldsymbol{\theta}, \end{aligned} \tag{23}$$

where $\boldsymbol{n}\in\mathbb{Z}^N, m\in\mathbb{N}$. It is not difficult to see using (5) and (6) that

$$\begin{aligned} v(0,\boldsymbol{n}) &= \sum_{\boldsymbol{j}\in\mathbb{Z}^N} G^\alpha_0(\boldsymbol{n}-\boldsymbol{j})\phi(\boldsymbol{j}) = \sum_{\boldsymbol{j}\in\mathbb{Z}^N} \frac{1}{(2\pi)^N} \int\limits_{[-\pi,\pi]^N} e^{-i(\boldsymbol{n}-\boldsymbol{j})\cdot\boldsymbol{\theta}} \phi(\boldsymbol{j}) d\boldsymbol{\theta} \\ &= \frac{1}{(2\pi)^N} \int\limits_{[-\pi,\pi]^N} e^{-i\boldsymbol{n}\cdot\boldsymbol{\theta}} \left(\sum_{\boldsymbol{j}\in\mathbb{Z}^N} e^{i\boldsymbol{j}\cdot\boldsymbol{\theta}} \phi(\boldsymbol{j}) \right) d\boldsymbol{\theta} \\ &= \frac{1}{(2\pi)^N} \int\limits_{[-\pi,\pi]^N} e^{-i\boldsymbol{n}\cdot\boldsymbol{\theta}} \hat{\phi}(\boldsymbol{\theta}) d\boldsymbol{\theta} = \phi(\boldsymbol{n}), \quad \boldsymbol{n}\in\mathbb{Z}^N. \end{aligned} \tag{24}$$

Combining this last equation with (21) and (23) we conclude that v is a solution of problem (17). □

Remark 3.3. We note that in the 1-dimensional and non-fractional case ($N=1$, $\alpha=1$), the heat equation takes the form

$$v(m+1,n) - v(m,n) = v(m+1,n+1) - 2v(m+1,n) + v(m+1,n-1)$$

which is slightly different from the more usual form $v(m+1,n) - v(m,n) = v(m,n+1) - 2v(m,n) + v(m,n-1)$. The solution is already known in case of continuous time and discrete space variable (see e.g. [26, Section 5.2]). In contrast, the representation given here for discrete space and discrete time is new. It is worth noting that the representation (18) reveals some qualitative behavior of the solution. For example, the asymptotic behavior $\lim\limits_{m\to\infty} v(m,\boldsymbol{n}) = 0$ can be deduced from the corresponding of the sequence $(1+(\sum_{j=1}^N 4\sin^2(\theta_j/2))^\alpha)^{-m}$ as $m\to\infty$.

Our next result shows that the N-dimensional discrete fractional Laplacian operator $(-\Delta_{d,N})^\alpha$ is related with the "α-root" of the forward difference operator D^1_Δ, namely, the operator $D^{1/\alpha}_\nabla$. We provide two results of this type. In the first one we consider the heat equation previously analyzed where we find a positive answer in case $\alpha = \frac{1}{p}$, $p \in \mathbb{N}$.

Theorem 3.4. *Given $\phi \in \ell^\infty(\mathbb{Z}^N)$, for each $\alpha = \frac{1}{p}$, $p \in \mathbb{N}$ the expression given by (18) solves the problems*

$$\begin{cases} D^1_\Delta v(m,\boldsymbol{n}) = -(-\Delta_{d,N})^{1/p} v(m+1,\boldsymbol{n}), \quad \boldsymbol{n} \in \mathbb{Z}^N, \quad m \in \mathbb{N}, \\ \\ \quad v(0,\boldsymbol{n}) = \phi(\boldsymbol{n}) \end{cases} \tag{25}$$

and

$$\begin{cases} D^p_\nabla v(m,\boldsymbol{n}) = (-1)^{p+1}\Delta_{d,N} v(m,\boldsymbol{n}), \quad \boldsymbol{n} \in \mathbb{Z}^N, \quad m \in \mathbb{N}, \\ \\ \quad v(0,\boldsymbol{n}) = \phi(\boldsymbol{n}) \\ \\ \quad v(-j,\boldsymbol{n}) = (Id + (-\Delta_{d,N})^{1/p})^j \phi(\boldsymbol{n}), \quad j \in \mathbb{N}. \end{cases} \tag{26}$$

Proof. Let $\alpha = \frac{1}{p}$. Observe the fact that expression given by (18) solves problem (25) is already proven in Theorem 3.2. It only remains to show that it also solves problem (26). Indeed, let $a_\theta := \sum_{j=1}^N (4\sin^2(\theta_j/2))$, then we have for every $\boldsymbol{n} \in \mathbb{Z}^N$ and $m \in \mathbb{N}$:

$$\begin{aligned} \Delta_{d,N} v(m,\boldsymbol{n}) &= \sum_{\boldsymbol{j}\in\mathbb{Z}^N} \Delta_{d,N} G^\alpha_m(\boldsymbol{n}-\boldsymbol{j})\phi(\boldsymbol{j}) \\ &= \sum_{\boldsymbol{j}\in\mathbb{Z}^N} \frac{1}{(2\pi)^N} \int\limits_{[-\pi,\pi]^N} \Delta_{d,N} e^{-i(\boldsymbol{n}-\boldsymbol{j})\cdot\boldsymbol{\theta}} (1+a^\alpha_\theta)^{-m} \phi(\boldsymbol{j}) d\boldsymbol{\theta} \\ &= \sum_{\boldsymbol{j}\in\mathbb{Z}^N} \frac{1}{(2\pi)^N} \int\limits_{[-\pi,\pi]^N} \sum_{k=1}^N (e^{-i(\boldsymbol{n}-\boldsymbol{j}+e_k)\cdot\boldsymbol{\theta}} - 2e^{-i(\boldsymbol{n}-\boldsymbol{j})\cdot\boldsymbol{\theta}} + e^{-i(\boldsymbol{n}-\boldsymbol{j}-e_k)\cdot\boldsymbol{\theta}}) \times \\ &\qquad\qquad \times (1+a^\alpha_\theta)^{-m} \phi(\boldsymbol{j}) d\boldsymbol{\theta} \\ &= \sum_{\boldsymbol{j}\in\mathbb{Z}^N} \frac{1}{(2\pi)^N} \int\limits_{[-\pi,\pi]^N} e^{-i(\boldsymbol{n}-\boldsymbol{j})\cdot\boldsymbol{\theta}} \sum_{k=1}^N (e^{-i\theta_k} - 2 + e^{i\theta_k})(1+a^\alpha_\theta)^{-m} \phi(\boldsymbol{j}) d\boldsymbol{\theta} \\ &= \sum_{\boldsymbol{j}\in\mathbb{Z}^N} \frac{1}{(2\pi)^N} \int\limits_{[-\pi,\pi]^N} e^{-i(\boldsymbol{n}-\boldsymbol{j})\cdot\boldsymbol{\theta}} \sum_{k=1}^N (2\cos\theta_k - 2)(1+a^\alpha_\theta)^{-m} \phi(\boldsymbol{j}) d\boldsymbol{\theta} \\ &= \sum_{\boldsymbol{j}\in\mathbb{Z}^N} \frac{1}{(2\pi)^N} \int\limits_{[-\pi,\pi]^N} e^{-i(\boldsymbol{n}-\boldsymbol{j})\cdot\boldsymbol{\theta}} \sum_{k=1}^N (-4\sin^2(\theta_k/2))(1+a^\alpha_\theta)^{-m} \phi(\boldsymbol{j}) d\boldsymbol{\theta} \\ &= \sum_{\boldsymbol{j}\in\mathbb{Z}^N} \frac{1}{(2\pi)^N} \int\limits_{[-\pi,\pi]^N} e^{-i(\boldsymbol{n}-\boldsymbol{j})\cdot\boldsymbol{\theta}} (-a_\theta)(1+a^\alpha_\theta)^{-m} \phi(\boldsymbol{j}) d\boldsymbol{\theta}. \end{aligned} \tag{27}$$

Also, we have:

$$D_\nabla^{1/\alpha} v(m,\boldsymbol{n}) = \sum_{\boldsymbol{j}\in\mathbb{Z}^N} D_\nabla^{1/\alpha} G_m^\alpha(\boldsymbol{n}-\boldsymbol{j})\phi(\boldsymbol{j}) \tag{28}$$
$$= \sum_{\boldsymbol{j}\in\mathbb{Z}^N} \frac{1}{(2\pi)^N} \int\limits_{[-\pi,\pi]^N} e^{-i(\boldsymbol{n}-\boldsymbol{j})\cdot\boldsymbol{\theta}} D_\nabla^{1/\alpha}(1+a_\theta^\alpha)^{-m}\phi(\boldsymbol{j})d\boldsymbol{\theta} \quad \boldsymbol{n}\in\mathbb{Z}^N, \quad m\in\mathbb{N}.$$

Define $q_\theta := 1 + a_\theta^\alpha$. Considering Definition 2.2 we have for every $m \in \mathbb{N}$:

$$D_\nabla^{1/\alpha} q_\theta^{-m} = \sum_{j=0}^{\infty} k^{-1/\alpha}(j) q_\theta^{-(m-j)} = q_\theta^{-m} \sum_{j=0}^{\infty} k^{-1/\alpha}(j) q_\theta^j. \tag{29}$$

Since $\alpha = \frac{1}{p}$ we obtain by Remark 2.1

$$D_\nabla^p q_\theta^{-m} = q_\theta^{-m} \sum_{j=0}^{p} (-1)^j \binom{p}{j} q_\theta^j = q_\theta^{-m}(1-q_\theta)^p = (1+a_\theta^\alpha)^{-m}(-1)^p(a_\theta^\alpha)^p = (1+a_\theta^\alpha)^{-m}(-1)^p a_\theta. \tag{30}$$

Using equality (30) in (28) we arrive to:

$$D_\nabla^p v(m,\boldsymbol{n}) = (-1)^p \sum_{\boldsymbol{j}\in\mathbb{Z}^N} \frac{1}{(2\pi)^N} \int\limits_{[-\pi,\pi]^N} e^{-i(\boldsymbol{n}-\boldsymbol{j})\cdot\boldsymbol{\theta}} a_\theta(1+a_\theta^\alpha)^{-m}\phi(\boldsymbol{j})d\boldsymbol{\theta} \quad \boldsymbol{n}\in\mathbb{Z}^N, \quad m\in\mathbb{N}. \tag{31}$$

Comparing (31) with (27) we arrive to the first equation in (26).

It only remains to show that v satisfies the initial conditions given in the third equation of (26). Indeed, let first compute:

$$G_{-m}^\alpha(\boldsymbol{n}) = \frac{1}{(2\pi)^N} \int\limits_{[-\pi,\pi]^N} e^{-i\boldsymbol{n}\cdot\boldsymbol{\theta}} (1 + (\sum_{k=1}^N 4\sin^2(\theta_k/2))^\alpha)^m d\boldsymbol{\theta}$$
$$= \sum_{s=0}^m \binom{m}{s} \frac{1}{(2\pi)^N} \int\limits_{[-\pi,\pi]^N} e^{-i\boldsymbol{n}\cdot\boldsymbol{\theta}} (\sum_{k=1}^N 4\sin^2(\theta_k/2))^{\alpha s} d\boldsymbol{\theta} = \sum_{s=0}^m \binom{m}{s} K^{\alpha s}(\boldsymbol{n}).$$

As a result, we obtain

$$v(-m,\boldsymbol{n}) = \sum_{\boldsymbol{j}\in\mathbb{Z}^N} G_{-m}^\alpha(\boldsymbol{n}-\boldsymbol{j})\phi(\boldsymbol{j}) = \sum_{s=0}^m \binom{m}{s} \sum_{\boldsymbol{j}\in\mathbb{Z}^N} K^{\alpha s}(\boldsymbol{n}-\boldsymbol{j})\phi(\boldsymbol{j})$$
$$= \sum_{s=0}^m \binom{m}{s} [(-\Delta_{d,N})^\alpha]^s \phi(\boldsymbol{n}) = (Id + (-\Delta_{d,N})^\alpha)^m \phi(\boldsymbol{n}) \quad \boldsymbol{n}\in\mathbb{Z}^N, \quad m\in\mathbb{N}, \tag{32}$$

where in the next-to-last equality we have employed property (13). $\square$

Remark 3.5. When $p = 1$ problems (25) and (26) coincide. Indeed, from (7)

$$D_\nabla^1 v(m,\boldsymbol{n}) = \sum_{j=0}^{\infty} (\delta_0 - \delta_1)(m-j) v(j,\boldsymbol{n}) = v(m,\boldsymbol{n}) - v(m-1,\boldsymbol{n})$$

and then equation

$$D^1_\nabla v(m,\boldsymbol{n}) = -\Delta_d v(m,\boldsymbol{n}),\ \boldsymbol{n}\in\mathbb{Z}^N,\ m\in\mathbb{N}$$

reduces to

$$D^1_\Delta v(m,\boldsymbol{n}) = -\Delta_d v(m+1,\boldsymbol{n}),\ \boldsymbol{n}\in\mathbb{Z}^N,\ m\in\mathbb{N}.$$

Example 1. In case $p=2$ and $N=1$ and using [15, Example 2.1] we have that equation (25) equals to

$$v(m+1,n)-v(m,n) = \frac{4}{\pi}\sum_{k\in\mathbb{Z}}\frac{v(m+1,n-k)}{(2k-1)(2k+1)}, \quad n\in\mathbb{Z},\ m\in\mathbb{N}, \tag{33}$$

with prescribed initial condition $v(0,n)$, whereas equation (26) reads

$$D^2_\nabla v(m,n) = -[v(m,n+1)-2v(m,n)+v(m,n-1)],\ m\in\mathbb{N},\ n\in\mathbb{Z}. \tag{34}$$

Using (10), we obtain that (34) is equivalent to

$$v(m,n) = v(m,n+1)+v(m,n-1)-2v(m-1,n)+v(m-2,n), \quad m\in\mathbb{N},\ n\in\mathbb{Z}, \tag{35}$$

with given initial conditions $v(0,n)$ and $v(-1,n)$. Note the presence of history (or memory) in the model (35) represented by the second initial condition $v(-1,n)$. In contrast, the history is represented by the fractional power $\alpha = 1/2$ (or right-hand term) in the model (33). By Theorem 3.4 we conclude that the equations (33) and (35) have the same solution. Of course, the last equation is computationally simpler than (33). This observation ratifies the claims about the advantages of the connections presented in this article, such as previously stated in references [13] and [16].

In our second result we consider a diffusion-like equation.

Theorem 3.6. *Given $\phi\in\ell^\infty(\mathbb{Z}^N)$ the expression given by:*

$$v(m,\boldsymbol{n}) = \sum_{\boldsymbol{j}\in\mathbb{Z}^N} H^\alpha_m(\boldsymbol{n}-\boldsymbol{j})\phi(\boldsymbol{j}) \quad \boldsymbol{n}\in\mathbb{Z}^N, \quad m\in\mathbb{Z},$$

where $H^\alpha_m(\boldsymbol{n}) := \dfrac{1}{(2\pi)^N}\displaystyle\int_{[-\pi,\pi]^N} e^{-i\boldsymbol{n}\cdot\boldsymbol{\theta}}(1-(\sum_{j=1}^N 4\sin^2(\theta_j/2))^\alpha)^{-m}d\boldsymbol{\theta}$ *solves the problem*

$$\begin{cases} D^1_\Delta v(m,\boldsymbol{n}) &= (-\Delta_{d,N})^\alpha v(m+1,\boldsymbol{n}), \quad \boldsymbol{n}\in\mathbb{Z}^N, \quad m\in\mathbb{N}, \\ v(0,\boldsymbol{n}) &= \phi(\boldsymbol{n}) \end{cases} \tag{36}$$

for any $\alpha>0$ and also the problem

$$\begin{cases} D^{1/\alpha}_\nabla v(m,\boldsymbol{n}) &= -\Delta_{d,N} v(m,\boldsymbol{n}), \quad \boldsymbol{n}\in\mathbb{Z}^N, \quad m\in\mathbb{N}, \\ v(0,\boldsymbol{n}) &= \phi(\boldsymbol{n}) \\ v(-j,\boldsymbol{n}) &= (Id-(-\Delta_{d,N})^\alpha)^j\phi(\boldsymbol{n}), \quad j\in\mathbb{N}, \end{cases} \tag{37}$$

whenever $0<\alpha\le\frac{1}{2+\log_2 N}$.

Proof. Recall that $a_\theta := \sum_{j=1}^{N}(4\sin^2(\theta_j/2))$. Replacing $(1+a_\theta^\alpha)$ by $(1-a_\theta^\alpha)$ in formulas (20), (21), (22) and (23) it is easy to see that $v(m,\boldsymbol{n}) = \sum_{\boldsymbol{j}\in\mathbb{Z}^N} H_m^\alpha(\boldsymbol{n}-\boldsymbol{j})\phi(\boldsymbol{j})$ is a solution of (36). Let now see that v also solves problem (37). Indeed, following the proof of Theorem 3.4 we have for every $\boldsymbol{n}\in\mathbb{Z}^N$ and $m\in\mathbb{N}$:

$$\Delta_{d,N} v(m,\boldsymbol{n}) = \sum_{\boldsymbol{j}\in\mathbb{Z}^N} \frac{1}{(2\pi)^N} \int\limits_{[-\pi,\pi]^N} e^{-i(\boldsymbol{n}-\boldsymbol{j})\cdot\boldsymbol{\theta}}(-a_\theta)(1-a_\theta^\alpha)^{-m}\phi(\boldsymbol{j})d\boldsymbol{\theta}. \tag{38}$$

Also, we have:

$$D_\nabla^{1/\alpha} v(m,\boldsymbol{n}) = \sum_{\boldsymbol{j}\in\mathbb{Z}^N} \frac{1}{(2\pi)^N} \int\limits_{[-\pi,\pi]^N} e^{-i(\boldsymbol{n}-\boldsymbol{j})\cdot\boldsymbol{\theta}} D_\nabla^{1/\alpha}(1-a_\theta^\alpha)^{-m}\phi(\boldsymbol{j})d\boldsymbol{\theta} \quad \boldsymbol{n}\in\mathbb{Z}^N, \quad m\in\mathbb{N}. \tag{39}$$

Let $q_\theta := 1-a_\theta^\alpha$. We claim that $|q_\theta|<1$ whenever $q_\theta \neq \pm 1$ (which imply $a_\theta^\alpha \neq 0$). Indeed, the hypothesis implies the inequality $2^{2\alpha+\alpha\log_2 N} \leq 2$, or, equivalently $2^2 N \leq 2^{1/\alpha}$. Hence $N - 2^{1/\alpha-1} \leq -N$. Since $-N \leq \sum_{j=0}^{N}\cos(\theta_j) \leq N$ we obtain the inequality $0 \leq N - \sum_{j=1}^{N}\cos(\theta_j) \leq 2^{1/\alpha-1}$. Therefore $0 \leq \sum_{j=1}^{N}(1-\cos(\theta_j)) \leq 2^{1/\alpha-1}$. Then, using the identity $1-\cos\theta = 2\sin^2(\theta/2)$, we obtain $0 \leq \sum_{j=0}^{N} 4\sin^2(\theta_j/2)) \leq 2^{1/\alpha}$. This shows that $0 \leq a_\theta^\alpha \leq 2$ and, consequently, $|q_\theta| = |a_\theta^\alpha - 1| < 1$ whenever $q_\theta \neq \pm 1$. This proves the claim.

Considering Definition 2.2 we have for every $m\in\mathbb{N}$:

$$D_\nabla^{1/\alpha} q_\theta^{-m} = \sum_{j=0}^{\infty} k^{-1/\alpha}(j) q_\theta^{-(m-j)} = q_\theta^{-m}\sum_{j=0}^{\infty} k^{-1/\alpha}(j) q_\theta^{j} = q_\theta^{-m}(1-q_\theta)^{1/\alpha} = (1-a_\theta^\alpha)^{-m} a_\theta, \tag{40}$$

where in the last equality we have used the generating formula given by (8). Note that the cases $q_\theta = \pm 1$ follow from Remark 2.1. Using equality (40) in (39) we arrive to:

$$D_\nabla^{1/\alpha} v(m,\boldsymbol{n}) = \sum_{\boldsymbol{j}\in\mathbb{Z}^N} \frac{1}{(2\pi)^N} \int\limits_{[-\pi,\pi]^N} e^{-i(\boldsymbol{n}-\boldsymbol{j})\cdot\boldsymbol{\theta}} a_\theta(1-a_\theta^\alpha)^{-m}\phi(\boldsymbol{j})d\boldsymbol{\theta} \quad \boldsymbol{n}\in\mathbb{Z}^N, \quad m\in\mathbb{N}. \tag{41}$$

It only remains to show that v satisfies the initial conditions given in the third equation of (37). Indeed, we proceed as in the last part of the proof of Theorem 3.4 obtaining this time

$$H_{-m}^\alpha(\boldsymbol{n}) = \frac{1}{(2\pi)^N} \int\limits_{[-\pi,\pi]^N} e^{-i\boldsymbol{n}\cdot\boldsymbol{\theta}}(1-(\sum_{k=1}^{N} 4\sin^2(\theta_k/2))^\alpha)^m d\boldsymbol{\theta} = \sum_{s=0}^{m}\binom{m}{s}(-1)^s K^{\alpha s}(\boldsymbol{n}).$$

Therefore

$$v(-m,\boldsymbol{n}) = \sum_{\boldsymbol{j}\in\mathbb{Z}^N} H_{-m}^\alpha(\boldsymbol{n}-\boldsymbol{j})\phi(\boldsymbol{j}) = (Id-(-\Delta_{d,N})^\alpha)^m\phi(\boldsymbol{n}) \quad \boldsymbol{n}\in\mathbb{Z}^N, \quad m\in\mathbb{N}, \tag{42}$$

where we have employed property (13). Combining (38) and (41) we have proven that v also solves problem (37) and then problems (36) and (37) have the same solution. □

Remark 3.7. It should be noted that one of the advantages of our analysis on complete lattices for fractional order operators, compared to continuous analysis, is that it allows to make the language of distribution theory, which is always present in the continuous variable case, more transparent through simpler computations, replacing the Dirac's delta by the Kronecker delta. This type of approach serves as a microstructural basis for the analysis of continuous fractional models and to describe the non-local properties of different types of media at the nanoscale and microscale [27,28].

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