



Contractions of Kannan-type and of Chatterjea-type on fuzzy quasi-metric spaces

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Abstract

We characterize the completeness of fuzzy quasi-metric spaces by means of a fixed point theorem of Kannan-type. Thus, we extend the classical characterization of metric completeness due to Subrahmanyam as well as recent results in the literature on the characterization of quasi-metric completeness and fuzzy metric completeness, respectively. We also introduce and discuss contractions of Chatterjea-type in this asymmetric context.

Keywords: Fuzzy quasi-metric space Quasi-metric space Complete Fixed point Kannan contraction Chatterjea contraction.

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1. Introduction

The problem of characterizing complete fuzzy metric spaces with the help of fixed point results has been recently discussed in [1, 16, 18, 19, 21, 22] as a natural prolongation of the classical problem of obtaining necessary and sufficient conditions for the metric completeness via fixed point theorems (see e.g. [11, 13, 20, 23, 24, 25]). In particular, Subrahmanyam proved in [23] that both the famous Kannan fixed point theorem [12] and its ‘companion’ Chatterjea fixed point theorem [4] provide nice characterizations of metric completeness. Quasi-metric and fuzzy metric extensions of Subrahmanyam characterization for the Kannan case were obtained in [2] and [18], respectively. In this paper we investigate the problem of extending that characterizations to the realm of complete fuzzy quasi-metric spaces. In Section 3 we observe that the quasi-metric generalization of Chatterjea’s theorem continues to be a good ‘companion’ of the quasi-metric generalization of Kannan’s theorem obtained in [2]. Section 4 is devoted to trying the close of this natural puzzle researching the fuzzy (quasi-)metric case. We will show that while a satisfactory answer is reached in the Kannan setting, the Chatterjea setting presents certain difficulties; despite this, a partial solution to this case is also presented.

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2. Background

Our main reference for quasi-metric spaces is [6] and for fuzzy quasi-metric spaces they are [10] and [5].

We remind that a quasi-metric on a set X is a function $d : X \times X \rightarrow [0, \infty)$ such that for all $x, y, z \in X$: (i) $x = y \Leftrightarrow d(x, y) = d(y, x) = 0$, and (ii) $d(x, z) \leq d(x, y) + d(y, z)$.

By a quasi-metric space we mean a pair (X, d) such that X is a set and d is a quasi-metric on X .

Let d be a quasi-metric on a set X . For each $x \in X$ and $\varepsilon > 0$ set $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$. Then, the family $\{B_d(x, \varepsilon) : x \in X, \varepsilon > 0\}$ is a base of open sets for a T_0 topology τ_d on X , called the topology induced by d .

Given a quasi-metric d on X , the function $d^s : X \times X \rightarrow [0, \infty)$ defined by $d^s(x, y) = \max\{d(x, y), d(y, x)\}$ for all $x, y \in X$, is a metric on X .

There exist many interesting instances of quasi-metric spaces in the literature, see e.g. [6, 7] (a few examples may also be found at the end of Section 3).

On the other hand, the lack of symmetry yields several different notions of Cauchy-ness and quasi-metric completeness which coincide with the classical notions when dealing with a metric space. In our context, a sequence $(x_n)_{n \in \mathbb{N}}$ in a quasi-metric space (X, d) will be called a Cauchy sequence if it is a Cauchy sequence in the metric space (X, d^s) , and we shall use the following very general notion of completeness:

A quasi-metric space (X, d) is complete provided that every Cauchy sequence $(x_n)_{n \in \mathbb{N}}$ is τ_d -convergent, i.e., if there exists some $x \in X$ such that $\lim_{n \rightarrow \infty} d(x, x_n) = 0$.

Let us recall [14] that a binary operation $* : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-norm provided that it satisfies the following conditions: (i) $*$ is associative and commutative; (ii) $*$ is continuous; (iii) $a * 1 = a$ for every $a \in [0, 1]$; (iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, with $a, b, c, d \in [0, 1]$.

It is well known that if $*$ is a continuous t-norm, then $* \leq \wedge$, where \wedge is the continuous t-norm given by $a \wedge b = \min\{a, b\}$.

In [10] (see also [5]) were introduced and discussed the following notions as a natural asymmetric generalization of the classical notions of fuzzy metric space in the senses of Kramosil and Michalek [15] and George and Veeramani [8, 9], respectively.

Definition 2.1. [5, 10]. A KM-fuzzy quasi-metric on a set X is a pair $(M, *)$ such that $*$ is a continuous t-norm and M is a fuzzy set in $X \times X \times [0, \infty)$ fulfilling the following four conditions for every $x, y, z \in X$:

$$(KM1) \quad M(x, y, 0) = 0;$$

$$(KM2) \quad x = y \text{ if and only if } M(x, y, t) = M(y, x, t) = 1 \text{ for all } t > 0;$$

$$(KM3) \quad M(x, z, t + s) \geq M(x, y, t) * M(y, z, s) \text{ for all } t, s \geq 0;$$

$$(KM4) \quad M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1] \text{ is left continuous.}$$

A KM-fuzzy quasi-metric $(M, *)$ on X fulfilling for every $x, y \in X$:

$$(KM5) \quad M(x, y, t) = M(y, x, t) \text{ for all } t > 0,$$

is said to be a KM-fuzzy metric on X .

Definition 2.2. [5, 10]. A KM-fuzzy (quasi-)metric space is a triple $(X, M, *)$ such that X is a set and $(M, *)$ is a KM-fuzzy (quasi-)metric on X .

Definition 2.3. [5, 10]. A GV-fuzzy quasi-metric on a set X is a pair $(M, *)$ such that $*$ is a continuous t-norm and M is a fuzzy set in $X \times X \times (0, \infty)$ fulfilling the following four conditions for every $x, y, z \in X$:

$$(GV1) \quad M(x, y, t) > 0 \text{ for all } t > 0;$$

$$(GV2) \quad x = y \text{ if and only if } M(x, y, t) = M(y, x, t) = 1 \text{ for some } t > 0;$$

$$(GV3) \quad M(x, z, t + s) \geq M(x, y, t) * M(y, z, s) \text{ for all } t, s > 0;$$

$$(GV4) \quad M(x, y, \cdot) : (0, \infty) \rightarrow [0, 1] \text{ is continuous.}$$

A GV-fuzzy quasi-metric $(M, *)$ on X fulfilling for every $x, y \in X$:

$$(GV5) \quad M(x, y, t) = M(y, x, t) \text{ for all } t > 0$$

is said to be a GV-fuzzy metric on X .

Definition 2.4. [5, 10]. A GV-fuzzy (quasi-)metric space is a triple $(X, M, *)$ such that X is a set and $(M, *)$ is a GV-fuzzy (quasi-)metric on X .

Remark 2.5. [5, 10]. If $(M, *)$ is a KM-fuzzy quasi-metric on a set X , then, for each $x, y \in X$ the function $M(x, y, \cdot)$ is nondecreasing.

Remark 2.6. It easily follows from Remark 2.5 that, given $x, y \in X$, if $M(x, y, t) > 1 - t$ for all $t > 0$, then $M(x, y, t) = 1$ for all $t > 0$.

Remark 2.7. Note that the class of KM-fuzzy metric spaces $(X, M, *)$ coincides with the class of fuzzy metric spaces in the sense of Kramosil and Michalek [15], with the exception of the condition $\lim_{t \rightarrow \infty} M(x, y, t) = 1$ for all $x, y \in X$, which is required in [15], whereas the class of GV-fuzzy metric spaces is exactly the class of fuzzy metric spaces in the sense of George and Veeramani [8, 9].

Analogous to the quasi-metric case, if $(M, *)$ is a KM-fuzzy quasi-metric (resp. a GV-fuzzy quasi-metric) on a set X , the pair $(M^{\min}, *)$ is a KM-fuzzy metric (resp. a GV-fuzzy metric) on X , where M^{\min} is the fuzzy set in $X \times X \times [0, \infty)$ (resp. in $X \times X \times (0, \infty)$) given by $M^{\min}(x, y, t) = \min\{M(x, y, t), M(y, x, t)\}$. We shall refer to $(X, M^{\min}, *)$ as the KM-fuzzy metric (resp. the GV-fuzzy metric) space induced by $(X, M, *)$.

On the other hand, and as in the quasi-metric setting, if $(M, *)$ is a KM-fuzzy quasi-metric on a set X and for each $x \in X$, $\varepsilon \in (0, 1)$ and $t > 0$ we put $B_M(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}$, then, the family $\{B_M(x, \varepsilon, t) : x \in X, \varepsilon \in (0, 1), t > 0\}$ is a base of open sets for a T_0 topology τ_M on X , called the topology induced by $(M, *)$ (see e.g. [10, p. 131]).

From the definition of the topology τ_M we deduce the following well-known and useful fact [10, Proposition 2.8]:

A sequence $(x_n)_{n \in \mathbb{N}}$ in a KM-fuzzy quasi-metric space $(X, M, *)$ τ_M -converges to an $x \in X$ if and only if $\lim_{n \rightarrow \infty} M(x, x_n, t) = 1$ for all $t > 0$. (By \mathbb{N} we denote the set of all positive integers).

Before to define the notions of Cauchy-ness and completeness for KM-fuzzy quasi-metric spaces that we will employ here, remind that a sequence $(x_n)_{n \in \mathbb{N}}$ in a KM-fuzzy metric space $(X, M, *)$ is a Cauchy sequence provided that for each $\varepsilon \in (0, 1)$ and each $t > 0$ there is an $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$.

Definition 2.8. A sequence in a KM-fuzzy quasi-metric space $(X, M, *)$ is said to be a Cauchy sequence if it is a Cauchy sequence in the fuzzy metric space $(X, M^{\min}, *)$.

Definition 2.9. A KM-fuzzy quasi-metric space $(X, M, *)$ is said to be complete if every Cauchy sequence is τ_M -convergent.

Remark 2.10. Each GV-fuzzy (quasi-)metric space $(X, M, *)$ can be considered as a KM-fuzzy (quasi-)metric space simply by defining $M(x, y, 0) = 0$ for all $x, y \in X$. Hence, any GV-fuzzy quasi-metric induces a topology defined as in the KM-case. Furthermore, the notions and properties for KM-fuzzy quasi-metric spaces given above hold for GV-fuzzy quasi-metric spaces.

The following is an important instance of a KM-fuzzy quasi-metric space which is not a GV-fuzzy quasi-metric space, in general.

Example 2.11. [3] Let (X, d) be a quasi-metric space. Then $(X, M_{d,01}, *)$ is a KM-fuzzy quasi-metric space for any continuous t -norm $*$, where M is the fuzzy set in $X \times X \times [0, \infty)$ defined as $M_{d,01}(x, y, t) = 0$ if $d(x, y) \geq t$ and $M_{d,01}(x, y, t) = 1$ if $d(x, y) < t$. Furthermore, the topologies τ_d and $\tau_{M_{d,01}}$ agree on X , and $(X, M_{d,01}, *)$ is complete if and only if (X, d) is complete.

We finish this section with a typical example of a GV-fuzzy quasi-metric space.

Example 2.12. [10]. Let (X, d) be a quasi-metric space. Then $(X, M_{d,s}, *)$ is a GV-fuzzy quasi-metric space for any continuous t -norm $*$, where $M_{d,s}$ is the fuzzy set in $X \times X \times (0, \infty)$ defined as

$$M_{d,s}(x, y, t) = \frac{t}{t + d(x, y)},$$

for all $t > 0$. $(X, M_{d,s}, *)$ is said to be the standard GV-fuzzy quasi-metric space of (X, d) . Furthermore, the topologies τ_d and $\tau_{M_{d,s}}$ agree on X , and $(X, M_{d,s}, *)$ is complete if and only if (X, d) is complete.

3. The quasi-metric setting

We begin this section by proposing the following natural quasi-metric generalizations of the notions of Kannan contraction and Chatterjea contraction for metric spaces.

Definition 3.1. We say that a self map T of a quasi-metric space (X, d) is a Kannan contraction (a d -Kannan mapping in [2]) if there is a constant $c \in (0, 1/2)$ such that

$$d(Tx, Ty) \leq c[d(x, Tx) + d(y, Ty)],$$

for all $x, y \in X$.

Definition 3.2. We say that a self map T of a quasi-metric space (X, d) is a Chatterjea contraction if there is a constant $c \in (0, 1/2)$ such that

$$d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)],$$

for all $x, y \in X$.

Remark 3.3. It is clear that every Kannan contraction (resp. every Chatterjea contraction) on a quasi-metric space (X, d) , is a Kannan contraction (resp. a Chatterjea contraction) on the metric space (X, d^s) . However, the reverse implications do not hold, in general (see Example 3.6 below).

The next result will be crucial.

Proposition 3.4. A quasi-metric space (X, d) is complete if each self map T of X satisfying

$$d^s(Tx, Ty) \leq \frac{1}{3} \min\{d(x, Tx) + d(y, Ty), d(x, Ty) + d(y, Tx)\},$$

for all $x, y \in X$, has a fixed point.

Proof. Suppose that (X, d) is not complete. Then, there is a non τ_d -convergent Cauchy sequence $(x_n)_{n \in \omega}$ in (X, d) . Following the proof of [2, Theorem 2.8] we can construct a self map T of X without fixed points and such that

$$d^s(Tx, Ty) \leq \frac{1}{5} [d(x, Tx) + d(y, Ty)],$$

for all $x, y \in X$. (The original proof of [2, Theorem 2.8] is given for $c = 1/4$, but it is also valid, for instance, for $c = 1/5$, without any change). From the triangle inequality it follows that

$$d^s(Tx, Ty) \leq \frac{1}{5} [d(x, Ty) + d^s(Ty, Tx) + d(y, Tx) + d^s(Tx, Ty)],$$

for all $x, y \in X$. Therefore

$$d^s(Tx, Ty) \leq \frac{1}{3} [d(x, Ty) + d(y, Tx)],$$

for all $x, y \in X$. Consequently

$$d^s(Tx, Ty) \leq \frac{1}{3} \min\{d(x, Tx) + d(y, Ty), d(x, Ty) + d(y, Tx)\},$$

for all $x, y \in X$. We have reached a contradiction, which concludes the proof. \square

Theorem 3.5. For a quasi-metric space (X, d) the following conditions are equivalent.

- (1) (X, d) is complete.
- (2) Every Kannan contraction on (X, d) has a (unique) fixed point.
- (3) Every Chatterjea contraction on (X, d) has a (unique) fixed point.

Proof. (1) \Leftrightarrow (2) [2, Theorem 2.8].

(1) \Rightarrow (3) Let T be a Chatterjea contraction on the complete quasi-metric space (X, d) . Then, there is $c \in (0, 1/2)$ such that

$$d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)],$$

for all $x, y \in X$. It immediately follows that

$$d^s(Tx, Ty) \leq c[d^s(x, Ty) + d^s(y, Tx)],$$

for all $x, y \in X$, so T is a Chatterjea contraction on the metric space (X, d^s) .

Fix an $x_0 \in X$. Then, the classical proof of Chatterjea's theorem shows that $(T^n x_0)_{n \in \mathbb{N}}$ is a Cauchy sequence in (X, d^s) . By completeness of (X, d) there exists $z \in X$ such that $\lim_{n \rightarrow \infty} d(z, T^n x_0) = 0$. We show that Tz is the unique fixed point of T . Indeed, we get

$$\begin{aligned} d(T^{n+1}x_0, Tz) &\leq c[d(T^n x_0, Tz) + d(z, T^{n+1}x_0)] \\ &\leq c[d(T^n x_0, T^{n+1}x_0) + d(T^{n+1}x_0, Tz) + d(z, T^{n+1}x_0)], \end{aligned}$$

for all $n \in \mathbb{N}$. Therefore

$$d(T^{n+1}x_0, Tz) \leq \frac{c}{1-c}[d(T^n x_0, T^{n+1}x_0) + d(z, T^{n+1}x_0)],$$

for all $n \in \mathbb{N}$, so that $\lim_{n \rightarrow \infty} d(T^{n+1}x_0, Tz) = 0$. By the triangle inequality $d(z, Tz) = 0$. Thus, we have

$$d(Tz, T^2z) \leq c[d(z, T^2z) + d(Tz, Tz)] \leq c[d(z, Tz) + d(Tz, T^2z)] = cd(Tz, T^2z),$$

so $d(Tz, T^2z) = 0$, and also

$$d(T^2z, Tz) \leq c[d(Tz, Tz) + d(z, T^2z)] \leq c[d(z, Tz) + d(Tz, T^2z)] = 0.$$

Hence $Tz = T^2z$.

Finally, let $u \in X$ such that $u = Tu$. Then

$$d^s(u, Tz) = d^s(Tu, T^2z) \leq c[d^s(u, T^2z) + d^s(Tz, Tu)] = 2cd^s(u, Tz).$$

Since $2c < 1$, $d^s(u, Tz) = 0$, i.e., $u = Tz$.

(3) \Rightarrow (1) Let T be a self map of X such that

$$d^s(Tx, Ty) \leq \frac{1}{3} \min\{d(x, Tx) + d(y, Ty), d(x, Ty) + d(y, Tx)\},$$

for all $x, y \in X$. Then

$$d(Tx, Ty) \leq \frac{1}{3}[d(x, Ty) + d(y, Tx)],$$

for all $x, y \in X$, so T is a Chatterjea contraction on (X, d) . By assumption T has a fixed point. Hence (X, d) is complete by Proposition 3.4. \square

Regarding our comment at the end of Remark 3.3 we present the following example.

Example 3.6. Let $X = [0, 2]$ and let d be the quasi-metric on X given by $d(x, y) = \max\{y - x, 0\}$ for all $x, y \in X$. Since d^s is the restriction to X of the usual metric on the set \mathbb{R} of all reals, it follows that (X, d^s) is a compact metric space, so (X, d) is a complete quasi-metric space. Now let $T : X \rightarrow X$ defined by $Tx = 0$ if $x \in [0, 1]$ and $Tx = x/4$ if $x \in (1, 2]$. We show that T is both a Kannan contraction and a Chatterjea contraction on the metric space (X, d^s) .

- If $x, y \in [0, 1]$ we have $d^s(Tx, Ty) = 0$.
- If $x \in [0, 1]$ and $y \in (1, 2]$ we get $d^s(Tx, Ty) = y/4$, $d^s(x, Tx) = x$, $d^s(y, Ty) = 3y/4$, $d^s(x, Ty) = |x - y/4|$ and $d^s(y, Tx) = y$, so

$$d^s(Tx, Ty) = \frac{1}{3}d^s(y, Ty) < \frac{1}{3}d^s(y, Tx).$$

- If $x, y \in (1, 2]$, we get $d^s(Tx, Ty) = |x - y|/4$, $d^s(x, Tx) = 3x/4$, $d^s(y, Ty) = 3y/4$, $d^s(x, Ty) = (4x - y)/4$ and $d^s(y, Tx) = (4y - x)/4$, so

$$d^s(Tx, Ty) < \frac{1}{3}d^s(x, Tx) < \frac{1}{3}[d^s(x, Ty) + d^s(y, Tx)].$$

Therefore T is both a Kannan contraction and a Chatterjea contraction on (X, d^s) , with $c = 1/3$. However, for $x = 0$ and $y \in (1, 2]$ we obtain $d^s(Tx, Ty) = d(x, Ty) = y/4$, and $d(x, Tx) = d(y, Ty) = d(y, Tx) = 0$, so T is not a Kannan contraction neither a Chatterjea contraction on (X, d) .

The following is an example of a self map T on a complete quasi-metric space (X, d) which is a Kannan contraction on (X, d) , and thus a Kannan contraction of (X, d^s) but is not a Chatterjea contraction on (X, d^s) , and thus not a Chatterjea contraction on (X, d) .

Example 3.7. Let $X = \{0, 1, 2\}$ and let d be the quasi-metric on X given by $d(x, x) = 0$ for all $x \in X$, $d(0, 1) = d(0, 2) = d(2, 0) = d(2, 1) = 1$, $d(1, 0) = 2$ and $d(1, 2) = 3$. Evidently (X, d) is complete because the Cauchy sequences in (X, d^s) are those that are eventually constant.

Now let $T : X \rightarrow X$ defined by $T0 = T2 = 0$ and $T1 = 2$.

We show that T is a Kannan contraction on (X, d) (and thus on (X, d^s)). To this end it suffices to check that

$$\begin{aligned} d^s(T0, T1) &= d^s(T2, T1) = d^s(0, 2) = 1 = \frac{1}{3}[d(0, T0) + d(1, T1)] \\ &< \frac{1}{3}[d(2, T2) + d(1, T1)]. \end{aligned}$$

Note also that T is not a Chatterjea contraction on (X, d^s) (and thus not on (X, d)) because

$$d^s(T2, T1) = 1 = \frac{1}{2}[0 + 2] = \frac{1}{2}[d^s(2, T1) + d^s(1, T2)].$$

We finish this section with an example of a self map of a complete quasi-metric space (X, d) which is a Chatterjea contraction on (X, d) , and, consequently, a Chatterjea contraction on the metric space (X, d^s) , but not a Kannan contraction on (X, d^s) , and, consequently, not a Kannan contraction on (X, d) .

Example 3.8. Let $X = [0, 1]$ and let d be the quasi-metric on X given by $d(x, y) = \max\{x - y, 0\}$. It is well known that (X, d) is a complete quasi-metric space (note that d^s is the usual metric on X).

Let T be the self map of X defined as $T1 = 1/3$, and $Tx = 0$ for all $x \in [0, 1)$.

We show that T is not a Kannan contraction on (X, d^s) and thus is not a Kannan contraction on (X, d) . Indeed, we get

$$d^s(T1, T0) = d^s(1/3, 0) = \frac{1}{3} = \frac{1}{2}\left[\frac{2}{3} + 0\right] = \frac{1}{2}[d^s(1, T1) + d^s(0, T0)].$$

However T is a Chatterjea contraction on (X, d) and thus on (X, d^s) . Indeed, for $x = 1$ and $y \in [0, 1)$ we get

$$d^s(Tx, Ty) = d^s(1/3, 0) = \frac{1}{3} \leq \frac{1}{3}[d(x, Ty) + d(y, Tx)].$$

4. The fuzzy quasi-metric setting

In this section KM-fuzzy quasi-metric will be simply called fuzzy quasi-metric spaces.

In [18] we introduced the notions of (1)-Kannan contraction and (1/2)-Kannan contraction for fuzzy metric spaces. We generalize that notions in a natural way as follows.

Definition 4.1. Let $(X, M, *)$ be a fuzzy quasi-metric space. We say that a self map T of X is a (1)-Kannan contraction on $(X, M, *)$ if there is a constant $c \in (0, 1)$ such that for any $x, y \in X$ and $t > 0$,

$$\min\{M(x, Tx, t), M(y, Ty, t)\} > 1 - t \Rightarrow M(Tx, Ty, ct) > 1 - ct. \quad (1Kn)$$

Analogous to the quasi-metric setting, we say that a self map T of X is a (1/2)-Kannan contraction on $(X, M, *)$ if there is a constant $c \in (0, 1/2)$ such that for any $x, y \in X$ and $t > 0$,

$$\min\{M(x, Tx, t), M(y, Ty, t)\} > 1 - t \Rightarrow M(Tx, Ty, ct) > 1 - ct.$$

Remark 4.2. If T is a (1)-Kannan contraction on a fuzzy quasi-metric space $(X, M, *)$ and we interchange x and y in condition (1Kn), then that condition can be reformulated as follows:

$$\min\{M(x, Tx, t), M(y, Ty, t)\} > 1 - t \Rightarrow M^{\min}(Tx, Ty, ct) > 1 - ct. \quad (1Knm)$$

Clearly, any (1/2)-Kannan contraction is a (1)-Kannan contraction. However the converse does not hold even for fuzzy metric spaces (see [18, Example 2]).

In the proof of our next theorem we use ideas and methods from [18, Theorem 3]. In particular, we shall apply the following auxiliary result whose demo is almost identical to the first part of the proof of [18, Theorem 3], so it is omitted.

Lemma 4.3. Let T be a (1)-Kannan contraction on a fuzzy quasi-metric space $(X, M, *)$. Fix $x_0 \in X$. Then $(T^n x_0)_{n \in \mathbb{N}}$ is a Cauchy sequence in $(X, M, *)$.

Theorem 4.4. Every (1)-Kannan contraction on a complete fuzzy quasi-metric space has a unique fixed point.

Proof. Let $(X, M, *)$ be a complete fuzzy quasi-metric metric space and let T be a (1)-Kannan contraction on X . Then, there is a constant $c \in (0, 1)$ for which condition (1Kn) is fulfilled. Furthermore T is a (1)-Kannan contraction on the fuzzy metric space $(X, M^{\min}, *)$ with constant of contraction c , by Remark 4.2.

Fix $t_0 > 1$ and $r, s > 0$ such that $c < s < r < 1$. For each $k \in \mathbb{N}$, define

$$A_{k,r,s} := \{\varepsilon \in (0, 1) : \varepsilon + sr^{k-1}t_0 < r^k t_0\}.$$

Now fix $x_0 \in X$. Applying Lemma 4.3 we get that $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $(X, M, *)$, where $x_n := T^n x_0$ for all $n \in \mathbb{N} \cup \{0\}$. So there is $z \in X$ such that the sequence $(x_n)_{n \in \mathbb{N}}$ converges to z in τ_M , i.e., $\lim_{n \rightarrow \infty} M(z, x_n, t) = 1$ for all $t > 0$.

Following the proof of [18, Theorem 3], joint with Remark 4.2, we deduce that $M(z, Tz, t) = 1$ for all $t > 0$ (the details are omitted).

In the sequel we show that Tz is a fixed point of T .

To reach it we first check, by mathematical induction, that for each $k \in \mathbb{N}$,

$$M^{\min}(Tz, T^2z, r^k t_0) \geq 1 - r^k t_0. \quad (\dagger)$$

where we assume, without loss of generality, that $r^k t_0 \leq 1$.

Indeed, since $M(Tz, T^2z, t_0) > 1 - t_0$ and $M(x_n, x_{n+1}, t_0) > 1 - t_0$, we deduce from Remarks 2.5 and 4.2 that

$$M^{\min}(T^2z, x_{n+1}, st_0) \geq M^{\min}(T^2z, x_{n+1}, ct_0) > 1 - ct_0 > 1 - st_0,$$

for all $n \in \mathbb{N} \cup \{0\}$.

Given $t > 0$ there is $n_t \in \mathbb{N}$ such that $M(x_n, x_{n+1}, t) > 1 - t$ for all $n \geq n_t$. Since $M(z, Tz, t) = 1$, it follows from condition (1Knm) that

$$M^{\min}(Tz, x_{n+1}, ct) > 1 - ct,$$

for all $n \geq n_t$. Hence, for each $\varepsilon \in A_{1,r,s}$ (taking $t = \varepsilon/c$) there is $n_\varepsilon \geq n_t$ such that $M^{\min}(Tz, x_{n_\varepsilon}, \varepsilon) > 1 - \varepsilon$. Therefore

$$\begin{aligned} M^{\min}(Tz, T^2z, rt_0) &\geq M^{\min}(Tz, x_{n_\varepsilon}, \varepsilon) * M^{\min}(x_{n_\varepsilon}, T^2z, st_0) \\ &\geq (1 - \varepsilon) * (1 - st_0) \geq (1 - \varepsilon) * (1 - rt_0). \end{aligned}$$

From the continuity of $*$ we get

$$M^{\min}(Tz, T^2z, rt_0) \geq 1 - rt_0.$$

So, we have proved the inequality (\dagger) for $k = 1$.

Now suppose that the inequality (\dagger) is true for $k = j$, $j \in \mathbb{N}$. We shall check that $M^{\min}(Tz, T^2z, r^{j+1}t_0) \geq 1 - r^{j+1}t_0$.

Since $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence, there is $n_j \in \mathbb{N}$ such that

$$M(x_n, x_{n+1}, r^j t_0) > 1 - r^j t_0,$$

for all $n \geq n_j$. This fact along with our induction hypothesis and condition (1Knm) implies that

$$M^{\min}(T^2z, x_{n+1}, cr^j t_0) > 1 - cr^j t_0,$$

for all $n \geq n_j$. Therefore

$$M^{\min}(T^2z, x_{n+1}, sr^j t_0) > 1 - sr^j t_0,$$

for all $n \geq n_j$.

Now let $\varepsilon \in A_{j+1,r,s}$. Then $\varepsilon + sr^j t_0 < r^{j+1} t_0$, and there is $n_\varepsilon > n_j$ for which $M^{\min}(Tz, x_{n_\varepsilon}, \varepsilon) > 1 - \varepsilon$. Hence

$$\begin{aligned} M^{\min}(Tz, T^2z, r^{j+1} t_0) &\geq M^{\min}(Tz, x_{n_\varepsilon}, \varepsilon) * M^{\min}(x_{n_\varepsilon}, T^2z, sr^j t_0) \\ &\geq (1 - \varepsilon) * (1 - sr^j t_0) \geq (1 - \varepsilon) * (1 - r^{j+1} t_0). \end{aligned}$$

Again, from the continuity of $*$ we deduce that

$$M^{\min}(Tz, T^2z, r^{j+1} t_0) \geq 1 - r^{j+1} t_0.$$

We conclude that inequality (\dagger) is true.

Take any $t > 0$. Then, there exists $k \in \mathbb{N}$ such that $r^k t_0 < t$, so

$$M^{\min}(Tz, T^2z, t) \geq M^{\min}(Tz, T^2z, r^k t_0) > 1 - r^k t_0 > 1 - t.$$

From Remark 2.6 it follows that $M^{\min}(Tz, T^2z, t) = 1$ for all $t > 0$. Hence $Tz = T^2z$, so Tz is a fixed point of T .

Finally, let $u \in X$ such that $u = Tu$. So $\min\{M(Tz, T^2z, t), M(u, Tu, t)\} = 1$, for all $t > 0$. By condition (1Knm) we deduce that $M^{\min}(T^2z, Tu, ct) > 1 - ct$ for all $t > 0$. Consequently $T^2z = Tu$, i.e., $Tz = u$, so Tz is the unique fixed point of T . \square

Corollary 4.5. *Every (1/2)-Kannan contraction on a complete fuzzy quasi-metric space has a unique fixed point.*

Remark 4.6. [18, Example 2] shows that Theorem 4.4 is a real generalization of Corollary 4.5.

The next is an example of a self map of a complete quasi-metric space (X, d) which is not a Kannan contraction on (X, d) but, instead, it is a (1)-Kannan contraction on the complete fuzzy quasi-metric space $(X, M_{d,01}, *)$ for any continuous t-norm $*$.

Example 4.7. Let (X, d) be the complete quasi-metric space where $X = [0, 1]$ and d is the quasi-metric on X given by $d(x, y) = \max\{x - y, 0\}$. It was shown in Example 3.8 above that the self map T of X defined as $T1 = 1/3$ and $Tx = 0$ for all $x \in [0, 1)$, is not a Kannan contraction on (X, d) .

Let $(X, M_{d,01}, *)$ be the complete fuzzy quasi-metric space as constructed in Example 2.11 above. We are going to prove that T is a (1)-Kannan contraction on $(X, M_{d,01}, *)$ for $c = 1/2$. Thus, it will verify the conditions of Theorem 4.4.

Indeed, let $x, y \in X$ and $t > 0$ such that $\min\{M_{d,01}(x, Tx, t), M_{d,01}(y, Ty, t)\} > 1 - t$.

Then, we shall check that $M_{d,01}(Tx, Ty, t/2) > 1 - t/2$.

If $M_{d,01}(Tx, Ty, t/2) = 1$, the conclusion is obvious. Hence, we will assume in the sequel that $M_{d,01}(Tx, Ty, t/2) = 0$. Thus, we get $d(Tx, Ty) \geq t/2$.

By the construction of T it suffices to consider two cases, namely:

Case 1. $x = 1, y \in [0, 1)$.

Case 2. $x \in [0, 1), y = 1$.

In Case 1, from $d(Tx, Ty) \geq t/2$ we get $d(1/3, 0) = 1/3 \geq t/2$, so $t \leq 2/3$.

If, in addition, $M_{d,01}(1, T1, t) = 0$, we deduce, by hypothesis, that $t > 1$, a contradiction.

Hence, we will have that $M_{d,01}(1, T1, t) = 1$. This implies that $d(1, 1/3) = 2/3 < t$, which yields again a contradiction.

In Case 2, from $d(Tx, Ty) \geq t/2$ we get $d(0, 1/3) = 0 \geq t/2$, a contradiction.

We conclude that $M_{d,01}(Tx, Ty, t/2) = 1$, and thus $M_{d,01}(Tx, Ty, t/2) > 1 - t/2$. Therefore T is a (1)-Kannan contraction on $(X, M_{d,01}, *)$.

Our main result (Theorem 4.10 below) provides a full quasi-metric extension of [18, Theorem 5]. In its proof we shall use the next quasi-metric generalization of an important result due to Radu [17, Proposition 2.1.1], which can be partially found in [3, Example 2] and in [5, Remark 7.6.1] (let us recall that the famous Łukasiewicz t-norm $*_L$ is the continuous t-norm defined by $a *_L b = \max\{a + b - 1, 0\}$ for all $a, b \in [0, 1]$).

Proposition 4.8. Let $(X, M, *)$ be a fuzzy quasi-metric space. For each $x, y \in X$ put

$$d_M(x, y) = \sup\{t \geq 0 : M(x, y, t) \leq 1 - t\}.$$

Then d_M satisfies the following condition

$$d_M(x, y) < t \Leftrightarrow M(x, y, t) > 1 - t, \tag{C1}$$

for all $t > 0$. Furthermore, if $*_L \leq *$, then d_M is a quasi-metric on X whose induced topology agrees with τ_M , and (X, d_M) is complete if and only if $(X, M, *)$ is complete.

We shall also use the following essentially well-known fact.

Lemma 4.9. If a Cauchy sequence $(x_n)_{n \in \mathbb{N}}$ in a fuzzy quasi-metric space $(X, M, *)$ has a subsequence which is τ_M -convergent to some $z \in X$, then $(x_n)_{n \in \mathbb{N}}$ is τ_M -convergent to z .

Proof. Let $(x_{n(k)})_{k \in \mathbb{N}}$ be a subsequence of $(x_n)_{n \in \mathbb{N}}$ for which there is some $z \in X$ such that $(x_{n(k)})_{k \in \mathbb{N}}$ τ_M -converges to z .

Choose an arbitrary $\varepsilon \in (0, 1)$. By the continuity of $*$ we can find a $\delta \in (0, \varepsilon/2)$ such that $(1 - \delta) * (1 - \delta) > 1 - \varepsilon$. Then, there exists $n_\delta \in \mathbb{N}$ such that $M^{\min}(x_n, x_m, \delta) > 1 - \delta$ for all $n, m \geq n_\delta$. Since for any $m \geq n_\delta$ there exists $k_0 \in \mathbb{N}$ with $n(k_0) \geq m$ and $M(z, x_{n(k_0)}, \delta) > 1 - \delta$, we deduce that, for $m \geq n_\delta$,

$$M(z, x_m, \varepsilon) \geq M(z, x_{n(k_0)}, \delta) * M(x_{n(k_0)}, x_m, \delta) \geq (1 - \delta) * (1 - \delta) > 1 - \varepsilon.$$

Consequently, the sequence $(x_n)_{n \in \mathbb{N}}$ τ_M -converges to z . □

Theorem 4.10. For a fuzzy quasi-metric space $(X, M, *)$ the following conditions are equivalent.

- (1) $(X, M, *)$ is complete.
- (2) Every (1)-Kannan contraction on X has a (unique) fixed point.
- (3) Every (1/2)-Kannan contraction on X has a (unique) fixed point.

Proof. (1) \Rightarrow (2) Theorem 4.4.

(2) \Rightarrow (3) Obvious.

(3) \Rightarrow (1) Suppose that $(X, M, *)$ is not complete. Then there exists a Cauchy sequence $(x_n)_{n \in \mathbb{N}}$ in $(X, M, *)$ which is not τ_M -convergent.

We divide the rest of the proof in the following claims.

Claim 1. For each $y \in X$ there is an $n(y) \in \mathbb{N}$ such that $d_M(y, x_n) > 0$ for all $n \geq n(y)$.

Indeed, suppose that there are $y \in X$ and a subsequence $(x_{n(i)})_{i \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ such that $d_M(y, x_{n(i)}) = 0$ for all $i \in \mathbb{N}$. By condition (C1) we get $M(y, x_{n(i)}, t) > 1 - t$ for all $t > 0$. Therefore $(x_n)_{n \in \mathbb{N}}$ τ_M -converges to y by Remark 2.6 and Proposition 4.8, a contradiction.

Claim 2. For each $y \in X$ there is a $k(y) \geq n(y)$ such that $d_M(y, x_n) \geq 1/k(y)$ for all $n \geq k(y)$.

Indeed, suppose that there is $y \in X$ such that for any $k \geq n(y)$ there exists $n_k > k$ satisfying $d_M(y, x_{n_k}) < 1/k$. By condition (C1) we get $M(y, x_{n_k}, 1/k) > 1 - 1/k$, which implies that $(x_{n_k})_{k \in \mathbb{N}}$ τ_M -converges to y . By Lemma 4.9, this yields a contradiction.

Now, since $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $(X, M, *)$, for each $y \in X$ there is a $j(y) \in \mathbb{N}$ such that $j(y) \geq k(y)$ and

$$M(x_n, x_m, 1/3j(y)) > 1 - 1/3j(y),$$

whenever $n, m \geq j(y)$. Hence $d_M(x_n, x_m) < 1/3j(y)$, whenever $n, m \geq j(y)$.

Define a self map T of X as follows:

$$Ty = x_{j(y)} \quad \text{for all } y \in X.$$

Claim 3. The self map T has no fixed points in X .

Indeed, put $F := \{x_n : n \in \mathbb{N}\}$. If $y \in X \setminus F$ is obvious that $Ty \neq y$. If $y \in F$ there is $i \in \mathbb{N}$ such that $y = x_i$. Since $j(y) \geq k(y) \geq n(y)$, it follows from Claim 1 that $d_M(y, x_{j(y)}) > 0$, so $y \neq x_{j(y)}$.

Claim 4. T is a (1/2)-Kannan contraction on $(X, M, *)$ (with constant $c = 1/3$).

Indeed, let $y, z \in X$ and $t > 0$ such that $\min\{M(y, Ty, t), M(z, Tz, t)\} > 1 - t$. Then $d_M(y, Ty) < t$ and $d_M(z, Tz) < t$.

If $j(y) < j(z)$ we obtain

$$\begin{aligned} d_M(Ty, Tz) &= d_M(x_{j(y)}, x_{j(z)}) < 1/3j(y) \leq 1/3k(y) \\ &\leq d_M(y, x_{j(y)})/3 = d_M(y, Ty)/3 < t/3. \end{aligned}$$

Therefore $M(Ty, Tz, t/3) > 1 - t/3$.

If $j(y) > j(z)$ we obtain

$$\begin{aligned} d_M(Ty, Tz) &= d_M(x_{j(y)}, x_{j(z)}) < 1/3j(z) \leq 1/3k(z) \\ &\leq d_M(z, x_{j(z)})/3 = d_M(z, Tz)/3 < t/3. \end{aligned}$$

Therefore $M(Ty, Tz, t/3) > 1 - t/3$.

We have constructed a (1/2)-Kannan contraction on $(X, M, *)$ without fixed points. This contradiction concludes the proof. \square

The last part of this paper is devoted to discuss the extension of Theorems 4.4 and 4.10 above to contractions of Chatterjea-type on complete fuzzy quasi-metric spaces. Although we have not been able to obtain results as resounding as such theorems, some partial results can be found in Propositions 4.13, 4.14 and 4.16 below.

Definition 4.11. Let $(X, M, *)$ be a fuzzy quasi-metric space. We say that a self map T of X is a (1)-Chatterjea contraction on $(X, M, *)$ if there is a constant $c \in (0, 1)$ such that for any $x, y \in X$ and $t > 0$,

$$\min\{M(x, Ty, t), M(y, Tx, t)\} > 1 - t \Rightarrow M(Tx, Ty, ct) > 1 - ct. \quad (1Ch)$$

By analogy with the quasi-metric setting, we say that a self map T of X is a (1/2)-Chatterjea contraction on $(X, M, *)$ if there is a constant $c \in (0, 1/2)$ such that for any $x, y \in X$ and $t > 0$,

$$\min\{M(x, Ty, t), M(y, Tx, t)\} > 1 - t \Rightarrow M(Tx, Ty, ct) > 1 - ct.$$

Remark 4.12. If T is a (1)-Chatterjea contraction on a fuzzy quasi-metric space $(X, M, *)$ and we interchange x and y in condition (1Ch), then that condition can be reformulated as follows:

$$\min\{M(x, Ty, t), M(y, Tx, t)\} > 1 - t \Rightarrow M^{\min}(Tx, Ty, ct) > 1 - ct. \quad (1Chm)$$

Clearly, any (1/2)-Chatterjea contraction is a (1)-Chatterjea contraction.

If T is a (1)-Chatterjea contraction on a fuzzy (quasi-)metric space $(X, M, *)$ it is possible to show that for any $x_0 \in X$, $(T^n x_0)_{n \in \mathbb{N}}$ is a Cauchy sequence in $(X, M, *)$ (see Proposition 4.13 below). Thus, if $(X, M, *)$ is complete, the sequence $(x_n)_{n \in \mathbb{N}}$ τ_M -converges to some $z \in X$. Unfortunately, and in contrast to the Kannan case, we don't know if $M(z, Tz, t) = 1$ for all $t > 0$ (compare with Theorem 4.4 above and [18, Theorem 3]). Nevertheless, and as we pointed out above, a partial result is provided in Proposition 4.14 below.

Proposition 4.13. Let T be a (1)-Chatterjea contraction on a fuzzy quasi-metric space $(X, M, *)$. Fix $x_0 \in X$. Then $(T^n x_0)_{n \in \mathbb{N}}$ is a Cauchy sequence in $(X, M, *)$.

Proof. Let $c \in (0, 1)$ for which condition (1Ch) in Definition 4.11 is fulfilled. Fix $t_0 > 1$. For any $x, y \in X$ we get $M(x, Ty, t_0) > 1 - t_0$ and $M(y, Tx, t_0) > 1 - t_0$.

Thus, by the contraction condition (1Ch), $M(Tx, Ty, ct_0) > 1 - ct_0$.

In particular, from $M(x, T^2y, t_0) > 1 - t_0$ and $M(Ty, Tx, t_0) > 1 - t_0$, it follows that

$$M(Tx, T^2y, ct_0) > 1 - ct_0.$$

Analogously,

$$M(Ty, T^2x, ct_0) > 1 - ct_0.$$

Then, it follows from Remark 4.12 that

$$M^{\min}(T^2x, T^2y, c^2t_0) > 1 - c^2t_0.$$

Repeating this process we obtain, for each $n \in \mathbb{N}$,

$$M^{\min}(T^n x, T^n y, c^n t_0) > 1 - c^n t_0.$$

Put $x_n := T^n x_0$ for all $n \in \mathbb{N} \cup \{0\}$. We see that $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $(X, M, *)$.

Indeed, given $t > 0$ and $\varepsilon \in (0, 1)$, there is $n_\varepsilon \in \mathbb{N}$ such that $c^n t_0 < \min\{\varepsilon, t\}$ for all $n \geq n_\varepsilon$. Let $m, n \geq n_\varepsilon$. Assume that $m > n$. Then $m = n + k$ for some $k \in \mathbb{N}$, and thus

$$\begin{aligned} M^{\min}(x_n, x_m, t) &= M^{\min}(T^n x_0, T^n T^k x_0, t) \geq M^{\min}(T^n x_0, T^n T^k x_0, c^n t_0) \\ &> 1 - c^n t_0 > 1 - \varepsilon. \end{aligned}$$

We conclude that $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $(X, M, *)$. □

Proposition 4.14. Every (1/2)-Chatterjea contraction on a complete fuzzy quasi-metric space $(X, M, *)$ such that $*_L \leq *$ has a unique fixed point.

Proof. Let T be a $(1/2)$ -Chatterjea contraction (with constant $c \in (0, 1/2)$) on $(X, M, *)$. We shall show that T is a Chatterjea contraction on the complete quasi-metric space (X, d_M) as constructed in Proposition 4.8.

Indeed, let $x, y \in X$ and let $t \geq 0$ such that $M(Tx, Ty, t) \leq 1 - t$. Then

$$\min\{M(x, Ty, t/c), M(y, Tx, t/c)\} \leq 1 - t/c,$$

so, by condition (C1) in Proposition 4.8,

$$\max\{d_M(x, Ty), d_M(y, Tx)\} \geq t/c.$$

Hence $t \leq c \max\{d_M(x, Ty), d_M(y, Tx)\}$. Thus

$$\begin{aligned} d_M(Tx, Ty) &= \sup\{t \geq 0 : M(Tx, Ty, t) \leq 1 - t\} \\ &\leq c \max\{d_M(x, Ty), d_M(y, Tx)\} \leq c[d_M(x, Ty) + d_M(y, Tx)]. \end{aligned}$$

We have proved that T is a Chatterjea contraction on (X, d_M) , so, by Theorem 3.5, it has a unique fixed point. \square

The following example illustrates Proposition 4.14.

Example 4.15. Let (X, d) the complete quasi-metric space where $X = [0, 1]$ and d is given by $d(x, y) = \max\{x - y, 0\}$ for all $x, y \in X$. Consider the complete fuzzy quasi-metric space $(X, M_{d,0.1}, *)$ and fix an $x_0 \in (0, 1/2)$. Then, we define the self map T of X given by $T1 = x_0$ and $Tx = 0$ otherwise.

We shall check that T is a $(1/2)$ -Chatterjea contraction on $(X, M_{d,0.1}, *)$ with constant $c = x_0$.

Indeed, let $x, y \in X$ and $t > 0$ such that $\min\{M_{d,0.1}(x, Ty, t), M_{d,0.1}(y, Tx, t)\} > 1 - t$.

Suppose that $M_{d,0.1}(Tx, Ty, ct) = 0$. Then $d(Tx, Ty) \geq ct = x_0t$.

It suffices to consider two cases.

Case 1. $x = 1, y \in [0, 1)$.

From $d(Tx, Ty) \geq x_0t$ we deduce that $x_0 \geq x_0t$, so $t \leq 1$. Since, by hypothesis, $M_{d,0.1}(x, Ty, t) > 1 - t$ we deduce that $M_{d,0.1}(x, Ty, t) = 1$, thus $d(1, 0) = 1 < t$, a contradiction.

Case 2. $x \in [0, 1)$ and $y = 1$. From $d(Tx, Ty) \geq ct$ we deduce that $d(0, x_0) = 0 \geq ct$, a contradiction.

We conclude that $M_{d,0.1}(Tx, Ty, ct) = 1 > 1 - ct$, so T is a $(1/2)$ -Chatterjea contraction on $(X, M_{d,0.1}, *)$, and all conditions of Proposition 4.14 are satisfied.

We finish with the following partial converse of Proposition 4.14.

Proposition 4.16. A fuzzy quasi-metric space $(X, M, *)$ such that $*_L \leq *$ is complete if every (1) -Chatterjea contraction has a fixed point.

Proof. Let T be a Chatterjea contraction (with constant $c \in (0, 1/2)$) on the quasi-metric space (X, d_M) . Then, for any $x, y \in X$ we have

$$d_M(Tx, Ty) \leq c[d_M(x, Ty) + d_M(y, Tx)].$$

We check that T is a (1) -Chatterjea contraction on $(X, M, *)$ (with constant $2c$).

Indeed, let $x, y \in X$ and $t > 0$ such that $\min\{M(x, Ty, t), M(y, Tx, t)\} > 1 - t$. By condition (C1), we get $\max\{d_M(x, Ty), d_M(y, Tx)\} < t$. Then

$$d_M(Tx, Ty) < 2ct,$$

and thus $M(Tx, Ty, 2ct) > 1 - 2ct$. By our assumption T has a fixed point. It follows from Theorem 3.5 that (X, d_M) is a complete quasi-metric space, so $(X, M, *)$ is complete by Proposition 4.8. \square

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