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Combinatorial Number Theory, Recurrence of Operators and Linear Dynamics

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Valencia, June 2023

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UNIVERSITAT POLITÈCNICA DE VALÈNCIA

DEPARTAMENTO DE MATEMÁTICA APLICADA
UNIVERSIDAD POLITÉCNICA DE VALENCIA

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Institut Universitari de Matemàtica Pura i Aplicada

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Resum

La tesi “*Teoria Combinatòria de Nombres, Recurrència d'Operadors i Dinàmica Lineal*” se situa dins de l'estudi de la dinàmica d'operadors lineals, o simplement *Dinàmica Lineal*. L'objectiu d'aquest treball és estudiar múltiples nocions de *recurrència*, que poden presentar els sistemes dinàmics lineals, i que classificarem mitjançant la *Teoria Combinatòria de Nombres*.

La Dinàmica Lineal estudia les òrbites generades per les iteracions d'una transformació lineal. Les propietats més estudiades en aquesta branca de les matemàtiques als darrers 30 anys han estat la hiperciclicitat (existència d'òrbites denses) i el caos (amb les seves múltiples definicions), sent aquesta una àrea de recerca molt activa i obtenint-se un considerable nombre de resultats profunds i interessants (vegeu [10, 55]). Nosaltres ens centrarem en la *recurrència*, propietat molt estudiada per a sistemes dinàmics clàssics no lineals (vegeu [41, 48]), però, pràcticament nova en Dinàmica Lineal doncs no és fins al 2014, amb l'article [30] de Costakis, Manoussos i Parissis titulat “*Recurrent linear operators*”, quan es comença a estudiar aquesta noció de manera sistemàtica en el context d'operadors actuant en espais de Banach.

La situació bàsica de la qual parteix el nostre estudi és la següent: $T : X \longrightarrow X$ serà un operador lineal i continu actuant sobre un F-espai X (és a dir, un espai vectorial topològic que admet una mètrica completa), encara que de vegades necessitarem que l'espai subjacent X siga un espai de Fréchet, de Banach o de Hilbert. Llavors, donat un vector $x \in X$ i un entorn U de x estudiarem el *conjunt de retorn* $N_T(x, U) := \{n \in \mathbb{N}_0 : T^n x \in U\}$ i depenent de la seva mida, observada des del punt de vista de la Teoria Combinatòria de Nombres, direm que el vector x presenta una o altra propietat de recurrència.

La memòria de la tesi s'ha realitzat per compendi d'articles i, seguint la normativa establerta per l'**Escola de Doctorat**, l'estructura és la següent:

– **Introducció**. Es presenten les nocions i definicions bàsiques necessàries, junt amb la notació utilitzada al llarg de la memòria i l'explicació de què conté cadascun dels capítols següents. Aquest capítol pretén ser el fil conductor del treball.

1. **Frequently recurrent operators**. Adaptació de la “versió d'autor” revisada de l'article [21]: *Journal of Functional Analysis*, **283** (12) (2022), article núm. 109713, 36 pàgines. En aquest es defineixen per primera vegada les nocions de *recurrència reiterada*, *$\mathcal{U}$-freqüent* i *freqüent*, i les seves propietats bàsiques (com les similituds amb les respectives nocions d'hiperciclicitat, les diferències entre els distints tipus de recurrència introduïts, la mida dels diversos conjunts de vectors recurrents, la relació d'aquests fenòmens amb propietats espectrals dels operadors, i els respectius teoremes del tipus “Ansari” i “León-Müller”) són estudiades. Finalment es generalitza l'estudi mitjançant el concepte de $\mathcal{F}$-recurrència, que es connecta amb la noció de $\mathcal{F}$-hiperciclicitat anteriorment estudiada en treballs com [6, 84, 16, 20].

2. **Recurrence properties: An approach via invariant measures**. Adaptació al format de la tesi de la "versió d'autor" revisada de l'article [50]: *Journal de Mathématiques Pures et Appliquées*, **169** (2023), 155–188. Es relaciona la recurrència d'operadors, que s'havia estudiat únicament des del punt de vista topològic, amb la Teoria Ergòdica i els sistemes dinàmics que conserven la mesura. Restringint l'espai subjacent a espais de Banach reflexius i espais de Hilbert s'obtenen fortes equivalències entre les diferents propietats de recurrència establertes a [21]: a partir de vectors amb recurrència feble es construeixen mesures invariants, i d'aquestes s'obtenen nocions de recurrència més fortes.

3. **Questions in linear recurrence: From the $T \oplus T$-problem to lineability**. Adaptació de la "versió d'autor" del preprint [51]. En aquest es resol un problema obert de l'any 2014 (vegeu [30, Question 9.6]): *Siga $T : X \longrightarrow X$ un operador recurrent. És cert que l'operador $T \oplus T$ és recurrent en $X \oplus X$?* Per resoldre'l introduïm la *quasi-rigidesa*, que serà, per a la recurrència, la noció anàloga a la propietat *feble-barrejant* (topològica) per a la transitivitat/hiperciclicitat; i després construïm operadors recurrents però no quasi-rígids en tot espai de Banach infinit-dimensional i separable. La quasi-rigidesa és posteriorment emprada per a estudiar la lineabilitat dels conjunts de vectors $\mathcal{F}$-recurrents.

4. **Recurrent subspaces in Banach spaces**. Adaptació de la "versió d'autor" del preprint [69]. S'inclou l'estudi de la propietat d'*espaiabilitat* (existència d'un subespai vectorial tancat i de dimensió infinita) per al conjunt de vectors recurrents. Emprant la Teoria Espectral com als treballs [66, 47] es caracteritzen els operadors quasi-rígids actuant en espais de Banach que admeten *subespais recurrents*, i s'obté el curiós resultat: *un operador feble-barrejant admet un subespai hipercíclic si, i només si, admet un subespai recurrent.*

– **Discussió general dels resultats**. Es discuteix la naturalesa dels diferents resultats que hem aconseguit. També hem inclòs alguns comentaris i resultats extra relacionats amb cada un dels capítols/articles que formen aquesta memòria.

– **Conclusions**. S'inclouen les conclusions del treball, analitzant l'impacte que pot tenir a la Dinàmica Lineal, i recollint les principals línies de recerca i problemes que queden oberts.

– **Apèndix**. Per aconseguir un caràcter auto-contingut hem afegit un apèndix amb resultats bàsics de *Teoria Combinatòria de Nombres* que es donen per suposats en els treballs que componen la memòria. S'inclouen: alguns conceptes de mida per a conjunts de nombres naturals relacionats amb les propietats de la compactació de Stone-Čech $\beta\mathbb{N}_0$; les definicions i propietats bàsiques d'algunes nocions de densitat per a conjunts de nombres naturals; el concepte de família de Furstenberg; i alguns exemples.

Resumen

La tesis "*Teoría Combinatoria de Números, Recurrencia de Operadores y Dinámica Lineal*" se sitúa dentro del estudio de la dinámica de operadores lineales, o *Dinámica Lineal*. El objetivo de este trabajo es estudiar múltiples nociones de *recurrencia*, que pueden presentar los sistemas dinámicos lineales, y que clasificaremos mediante la *Teoría Combinatoria de Números*.

La Dinámica Lineal estudia las órbitas generadas por las iteraciones de una transformación lineal. Las propiedades más estudiadas en esta rama durante los últimos 30 años han sido la hiperciclicidad (existencia de órbitas densas) y el caos (con sus múltiples definiciones), siendo esta un área de investigación muy activa y obteniéndose un considerable número de resultados profundos e interesantes (véase [10, 55]). Nosotros nos centraremos en la *recurrencia*, propiedad muy estudiada para sistemas dinámicos clásicos no lineales (véase [41, 48]), pero prácticamente nueva en Dinámica Lineal pues no es hasta 2014, con el artículo [30] de Costakis, Manoussos y Parissis titulado "*Recurrent linear operators*", cuando se empieza a estudiar esta noción de manera sistemática en el contexto de operadores actuando en espacios de Banach.

La situación básica de la que parte nuestro estudio es la siguiente: $T : X \longrightarrow X$ será un operador lineal y continuo actuando sobre un F-espacio X (es decir, un espacio vectorial topológico que admite una métrica completa), aunque a veces necesitaremos que el espacio subyacente X sea un espacio de Fréchet, de Banach o de Hilbert. Dado un vector $x \in X$ y un entorno U de x estudiaremos el *conjunto de retorno* $N_T(x, U) := \{n \in \mathbb{N}_0 : T^n x \in U\}$ y dependiendo de su tamaño, observado mediante la Teoría Combinatoria de Números, diremos que el vector x presenta una propiedad de recurrencia u otra.

La memoria de la tesis se ha realizado por compendio de artículos y, siguiendo la normativa establecida por la **Escuela de Doctorado**, la estructura es la siguiente:

- **Introducción**. Se presentan las nociones y definiciones básicas necesarias, junto con la notación utilizada a lo largo de la memoria y la explicación de qué contiene cada uno de los capítulos siguientes. Este capítulo pretende ser el hilo conductor del trabajo.

1. **Frequently recurrent operators**. Adaptación de la "versión de autor" del artículo [21]: *Journal of Functional Analysis*, **283** (12) (2022), artículo núm. 109713, 36 páginas. En este se definen por primera vez las fuertes nociones de *recurrencia reiterada*, *$\mathcal{U}$-frecuente* y *frecuente*, y sus propiedades básicas (como las similitudes con las respectivas nociones de hiperciclicidad, las diferencias entre los distintos tipos de recurrencia, el tamaño de los varios conjuntos de vectores recurrentes, la relación de estos fenómenos con propiedades espectrales, y los respectivos teoremas del tipo "Ansari" y "León-Müller") son estudiadas. Finalmente se generaliza el estudio mediante el concepto de $\mathcal{F}$-recurrencia, que se conecta con la noción de $\mathcal{F}$-hiperciclicidad anteriormente estudiada en trabajos como [6, 84, 16, 20].

2. **Recurrence properties: An approach via invariant measures**. Adaptación al formato de la tesis de la "versión de autor" revisada del artículo [50]: *Journal de Mathématiques Pures et Appliquées*, **169** (2023), 155–188. En este se relaciona la recurrencia de operadores, que se había estudiado únicamente desde el punto de vista topológico, con la Teoría Ergódica y los sistemas dinámicos que conservan la medida. Restringiendo el espacio subyacente a espacios de Banach reflexivos y espacios de Hilbert se obtienen fuertes equivalencias entre las distintas propiedades de recurrencia establecidas en [21]: a partir de vectores con recurrencia débil se construyen medidas invariantes, y de estas se obtienen nociones de recurrencia más fuertes.

3. **Questions in linear recurrence: From the $T \oplus T$-problem to lineability**. Adaptación de la "versión de autor" del preprint [51]. Se resuelve negativamente un problema abierto de 2014 (véase [30, Question 9.6]): *Sea $T : X \longrightarrow X$ un operador recurrente. ¿Es cierto que el operador $T \oplus T$ es recurrente en $X \oplus X$?* Para resolverlo introducimos la *casi-rigidez*, que será, para la recurrencia, la noción análoga a la propiedad *débil-mezclante* (topológica) para la transitividad/hiperciclicidad; y luego construimos operadores recurrentes pero no casi-rígidos en todo espacio de Banach infinito-dimensional y separable. La casi-rigidez es posteriormente utilizada para estudiar la lineabilidad de los conjuntos de vectores $\mathcal{F}$-recurrentes.

4. **Recurrent subspaces in Banach spaces**. Adaptación de la "versión de autor" revisada del preprint [69]. En este se estudia la propiedad de *espaciabilidad* (existencia de un subespacio vectorial cerrado y de dimensión infinita) para el conjunto de vectores recurrentes. Usando la Teoría Espectral como en [66, 47] se caracterizan los operadores casi-rígidos que admiten *subespacios recurrentes*, y se obtiene el curioso resultado: *un operador débil-mezclante admite un subespacio hipercíclico si, y solamente si, admite un subespacio recurrente.*

– **Discusión general de los resultados**. Se discute la naturaleza de los diferentes resultados conseguidos. También hemos incluido algunos comentarios y resultados extra relacionados con cada uno de los capítulos/artículos que forman esta memoria.

– **Conclusiones**. Se incluyen las conclusiones del trabajo, analizando el impacto que puede tener en el área de la Dinámica Lineal, y recogiendo las principales líneas de investigación y problemas que quedan abiertos.

– **Apéndice**. Para conseguir un carácter auto-contenido hemos añadido un apéndice con los resultados básicos de *Teoría Combinatoria de Números* que se han utilizado en los trabajos que componen la memoria. Se incluyen: algunos conceptos de tamaño para conjuntos de números naturales relacionados con las propiedades de la compactación de Stone-Čech $\beta\mathbb{N}_0$; las definiciones y propiedades básicas de algunas nociones de densidad para conjuntos de números naturales; el concepto de familia de Furstenberg; y algunos ejemplos.

Summary

The thesis "*Combinatorial Number Theory, Recurrence of Operators and Linear Dynamics*" is part of the study of the dynamics of linear operators, simply called *Linear Dynamics*. The objective of this work is to study multiple notions of *recurrence*, that linear dynamical systems can present, and which will be classified through *Combinatorial Number Theory*.

Linear Dynamics studies the orbits generated by the iterations of a linear transformation. The two most studied properties in this branch of mathematics during the last 30 years have been hypercyclicity (existence of dense orbits) and chaos (with its multiple definitions), being this a very active research area with a considerable number of exceptionally deep but also interesting results (see [10, 55]). We will focus on *recurrence*, a property widely studied in the classical setting of non-linear dynamical systems (see [41, 48]), but practically new with respect to Linear Dynamics since it was not until 2014, with the article [30] by Costakis, Manoussos and Parissis entitled "*Recurrent linear operators*", when this notion started to be systematically studied in the context of operators acting on Banach spaces.

The basic situation from which our study starts is the following: $T : X \longrightarrow X$ will be a continuous linear operator (sometimes simply called *linear operator* or just *operator*) acting on an F-space X (that is, a completely metrizable topological vector space), although sometimes we will need the underlying space X to be a Fréchet, Banach or Hilbert space. Given a vector $x \in X$ and a neighbourhood U of x we will study the *return set* $N_T(x, U) := \{n \in \mathbb{N}_0 : T^n x \in U\}$ and depending on its size, observed from the Combinatorial Number Theory point of view, we will say that the vector x presents one property of recurrence or another.

The thesis memoir is a compendium of articles and, following the regulations established by the **Doctoral School**, the structure is the following:

– **Introduction**. We present the basic notions, definitions and notation used throughout the memoir, together with the explanation of what contains each of the following chapters. This chapter is intended to be the common thread of the work.

1. **Frequently recurrent operators**. Adaptation of the revised "author version" of article [21]: *Journal of Functional Analysis*, **283** (12) (2022), paper no. 109713, 36 pages. Here, the strong notions of *reiterative*, *$\mathcal{U}$-frequent* and *frequent recurrence* are defined for the first time, and their basic properties (such as the similarities with the respective notions of hypercyclicity, the differences between each type of recurrence, the size of the various sets of recurrent vectors, the interplay between recurrence and spectral properties, and the respective "Ansari" and "León-Müller" type theorems) are studied. The theory is finally generalized through the concept of $\mathcal{F}$-recurrence, which is connected to the notion of $\mathcal{F}$-hypercyclicity previously studied in many interesting works such as [6, 84, 16, 20].

2. **Recurrence properties: An approach via invariant measures**. Adaptation of the revised "author version" of article [50]: *Journal de Mathématiques Pures et Appliquées*, **169** (2023), 155–188. In this chapter the recurrence properties for linear operators, which had been studied only from the topological point of view, are related to Ergodic Theory and measure preserving systems. Considering reflexive Banach spaces and Hilbert spaces we obtain strong equivalences between the different recurrence properties established in [21]: we construct invariant measures from vectors with a rather weak recurrence property, and from these measures we get stronger recurrence notions.

3. **Questions in linear recurrence: From the $T \oplus T$-problem to lineability**. Adaptation of the revised "author version" of the preprint [51]. We solve in the negative an open problem posed in 2014 (see [30, Question 9.6]): *Let $T : X \longrightarrow X$ be a recurrent operator. Is it true that the operator $T \oplus T$ is recurrent on $X \oplus X$?* In order to do that we establish the analogous notion, for recurrence, to that of (topological) *weak-mixing* for transitivity/hypercyclicity, namely *quasi-rigidity*; and then we construct recurrent but not quasi-rigid operators on every separable infinite-dimensional Banach space. The concept of quasi-rigidity is then used to study some lineability properties for the sets of $\mathcal{F}$-recurrent vectors.

4. **Recurrent subspaces in Banach spaces**. Adaptation of the revised "author version" of the preprint [69]. In this chapter we study the *spaceability* (existence of an infinite-dimensional closed subspace) for the set of recurrent vectors. Using Spectral Theory as in [66, 47] we characterize the quasi-rigid operators acting on Banach spaces that admit *recurrent subspaces*, and the following curious result is obtained: *a weakly-mixing operator admits a hypercyclic subspace if and only if it admits a recurrent subspace.*

– **General discussion of the results**. We discuss the nature of the different results achieved. We have also included some remarks and further results related to each of the chapters/articles forming this memoir.

– **Conclusions**. The conclusions are included, analysing the impact that this work can have in Linear Dynamics, and collecting the main lines of research and problems that remain open.

– **Appendix**. Looking for a self-contained text we have added an appendix with some of the basic *Combinatorial Number Theory* results that are taken for granted along the different chapters/articles forming this memoir. Included are: some size concepts for infinite sets of natural numbers related to the properties of the Stone-Čech compactification $\beta\mathbb{N}_0$; the definitions and basic properties of some notions of density for sets of natural numbers; the concept of Furstenberg family; and some examples.

Contents

Introduction

The *Theory of Dynamical Systems* is the branch of mathematics that studies the long-term behaviour of a system, that is, the evolution of a "complex object" (the system) whose parts or components "change" with the pass of time. Some examples would be: a population of an animal species (we could measure its size, or the amount of individuals fulfilling some property); an economic system (we may analyse its acceleration, stability or recession); or a physical system (where the position and speed of some particles could be computed). See Figure 1.

In general, we consider a non-empty set X as the collection of all possible states of a system (biological, economic, physical, ...), which is usually called the *phase space*, and we assume that the evolution of the system is given by a *transformation* or an *application* $T : X \longrightarrow X$, so that if $x_n \in X$ is the state of the system at time $n \geq 0$ then

$$x_{n+1} := T(x_n), \qquad n = 0, 1, 2, ...$$

The pair (X, T) is usually called a (discrete) *dynamical system*, and given an *initial state* of the system $x_0 \in X$, observing its evolution is equivalent to studying the *orbit* of this point:

$$\mathrm{Orb}(x_0, T) := \{x_0, T(x_0), T^2(x_0), ...\} \quad \text{where} \quad T^n := T \circ ... \circ T \quad (n \text{ times}).$$

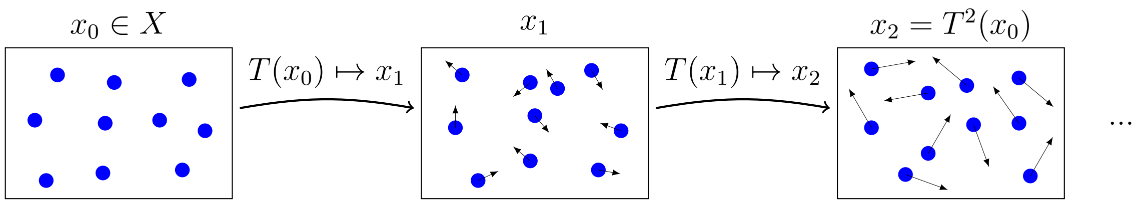

Figure 1: A *physical system* (particles moving in a box), initial state x_0, and its evolution.

In order to analyse the behaviour of a system (X, T) one usually requires the existence of a structure on the set X and certain restrictions on the application T. As set out in [80], the three most studied cases throughout history are when:

- X is a *differentiable manifold* and T is a *diffeomorphism*, the case of *Differentiable Dynamics*;
- X is a *measure space* and T is a *measurable map*, the case of *Ergodic Theory*;
- X is a *topological space* and T is a *continuous map*, the case of *Topological Dynamics*.

We will focus on Topological Dynamics, but the three branches overlap in many examples giving different points of view to the same system via the interaction between both topological and measurable hypothesis (see Section 3 of this Introduction or Chapter 2). From now on a *dynamical system* will be a pair (X, T) formed by a (usually *Hausdorff*) topological space X and a continuous map $T : X \longrightarrow X$.

The study of a mathematical object is often simplified by decomposing it into smaller parts to study them separately. If this is not possible, then the object is said to be *irreducible*. For us this idea will be represented by the well-known notion of *topological transitivity*: a dynamical system (X, T) is said to be *topologically transitive* if for every pair U, V of non-empty open subsets of X there exists a natural number $n \geq 0$ such that $T^n(U) \cap V \neq \varnothing$. See Figure 2.

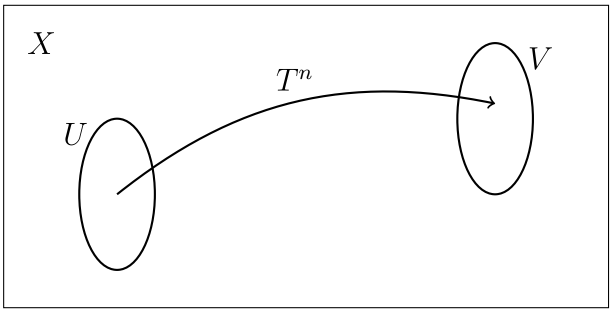

Figure 2: Topological transitivity.

Transitivity is a global property that implies the connection between all the non-trivial parts of a dynamical system and, as mentioned by Kolyada and Snoha [63], it was already employed in 1920 by Birkhoff in the context of continuous applications acting on compact subsets of $\mathbb{R}^n$. The *Birkhoff transitivity theorem* states that the above property is equivalent to the existence of a dense orbit when X is a complete metric space without isolated points (see [55, Theorem 1.16]):

Birkhoff Transitivity Theorem: *Let $T : X \longrightarrow X$ be a continuous map on a separable complete metric space X without isolated points. The following statements are equivalent:*

(i) *the system (X, T) is topologically transitive;*

(ii) *there exists some $x \in X$ such that $\mathrm{Orb}(x, T) = \{T^n(x) : n \geq 0\}$ is dense in X.*

If one of these conditions holds then the set of points in X with dense orbit is a dense G_δ-set.

When X is a metric space we can measure distances between points and know how far the predictions obtained (for example calculating an orbit) are from a certain fixed state of the system $x \in X$. Thus, it is not surprising that the initial topological space X becomes a (complete) metric space. In this work we treat Topological Dynamics from the *Linear Dynamics* point of view and from now on a *linear dynamical system* will be a pair (X, T) where:

– the space X is a *separable* (usually *infinite-dimensional*) *F-space*;

– and the map T is a *continuous linear operator*, also called *linear operator* or simply *operator*.

We denote by $\mathcal{L}(X)$ the *set of continuous linear operators* acting on such a space X, and we will use the Operator Theory notation writing "Tx" instead of "$T(x)$" for each $x \in X$.

Recall that a *topological vector space* X is a vector space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, of real or complex numbers, endowed with a topology such that the adding and scalar multiplication,

$$+ \ : \ X \times X \longrightarrow X, \quad (x, y) \mapsto x + y,$$
$$\cdot \ : \ \mathbb{K} \times X \longrightarrow X, \quad (\lambda, x) \mapsto \lambda x,$$

are continuous. Recall also that a non-negative functional $\| \cdot \| : X \longrightarrow [0, \infty[$ is called an *F-norm* if for each $x, y \in X$ and $\lambda \in \mathbb{K}$ the following properties hold:

(FN1) $\|x + y\| \leq \|x\| + \|y\|$;

(FN2) $\|\lambda x\| \leq \|x\|$ whenever $|\lambda| \leq 1$;

(FN3) $\lim_{\lambda \to 0} \|\lambda x\| = 0$;

(FN4) $\|x\| = 0$ if and only if $x = 0$.

This functional defines a (translation invariant) metric in X via the formula $d(x, y) = \|x - y\|$ for each $x, y \in X$. The properties above easily imply that (X, d) is a topological vector space, which is called an *F-space* when the metric d is complete. Sometimes we will need X to be a *Fréchet*, a *Banach* or a *Hilbert space*, which are the *locally convex* F-spaces that appear when the topology is endowed by a family of *semi-norms* or a single *norm*. We refer the reader to the textbooks [37, 60, 72] for any unexplained but standard notion related to this spaces.

Linear Dynamics connects Functional Analysis and Topological Dynamics. The origin of this branch is found in *hypercyclicity*, and from there different lines of work have been derived such as recurrence in hypercyclicity (topic that we cover in depth, see Section 2), mixing properties, chaos, disjoint-hypercyclicity, etc. Let us recall that hypercyclicity is the study of linear operators admitting a so-called *hypercyclic* vector, that is, a vector which has dense orbit. See Figure 3.

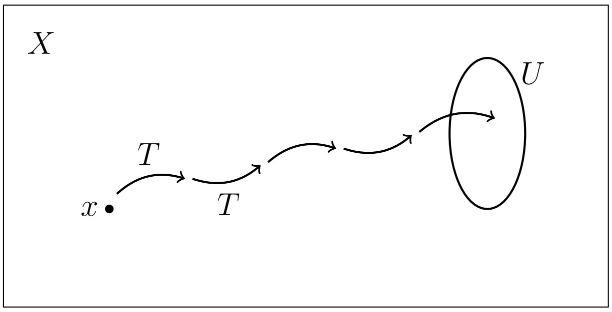

Figure 3: The orbit of $x \in X$ visits every non-empty open subset U of X, i.e. $X = \overline{\mathrm{Orb}(x, T)}$.

In our linear setting the Birkhoff transitivity theorem reads as follows (see [10, Theorem 1.2]):

Linear Transitivity Theorem: *A continuous linear operator $T \in \mathcal{L}(X)$ acting on a separable F-space X is hypercyclic if and only if it is topologically transitive. In that case, the set* $\mathrm{HC}(T)$ *of hypercyclic vectors for T is a dense G_δ-set.*

Birkhoff (in 1929 [19]), MacLane (in 1952 [70]) and Rolewicz (in 1969 [83]) found the first examples of hypercyclic linear operators, but we can fix the birth of Linear Dynamics in 1982 with Kitai's thesis [62] and the obtaining of Kitai's criterion (see [55, Theorem 3.4]). Since then many mathematicians have contributed to the development of this branch of Analysis, and during the last 30 years Linear Dynamics has become a very active research area.

Researchers in Operator Theory became interested in hypercyclicity by the famous *invariant subspace problem* (solved in Banach spaces by Enflo [35] with a counterexample proposed in 1975 but published in 1987 due to its complexity), and by the *invariant subset problem*: an operator has no non-trivial invariant closed subset if every non-zero vector is hypercyclic. Read showed in [82] that such an operator exists in classical Banach spaces such as ℓ^1, leaving both problems open for Hilbert spaces. The works of Kitai [62], Gethner and Shapiro [42], Godefroy and Shapiro [46] and Herrero [56, 57] established the basis of the *hypercyclicity theory*.

The monographs of Bayart and Matheron [10], Grosse-Erdmann and Peris [55] and also the recent survey of Gilmore [43] represent a good compendium of the latest advances in the theory of hypercyclicity and *linear chaos*. The notion of *chaos* is basic in every theory of dynamical systems and, among the existent alternative definitions, Devaney suggested in [33] the following: a continuous map $T : X \longrightarrow X$ acting on a metric space (X, d) is said to be *Devaney chaotic* (for us simply *chaotic*) if the next conditions are satisfied

(DC1) **Long term unpredictability**: sometimes called the *butterfly effect*, and captured by the notion of *sensitive dependence on initial conditions*, i.e. the property that there exists some *sensitivity constant* $\delta > 0$ such that for all $\varepsilon > 0$ and $x \in X$ there exists $y \in X$ and $n \in \mathbb{N}$ fulfilling that $d(x, y) < \varepsilon$ and $d(T^n x, T^n y) > \delta$. See Figure 4.

(DC2) **Irreducibility of the system**: represented by the introduced *topological transitivity*.

(DC3) **Some regularity**: demanding the existence of a dense set of periodic points (a point $x \in X$ is called *periodic* if there exists some positive integer $p \in \mathbb{N}$ such that $T^p x = x$).

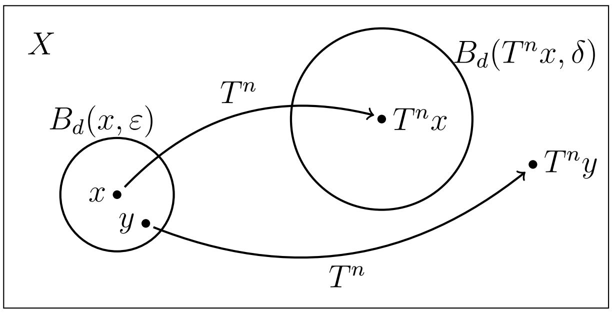

Figure 4: Sensitive dependence on initial conditions.

It was later shown by Banks, Brooks, Cairns, Davis and Stacey that if both (DC2) and (DC3) hold they imply (DC1); see [5]. The notion of chaos is usually seen as a non-linear phenomenon, although it is by now well established that many linear systems may present *linear chaos*. Indeed, in the linear setting hypercyclicity, i.e. (DC2), is enough to obtain (DC1); see [55, Proposition 2.30]. The concept of chaos introduced a new aspect in hypercyclicity and followed by some more extensions, such as the study of hypercyclic semigroups or supercyclic operators, the *Theory of Linear Dynamical Systems* was consolidated.

The dynamical notion that we study in this memoir is that of *recurrence*: given a dynamical system (X,T) we say that a point $x \in X$ is *recurrent* if $x \in \overline{\mathrm{Orb}(Tx,T)}$, that is, if x belongs to the closure of its forward orbit. Equivalently, if for every neighbourhood U of x there exists some natural number $n \geq 1$ such that $T^n x \in U$ (see Figure 5). Moreover, when X is a metric (or first-countable) space (as it happens for F-spaces), the previous definition can be rewritten in terms of sequences: a point $x \in X$ is recurrent if and only if there exists an increasing sequence of positive integers $(n_k)_{k\in\mathbb{N}}$ such that $T^{n_k}x \to x$ as $k \to \infty$.

We say that a dynamical system (X,T) is (pointwise) *recurrent* if its set $\mathrm{Rec}(T)$ of recurrent points is *dense* in X. Since we will study linear systems, we would like to remark on the necessity of this *density* assumption: if X is a vector space then the zero-vector $0 \in X$ is always a fixed point (and hence recurrent) for every continuous linear operator $T \in \mathcal{L}(X)$. Therefore, to say that a map T has a *recurrent behaviour* it is not enough to assume that $\mathrm{Rec}(T) \neq \varnothing$ as it is done in the hypercyclicity case with the set of hypercyclic vectors $\mathrm{HC}(T)$.

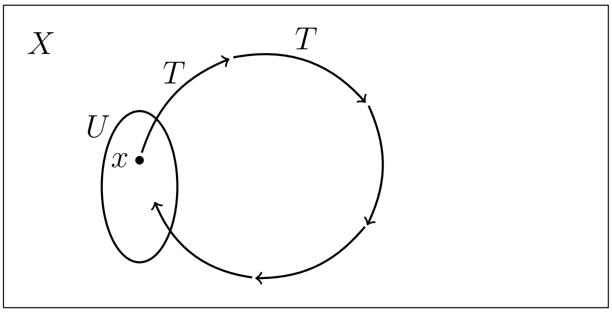

Figure 5: The forward orbit of $x \in X$ visits every neighbourhood U of x, i.e. $x \in \overline{\mathrm{Orb}(Tx,T)}$.

Even though the recurrence property is not equal to hypercyclicity, neither periodicity, we have the following interesting relations between these notions:

– every dense orbit is recurrent, so that every hypercyclic vector/operator is indeed recurrent;

– the periodic points are recurrent, and in a really strong sense (see Chapter 3, Section 4.2).

This implies that, somehow, we will be looking at a dynamical phenomenon related with the already introduced, well-known and interesting properties of hypercyclicity and linear chaos.

As it happened for chaos, *recurrence* has been historically studied for non-linear systems and it is one of the fundamental, oldest and most investigated concepts in dynamics. We can date the beginning of the theory at the end of the 19th century when Poincaré introduced the later called *Poincaré recurrence theorem* in the context of Ergodic Theory (see Section 3). If we focus on Topological Dynamics the appearance of recurrence goes back to the works of Gottschalk and Hedlund in 1955 [48], and of Furstenberg in 1981 [41], together with some more recent advances by many authors such as Banks [4], Glasner [45] and Oprocha and Zhang [79].

In spite of the great non-linear dynamical knowledge existing in this area, recurrence in Linear Dynamics has only recently been systematically studied since 2014 with the fundamental paper of Costakis, Manoussos and Parissis [30] (see also [31]). As one can naturally expect, lots of questions in *linear recurrence* remain open. Our objective in this work has been to study these problems together with various different linear recurrence-kind properties, with the aim of following the natural evolution line of the *Theory of Linear Dynamical Systems*.

In the remainder of this Introduction we recall the main results from the Costakis, Manoussos and Parissis work [30], together with the definitions and pertinent aspects of Linear Dynamics that we need and treat along the different chapters/articles forming this memoir.

1 Linear recurrence: the state of the art

The work of Costakis, Manoussos and Parissis [30] will be the starting point of our study. In this first section we include a sketch of the main results obtained there, which will be used in future parts of this memoir to deepen the study of recurrence.

The first thing to note is that there is a recurrence version of the Birkhoff transitivity theorem, which we call the *Costakis-Manoussos-Parissis theorem* (see [30, Proposition 2.1]). In order to introduce this result, let us start by recalling the following definition: a dynamical system (X, T) is said to be *topologically recurrent* if for every non-empty open subset U of X there exists a natural number $n \geq 1$ such that $T^n(U) \cap U \neq \varnothing$. Two things should be noted:

(1) there is a clear symmetry between the notions of topological recurrence and transitivity (the only difference being the quantity of open sets involved in the definition);

(2) topologically recurrent systems have been called *non-wandering* in the classical literature of non-linear dynamics (see [41, Chapter 1, Section 8]). This name comes from the fact that the subsets $U \subset X$ for which the collection $\{U, T(U), T^2(U), ...\}$ is pairwise disjoint have been called *wandering* sets.

We have the initial symmetry between hypercyclicity and recurrence: topological recurrence is, for (pointwise) recurrence, the analogous notion to that of topological transitivity for the concept of hypercyclicity:

Costakis-Manoussos-Parissis Theorem: *Let $T : X \longrightarrow X$ be a continuous map acting on a complete metric space X. The following statements are equivalent:*

(i) *the dynamical system (X, T) is topologically recurrent;*

(ii) *the set $\mathrm{Rec}(T) = \left\{x \in X : x \in \overline{\mathrm{Orb}(Tx, T)}\right\}$ is dense in X.*

If one of these conditions holds then the set $\mathrm{Rec}(T)$ of recurrent points for T is a dense G_δ-set.

This result was already well-known in non-linear dynamics. However, the proof given by Costakis et al. [30, Proof of Proposition 2.1] is really similar to that of the Birkhoff transitivity theorem (intersecting countably many open dense subsets to achieve the G_δ-set condition), while the proof given by Furstenberg in [41, Theorem 1.27] considers the semi-continuous function

$$F(x) := \inf_{n \geq 1} d(x, T^n x) \quad \text{for each } x \in X,$$

(where d is the metric of X), it shows that the points of continuity for F are those where the function vanishes, and then it uses the fact that every semi-continuous function possesses a residual set of continuity points.

After exhibiting this first equivalence between "pointwise" and topological recurrence in the general context of non-linear dynamical systems, the linearity is used in [30] to establish the recurrence version of the *Ansari-* and *León-Müller-type theorems*, showing a second parallelism between hypercyclicity and recurrence. We recall that the Ansari and León-Müller theorems state that powers and unimodular multiples of an operator still have the same set of hypercyclic vectors (see [2] and [67]):

Ansari and León-Müller Theorems: *Let $T \in \mathcal{L}(X)$. The following statements hold:*

(a) *For any $p \in \mathbb{N}$ we have the equality* $\mathrm{HC}(T) = \mathrm{HC}(T^p)$.

(b) *For any $\lambda \in \mathbb{K}$ with $|\lambda| = 1$ we have the equality* $\mathrm{HC}(T) = \mathrm{HC}(\lambda T)$.

The *linearity* is clearly needed in statement (b), but it is also completely necessary to prove statement (a) since a connectedness argument is used (there exist transitive non-linear systems with powers that are not topologically transitive, see [55, Exercise 1.2.11]). The next result was then proved in [30, Proposition 2.3]:

Ansari-León-Müller Recurrence Theorem: *Let $T \in \mathcal{L}(X)$. The following statements hold:*

(a) *For any $p \in \mathbb{N}$ we have the equality* $\mathrm{Rec}(T) = \mathrm{Rec}(T^p)$.

(b) *For any $\lambda \in \mathbb{K}$ with $|\lambda| = 1$ we have the equality* $\mathrm{Rec}(T) = \mathrm{Rec}(\lambda T)$.

It is worth mentioning that the recurrence version of statement (a) is still true for non-linear dynamical systems. We have proved this rather general fact together with statement (b) in [51] (see Chapter 3, Section 4.2) via a much easier proof than that given in [30, Proposition 2.3].

Another topic treated in [30], just related to linearity and that shows again a symmetry between hypercyclicity and recurrence, is the study of spectral properties. In particular, they showed that the following statements hold for any $T \in \mathcal{L}(X)$ on a **complex** Banach space X:

– *if the spectral radius $r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$ is less than 1, then T is not a recurrent operator*;

– *if T is a recurrent operator, then every component of the spectrum $\sigma(T)$ of T intersects the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$;*

– *if T is a compact operator, then T is not a recurrent operator.*

These properties are well-known to be possessed by hypercyclic operators so that *hypercyclicity* was not the "exactly necessary dynamical condition" for an operator to present this behaviour since *recurrence* is a weaker notion. A first difference between these two properties appears when we look at the spectrum of the adjoint operator T^* for a given recurrent/hypercyclic $T \in \mathcal{L}(X)$. Indeed the following can be found in [55, Lemma 2.53]:

– *If T is hypercyclic, then $T - \lambda$ has dense range for every $\lambda \in \mathbb{C}$, hence $\sigma_p(T^*) = \varnothing$.*

In contrast, by [30, Proposition 2.14] we have that:

– *If T is recurrent, then $T - \lambda$ has dense range for every $\lambda \in \mathbb{C} \setminus \mathbb{T}$, hence $\sigma_p(T^*) \subset \mathbb{T}$.*

We have extended the *dense range* part to the context of recurrent continuous linear operators acting on Hausdorff topological vector spaces in [51] (see Chapter 3, Section 2.2). An extreme case of the previous result is when given any complex number $\lambda \in \mathbb{T}$ we consider the operator $T := \lambda I : X \longrightarrow X$ (where $I : X \longrightarrow X$ is the identity operator on X). Then we have the equality $X = \operatorname{Rec}(T)$ as we argue in the following paragraph, so that the operator T is recurrent while $T - \lambda$ is precisely the null operator, which clearly does not have dense range.

In the previous reasoning we are using the concept of unimodular eigenvector: a non-zero vector $x \in X \setminus \{0\}$ is called a *unimodular eigenvector* for an operator $T \in \mathcal{L}(X)$ precisely when $Tx = \lambda x$ for some $\lambda \in \mathbb{T}$. We will denote the *set of unimodular eigenvectors* by

$$\mathcal{E}(T) = \{x \in X \setminus \{0\} : Tx = \lambda x \text{ for some } \lambda \in \mathbb{T}\}.$$

In [30, Lemma 2.16] it was proved that:

– *If $T \in \mathcal{L}(X)$ fulfills that* $\operatorname{span}(\mathcal{E}(T))$ *is dense in X, then T is a recurrent operator.*

We will see in future sections the importance of unimodular eigenvectors, not just in recurrence studied from the Topological Dynamics point of view but also in Ergodic Theory, and we will show in [21] and [50] (see Chapter 1, Section 7; and Chapter 2, Sections 1.3 and 4), that the hypothesis "$\operatorname{span}(\mathcal{E}(T))$ *is dense in X*" (also called "*having a discrete spectrum*") implies a much stronger recurrence notion, which is close to *periodicity*: we will denote by $\operatorname{Per}(T)$ the set of periodic vectors for an operator $T \in \mathcal{L}(X)$. Recall that, when X is a **complex** space, the following property holds (see [55, Proposition 2.33]):

$$\operatorname{Per}(T) = \operatorname{span}\{x \in X : Tx = e^{\alpha\pi i}x \text{ for some } \alpha \in \mathbb{Q}\} \subset \operatorname{span}(\mathcal{E}(T)).$$

A really strong difference between recurrence and hypercyclicity is that one can consider both *power-bounded* but also *finite-dimensional* recurrent operators, as the identity map is on any (finite-dimensional) space. For the power-boundedness case [30, Lemma 3.1] shows that:

– *If $T \in \mathcal{L}(X)$ is a power-bounded operator then* $\operatorname{Rec}(T)$ *is a closed set. As a consequence, if the operator T is power-bounded and recurrent then* $X = \operatorname{Rec}(T)$.

We extend this result to other recurrence notions in [21] (see Chapter 1, Section 3). If we turn to operators acting on finite-dimensional spaces recall that these are matrices, and recurrence is then reduced to the existence of a basis formed by unimodular eigenvectors:

– *A complex matrix $T : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is recurrent if and only if it is similar to a diagonal matrix with unimodular entries.*

A very similar result holds for real-valued matrices, and in the proof of both cases the Jordan decomposition plays a fundamental role (see [30, Theorems 4.1 and 4.2]).

The rest of [30] is devoted to exhibit examples of recurrent operators (they focus on unilateral and bilateral backward shifts, composition and multiplication operators) but also to study some other properties such as *rigidity* and *uniform rigidity*. However, these last two notions are not completely relevant for this memoir so we will not elaborate further on them here.

They also state some open problems and, among them, the following was (due to the high complexity presented by the respective hypercyclicity version) at least striking:

[**30, Question 9.6**]: *Let $T \in \mathcal{L}(X)$ be a recurrent operator. Is it true that the direct sum operator $T \oplus T$ is recurrent on the direct sum space $X \oplus X$?*

We dedicate a complete section of this Introduction to talk about these $T \oplus T$-type problems (see Section 4) and we solve this particular case in [51] (see Chapter 3, Section 3).

The results stated until now form the known theory of *linear recurrence* from which our study started. From now on we will introduce the different concepts of Linear Dynamics needed and used along the chapters/articles forming this memoir. Our objective is to summarize the work presented by explaining what has been done in each chapter of this document.

2 Hypercyclicity, recurrence and Furstenberg families

The questions considered in this section relate to article [21] (see Chapter 1), even though the main definitions and results that we are about to present have been also used in [50] and [51] (see Chapters 2 and 3). We will restrict ourselves to the linear case of the theory, although in some of the chapters/articles we have employed the same notation for non-linear dynamics.

In the last two decades hypercyclicity has been studied from the *frequency of visits* point of view: instead of just studying the density of an orbit, one investigates "how often" the orbit of a vector returns to every open subset of the space. In order to properly understand and introduce this concept we will use the following notation: given an operator $T \in \mathcal{L}(X)$, a vector $x \in X$ and any non-empty subset U of X, the *return set* from x to U will be the set

$$N_T(x, U) := \{n \geq 0 : T^n x \in U\},$$

which will be denoted by $N(x, U)$ if no confusion with the map studied arises. It is then trivial to check that a vector $x \in X$ is hypercyclic for T if and only if for every non-empty open subset U of X the return set $N(x, U)$ is an infinite subset of the natural numbers $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Looking at the size of these return sets we can generalize the concept of hypercyclicity in terms of "how often" a hypercyclic vector visits every open subset of the space (see Figure 6). Note that, since the concept of "usual hypercyclicity" demands the return sets to be infinite, we will need to measure and classify the size of infinite subsets of natural numbers. This will be done by using the *Combinatorial Number Theory*: we have included an extensive Appendix to cover different concepts of size that we use in the following lines and we recommend to look at Section 2, of the mentioned Appendix, where we cover in detail the definitions and basic properties of the different densities that are about to appear.

The first generalization of this type was the appearance of *frequent hypercyclicity* introduced in 2006 by Bayart and Grivaux [6]: we say that a vector $x \in X$ is *frequently hypercyclic* for $T \in \mathcal{L}(X)$ if for every non-empty open subset U of X the return set $N(x, U)$ has positive *lower density*, i.e.

$$\underline{\text{dens}}(N(x, U)) := \liminf_{N \to \infty} \frac{\#(N(x, U) \cap [0, N])}{N + 1} > 0.$$

Moreover, the set of frequently hypercyclic vectors for T will be denoted by $\text{FHC}(T)$, and the operator T is called *frequently hypercyclic* whenever it admits a frequently hypercyclic vector.

As set out in [6, Proposition 3.12], [10, Proposition 6.23] and [55, Section 9.1], this concept appears naturally from Ergodic Theory (see also Section 3 of this Introduction).

In 2009 Shkarin [84] introduced the concept of $\mathcal{U}$*-frequent hypercyclicity*: a vector $x \in X$ is called $\mathcal{U}$*-frequently hypercyclic* for $T \in \mathcal{L}(X)$ if for every non-empty open subset U of X the return set $N(x, U)$ has positive *upper density*, i.e.

$$\overline{\mathrm{dens}}(N(x,U)) := \limsup_{N\to\infty} \frac{\#(N(x,U) \cap [0,N])}{N+1} > 0.$$

We denote by $\mathrm{UFHC}(T)$ the set of $\mathcal{U}$-frequently hypercyclic vectors for T, and again the operator is called $\mathcal{U}$*-frequently hypercyclic* whenever it admits a $\mathcal{U}$-frequently hypercyclic vector.

A final example before going through the general case is that of *reiterative hypercyclicity* coined in 2016 by Bès et al. [16], which uses the Banach density (see Section 2 of the Appendix): a vector $x \in X$ is called *reiteratively hypercyclic* for $T \in \mathcal{L}(X)$ if for every non-empty open subset U of X the return set $N(x, U)$ has positive *upper Banach density*, i.e.

$$\overline{\mathrm{Bd}}(N(x,U)) := \limsup_{N\to\infty} \left(\sup_{m\geq 0} \frac{\#(N(x,U) \cap [m,m+N])}{N+1} \right) > 0.$$

We denote by $\mathrm{RHC}(T)$ the set of reiteratively hypercyclic vectors for the operator T, and again T is called *reiteratively hypercyclic* whenever $\mathrm{RHC}(T) \neq \varnothing$. The relation between the densities above imply that, for every operator $T \in \mathcal{L}(X)$, the following inclusions hold

$$\mathrm{FHC}(T) \subset \mathrm{UFHC}(T) \subset \mathrm{RHC}(T) \subset \mathrm{HC}(T).$$

In general we will work with the concept of $\mathcal{F}$-hypercyclicity for a family $\mathcal{F}$, which will represent the *frequency*. Even though we have used a slightly different definition of *Furstenberg family* in each chapter (see Chapter 1, Section 8; Chapter 2, Section 1.2; Chapter 3, Section 4) we justify the consistency of this (alternative) choices in Section 3 of the Appendix. Here we will use the most complete definition in order to avoid unnecessary difficulties: a collection of sets $\mathcal{F} \subset \mathscr{P}(\mathbb{N}_0)$ is said to be a *Furstenberg family* (a *family* for short) provided that: each set $A \in \mathcal{F}$ is infinite; $\mathcal{F}$ is hereditarily upward (i.e. $B \in \mathcal{F}$ whenever $A \in \mathcal{F}$ and $A \subset B$); and also that $A \cap [n, \infty[\in \mathcal{F}$ for all $A \in \mathcal{F}$ and $n \in \mathbb{N}$. Following [16, 20, 23, 24, 25] we now define:

Definition 2.1 (**$\mathcal{F}$-hypercyclicity**). Consider $T \in \mathcal{L}(X)$ and let $\mathcal{F}$ be a Furstenberg family. A vector $x \in X$ is said to be $\mathcal{F}$*-hypercyclic* for T if $N_T(x, U) \in \mathcal{F}$ for every non-empty open subset U of X. We will denote by $\mathcal{F}\mathrm{HC}(T)$ the set of $\mathcal{F}$-hypercyclic vectors for T, and we will say that the operator T is $\mathcal{F}$*-hypercyclic* whenever the set $\mathcal{F}\mathrm{HC}(T)$ is non-empty.

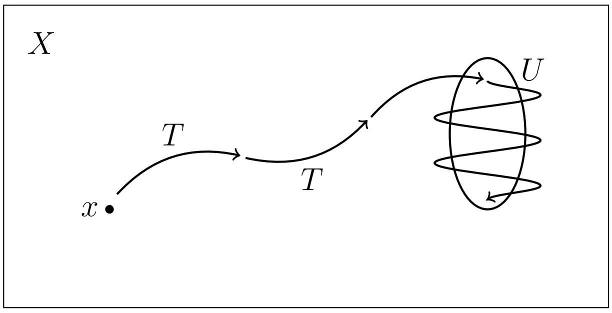

Figure 6: The orbit of $x \in X$ visits frequently every non-empty open set, i.e. $N_T(x, U) \in \mathcal{F}$.

After having introduced in previous sections the historical evolution of Linear Dynamics, starting from the appearance of *hypercyclicity* and arriving to (pointwise) *recurrence*, it is now completely natural to define:

Definition 2.2 (**$\mathcal{F}$-recurrence**). Consider $T \in \mathcal{L}(X)$ and let $\mathcal{F}$ be a Furstenberg family. A vector $x \in X$ is called *$\mathcal{F}$-recurrent* for T if $N_T(x, U) \in \mathcal{F}$ for every neighbourhood U of x. We will denote by $\mathcal{F}\mathrm{Rec}(T)$ the set of $\mathcal{F}$-recurrent vectors for T, and we will say that the operator T is *$\mathcal{F}$-recurrent* whenever the set $\mathcal{F}\mathrm{Rec}(T)$ is dense in X.

Choosing now the right Furstenberg family we can recover the previously introduced and very well-known hypercyclicity notions. In particular:

- we recover the concepts of "usual" recurrence/hypercyclicity when we consider the notions of $\mathcal{F}$-recurrence/$\mathcal{F}$-hypercyclicity for the family $\mathcal{F} = \mathcal{I}$ formed by the infinite subsets of $\mathbb{N}_0$;

- for the family of *sets with positive upper Banach density* $\overline{\mathcal{BD}} := \{A \subset \mathbb{N}_0 : \overline{\mathrm{Bd}}(A) > 0\}$, the notion of $\overline{\mathcal{BD}}$-hypercyclicity equals to reiterative hypercyclicity, the $\overline{\mathcal{BD}}$-recurrence will be called *reiterative recurrence* and $\mathrm{RRec}(T)$ will be the set of reiteratively recurrent vectors;

- for the family of *sets with positive upper density* $\overline{\mathcal{D}} := \{A \subset \mathbb{N}_0 : \overline{\mathrm{dens}}(A) > 0\}$, the notion of $\overline{\mathcal{D}}$-hypercyclicity coincides with that of $\mathcal{U}$-frequent hypercyclicity, the $\overline{\mathcal{D}}$-recurrence is called *$\mathcal{U}$-frequent recurrence* and $\mathrm{UFRec}(T)$ will be the set of $\mathcal{U}$-frequently recurrent vectors;

- and for the family of *sets with positive lower density* $\underline{\mathcal{D}} := \{A \subset \mathbb{N}_0 : \underline{\mathrm{dens}}(A) > 0\}$, the notion of $\underline{\mathcal{D}}$-hypercyclicity coincides with that of frequent hypercyclicity, the notion of $\underline{\mathcal{D}}$-recurrence is called *frequent recurrence* and $\mathrm{FRec}(T)$ will denote the set of frequently recurrent vectors.

The families $\mathcal{F}$ for which there exist $\mathcal{F}$-hypercyclic operators are by far less common than those for which $\mathcal{F}$-recurrence exists: having an orbit distributed around the whole space is much more complicated than having it just around the initial point of the orbit. Some families associated just to recurrence are the *family of syndetic sets* $\mathcal{S}$ (a set is syndetic if the differences between its consecutive elements are bounded, see Section 1 of the Appendix); or the families $\mathcal{IP}^*$ and Δ^* which are even filters. We will not elaborate further on the $\mathcal{IP}^*$ and Δ^**-recurrence* notions here (see Chapter 1, Section 6; Chapter 2, Section 1.3; and Section 4 of the Appendix for more on the role of the families $\mathcal{IP}$, Δ, $\mathcal{IP}^*$ and Δ^* in dynamics).

However, we would like to mention here that the notion of *$\mathcal{S}$-recurrence* is specially important in both articles [21] and [50] (see Chapters 1 and 2). In the literature this notion has been called *uniform recurrence*. We will denote by $\mathrm{URec}(T)$ the set of uniformly recurrent vectors, and an operator $T \in \mathcal{L}(X)$ is called *uniformly recurrent* if the set $\mathrm{URec}(T)$ is dense in X.

The following inclusions are then checked along the chapters/articles and Appendix:

$$\mathrm{Per}(T) \subset \mathrm{span}(\mathcal{E}(T)) \subset \mathrm{URec}(T) \subset \mathrm{FRec}(T) \subset \mathrm{UFRec}(T) \subset \mathrm{RRec}(T) \subset \mathrm{Rec}(T).$$

In [21] (see Chapter 1) we have deeply studied the $\mathcal{F}$-recurrence notions mentioned above. Note that almost every result and question from Costakis et al. [30], obtained for and related to "usual" recurrence, makes sense in this broader context of $\mathcal{F}$-recurrence and, indeed, for each different Furstenberg family $\mathcal{F}$, we have addressed the following problems:

(a) Which is the size of the set of $\mathcal{F}$-recurrent vectors?

(b) How are the notions of $\mathcal{F}$-hypercyclicity and $\mathcal{F}$-recurrence related?

(c) Are there natural assumptions under which $\mathcal{F}$-recurrence implies $\mathcal{F}$-hypercyclicity?

(d) Are the Ansari- and León-Müller-type theorems true for $\mathcal{F}$-recurrence?

(e) Which is the structure of the spectrum of an $\mathcal{F}$-recurrent operator?

(f) Which kind of $\mathcal{F}$-recurrence present the periodic and unimodular eigenvectors?

(g) How does power-boundedness interact with $\mathcal{F}$-recurrence?

(h) Which kind of $\mathcal{F}$-recurrence present the operators acting on finite-dimensional spaces?

(i) Are there known natural classes of $\mathcal{F}$-recurrent operators?

(j) Can we distinguish the different notions of $\mathcal{F}$-recurrence introduced above?

The interested reader is now invited to look through Chapter 1 (see [21]).

The answers are located in Chapter 1 as follows: questions (a), (b) and (c) are treated in Section 2; questions (d) and (e) are treated in Section 4; question (f) is treated in Section 7 (see also Chapter 2, Sections 1.3 and 4; and Chapter 3, Sections 4.2 and 5.1); question (g) is treated in Section 3; and questions (h), (i) and (j) are treated in Sections 5 and 7.

One of the most powerful tools and main ideas in the framework of Chapter 1 is that, under some natural assumptions on the family $\mathcal{F}$, the *$\mathcal{F}$-hypercyclicity* behaviour of a vector can be decomposed in two necessary ingredients:

(1) *usual hypercyclicity*, we have to require a dense orbit;

(2) *$\mathcal{F}$-recurrence*, the vector is just assumed to return with frequency $\mathcal{F}$ to its neighbourhoods.

This allows us to prove results such as the *Ansari-* and *León-Müller-$\mathcal{F}$-hypercyclicity theorems*, by proving them for "usual hypercyclicity", which are already well-known, and then proving them for the respective $\mathcal{F}$-recurrence notion (see Chapter 1, Sections 4 and 8).

We have also studied $\mathcal{F}$-recurrence for other Furstenberg families $\mathcal{F}$ than those included in this section (see Chapter 1, Sections 6 and 8; and Chapter 2, Section 1.3). In particular, the list included in Chapter 3, Section 4, Example 4.2, together with the notions that have been treated in other $\mathcal{F}$-hypercyclicity works such as [20, 36], provides a considerable source of interesting examples.

See Section 2.1 of the *General discussion of the results* for more on Chapter 1.

3 Ergodicity and measure preserving systems

The problems considered in this section relate to article [50] (see Chapter 2), although some of the questions that we treat here were originally stated in article [21] (see Chapter 1).

Until now we have focused on Topological Dynamics: hypercyclicity and recurrence have been described in terms of open sets and neighbourhoods. However, as we advanced at the beginning of this Introduction, we can study dynamical systems from the Ergodic Theory point of view, and our objective in Chapter 2 was to exhibit a symmetry between both topological and measurable theories. The Ergodic Theory set up needs a *measure space* $(X, \mathscr{A}, \mu)$ where:

– the collection of sets $\mathscr{A} \subset \mathscr{P}(X)$ is a *σ-algebra* on the non-empty set X;

– and $\mu : \mathscr{A} \longrightarrow [0, \infty[$ is a (usually *probability* or at least *positive finite*) *measure*.

If now we consider a map $T : X \longrightarrow X$, which is assumed to be *$\mathscr{A}$-measurable*, then the tuple $(X, \mathscr{A}, \mu, T)$ is called a *measure dynamical system* (see [29], [64] and [86] for more details).

The main property in this context is that of *ergodicity* but, in order to introduce it, we need to start by the concept of *invariance*: a measure dynamical system $(X, \mathscr{A}, \mu, T)$ is called a *measure preserving system* if for every $A \in \mathscr{A}$ the following holds

$$\mu(A) = \mu(T^{-1}(A)).$$

Both the transformation T and the measure μ interact in this definition, and sometimes it is said that T is a *μ-invariant transformation* or that μ is a *T-invariant measure*. We will simply say indistinctly that the transformation or the measure is *invariant*. This property can be compared with *recurrence*: in fact, as stated and proved in [41, Introduction, Section 4], the following result is one of the first theorems studying a recurrence-kind property, which was discussed by Poincaré in 1890 and proved by Carathéodory using Measure Theory in 1919:

Poincaré Recurrence Theorem: *Let $(X, \mathscr{A}, \mu, T)$ be a measure preserving system and let $A \in \mathscr{A}$ be a set with positive measure $\mu(A) > 0$. Then we have that*

$$\mathrm{Orb}(Tx, T) \cap A \neq \varnothing \quad \textit{for } \mu\textit{-a.e. point } x \in A.$$

A measure preserving system $(X, \mathscr{A}, \mu, T)$ is called *ergodic* if for each $A \in \mathscr{A}$ fulfilling that $A = T^{-1}(A)$ then $\mu(A) \in \{0, 1\}$. This property can be compared with *topological transitivity* since it is equivalent to the fact that for each pair of sets $A, B \in \mathscr{A}$ with $\mu(A), \mu(B) > 0$ there exists a natural number $n \geq 0$ such that $\mu\left(A \cap T^{-n}(B)\right) > 0$; see [86, Theorem 1.5].

Let us also recall two well-known, important and crucial results regarding ergodicity. The first one is the so-called *Birkhoff pointwise ergodic theorem* (see [44, Theorem 3.41]):

Birkhoff Pointwise Ergodic Theorem: *Let $(X, \mathscr{A}, \mu, T)$ be an ergodic system and consider any $f \in L^1(X, \mathscr{A}, \mu)$. Then we have that*

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} f(T^k x) = \int_X f \, d\mu \quad \textit{for } \mu\textit{-a.e. point } x \in X.$$

The second important ergodic result, for this memoir, is the *ergodic decomposition theorem*, which links invariance and ergodicity in a very strong sense (see [44, Theorem 3.42]):

Ergodic Decomposition Theorem: *For each measure preserving system* $(X, \mathscr{A}, \mu, T)$ *there exists an abstract probability measure space* $(\mathcal{M}, \tau)$ *formed by measures on* $(X, \mathscr{A})$ *for which*

$$\mu(A) = \int_{\mathcal{M}} \nu(A) \ d\tau(\nu) \quad \text{for each } A \in \mathscr{A},$$

and such that for τ*-a.e. measure* $\nu \in \mathcal{M}$ *the system* $(X, \mathscr{A}, \nu, T)$ *is ergodic.*

There exist many other ergodic results, such as the *mean ergodic theorem* (proved in 1932 by von Neumann [77] and applicable to contractions on Hilbert spaces), but we will concentrate on those stated here to connect the introduced measurable properties with Linear Dynamics. Given a continuous linear operator $T \in \mathcal{L}(X)$ acting on an separable F-space X we will consider $\mathscr{B}(X)$, the *σ-algebra of Borel sets* on X, and a *Borel probability measure* μ on $(X, \mathscr{B}(X))$, which will sometimes have *full support* (i.e. $\mu(U) > 0$ for every non-empty open subset U of X). The continuity of T implies its $\mathscr{B}(X)$-measurability, and we will usually omit the word "Borel".

For every operator $T \in \mathcal{L}(X)$ there always exists at least one invariant measure: the atomic *Dirac mass* δ_0 at $0_X \in X$ (i.e. $\delta_0(A) = 1$ if $0_X \in A$ and 0 otherwise, for each $A \subset X$) is always invariant by linearity. However, this is not a really interesting case and we will say that an invariant probability measure μ is *non-trivial* if it differs from δ_0. There also exist natural examples of ergodic linear systems: the study of Ergodic Theory in Linear Dynamics started with the pioneering work of Flytzanis (see [39, 40]), and was then further developed in the papers [6], [7], [52] and [11], among others (see also the textbooks [28, 34, 10]).

The first relation between Linear Dynamics and Ergodic Theory that we want to remark on is the one introduced in 2006 by Bayart and Grivaux in [6, Proposition 3.12] for operators acting on separable F-spaces (see also [10, Proposition 6.23] and [55, Section 9.1]):

– *If* $T \in \mathcal{L}(X)$ *is* μ*-ergodic and* μ *has full support, then* T *is frequently hypercyclic.*

This is a consequence of the *Birkhoff pointwise ergodic theorem*: if $(X, \mathscr{B}(X), \mu, T)$ is ergodic and μ has full support, applying the ergodic theorem to each indicator function $\mathbb{1}_{U_n}$ for a countable basis of the topology $(U_n)_{n\in\mathbb{N}}$ we obtain that μ-a.e. vector $x \in X$ fulfills the equality

$$\operatorname{dens}(N(x, U_n)) = \lim_{N\to\infty} \frac{\#(N(x, U_n) \cap [0, N])}{N+1} = \lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \mathbb{1}_{U_n}(T^k x) = \int_X \mathbb{1}_{U_n} d\mu = \mu(U_n),$$

i.e., for each $n \in \mathbb{N}$ we have $\underline{\operatorname{dens}}(N(x, U_n)) = \overline{\operatorname{dens}}(N(x, U_n)) = \operatorname{dens}(N(x, U_n)) = \mu(U_n) > 0$ for every x in a set $A_n \subset X$ with $\mu(A_n) = 1$. It follows that each $x \in \bigcap_{n\in\mathbb{N}} A_n$ is a frequently hypercyclic vector for T, and this shows that *ergodicity + full support* $\Rightarrow$ *frequent hypercyclicity.* The second relation we want to recall here is the following well-known fact:

– *If* $T \in \mathcal{L}(X)$ *is* μ*-invariant and* μ *has full support, then* T *is recurrent.*

The *Poincaré recurrence theorem* implies that T is topologically recurrent, and hence recurrent by the *Costakis-Manoussos-Parissis theorem* (in [41, Theorem 3.3] it is even shown that μ-a.e. vector is recurrent for T). We have checked that *invariance + full support* $\Rightarrow$ *recurrence.*

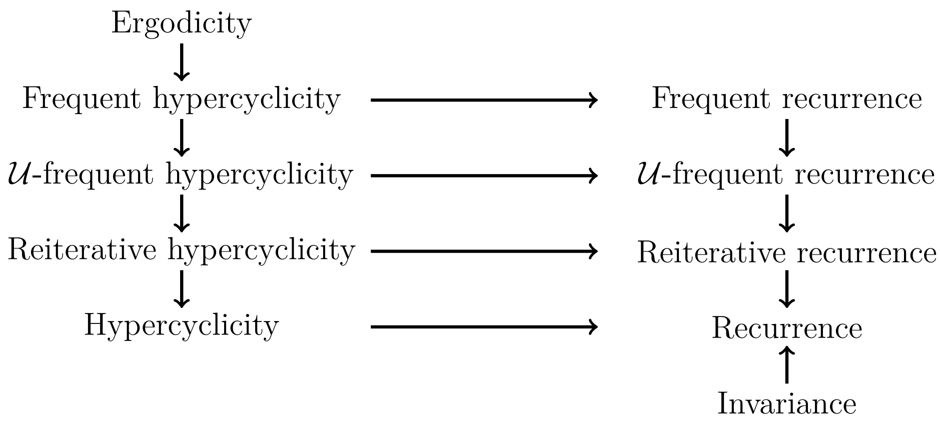

Figure 7: Previously known relations between Topological and Measurable Dynamics.

It is then natural to ask if the *invariance* property implies some stronger recurrence notion. The answer is yes, and we have proved it in [50] (see Chapter 2, Section 3, Lemma 3.1). Indeed, if given a probability measure μ on X we define its *support*, $\operatorname{supp}(\mu)$, as the complementary of the union of all null open sets (i.e. $\operatorname{supp}(\mu) = X \setminus \bigcup_{U \text{ open}, \mu(U)=0} U$, which coincides with the smallest closed set of full measure) then we have the following:

Chapter 2, Lemma 3.1: *Let $T \in \mathcal{L}(X)$ be an operator acting on a separable F-space X and suppose that μ is an invariant probability measure on $(X, \mathscr{B}(X))$. Then*

$$\mu(\operatorname{FRec}(T)) = 1 \quad \textit{and hence} \quad \operatorname{supp}(\mu) \subset \overline{\operatorname{FRec}(T)}.$$

The proof uses in a crucial way the *ergodic decomposition theorem* previously stated, and then the *Birkhoff pointwise ergodic theorem* as in the implication *ergodicity* $\Rightarrow$ *frequent recurrence*. The implications between Topological and Measurable Dynamics are now complete in Figure 8.

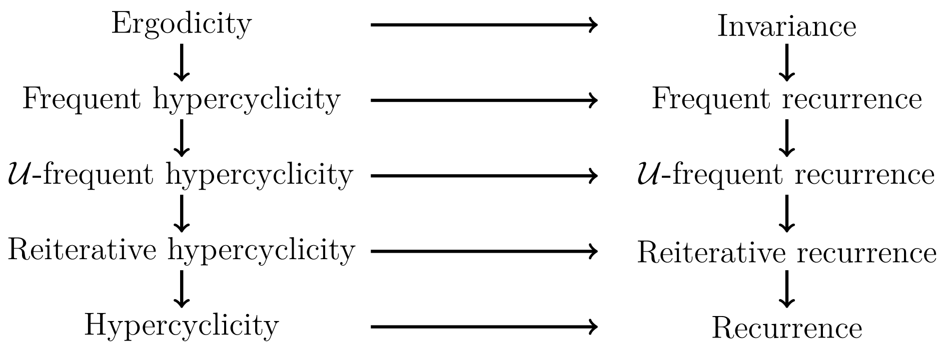

Figure 8: Relations between topological and measurable dynamical properties.

Once we know that invariance implies frequent recurrence one may wonder: *Under which conditions can we guarantee the existence of a (non-trivial) invariant measure?* Our main line of thought in article [50] (see Chapter 2) was to connect various notions of $\mathcal{F}$-recurrence from those defined and introduced in [21] via invariant measures, and proceeding in two steps:

(1) If $T \in \mathcal{L}(X)$ admits vectors with the "weak" *reiterative recurrence* property (much weaker, at least formally, than *frequent recurrence*), prove that then T admits a non-trivial invariant measure, perhaps with full support (see Chapter 2, Section 2, Theorem 2.3).

(2) If $T \in \mathcal{L}(X)$ admits a non-trivial invariant measure (perhaps with full support), prove that then T admits vectors with some strong recurrence property such as *frequent recurrence* (see Chapter 2, Section 3, Lemma 3.1 for the proof of *invariance* $\Rightarrow$ *frequent recurrence*), but also *unimodular eigenvectors* (which enjoy a much stronger recurrence property than that of *frequent recurrence*) when T acts on a Hilbert space and the invariant measure fulfills some extra good properties (see Chapter 2, Section 4, Lemma 4.4).

This approach in Linear Dynamics comes from the paper [52], which extends to the linear setting some results in the context of *compact dynamical systems* (see [41, Chapter 3 and Lemma 3.17]).

The interested reader is now invited to look through Chapter 2 (see [50]).

It is completely necessary to remark that the theory developed in Chapter 2 has some restrictions since it is not valid for all operators acting on arbitrary F-spaces. This is something natural because the results sought in this work have a really general character: we always start with a continuous linear operator $T \in \mathcal{L}(X)$ acting on an F-space, and we do not have any other parameters or initial conditions more than the assumptions we make on the underlying space X or on the operator T. Let us comment on the restrictions used in Chapter 2:

The existence of an invariant measure with full support implies frequent recurrence for every operator (and even for non-linear systems acting on second-countable spaces), but constructing invariant measures from reiteratively recurrent vectors is not that easy. The result that allows us to find invariant measures in Chapter 2 adapted to the linear setting is the following:

Chapter 2, Theorem 2.3: *Let $T \in \mathcal{L}(X)$ be an adjoint operator acting on a separable dual Banach space X and let w^* be the respective weak-star topology on X. Given $x_0 \in \mathrm{RRec}(T)\setminus\{0\}$ one can find a (non-trivial) invariant probability measure μ_{x_0} on $(X, \mathscr{B}(X))$ such that*

$$x_0 \in \mathrm{supp}(\mu_{x_0}) \subset \overline{\mathrm{Orb}(x_0,T)}^{w^*}.$$

Moreover, if $\mathrm{RRec}(T)$ is dense then T admits an invariant probability measure with full support.

This result together with the implication *invariance* $\Rightarrow$ *frequent recurrence* proves some really surprising equivalences (see Chapter 2, Section 1.2, Theorem 1.3): *for an adjoint operator $T \in \mathcal{L}(X)$ acting on a dual Banach space X we have that $\overline{\mathrm{FRec}(T)} = \overline{\mathrm{UFRec}(T)} = \overline{\mathrm{RRec}(T)}$, and hence the following statements are equivalent:*

(i) *T is frequently recurrent;*

(ii) *T is $\mathcal{U}$-frequently recurrent;*

(iii) *T is reiteratively recurrent.*

Note that the result is valid on reflexive spaces, so it applies for every operator acting on an ℓ^p-space, with $1 < p < \infty$. These equivalences do not hold outside the adjoint/dual/reflexive setting in general, as we proved in article [21] (see Chapter 1, Section 5, Corollary 5.8).

To obtain *unimodular eigenvectors* we have to impose more restrictions: the operator has to act on a *complex Hilbert space* and we need the reiteratively recurrent vectors, from which we construct the measures, to have bounded orbit (see Chapter 2, Section 1.3, Theorem 1.7). The invariant measures are obtained by the "adjoint operator's result" stated above since Hilbert spaces are reflexive, while getting unimodular eigenvectors from the measures constructed presents some difficulties as we discuss in article [50] (see Chapter 2, Section 4, Lemma 4.4):

Chapter 2, Lemma 4.4: *Let $T \in \mathcal{L}(H)$ be an operator acting on a complex separable Hilbert space H and suppose that μ is a (non-trivial) invariant probability measure on $(H, \mathscr{B}(H))$ such that $\int_H \|z\|^2 d\mu(z) < \infty$. Then we have the inclusions*

$$\operatorname{supp}(\mu) \subset \overline{\operatorname{span}(\operatorname{supp}(\mu))} \subset \overline{\operatorname{span}(\mathcal{E}(T))}.$$

This result is inspired by the work of Flytzanis [40]. The proof is based on interchanging the original measure μ by a proper Gaussian measure: a Borel probability measure m on a complex Banach space X is called a (*centered*) *Gaussian measure* if each continuous linear functional $x^* \in X^*$ has a complex (centered) Gaussian distribution when considered as a random variable on $(X, \mathscr{B}(X), m)$. See Figure 9. We refer the reader to [28] and [34] for more about Gaussian measures on Banach spaces, and to [10] and [11] for more on their role in Linear Dynamics.

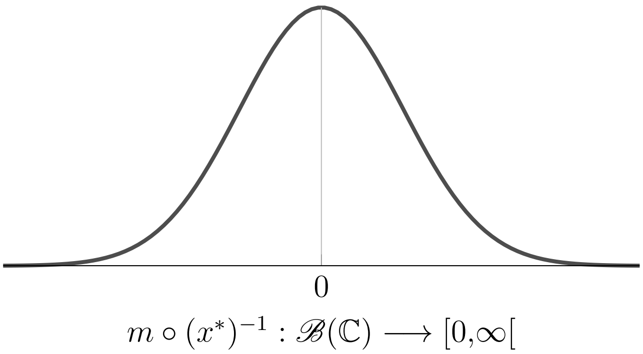

Figure 9: Distribution of a functional $x^* \in X^*$ with respect to a centered Gaussian measure m.

With the approach via invariant measures given in Chapter 2 we have been able to get an easy positive answer, in our adjoint/dual/reflexive setting, for the following two questions regarding *inverse* and *product systems*, which remain open in the general case:

[21, Question 2.14]: *Let $T \in \mathcal{L}(X)$ be an invertible $\mathcal{F}$-recurrent operator. Is it true that the inverse operator T^{-1} is again $\mathcal{F}$-recurrent?*

For a positive answer in our adjoint/dual/reflexive setting see Section 6 of Chapter 2. The second question has been already stated for "usual recurrence" in Section 1 of this Introduction, and we dedicate the next pages to a discussion about these $T \oplus T$-type problem:

[50, Question 8.7]: *Let $T \in \mathcal{L}(X)$ be an $\mathcal{F}$-recurrent operator. Is it true that the direct sum operator $T \oplus T$ is $\mathcal{F}$-recurrent on the direct sum space $X \oplus X$?*

See Section 2.2 of the *General discussion of the results* for more on Chapter 2.

4 The $T \oplus T$-type problems

The questions considered in this section relate to article [51] (see Chapter 3), even though we also recall some of the results obtained in article [50] (see Chapter 2). As in previous sections we will restrict ourselves to the linear case of the theory although in Chapter 3 we have used the same definitions and notation in the non-linear setting.

Given an operator $T \in \mathcal{L}(X)$ with some dynamical *"property"* (hypercyclicity, chaos, etc.), it is natural to ask whether the direct sum operator $T \oplus T : X \oplus X \longrightarrow X \oplus X$, acting as $T \oplus T(x_1, x_2) := (Tx_1, Tx_2)$ on the direct sum space $X \oplus X$, presents that *"property"*. This question will be called the *$T \oplus T$-"property" problem*. Studied cases in Linear Dynamics are:

- **Hypercyclicity**. The *$T \oplus T$-hypercyclicity problem* was a really long-standing problem, posed in 1992 by D. Herrero [56] and finally answered negatively in 2006 by De La Rosa and Read [32] (see the rest of this section or Section 1 of Chapter 3 for more on this problem).

- **Reiterative and $\mathcal{U}$-frequent hypercyclicity**. Both cases have been solved in the positive by Ernst, Esser and Menet in a recent 2021 paper (see [36, Theorem 2.5]).

- **Frequent hypercyclicity**. As far as we know the *$T \oplus T$-frequent-hypercyclicity problem* remains open although the notion of frequently hypercyclic operator appeared in 2006, before those of reiterative and $\mathcal{U}$-frequent hypercyclicity (see [43, Question 5]).

- **Devaney Chaos**. The *$T \oplus T$-chaos problem* admits an even stronger solution since for every (possibly different) couple of chaotic operators $T_1 \in \mathcal{L}(X_1)$ and $T_2 \in \mathcal{L}(X_2)$ acting on the couple of F-spaces X_1, X_2, the direct sum operator $T_1 \oplus T_2$ is again chaotic on the direct sum space $X_1 \oplus X_2$ (see for instance [55, Exercise 2.5.7]).

Once we have a (positive or negative) answer for a fixed *$T \oplus T$-type problem*, it is then natural to go a step further and look at the *N-fold direct sum* system

$$\underbrace{T \oplus \cdots \oplus T}_{N} : \underbrace{X \oplus \cdots \oplus X}_{N} \longrightarrow \underbrace{X \oplus \cdots \oplus X}_{N},$$

for each $N \in \mathbb{N}$, which to shorten will be denoted by

$$T_{(N)} := \underbrace{T \oplus \cdots \oplus T}_{N} \quad \text{and} \quad X^N = \underbrace{X \oplus \cdots \oplus X}_{N},$$

and where

$$T_{(N)}(x_1, x_2, ..., x_N) := (Tx_1, Tx_2, ..., Tx_N) \quad \text{for each } N\text{-tuple } (x_1, x_2, ..., x_N) \in X^N.$$

If the answer to the considered *$T \oplus T$-type problem* is:

(a) **Positive**: then one asks if the studied property is also fulfilled by $T_{(N)}$ for every $N \in \mathbb{N}$, provided that T presents the property;

(b) **Negative**: then one asks if the studied property is also fulfilled by $T_{(N)}$ for every $N \in \mathbb{N}$, provided that the 2-fold direct sum operator $T \oplus T$ presents the property.

With respect to the already mentioned cases:

- **Hypercyclicity**. Although the *$T \oplus T$-hypercyclicity problem* has a negative answer, once $T \oplus T$ is hypercyclic then so is $T_{(N)}$, for every $N \in \mathbb{N}$, by a well-known Furstenberg theorem (see for instance [55, Theorem 1.51]).
- **Reiterative and $\mathcal{U}$-frequent hypercyclicity**. This has been also positively solved by Ernst, Esser and Menet in 2021 (see [36, Corollaries 2.7 and 2.8]). We have given a shorter proof for the reiteratively hypercyclic case in [51] (see Chapter 3, Section 5.3, Corollary 5.17).
- **Frequent hypercyclicity**. Even if the direct sum operator $T \oplus T$ is frequently hypercyclic, it is still open if $T_{(N)}$ is frequently hypercyclic or not for any $N \geq 3$.
- **Devaney Chaos**. This case also admits a solution similar to that of the *$T \oplus T$-chaos problem*: for every N (possibly different) chaotic operators $T_j \in \mathcal{L}(X_j)$, $j = 1, ..., N$, acting on the F-spaces X_j, $j = 1, 2, ..., N$, then the direct sum operator $T_1 \oplus T_2 \oplus \cdots \oplus T_N$ is chaotic on the direct sum space $X_1 \oplus X_2 \oplus \cdots \oplus X_N$ (see again [55, Exercise 2.5.7]).

The *$T \oplus T$-type problems* are considered to be complicated at least for two reasons: to obtain a negative answer one has to construct a (usually non-trivial) counterexample; but also famous notions (such as frequent hypercyclicity) still have their respective $T \oplus T$-problem open. It is particularly important for us to remark the difficulty found on the *$T \oplus T$-hypercyclicity problem* that, as we have already mentioned, was posed in 1992 by D. Herrero [56]:

The $T \oplus T$-hypercyclicity problem: *Let $T \in \mathcal{L}(X)$ be a hypercyclic operator acting on an F-space X. Is the direct sum operator $T \oplus T$, acting on $X \oplus X$, hypercyclic?*

An operator $T \in \mathcal{L}(X)$ is called (topologically) *weakly-mixing* (from now on *weakly-mixing*) whenever $T \oplus T$ is topologically transitive, so the question above asks whether there exists any hypercyclic but not weakly-mixing operator. In 1999 Bès and Peris [18] showed that:

- *A continuous linear operator $T \in \mathcal{L}(X)$ is weakly-mixing if and only if it satisfies the so-called Hypercyclicity Criterion.*

In other words, $T \oplus T \in \mathcal{L}(X \oplus X)$ is a hypercyclic operator if and only if T satisfies the hypothesis of the following really well-known and practical result (see [55, Chapter 3]):

Hypercyclicity Criterion: *Let $T \in \mathcal{L}(X)$ be an operator acting on an F-space X. If there exist two dense subsets $X_0, Y_0 \subset X$, an increasing sequence of positive integers $(n_k)_{k\in\mathbb{N}}$, and a family of (not necessarily continuous) mappings $S_{n_k} : Y_0 \longrightarrow X$ such that*

(i) *$T^{n_k} x \to 0$ for each $x \in X_0$;*

(ii) *$S_{n_k} y \to 0$ for each $y \in Y_0$;*

(iii) *$T^{n_k} S_{n_k} y \to y$ for each $y \in Y_0$;*

then T is weakly mixing, and in particular hypercyclic.

The $T\oplus T$-*hypercyclicity problem* was finally answered negatively in 2006 by De La Rosa and Read (see [32]). The techniques used there were later refined by Bayart and Matheron in order to construct hypercyclic but not weakly-mixing operators on each Banach space admitting a normalized unconditional basis whose associated forward shift is continuous, as for instance on the classical $c_0(\mathbb{N})$ and $\ell^p(\mathbb{N})$ spaces with $1 \leq p < \infty$ (see [8, 9]).

Our main objective in article [51] (see Chapter 3) has been to study the previous problem, and to develop the corresponding theory, for the notion of *recurrence*. As we mentioned at the end of Section 1 of this Introduction, the $T \oplus T$-*recurrence problem* was posed by Costakis, Manoussos and Parissis in their 2014 fundamental recurrence paper (see [30, Question 9.6]):

> **The $T\oplus T$-recurrence problem:** *Let $T \in \mathcal{L}(X)$ be a recurrent operator on an F-space X. Is the direct sum operator $T \oplus T$, acting on $X \oplus X$, recurrent?*

In order to properly study and answer this question, in article [51] (Chapter 3) we have proceed in three steps:

(1) We first characterize the operators $T \in \mathcal{L}(X)$ such that $T_{(N)}$ is recurrent for every $N \in \mathbb{N}$. This characterization is given in terms of the (as far as we know) new notion of *quasi-rigidity*, which will be for recurrence, the analogous property to that of *weak-mixing/satisfying the Hypercyclicity Criterion* for hypercyclicity (see Chapter 3, Section 2, Theorem 2.5): an operator $T \in \mathcal{L}(X)$ is said to be *quasi-rigid* with respect to the sequence $(n_k)_{k\in\mathbb{N}}$ if there exists a dense subset Y of X such that $T^{n_k}x \to x$, as $k \to \infty$, for every $x \in Y$.

(2) We then solve the introduced $T\oplus T$-*recurrence problem* giving a negative answer: we show that in **every** (real or complex) separable infinite-dimensional Banach space X there exists a recurrent operator $T \in \mathcal{L}(X)$ such that $T\oplus T$ is not recurrent in $X\oplus X$. Note that these operators are *recurrent* but not *quasi-rigid*.

(3) We finally look at the N-fold direct sum operator $T_{(N)}$ for every $N \in \mathbb{N}$, which is the next step after obtaining a positive/negative answer to a $T \oplus T$-*type problem* as mentioned in this section, and we also show that $T_{(N)}$ is not necessarily recurrent even if the 2-fold direct sum operator $T\oplus T$ is recurrent. This represents a huge difference between recurrence and hypercyclicity and, moreover, we have proved the following stronger result:

> **Chapter 3, Theorem 3.2:** *Let X be any (real or complex) separable infinite-dimensional Banach space. For each $N \in \mathbb{N}$ there exists an operator $T \in \mathcal{L}(X)$ such that*
>
> $$T_{(N)} : X^N \longrightarrow X^N \textit{ is recurrent, and even } \operatorname{Rec}(T_{(N)}) = X^N,$$
>
> *but for which $T_{(N+1)} : X^{N+1} \longrightarrow X^{N+1}$ (and hence $T_{(J)}$ for all $J > N$) is not recurrent.*

The interested reader is now invited to look through Chapter 3 (see [51]).

We have also studied the $T \oplus T$-*type problem* for the main $\mathcal{F}$-recurrence notions introduced along the work. However, we have just obtained partially positive answers that we would like to briefly summarize here:

Starting by the adjoint/dual/reflexive Banach setting we used in article [50] (see Chapter 2) we obtain positive results in a really strong sense (as it happens for *Devaney chaos*). Indeed, the linear version of Chapter 2, Section 5, Theorem 5.1 reads as follows:

Chapter 2, Theorem 5.1: *Let $N \in \mathbb{N}$ and suppose that for each $1 \leq j \leq N$ there exists an adjoint operator $T_j \in \mathcal{L}(X_j)$ acting on a dual Banach space X_j. Then for the direct sum operator $T := T_1 \oplus T_2 \oplus \cdots \oplus T_N$, acting on the direct sum space $X_1 \oplus X_2 \oplus \cdots \oplus X_N$, we have the equality $\overline{\mathrm{FRec}(T)} = \bigoplus_{j=1}^{N} \overline{\mathrm{RRec}(T_j)}$. In particular, the following statements are equivalent:*

(i) *T is frequently recurrent;*

(ii) *T is $\mathcal{U}$-frequently recurrent;*

(iii) *T is reiteratively recurrent;*

(iv) *T_j is reiteratively recurrent for every $1 \leq j \leq N$.*

This result has some implications on the respective $\mathcal{F}$-hypercyclicity notions for operators acting on reflexive Banach spaces (see Chapter 2, Section 5). Finally, when the $\mathcal{F}$-recurrent operator $T \in \mathcal{L}(X)$ acts on an arbitrary F-space we have just obtained some natural sufficient conditions to ensure that the N-fold direct sum operator $T_{(N)}$ is again $\mathcal{F}$-recurrent for every number $N \in \mathbb{N}$. See Chapter 3, Section 5.2, Proposition 5.10. In particular, the result holds for every $\mathcal{F}$-recurrent operator admitting a cyclic vector.

See Section 2.3 of the *General discussion of the results*, but also the following section of this *Introduction*, for more on Chapter 3.

5 Lineability, dense lineability and spaceability

The questions considered in this section relate to articles [51] and [69] (see Chapters 3 and 4).

When we study a dynamical notion whose definition involves a set of vectors fulfilling some kind of property, one of the things to look at is the structure of such a set. One way to do this is by examinating the size of the sets in the *Baire category* sense, and for instance:

- The *Birkhoff transitivity theorem* states that the set of hypercyclic vectors $\mathrm{HC}(T)$ is always a dense G_δ (and hence a *residual*, also called *co-meager*) subset of X. The same happens for the set of recurrent vectors $\mathrm{Rec}(T)$ by the so-called *Costakis-Manoussos-Parissis theorem*.

- The sets of reiteratively and $\mathcal{U}$-frequently hypercyclic vectors $\mathrm{RHC}(T)$ and $\mathrm{UFHC}(T)$ are always *residual* when they are non-empty (see [12, 16, 20]), while the set $\mathrm{FHC}(T)$ of frequently hypercyclic vectors is always *meager* (also called of *first category*); see [76, 12].

- For the sets $\mathcal{F}\mathrm{Rec}(T)$ we show in [21] (see Chapter 1, Section 2) that: when T is hypercyclic then $\mathrm{RRec}(T)$ is *residual*, $\mathrm{UFRec}(T)$ can be either *residual* or *meager* (depending on if T is $\mathcal{U}$-frequently hypercyclic or not), and $\mathrm{FRec}(T)$ is always *meager*. However, $\mathcal{F}\mathrm{Rec}(T)$ is either *meager* or *co-meager* when T is not hypercyclic (see Chapter 1, Section 2, Example 2.4).

– In article [50] (see Chapter 2, Section 7) we have even looked at the "Baire category size" for the **set of reiteratively recurrent operators** acting on a complex separable Hilbert space with respect to the *Strong* and *Strong* Operator Topologies*.

In our linear setting other "largeness" notions can be used: *Is the set* $\mathrm{Rec}(T)$ *of recurrent vectors for* $T \in \mathcal{L}(X)$ *a vector subspace of* X*?* The dense G_δ condition implies the next fact:

Proposition 5.1 (**Adaptation of [55, Proposition 2.52]**). *Let* $T \in \mathcal{L}(X)$ *be a recurrent operator acting on an F-space* X*. Then*

$$X = \mathrm{Rec}(T) + \mathrm{Rec}(T),$$

that is, every vector $x \in X$ *can be written as the sum of two recurrent vectors.*

Proof. Given $x \in X$ both $\mathrm{Rec}(T)$ and $x - \mathrm{Rec}(T)$ are dense G_δ-sets, so their intersection must be non-empty by the Baire category theorem. The later implies that $x \in \mathrm{Rec}(T) + \mathrm{Rec}(T)$. □

As a consequence, the set $\mathrm{Rec}(T)$ itself can only be a vector subspace if every single vector is recurrent, that is if $X = \mathrm{Rec}(T)$. This is the case for *recurrent* and *power-bounded* operators (which includes the case of *recurrent* operators *acting on finite-dimensional spaces*) as showed in [30, Lemma 3.1], but not just for "usual recurrence" since we have shown that $X = \mathcal{F}\mathrm{Rec}(T)$ as soon as T is power-bounded and $\mathcal{F}$-recurrent (see Chapter 1, Section 3, Theorem 3.1).

Weakening the requirement it is natural to ask if, for a general (not power-bounded and hence acting on an infinite-dimensional space) $\mathcal{F}$-recurrent operator $T \in \mathcal{L}(X)$, the set $\mathcal{F}\mathrm{Rec}(T)$ contains a "large" vector subspace. We will interpret "largeness" in three different ways:

– **Lineability**: Given a subset of vectors $Y \subset X$ we say that Y is *lineable* if there exists an infinite-dimensional vector subspace $Z \subset X$ such that $Z \setminus \{0\} \subset Y$.

– **Dense lineability**: Given a subset of vectors $Y \subset X$ we say that Y is *dense lineable* if there exists a dense infinite-dimensional vector subspace $Z \subset X$ such that $Z \setminus \{0\} \subset Y$.

– **Spaceability**: Given a subset of vectors $Y \subset X$ we say that Y is *spaceable* if there exists an infinite-dimensional closed vector subspace $Z \subset X$ such that $Z \setminus \{0\} \subset Y$.

Two things should be noted:

(1) the structural notions above turn stronger as one goes down. Studying *lineability* is easier than *dense lineability*, which in its turn will be much easier than studying *spaceability*;

(2) in our case Y will be some set of $\mathcal{F}$-recurrent vectors $\mathcal{F}\mathrm{Rec}(T)$, and since we always have that the zero-vector $0_X \in X$ belongs to $\mathcal{F}\mathrm{Rec}(T)$ for every Furstenberg family $\mathcal{F}$, then the possible infinite-dimensional vector subspace $Z \subset X$ will be fully included in our set Y without having to take out the zero-vector.

Let us start by the first two (lineability and dense lineability) properties: by the well-known *Herrero-Bourdon theorem* (see [55, Theorem 2.55]), every hypercyclic operator admits a dense vector subspace in which every non-zero element is hypercyclic, i.e. $\mathrm{HC}(T)$ is *dense lineable* as soon as T is hypercyclic. Our objective in article [51, Section 5] (see Chapter 3, Section 5) has been to study these lineability properties for the sets of recurrent vectors:

- We prove in [51] (see Chapter 3, Section 5) that the set $\mathrm{Rec}(T)$ is always *lineable* when T is recurrent. This easily extends to the set $\mathcal{F}\mathrm{Rec}(T)$ under a very weak assumption on the family $\mathcal{F}$: we have to guarantee that unimodular eigenvectors present a stronger recurrence notion than $\mathcal{F}$-recurrence, i.e. we have to ensure that $\mathrm{span}(\mathcal{E}(T)) \subset \mathcal{F}\mathrm{Rec}(T)$. This condition is indeed satisfied by all the interesting $\mathcal{F}$-recurrence notions considered in this memoir.

- On the other hand, we do not know if $\mathrm{Rec}(T)$ is always *dense lineable*, and the problem is also left open for the sets $\mathcal{F}\mathrm{Rec}(T)$ of $\mathcal{F}$-recurrent vectors when the Furstenberg family $\mathcal{F}$ is not a filter. Nevertheless, we are able to give some natural sufficient conditions that guarantee this dense lineability, one of them being the cyclicity of T (see Chapter 3, Section 5).

The notion of *quasi-rigidity* (mentioned in Section 4 of this Introduction, and analogous for *recurrence* to the *weak-mixing* notion for *hypercyclicity*) can be expressed as a very particular kind of $\mathcal{F}$-recurrence (see Chapter 3, Section 4.1, Proposition 4.5), and this motivates those sufficient conditions obtained for *dense lineability*: indeed, for every *quasi-rigid* operator the set of recurrent vectors $\mathrm{Rec}(T)$ is dense lineable (see Chapter 3, Section 2.2, Proposition 2.7).

The interested reader is now invited to look through Section 5 of Chapter 3.

Quasi-rigidity is even more important for *spaceability*. In article [69] (see Chapter 4) we again justify why this property is, for recurrence, the analogous notion to that of *weak-mixing* or *satisfying the Hypercyclicity Criterion* for hypercyclicity: the known Banach space theory about the spaceability of the set of hypercyclic vectors can be fully rewritten and adapted to the recurrence setting by exchanging the *weak-mixing* assumption by that of *quasi-rigidity*.

Our results hold for "usual recurrence" on Banach spaces. We show deep relations between the well-known *hypercyclic spaceability theory* and the rather new *recurrent spaceability theory*, characterizing those quasi-rigid operators $T \in \mathcal{L}(X)$ that present a so-called *recurrent subspace* (i.e. an infinite-dimensional closed vector subspace of the set $\mathrm{Rec}(T)$), and establishing the curious equivalence, for weakly-mixing operators, between having a hypercyclic and a recurrent subspace (see Chapter 4, Sections 2 and 3, and Corollary 3.5):

Chapter 4, Corollary 3.5: *Let $T \in \mathcal{L}(X)$ be a weakly-mixing operator on a (real or complex) separable Banach space X. Then the following statements are equivalent:*

(i) *T has a hypercyclic subspace;*

(ii) *T has a recurrent subspace.*

The interested reader is now invited to look through Chapter 4 (see [69]).

Spectral Theory is our main tool when dealing with *spaceability*: many operators present recurrent subspaces (such as quasi-rigid compact perturbations of the identity), but there are also plenty of counterexamples (such as the Rolewicz operators). Moreover, using the developed theory we have *easily* shown that every C-type operator, as defined in [73, 53], has a hypercyclic subspace (see Chapter 4, Section 7, Example 7.5).

See Section 2.4 of the *General discussion of the results* for more on Chapter 4.

We close here this introductory chapter. For some extra remarks, further results and open problems we refer to the ***General discussion of the results*** and ***Conclusions*** chapters.

Chapter 1

Frequently recurrent operators

This chapter is an adaptation of the revised "author version" of the article:

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Adaptation: The general theoretic results included in the original version of the article were stated for *continuous linear operators acting on Fréchet spaces*. Here we have stated them in the broader context of *continuous linear operators acting on F-spaces* with the aim of pursuing the maximum generality shown in other parts of this memoir such as the Introduction or Chapter 3. Moreover, the notation has been slightly modified to use similar symbols in all chapters.

Abstract

Motivated by a recent investigation of Costakis et al. on the notion of recurrence in Linear Dynamics, we study various stronger forms of recurrence for linear operators, in particular that of frequent recurrence. We study, among other things, the relationship between each type of recurrence and the corresponding notion of hypercyclicity, the influence of power-boundedness, and the interplay between recurrence and spectral properties. We obtain, in particular, Ansari- and Léon-Müller-type theorems for $\mathcal{F}$-recurrence under very weak assumptions on the Furstenberg family $\mathcal{F}$. This allows us, as a by-product, to deduce Ansari- and Léon-Müller-type theorems for $\mathcal{F}$-hypercyclicity.

1 Introduction

The notion of recurrence for a dynamical system has a very long history, whose systematic study goes back to the classical works of Gottschalk and Hedlund [GH55] and Furstenberg [Fur81] (see also [Gla04] and [OZ13] for recent advances). In Linear Dynamics, however, recurrent operators have only recently been studied systematically in a fundamental paper by Costakis, Manoussos and Parissis [CMP14]; see also [CP12].

The literature on (non-linear) dynamical systems abounds with notions that are similar to recurrence. Of course, periodicity is a very strong form of recurrence, and it is fundamental in any dynamical theory. But some other forms of recurrence have also recently been looked at in Linear Dynamics, see [GMJPO15], [YW18], [HHY18], [GMM21], [CM22a].

The aim of this paper is to study various recurrence notions in Linear Dynamics. The appropriate framework is that of $\mathcal{F}$-recurrence for arbitrary Furstenberg families $\mathcal{F}$. However, for better readability we will mainly concentrate on those types of recurrence that deserve the greatest interest from the point of view of Linear Dynamics. We will discuss the general notion of $\mathcal{F}$-recurrence in Section 8.

Throughout Sections 1 to 7, X will denote a (separable) *F-space* (i.e. X is a completely metrizable topological vector space), that in some particular cases we will assume to be a Fréchet, Banach or Hilbert space, and $T : X \longrightarrow X$ will be a (continuous, linear) *operator*, briefly $T \in \mathcal{L}(X)$. A vector $x \in X$ is called *recurrent* for T if there exists a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers such that

$$T^{n_k}x \to x \quad \text{as } k \to \infty.$$

We will denote by $\operatorname{Rec}(T)$ the *set of recurrent vectors* for T, and T is called *recurrent* if $\operatorname{Rec}(T)$ is dense in X. The latter differs from, but is equivalent to the definition of recurrence given by Costakis et al., see [CMP14, Proposition 2.1 with Remark 2.2] and Remark 8.2 below.

A vector x is called *periodic* for T if there is some $p \geq 1$ such that $T^p x = x$. The *set of periodic points* of T will be denoted by $\operatorname{Per}(T)$. The vector x is called *uniformly recurrent* for T if, for any neighbourhood U of x, the *return set*

$$N(x, U) = \{n \geq 0 : T^n x \in U\}$$

is syndetic, that is, has bounded gaps. The *set of uniformly recurrent vectors* will be denoted by $\operatorname{URec}(T)$, and again, the operator T is called *uniformly recurrent* if this set is dense in X. Uniformly recurrent vectors are often called almost periodic in the literature, see [GH55], but also syndetically recurrent or strongly recurrent, see [BK04], [KS09].

In addition, we fix the following terminology as suggested by recent work in Linear Dynamics:

Definition 1.1. Let $T \in \mathcal{L}(X)$. A vector $x \in X$ is *frequently* (*$\mathcal{U}$-frequently* or *reiteratively*) *recurrent* for T if, for any neighbourhood U of x, the return set

$$N(x, U) = \{n \geq 0 : T^n x \in U\}$$

has positive lower density (upper density or upper Banach density, respectively). The set of such vectors is denoted by $\operatorname{FRec}(T)$ ($\operatorname{UFRec}(T)$ or $\operatorname{RRec}(T)$, respectively). If this set is dense in X then the operator T is called *frequently* (*$\mathcal{U}$-frequently* or *reiteratively*) *recurrent.*

We recall that, for a subset A of $\mathbb{N}_0$, its *lower density* is defined as

$$\underline{\text{dens}}(A) = \liminf_{N\to\infty} \frac{\#(A \cap [0, N])}{N+1},$$

its *upper density* as

$$\overline{\text{dens}}(A) = \limsup_{N\to\infty} \frac{\#(A \cap [0, N])}{N+1},$$

and its *upper Banach density* as

$$\overline{\text{Bd}}(A) = \lim_{N\to\infty} \left(\sup_{m\geq 0} \frac{\#(A \cap [m, m+N])}{N+1} \right);$$

see [GTT10] for other, equivalent definitions of the upper Banach density; see also [BGE18].

The notion of $\mathcal{U}$-frequent recurrence was first introduced by Costakis and Parissis [CP12], while Grivaux and Matheron [GM14] have introduced a concept of frequent recurrence that is (at least formally) weaker than ours[A].

In non-linear dynamics, frequently recurrent points have been called weakly almost periodic, $\mathcal{U}$-frequently recurrent points have been called quasi-weakly almost periodic, and reiteratively recurrent points have been called positive Banach upper density points, Banach recurrent points, or essentially recurrent points, see [HYZ13], [Li12], [YZ12], [HW18], [BD08].

As pointed out by the referee, there exists the notion of (topological) *multiple recurrence* studied for linear operators in [CP12]. It was observed in the recent article [CM22b] that it is equivalent to *$\mathcal{AP}$-recurrence*, where $\mathcal{AP}$ is the Furstenberg family consisting of those subsets of natural numbers containing arbitrarily long arithmetic progressions[B]. Let $\mathcal{AP}\text{Rec}(T)$ be the *set of $\mathcal{AP}$-recurrent vectors* for an operator T. We have the following inclusions, which are obvious, except $\text{RRec}(T) \subset \mathcal{AP}\text{Rec}(T)$, which was observed in [CM22b][C]:

$$\text{Per}(T) \subset \text{URec}(T) \subset \text{FRec}(T) \subset \text{UFRec}(T) \subset \text{RRec}(T) \subset \mathcal{AP}\text{Rec}(T) \subset \text{Rec}(T). \quad (1.1)$$

The paper is organized as follows. In Section 2 we compare the recurrence properties with their corresponding notions of hypercyclicity. In Section 3 we study how power-boundedness influences on recurrence. Section 4 is devoted to some structural properties of recurrence; in particular, we solve a problem of Grivaux et al. [GMM21, Question 7.11]. Weighted backward shifts are studied in Section 5, where we also show that the inclusions in (1.1) are strict, in a rather strong sense. Further operators are considered in Section 7; a common feature of many of these operators is a large supply of unimodular eigenvectors, which implies $\mathcal{IP}^*$-recurrence, an interesting strengthening of uniform recurrence. Thus, as a preparation, we briefly discuss $\mathcal{IP}^*$-recurrence in Section 6. In the final Section 8 we introduce and discuss the general notion of $\mathcal{F}$-recurrence for operators on general topological vector spaces. As a by-product of our work we obtain Ansari- and León-Müller-type results for $\mathcal{F}$-hypercyclicity, see Theorem 8.8.

We finish this introduction with two comments: our investigations have led to several open problems, see Questions 2.9, 2.11, 2.13, 2.14, 4.10[D], 5.3, and 6.3; and for any unexplained but standard notions from Linear Dynamics we refer to the textbooks [BM09] and [GEP11].

[A] See Sections 1.2 and 2.3 of Chapter 2 for more on this alternative *frequent recurrence* notion.
[B] See Section 2.1 of the *General discussion of the results* for more on *$\mathcal{AP}$-recurrence.*
[C] See also statement (d) of Proposition 2.10 in Section 2 of the Appendix.
[D] Question 4.10 has recently been solved in the negative; see [CM].

2 Recurrence, hypercyclicity, and size of recurrent sets

The central notion in Linear Dynamics is that of a *hypercyclic vector*, that is, a vector with dense orbit. In a similar vein, a vector $x \in X$ is called *frequently* (*$\mathcal{U}$-frequently* or *reiteratively*) *hypercyclic* for T if, for every non-empty open subset U of X, the return set $N(x,U)$ has positive lower density (positive upper density or positive upper Banach density, respectively). An operator that possesses such a vector is called *frequently* (*$\mathcal{U}$-frequently* or *reiteratively*) *hypercyclic*, see [BG06], [Shk09], [BMPP16], [BGE18], and the textbooks [BM09], [GEP11]. Note that uniform recurrence admits no hypercyclic analogue, see [BMPP16, Proposition 2].

Trivially, every notion of hypercyclicity implies the corresponding notion of recurrence. The converse, of course, is not true as seen by the identity operator. In this section we ask under which additional assumptions on the operator the converse does become true.

Our first result elaborates on [BMPP16, Theorem 14].

Theorem 2.1. *Let $T \in \mathcal{L}(X)$. Then the following assertions are equivalent:*

(i) *T is reiteratively hypercyclic;*

(ii) *T is hypercyclic, and* $\operatorname{RRec}(T)$ *is a residual set;*

(iii) *T is hypercyclic, and* $\operatorname{RRec}(T)$ *is of second category;*

(iv) *T admits a hypercyclic and reiteratively recurrent vector;*

(v) *T is hypercyclic and reiteratively recurrent;*

(vi) *T is hypercyclic, and every hypercyclic vector is reiteratively hypercyclic.*

In that case the set of hypercyclic and reiteratively recurrent vectors is residual.

Proof. (i) ⇒ (ii): By [BMPP16, Theorem 14] every hypercyclic vector is also reiteratively hypercyclic when T is reiteratively hypercyclic; and the set of hypercyclic vectors is always residual. (ii) ⇒ (iii): This is trivial. (iii) ⇒ (iv): This follows from the fact that the set of hypercyclic vectors is residual. (iv) ⇒ (v): Since T admits a hypercyclic reiteratively recurrent vector x, then each element of the orbit of x is also a hypercyclic reiteratively recurrent vector. Thus T is hypercyclic and reiteratively recurrent. (v) ⇒ (vi): This was essentially shown in the proof of [BMPP16, Theorem 14]. We repeat the argument for the sake of completeness:

Let x be a hypercyclic vector and U a non-empty open set. By hypothesis there is a reiteratively recurrent vector $y \in U$. Thus, $N(y,U) = \{n \geq 0 : T^n y \in U\}$ has positive upper Banach density. Now let $n \geq 0$. Then $U_n = \bigcap_{j \in N(y,U) \cap [0,n]} T^{-j}(U)$ is a non-empty open set containing y. By hypercyclicity of x there is then some $k_n \geq 0$ such that $T^{k_n} x \in U_n$, thus $T^{k_n + j} x \in U$ for every $j \in N(y,U) \cap [0,n]$. In other words, for every $n \geq 0$ there exists $k_n \geq 0$ such that

$$N(x,U) \supset k_n + (N(y,U) \cap [0,n]).$$

This easily implies that $N(x,U)$ has positive upper Banach density[E]. That is, x is reiteratively hypercyclic so we get (vi). (vi) ⇒ (i): This is trivial. □

[E]See Lemma 2.12 in Section 2 of the Appendix.

Since periodic points are reiteratively recurrent, we have the following result of Menet:

Corollary 2.2 ([Men17]). *Every chaotic operator is reiteratively hypercyclic.*

In view of the theorem one might wonder if a single non-zero reiteratively recurrent vector, for a hypercyclic operator, suffices to make it reiteratively hypercyclic. This is not the case:

Example 2.3. By [BMPP16, Theorem 13], there exists a mixing operator S on $\ell^2(\mathbb{N})$ that is not reiteratively hypercyclic. Let T be a mixing and chaotic operator on $\ell^2(\mathbb{N})$, for example twice the backward shift, $2B$ (see [GEP11, Example 3.2]). The operator $S \oplus T$ on $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ is also mixing (see [GEP11, Proposition 1.42] for a standard argument), $S \oplus T$ cannot be reiteratively hypercyclic because S is not, and $(0, y)$ is periodic for $S \oplus T$ if $y \in \mathrm{Per}(T)$. So we even have a mixing operator with a non-zero periodic point that is not reiteratively hypercyclic.

If T is recurrent, the set of recurrent vectors for T is residual, see [CMP14]. Also, if T is a reiteratively hypercyclic operator, then the set of reiteratively hypercyclic vectors is residual, see [BMPP16]. However:

Example 2.4. There is a reiteratively recurrent operator for which the set of reiteratively recurrent vectors is of first category. To see this, let $X = \ell^p(\mathbb{N})$, $1 \leq p < \infty$, or $c_0(\mathbb{N})$. We consider the operator $T : X \longrightarrow X$ that is defined by $Te_1 = e_1$ and

$$Te_k = \begin{cases} 2e_{k+1} & \text{if } 2^j \leq k < 2^{j+1} - 1, \\ \frac{1}{2^{(2^j-1)}} e_{2^j} & \text{if } k = 2^{j+1} - 1 \end{cases}$$

for $j \geq 1$, where $e_k = (\delta_{k,n})_{n \geq 1}$ denotes the k-th canonical unit sequence. Each vector e_k is periodic for T, so that T admits a dense set of periodic points. In particular, T is reiteratively recurrent. We now prove that $\mathrm{RRec}(T)$ is of first category: it suffices to show that

$$G = \left\{ x = (x_n)_{n \geq 1} \in X : |x_{2^j}| > \tfrac{1}{j} \text{ for infinitely many } j \geq 1 \right\}$$

is a residual set that does not contain any reiteratively recurrent vector. It is easily checked that G is dense, and since

$$G = \bigcap_{J \geq 1} \bigcup_{j \geq J} \left\{ x \in X : |x_{2^j}| > \tfrac{1}{j} \right\},$$

it is a dense G_δ-set, hence residual. Let now $x = (x_n)_{n \geq 1} \in G$ and $U = \{y \in X : \|y - x\| < \frac{1}{2}\}$. Since $x \in X$ there is some $N_0 \geq 0$ such that $|x_n| < \frac{1}{2}$ for every $n \geq N_0$. Consequently we have that if $y \in U$ then $|y_n| < 1$ for every $n \geq N_0$. Moreover, since $x \in G$ there is an infinite set $A \subset \mathbb{N}$ such that $2^j > N_0$ and $|x_{2^j}| > \frac{1}{j}$ for each $j \in A$. For each of such $j \in A$ and each $n = \ell 2^j + k$ with $\ell \geq 0$ and $j \leq k \leq 2^j - 1$ we have that

$$\left| [T^n x]_{2^j + k} \right| = 2^k \, |x_{2^j}| > \tfrac{2^k}{j} \geq \tfrac{2^j}{j} > 1,$$

so that $T^n x \notin U$. This implies that, for any $m \geq 0$, $\#(N(x, U) \cap [m, m + 2^j - 1]) \leq j$, hence

$$\overline{\mathrm{Bd}}(N(x, U)) = \lim_{A \ni j \to \infty} \sup_{m \geq 0} \frac{\#(N(x, U) \cap [m, m + 2^j - 1])}{2^j} \leq \lim_{A \ni j \to \infty} \tfrac{j}{2^j} = 0,$$

so $x \notin \mathrm{RRec}(T)$ and the set of reiteratively recurrent vectors is of first category.

We will now see that for $\mathcal{U}$-frequently recurrent operators the situation is a little different from that for reiterative recurrence found in Theorem 2.1. We start with a partial analogue.

Theorem 2.5. *Let $T \in \mathcal{L}(X)$. Then the following assertions are equivalent:*

(i) *T is $\mathcal{U}$-frequently hypercyclic;*

(ii) *T is hypercyclic, and* $\mathrm{UFRec}(T)$ *is a residual set;*

(iii) *T is hypercyclic, and* $\mathrm{UFRec}(T)$ *is of second category;*

(iv) *T admits a hypercyclic and $\mathcal{U}$-frequently recurrent vector.*

In that case the set of hypercyclic and $\mathcal{U}$-frequently recurrent vectors is residual. Moreover, every hypercyclic and $\mathcal{U}$-frequently recurrent vector is $\mathcal{U}$-frequently hypercyclic.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv): These follow from the fact that the set of $\mathcal{U}$-frequently hypercyclic vectors is either empty or residual (see [BR15] or [BGE18]), and that the same is true for the set of hypercyclic vectors. (iv) $\Rightarrow$ (i): Let x be a hypercyclic and $\mathcal{U}$-frequently recurrent vector, and let U be a non-empty open set. By the hypercyclicity of x there is some integer $m \geq 0$ such that $T^m x \in U$. By the continuity of T there is some neighbourhood V of x such that $T^m(V) \subset U$. Since x is $\mathcal{U}$-frequently recurrent, the set

$$N(x, V) = \{n \geq 0 : T^n x \in V\}$$

has positive upper density, and so has $N(x, V) + m$. But $N(x, V) + m \subset \{n \geq 0 : T^n x \in U\}$. This shows that x is $\mathcal{U}$-frequently hypercyclic. This also proves the additional claim. □

However, the analogue of Theorem 2.1, statement (v), breaks down for $\mathcal{U}$-frequent recurrence:

Example 2.6. Menet has constructed, on $c_0(\mathbb{N})$ and on each $\ell^p(\mathbb{N})$ space with $1 \leq p < \infty$, a chaotic operator T that is not $\mathcal{U}$-frequently hypercyclic; see [Men17]. Since every periodic point is $\mathcal{U}$-frequently recurrent, the operator T is hypercyclic and $\mathcal{U}$-frequently recurrent without being $\mathcal{U}$-frequently hypercyclic. Statement (iii) of Theorem 2.5 implies that the set $\mathrm{UFRec}(T)$ must be of first category. This is in sharp contrast to the fact that the set of $\mathcal{U}$-frequently hypercyclic vectors is always either empty or residual, see [BR15].

In the case of frequent hypercyclicity we have even fewer equivalent conditions. The proof is identical to that of the corresponding part in Theorems 2.1 or 2.5.

Theorem 2.7. *Let $T \in \mathcal{L}(X)$. Then the following assertions are equivalent:*

(i) *T is frequently hypercyclic;*

(ii) *T admits a hypercyclic and frequently recurrent vector.*

Moreover, every hypercyclic and frequently recurrent vector is frequently hypercyclic.

There is a striking difference in hypercyclicity when passing from lower to upper densities: while the set of frequently hypercyclic vectors is always of first category ([Moo13], [BR15]), the set of $\mathcal{U}$-frequently hypercyclic vectors is residual unless empty ([BR15]). We have just seen that we lose the latter property for $\mathcal{U}$-frequent recurrence. For frequent recurrence we collect here some cases where $\mathrm{FRec}(T)$ is of first category (even though some of the statements remain true for operators acting on F-spaces, we will state the result for Fréchet spaces to avoid unnecessary complications). Let us first recall the following notions:

When X is a Fréchet space, the orbit of a vector $x \in X$ for an operator $T \in \mathcal{L}(X)$ is said to be *distributionally near to zero* (respectively *distributionally unbounded*) if there is a set $A \subset \mathbb{N}_0$ with $\overline{\mathrm{dens}}(A) = 1$ such that $T^n x \to 0$ as $A \ni n \to \infty$ (respectively $p(T^n x) \to \infty$ as $A \ni n \to \infty$ for some continuous semi-norm $p(\cdot)$ on X). These two properties, put together, define the notion of a *distributionally irregular vector*, see [BBMP08].

Theorem 2.8. *Let X be a Fréchet space and suppose that $T \in \mathcal{L}(X)$ fulfills any of the following:*

(a) *T is hypercyclic;*

(b) *T has a distributionally unbounded orbit;*

(c) *T has a dense set of vectors whose orbits are distributionally near to zero;*

(d) *T has a dense set of vectors $x \in X$ such that $T^n x \to 0$ as $n \to \infty$.*

Then the set $\mathrm{FRec}(T)$ *is of first category.*

Proof. (a): If T is hypercyclic and $\mathrm{FRec}(T)$ is of second category then so is the set of hypercyclic and frequently recurrent vectors, which are frequently hypercyclic by the previous theorem. This is a contradiction with [Moo13, Theorem 1]. (b): By [BBMP08, Proposition 7], the hypothesis implies that there exists a residual subset of vectors in X with distributionally unbounded orbit. But none of these vectors can be frequently recurrent. (c): Argue as in (b) but using [BBMP08, Proposition 9]. (d): This is a special case of (c). □

The identity operator tells us that $\mathrm{FRec}(T)$ can be all of X, so it is natural to ask:

Question 2.9. Do we always have that either $\mathrm{FRec}(T) = X$ or $\mathrm{FRec}(T)$ is of first category?

When we now try to look in the same way at uniformly recurrent vectors then we have gone too far: such vectors can never be hypercyclic. This is obvious on Banach spaces but also valid in general, as follows from a classical result of Furstenberg [Fur81, Theorem 1.17]: the closure of the orbit of any uniformly recurrent vector is a minimal set (i.e. it does not contain any proper closed invariant subset). Thus, no periodic point can be an accumulation point of the orbit of a uniformly recurrent vector. We give here the proof of this conclusion for the sake of completeness (recently used in [BMPP16, Proposition 2]):

Theorem 2.10. *No periodic point for $T \in \mathcal{L}(X)$ is an accumulation point of the orbit of a uniformly recurrent vector. In particular, no uniformly recurrent vector for T is hypercyclic.*

Proof. Suppose on the contrary that a periodic point y is an accumulation point of the orbit of a uniformly recurrent vector x. Then x cannot belong to the (finite) orbit of y under T, so that there are disjoint open sets U and V containing x and the orbit of y, respectively. Let m be the maximum gap in the return set $N(x, U)$. Then there is a neighbourhood W of y such that $T^j(W) \subset V$ for $j = 0, \ldots, m$. By assumption there is some $n \geq 0$ such that $T^n x \in W$. But then $T^k x$ belongs to V and therefore not to U for the $m+1$ exponents $k = n, \ldots, n+m$, which is a contradiction. Since 0 is a periodic point of every operator, the final conclusion follows. □

Theorem 2.10 only leaves the possibility to study hypercyclic operators that *also* have a dense set of uniformly recurrent vectors (or, for that matter, a dense set of $\mathcal{U}$-frequently or frequently recurrent vectors). We will not pursue this here. Let us slightly modify Question 2.9 to ask the following:

Question 2.11. Do we always have that either $\mathrm{URec}(T) = X$ or $\mathrm{URec}(T)$ is of first category?

We will get a partial positive answer in Section 3. Note that, for periodic points, the corresponding property holds. It is a simple consequence of the Baire category theorem that either $\mathrm{Per}(T)$ is of first category or else $T^n = I$ for some $n \geq 1$ (and hence $\mathrm{Per}(T) = X$).

Our next result was motivated by [GMM21, Corollary 5.20]. The authors there show that if an operator T is uniformly recurrent, and if there is a dense set of vectors $x \in X$ such that $T^n x \to 0$ as $n \to \infty$, then T is $\mathcal{U}$-frequently hypercyclic. They call this result *somewhat unexpected*. We can give here a more natural (and improved) version of their finding.

Theorem 2.12. *Let $T \in \mathcal{L}(X)$. Suppose that there is a dense set of vectors $x \in X$ such that $T^n x \to 0$ as $n \to \infty$. Then we have the following:*

(a) *If T is recurrent then it is hypercyclic.*

(b) *If T is reiteratively recurrent then it is reiteratively hypercyclic.*

(c) *If T is $\mathcal{U}$-frequently recurrent then it is $\mathcal{U}$-frequently hypercyclic.*

Proof. For $\mathcal{U}$-frequent hypercyclicity, it suffices by [BGE18, Corollary 3.4] to show that, for any non-empty open set V in X there is some $\delta > 0$ such that, for any non-empty open set U in X, there is some $x \in U$ such that

$$\overline{\mathrm{dens}}\{n \geq 0 : T^n x \in V\} > \delta. \tag{1.2}$$

Such a set V contains a $\mathcal{U}$-frequently recurrent vector v. Choose open neighbourhood V_0 of v and W of zero such that $V_0 + W \subset V$. Then the set

$$A := \{n \geq 0 : T^n v \in V_0\}$$

has positive upper density. Choose $0 < \delta < \overline{\mathrm{dens}}(A)$. Now let U be a non-empty open set. By hypothesis there is some $y \in U - v$ such that $T^n y \to 0$ as $n \to \infty$. Then the vector $x := y + v$ belongs to U, and we have that

$$T^n x = T^n y + T^n v \in W + V_0 \subset V$$

whenever $n \in A$ is sufficiently large, which implies (1.2). The proof for reiterative recurrence and recurrence is similar, and one can even obtain that T is weakly-mixing. See also Theorem 8.5 below and [BGE18, Theorem 3.1]. □

The proof, however, breaks down for frequent hypercyclicity.

Question 2.13. Let T be a frequently recurrent operator (or even a chaotic operator) such that $T^n x \to 0$ as $n \to \infty$ for all x from a dense subset of X. Does it follow that T is frequently hypercyclic? It seems to be even open whether every chaotic operator with a dense generalized kernel (that is, $\overline{\bigcup_{n\geq 0} \mathrm{Ker}(T^n)} = X$) is frequently hypercyclic.

We include one more natural question. Does the dynamical properties of an invertible operator T pass to its inverse? This is well-known to be the case for hypercyclicity, reiterative hypercyclicity (see [BGE18]) and recurrence ([CMP14]). However, Menet [Men20], [Men22] has recently shown that the corresponding results are false for ($\mathcal{U}$-)frequent hypercyclicity.

Question 2.14. [F] Let T be an invertible operator. If T is reiteratively recurrent ($\mathcal{U}$-frequently recurrent, frequently recurrent, uniformly recurrent), does T^{-1} have the same property?

3 Recurrence and power-boundedness

Not surprisingly, power-boundedness influences strongly the dynamical properties of a linear operator: we say that $T \in \mathcal{L}(X)$ is *power-bounded* if the sequence $(T^n)_{n\geq 0}$ is equicontinuous, that is, if for any 0-neighbourhood W_1 there is a 0-neighbourhood W_2 such that, for any $n \geq 0$,

$$T^n(W_2) \subset W_1;$$

by the Banach-Steinhaus theorem, this is equivalent to saying that every orbit under T is bounded, see [Rud91]. The following is then obvious; see also [CMP14, Lemma 3.1].

Theorem 3.1. *Let $T \in \mathcal{L}(X)$. If T is power-bounded, then the sets* $\mathrm{URec}(T)$, $\mathrm{FRec}(T)$, $\mathrm{UFRec}(T)$, $\mathrm{RRec}(T)$ *and* $\mathrm{Rec}(T)$ *are closed.*

Proof. We only consider uniform recurrence. Let $x \in \overline{\mathrm{URec}(T)}$ and W be a 0-neighbourhood. Choose a 0-neighbourhood W_1 such that $W_1 + W_1 + W_1 \subset W$. By power-boundedness, there is a 0-neighbourhood $W_2 \subset W_1$ such that $T^n(W_2) \subset W_1$ for all $n \geq 1$. There is some $y \in \mathrm{URec}(T)$ such that $x - y, y - x \in W_2$. The set $A := \{n \geq 0 : T^n y - y \in W_1\}$ is syndetic and

$$T^n x - x = T^n(x - y) + T^n y - y + y - x \in W_1 + W_1 + W_1 \subset W \quad \text{for each } n \in A.$$

Since W is arbitrary, x is uniformly recurrent. □

This shows that, for every power-bounded operator, recurrence of the operator implies that every vector is recurrent; and similarly for the other notions of recurrence. On the other hand, for an operator acting on a Banach space, every uniformly recurrent vector has a bounded orbit. Thus we immediately obtain the following partial answer to Question 2.11:

Corollary 3.2. *Let X be a Banach space and $T \in \mathcal{L}(X)$ a uniformly recurrent operator. Then either* $\mathrm{URec}(T)$ *is of first category, or* $\mathrm{URec}(T) = X$.

[F] This question has recently been partially solved in the positive; see Section 6 of Chapter 2.

Proof. Suppose that the set $\mathrm{URec}(T)$ is of second category. Then so is the set of vectors with bounded orbit under T, which is then power-bounded by the Banach-Steinhaus theorem. By the previous theorem $\mathrm{URec}(T)$ is closed, and dense by hypothesis, so that $\mathrm{URec}(T) = X$. □

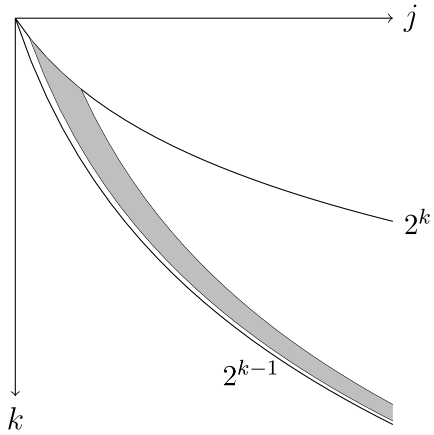
For F-space and Fréchet space operators, a uniformly recurrent orbit is not necessarily bounded, so that one cannot argue as in the proof of Corollary 3.2. An example is given by the backward shift on the space $\mathbb{K}^{\mathbb{N}}$ of all (real or complex) sequences, see [GMJPO15, Example 1]. We give here an example on a Fréchet space with a continuous norm. The type of operator considered in this example might also be of independent interest.

Example 3.3. Let X be the space of doubly indexed sequences $x = (x_{k,j})_{k\geq 0, 0\leq j<2^k}$ such that

$$p_n(x) := \sum_{k=0}^{\infty} \frac{1}{2^k} \max_{0\leq j<2^k} |x_{k,j}| + \sum_{k=2}^{\infty} k \max_{\substack{1\leq m\leq n \\ m<2^{k-1}}} |x_{k,2^{k-1}+m}| < \infty \qquad \text{for each } n \geq 1.$$

Figure 1.1 indicates the area of indices that is involved in the second sum. When endowed with the increasing sequence of (semi-)norms $(p_n)_{n\geq 1}$, it obviously becomes a Fréchet space.

Figure 1.1: Indices for the semi-norms p_n.



We consider the operator T on X given by

$$T(x_{k,j})_{k,j} = (x_{k,j+1 (\mathrm{mod}\ 2^k)})_{k,j},$$

that is, a simple row-wise rotation. To see that T is continuous, fix $n \geq 1$. Choose $l \geq 2$ so that $2^l \geq 2(n+2)$, which implies that $n+1 < 2^{k-1}$ for all $k \geq l$. Then we have for $x \in X$ that

$$\begin{aligned} p_n(Tx) &= \sum_{k=0}^{\infty} \frac{1}{2^k} \max_{0\leq j<2^k} |[Tx]_{k,j}| + \sum_{k=2}^{\infty} k \max_{\substack{1\leq m\leq n \\ m<2^{k-1}}} |[Tx]_{k,2^{k-1}+m}| \\ &= \sum_{k=0}^{\infty} \frac{1}{2^k} \max_{0\leq j<2^k} |x_{k,j}| + \sum_{k=2}^{l-1} k \max_{\substack{1\leq m\leq n \\ m<2^{k-1}}} |[Tx]_{k,2^{k-1}+m}| + \sum_{k=l}^{\infty} k \max_{1\leq m\leq n} |x_{k,2^{k-1}+m+1}| \\ &\leq \sum_{k=0}^{\infty} \frac{1}{2^k} \max_{0\leq j<2^k} |x_{k,j}| + (l-1)2^{l-1} \sum_{k=2}^{l-1} \frac{1}{2^k} \max_{0\leq j<2^k} |x_{k,j}| + \sum_{k=l}^{\infty} k \max_{1\leq m\leq n+1} |x_{k,2^{k-1}+m}| \\ &\leq (1 + (l-1)2^{l-1}) p_{n+1}(x), \end{aligned}$$

which proves continuity.

Now, consider the vector $x = (x_{j,k}) \in X$ given by $x_{k,0} = 1$ for $k \geq 0$ and all other $x_{k,j} = 0$. Then x is uniformly recurrent for T. Indeed, let $n \geq 1$ and $\varepsilon > 0$. Choose $l \geq 0$ such that $2^l > \max(n, 1/\varepsilon)$. Let $\nu \geq 0$. First we observe that

$$[T^{\nu 2^l} x]_{k,j} = x_{k,j}, \quad k \leq l, 0 \leq j < 2^k.$$

On the other hand, for $k > l$, the fact that $x_{k,j} = 0$ for all $j \neq 0$ implies that $[T^{\nu 2^l} x]_{k,j} = 0$ whenever j is not a multiple of 2^l. Now since, for these k, 2^{k-1} is a multiple of 2^l and $n < 2^l$, we have that

$$[T^{\nu 2^l} x]_{k,2^{k-1}+m} = 0, \quad k > l, 1 \leq m \leq n;$$

note that $m < 2^{k-1}$ is automatic. Thus we have for any $\nu \geq 0$

$$p_n(T^{\nu l} x - x) = \sum_{k>l} \frac{1}{2^k} = \frac{1}{2^l} < \varepsilon.$$

This shows that x is uniformly recurrent. On the other hand, by construction, the orbit of x is unbounded. It suffices to observe that for $k \geq 2$

$$[T^{2^{k-1}+1} x]_{k,2^{k-1}+1} = 1,$$

so that $p_1(T^{2^{k-1}+1} x) \geq k$.

The vector x considered above is not periodic, but for any neighbourhood U of x there is some $k \geq 1$ such that $T^{nk} x \in U$ for all $n \geq 0$. Such points have been called regularly recurrent (or regularly almost periodic) in non-linear dynamics, see [GH55], [BK04].

The examples show that, for Fréchet spaces, the proof for Corollary 3.2 breaks down at a very early stage. One may wonder what kind of (weak) boundedness the orbit of a uniformly recurrent vector possesses in the setting of Fréchet spaces, or even in F-spaces. On the other hand, for power-bounded operators we have a strong form of boundedness:

Theorem 3.4. *Let $T \in \mathcal{L}(X)$ be power-bounded. If x is a uniformly recurrent vector for T then the closure of its orbit is compact.*

Proof. We show that the orbit of x is totally bounded, that is, for any 0-neighbourhood W there are finitely many points $x_0, \ldots, x_N$ such that the orbit is contained in $\bigcup_{n=0}^N (x_n + W)$. Thus let W be a 0-neighbourhood. By power-boundedness there is a 0-neighbourhood W_0 such that $T^n(W_0) \subset W$ for all $n \geq 0$. Let N be the maximum gap in the return set $N(x, x + W_0)$. Then we have that

$$\{T^k x : k \geq 0\} \subset \bigcup_{n=0}^N T^n(x + W_0) \subset \bigcup_{n=0}^N (T^n x + W),$$

which implies the claim. □

We already recalled Furstenberg's result which says that the closure of the orbit of any uniformly recurrent vector is a minimal set. The dynamics on minimal compact sets (like irrational rotations on the torus) is a matter of study in non-linear dynamics.

4 Recurrence, the unit circle, and the spectrum

In this section we study some properties of recurrence whose hypercyclic analogues belong to the fundamental results in Linear Dynamics. Costakis et al. [CMP14] have obtained the following: for any recurrent operator T,

- T^p is recurrent for any $p \geq 1$; in fact, $\mathrm{Rec}(T^p) = \mathrm{Rec}(T)$;
- λT is recurrent whenever $|\lambda| = 1$; in fact, $\mathrm{Rec}(\lambda T) = \mathrm{Rec}(T)$;

moreover, if X is a complex Banach space, then

- every component of the spectrum $\sigma(T)$ meets the unit circle;
- the point spectrum $\sigma_p(T^*)$ of its *adjoint operator* T^* is contained in the unit circle.

We start by looking at the first two properties for other notions of recurrence. Our approach uses in a crucial way an idea of Bayart and Matheron [BM09, Section 6.3.3]. Let us say that a family $\mathcal{F}$ of subsets of $\mathbb{N}_0$ has the *Cut-Shift-and-Paste* property (called *CuSP* for short) if for any $A \in \mathcal{F}$, any $I_1, \ldots, I_q \subset \mathbb{N}_0$ with $A \subset \bigcup_{j=1}^q I_j$, and any $n_1, \ldots, n_q \in \mathbb{N}_0$ $(q \geq 1)$, we have that

$$\bigcup_{j=1}^{q} (n_j + A \cap I_j) \in \mathcal{F}.$$

Then [BM09, Lemma 6.29] says that the family of positive lower density sets has the CuSP.

Lemma 4.1. *The following families have the CuSP property: the syndetic subsets and the infinite subsets of* $\mathbb{N}_0$*, and the sets of positive lower, upper and upper Banach density.*

Proof. The positive lower density case is proved in [BM09, Lemma 6.29]; the same proof also covers the positive upper density case. The result is obvious for the family of infinite subsets. For the remaining cases, fix a set $A \subset \mathbb{N}_0$, fix some $I_1, \ldots, I_q \subset \mathbb{N}_0$ with $A \subset \bigcup_{j=1}^q I_j$ and let $n_1, \ldots, n_q \in \mathbb{N}_0$, where $q \geq 1$. Set $M = \max(n_1, \ldots, n_q)$.

Assume first that $\overline{\mathrm{Bd}}(A) > \delta > 0$. Fix $N_0 \geq 0$. Then there is some $N_1 \geq \max(N_0, M)$ such that, for every $N \geq N_1$, there is some $m \geq 0$ such that

$$\frac{\#(A \cap [m, m+N])}{N+1} > \delta.$$

Now, for some k, $1 \leq k \leq q$, we have that

$$\#(A \cap I_k \cap [m, m+N]) \geq \tfrac{1}{q} \cdot \#(A \cap [m, m+N]);$$

moreover, $\#\Big((n_k + A \cap I_k) \cap [m, m+N+M]\Big) \geq \#(A \cap I_k \cap [m, m+N])$. Then we obtain that

$$\frac{\#\left(\left(\bigcup_{j=1}^q (n_j + A \cap I_j)\right) \cap [m, m+N+M]\right)}{N+M+1} \geq \frac{1}{q} \cdot \frac{N+1}{N+M+1} \cdot \frac{\#(A \cap [m, m+N])}{N+1} \geq \frac{1}{2q}\delta,$$

which shows that $\bigcup_{j=1}^q (n_j + A \cap I_j)$ has positive upper Banach density.

Finally, assume that A is a syndetic set. Then there is some $N \geq 1$ such that every integer interval of length N contains some element of A. Let $J = [m_1, m_2]$ be an integer interval of length $N+M$. Then the interval $[m_1, m_1+N]$ contains an element $m \in A$. By assumption there is some k, $1 \leq k \leq q$, such that $m \in A \cap I_k$. But then $m + n_k$ is an element of $\bigcup_{j=1}^q (n_j + A \cap I_j)$ that belongs to J. Thus the set $\bigcup_{j=1}^q (n_j + A \cap I_j)$ is syndetic. □

As a first application we deduce an Ansari-type result for various forms of recurrence. Recall that Ansari proved that, for any $p \geq 1$, the operators T and T^p have the same hypercyclic vectors [Ans95]. Her proof uses in an essential way a connectedness argument; for recurrence the argument is simpler.

Theorem 4.2. *Let $T \in \mathcal{L}(X)$ and consider any $p \geq 1$. Then the operators T and T^p have the same uniformly (frequently, $\mathcal{U}$-frequently or reiteratively) recurrent vectors. In particular, if T is uniformly (frequently, $\mathcal{U}$-frequently or reiteratively) recurrent, then so is T^p.*

Proof. Let $p \geq 1$ be given. We will show that $\mathrm{URec}(T) = \mathrm{URec}(T^p)$, where we only use two properties of the family of syndetic sets: the CuSP and the fact that $A \subset \mathbb{N}_0$ is syndetic if and only if $pA = \{pn : n \in A\}$ is. The remaining assertions can be proved in the same way.

It suffices to show that $\mathrm{URec}(T) \subset \mathrm{URec}(T^p)$. We may also suppose that p is a prime number. Thus, let x be a uniformly recurrent vector for T. Let $(U_k)_{k\geq 1}$ be a decreasing sequence of neighbourhoods of x that forms a local base. For $k \geq 1$, we define

$$J_k = \{j \in \{0, \dots, p-1\} : \text{there exists } n \geq 0 \text{ with } n = j \pmod p \text{ and } T^n x \in U_k\}.$$

Then $(J_k)_{k\geq 1}$ is a decreasing sequence of non-empty finite sets, which therefore stabilizes. That is, there is a non-empty set $J \subset \{0, \dots, p-1\}$ and some $k_0 \geq 1$ such that $J_k = J$ for all $k \geq k_0$. We claim that J is a subgroup of $\mathbb{Z}/p\mathbb{Z}$. Indeed, let $j, j' \in J$. First, there is some $n \geq 0$ with $n = j \pmod p$ such that $T^n x \in U_{k_0}$. By continuity there is some $l \geq k_0$ such that $T^n(U_l) \subset U_{k_0}$. Now, since $j' \in J_l$, there is then some $n' \geq 0$ with $n' = j' \pmod p$ such that $T^{n'} x \in U_l$. Altogether we have that $T^{n+n'} x = T^n(T^{n'} x) \subset U_{k_0}$, hence, by definition, $j + j' \pmod p \in J_{k_0} = J$. Since p is prime, $\mathbb{Z}/p\mathbb{Z}$ only has two subgroups:

(a) We first assume that $J = \{0\}$. Then the sets

$$A_k := \{n \geq 0 : T^n x \in U_k\}, \quad k \geq k_0$$

only consist of multiples of p, and they are syndetic by hypothesis. Thus the sets $\frac{1}{p}A_k$ are syndetic, and $(T^p)^n x \in U_k$ for all $n \in \frac{1}{p}A_k$. This shows that x is uniformly recurrent for T^p.

(b) Now assume that $J = \{0, \dots, p-1\}$, hence $J_k = \{0, \dots, p-1\}$ for all $k \geq 1$. Let $k \geq 1$. For any $j \in J$ we can find some $n_j = p - j \pmod p$ such that $T^{n_j} x \in U_k$. By continuity there is some $l \geq 1$ such that, for any $j \in J$, $T^{n_j}(U_l) \subset U_k$. By our hypothesis, the set $A_l = \{n \geq 0 : T^n x \in U_l\}$ is syndetic. Set

$$I_j = \{n \geq 0 : n = j \pmod p\} \quad \text{for each } j \in J.$$

Note that given $n \in A_l \cap I_j$, for any $j \in J$, we have that $T^{n_j+n} x = T^{n_j}(T^n x) \in U_k$. In other words,

$$A := \bigcup_{j \in J} (n_j + A_l \cap I_j) \subset \{n \geq 0 : T^n x \in U_k\}.$$

Since $\{I_j : j \in J\}$ is a covering of $\mathbb{N}_0$, we deduce that A is a syndetic set by Lemma 4.1.

Moreover, if $m \in A$ then there are $j \in J$ and $n \geq 0$, $n = j \pmod p$ such that

$$m = n_j + n = p - j + j \pmod p = 0 \pmod p.$$

Thus the set $\frac{1}{p}A$ is syndetic and $(T^p)^n x \in U_k$ for all $n \in \frac{1}{p}A$. Since $k \geq 1$ was arbitrary, we see that x is uniformly recurrent for T^p. □

Remark 4.3. Theorem 4.2 holds for any continuous map on any topological space, and in particular for operators acting on topological vector spaces. It is enough to replace the countable local base $(U_k)_{k\geq 1}$ by the filter of all neighbourhoods at x. See also [CMP14, Remark 2.4].

As usual, the λT-problem is closely related to the T^p-problem: León and Müller [LSM04] showed that, for any scalar λ of modulus 1, T and λT have the same hypercyclic vectors. As it happens for hypercyclicity, the proof in the λT-case for recurrence requires somewhat more work than in the T^p-case showed in the previous theorem.

Theorem 4.4. *Let $T \in \mathcal{L}(X)$, and let λ be a scalar with $|\lambda| = 1$. Then the operators T and λT have the same uniformly (frequently, $\mathcal{U}$-frequently or reiteratively) recurrent vectors. In particular, if T is uniformly (frequently, $\mathcal{U}$-frequently or reiteratively) recurrent, then so is λT.*

Proof. This time we only use the CuSP property, so it again suffices to do the uniformly recurrent case. The real scalar case already follows from Theorem 4.2 because $(-T)^2 = T^2$. Thus we need only consider complex scalars. Alternatively, one can also repeat the following proof for $\mathbb{R}$ instead of $\mathbb{C}$.

It obviously suffices to show that $\mathrm{URec}(T) \subset \mathrm{URec}(\lambda T)$ whenever $|\lambda| = 1$. Thus, let $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, and let x be a uniformly recurrent vector for T. Let $(U_k)_{k\geq 1}$ be a decreasing sequence of neighbourhoods of x that forms a local base. For $k \geq 1$, we define

$$\Lambda_k = \{\mu \in \mathbb{T} : \text{there exists } n \geq 0 \text{ with } \lambda^n = \mu \text{ and } T^n x \in U_k\},$$

where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ denotes the unit circle. Then $(\Lambda_k)_{k\geq 1}$ is a decreasing sequence of non-empty subsets of $\mathbb{T}$.

Let

$$\Lambda = \bigcap_{k=1}^{\infty} \overline{\Lambda_k}.$$

Since Λ is the intersection of a decreasing sequence of non-empty closed sets, it is a non-empty closed subset of $\mathbb{T}$. We now claim that Λ is a subsemigroup of the multiplicative group $\mathbb{T}$. To see this, let $\mu, \mu' \in \Lambda$. Let $k \geq 1$ and $\varepsilon > 0$. Then there is some $\mu_k \in \Lambda_k$ such that $|\mu - \mu_k| < \varepsilon$. This implies that there is some $n_k \geq 0$ such that $\lambda^{n_k} = \mu_k$ and $T^{n_k} x \in U_k$. By continuity there is some $l \geq 1$ such that $T^{n_k}(U_l) \subset U_k$. We then find some $\mu'_l \in \Lambda_l$ such that $|\mu' - \mu'_l| < \varepsilon$ and hence some $n'_l \geq 0$ such that $\lambda^{n'_l} = \mu'_l$ and $T^{n'_l} x \in U_l$. Altogether we get that

$$T^{n_k + n'_l} x \in T^{n_k}(U_l) \subset U_k.$$

Since $\lambda^{n_k + n'_l} = \mu_k \mu'_l$, we deduce that $\mu_k \mu'_l \in \Lambda_k$. On the other hand,

$$|\mu\mu' - \mu_k \mu'_l| \leq |\mu - \mu_k| \cdot |\mu'| + |\mu_k| \cdot |\mu' - \mu'_l| < 2\varepsilon.$$

Since k and $\varepsilon > 0$ were arbitrary, $\mu\mu' \in \overline{\Lambda_k}$ for all $k \geq 1$. Thus $\mu\mu' \in \Lambda$, as had to be shown.

As a consequence, there are only two possibilities for Λ, see [GEP11, pages 170 and 171]:

(a) This time it is easier to start with the full group: $\Lambda = \mathbb{T}$. Let U be a neighbourhood of x. By continuity of scalar multiplication there is some $\varepsilon > 0$ and some $k \geq 1$ such that $B(1,\varepsilon)U_k \subset U$, where $B(z_0,\varepsilon) = \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$. Since $\Lambda = \mathbb{T}$, the set Λ_k is dense in $\mathbb{T}$. Using compactness there are $N \geq 1$ and $n_j \geq 0$ with $T^{n_j}x \in U_k$, $j = 1,\dots,N$, such that

$$\mathbb{T} \subset \bigcup_{j=1}^{N} B(\lambda^{n_j},\varepsilon). \tag{1.3}$$

By continuity there exists $l \geq 1$ such that, for $j = 1,\dots,N$, $T^{n_j}(U_l) \subset U_k$, and we have that the set $A_l := \{n \geq 0 : T^n x \in U_l\}$ is syndetic. Also, it follows from (1.3) that the sets

$$I_j := \{n \geq 0 : \lambda^{n_j+n} \in B(1,\varepsilon)\}, \quad j = 1,\dots,N,$$

form a cover of $\mathbb{N}_0$. Now, if $n \in A_l \cap I_j$, $j = 1,\dots,N$, then

$$(\lambda T)^{n_j+n}x = \lambda^{n_j+n}T^{n_j}(T^n x) \in B(1,\varepsilon)T^{n_j}(U_l) \subset B(1,\varepsilon)U_k \subset U.$$

This shows that

$$A := \bigcup_{j=1}^{N}(n_j + A_l \cap I_j) \subset \{n \geq 0 : (\lambda T)^n x \in U\},$$

and it follows from Lemma 4.1 that $\{n \geq 0 : (\lambda T)^n x \in U\}$ is syndetic. Thus x is uniformly recurrent for λT.

(b) It remains the case when there is some $N \geq 1$ such that $\Lambda = \{e^{2\pi i \frac{j}{N}} : j = 1,\dots,N\}$. Let U be a neighbourhood of x, and then $\varepsilon > 0$ and $k' \geq 1$ such that $B(1,\varepsilon)U_{k'} \subset U$. It follows from a simple compactness argument that there is some $k \geq k'$ such that

$$\Lambda_k \subset \bigcup_{j=1}^{N} B\Big(e^{2\pi i \frac{j}{N}}, \tfrac{\varepsilon}{2}\Big).$$

Since $e^{2\pi i \frac{-j}{N}} \in \Lambda \subset \overline{\Lambda_k}$, $j = 1,\dots,N$, there are $n_j \geq 0$ with $T^{n_j}x \in U_k$ and $|\lambda^{n_j} - e^{2\pi i \frac{-j}{N}}| < \frac{\varepsilon}{2}$. As before there is some $l \geq k$ such that, for $j = 1,\dots,N$ we have that $T^{n_j}(U_l) \subset U_k$, and the set $A_l = \{n \geq 0 : T^n x \in U_l\}$ is syndetic. Now, let $n \in A_l$. Then $\lambda^n \in \Lambda_l \subset \Lambda_k$, so that there is some $j \in \{1,\dots,N\}$ such that $|\lambda^n - e^{2\pi i \frac{j}{N}}| < \frac{\varepsilon}{2}$, and hence

$$|\lambda^{n_j}\lambda^n - 1| \leq \left|\lambda^{n_j} - e^{2\pi i \frac{-j}{N}}\right| \cdot |\lambda^n| + \left|e^{2\pi i \frac{-j}{N}}\lambda^n - 1\right| < \varepsilon.$$

This shows that the sets $I_j := \{n \geq 0 : \lambda^{n_j+n} \in B(1,\varepsilon)\}$, $j = 1,\dots,N$, cover A_l. From here the proof can be finished as in case (a) by using Lemma 4.1 again. □

Remark 4.5. Again, by considering the neighbourhood filter instead of the countable local base $(U_k)_{k\geq 1}$, Theorem 4.4 holds for any operator on any topological vector space.

Our proofs for the T^p- and λT-problems work equally well for recurrent operators and therefore provide alternative, if longer, proofs to those of Costakis et al. [CMP14][G].

[G]See Chapter 3, Section 4.2, Proposition 4.7 for a shorter proof in the usual recurrence case.

Combining Theorems 4.2 and 4.4 with Theorems 2.5 and 2.7 we obtain new proofs for the “Ansari-” and “León-Müller-type” theorems with respect to the $\mathcal{U}$-frequent hypercyclicity and frequent hypercyclicity notions, that are originally due to Shkarin [Shk09] and Bayart, Grivaux and Matheron [BG06], [BM09], respectively. The corresponding results for the notion of reiterative hypercyclicity follow from Theorem 2.1 and seem to be new. Note, however, that reiterative hypercyclicity passes directly to T^p and λT since the sets of reiteratively hypercyclic and hypercyclic vectors coincide for a reiteratively hypercyclic operator (see [BMPP16]).

Corollary 4.6. *Let $T \in \mathcal{L}(X)$ be reiteratively hypercyclic.*

(a) *If $p \geq 1$, then T^p is reiteratively hypercyclic.*

(b) *If λ is a scalar with $|\lambda| = 1$, then λT is reiteratively hypercyclic.*

We have another interesting application of Theorem 4.4. In [GMM21, Question 7.11], the authors ask whether any Banach space operator with a dense set of uniformly recurrent vectors must have a non-zero periodic point. Since the operator λI with $\lambda \in \mathbb{T}$ not a root of unity provides a counterexample to this question, see also Remark 4.8 below, the authors probably were only interested in hypercyclic operators. Still, a negative answer follows form the theorem above and an important counterexample of Bayart and Bermúdez [BB09].

Corollary 4.7. *There exists a hypercyclic operator on Hilbert space that admits a dense set of uniformly recurrent vectors but no non-zero periodic points.*

Proof. In [BB09, Theorem 3.1] it is proved that there exists a chaotic operator T on complex Hilbert space such that λT is not chaotic for some $\lambda \in \mathbb{T}$. Indeed, the proof even shows that the point spectrum of λT contains no root of unity, so that λT has no non-zero periodic points, see [GEP11, Proposition 2.33]. Now since periodic points are uniformly recurrent vectors, the operator T is uniformly recurrent, and then so is λT by Theorem 4.4. □

Remark 4.8. Let us mention that the corollary can be proved without Theorem 4.4. Indeed, if x is a periodic point for an operator T, then it follows rather directly that x is uniformly recurrent for λT for any $\lambda \in \mathbb{T}$. To see this, suppose that $T^N x = x$ for some $N \geq 1$, and let $\lambda \in \mathbb{T}$. Let $\varepsilon > 0$. It is well known (see also Lemma 7.1 below) that there is then a syndetic set $A \subset \mathbb{N}_0$ such that $|(\lambda^N)^n - 1| < \frac{\varepsilon}{\|x\|}$ for all $n \in A$; of course we may assume that $x \neq 0$. Hence $|(\lambda T)^{nN}x - x| = |(\lambda^N)^n - 1| \cdot \|x\| < \varepsilon$ for all $n \in A$, and x is uniformly recurrent for λT.

We turn to the spectrum of recurrent operators when the underlying space is a complex Banach space. By Costakis et al. [CMP14] we know that every component meets the unit circle. We have additional information when T is $\mathcal{U}$-frequently recurrent. Shkarin showed in [Shk09, Theorem 1.2 and its proof] that the spectrum of a $\mathcal{U}$-frequently hypercyclic operator cannot have isolated points. His argument also serves to show the following:

Theorem 4.9. *Let X be a complex Banach space and suppose that $T \in \mathcal{L}(X)$ is a $\mathcal{U}$-frequently recurrent operator.*

(a) *If $\sigma(T) = \{\lambda\}$ is a singleton, then $|\lambda| = 1$ and $T = \lambda I$.*

(b) *If $\sigma(T)$ has an isolated point $\lambda \in \mathbb{C}$, then $|\lambda| = 1$ and there are non-trivial T-invariant closed subspaces M_1 and M_2 of X such that $X = M_1 \oplus M_2$ and $T|_{M_1} = \lambda I|_{M_1}$.*

In particular, in either case T cannot also be hypercyclic; indeed, in (b), a typical argument (see [GEP11, Proposition 2.25]) would imply that $T|_{M_1} = \lambda I|_{M_1}$ was hypercyclic, which is absurd. Note that this result also contains the well known fact that the spectrum of any chaotic operator has no isolated points, see [BMGP01] and [BM09, Proposition 6.37].

Proof of Theorem 4.9. (a): By the result of Costakis et al. [CMP14] mentioned above we have that $|\lambda| = 1$. Let $S = \lambda^{-1}T$, which is also $\mathcal{U}$-frequently recurrent by Theorem 4.4. On the other hand, an analysis of Shkarin's argument, see [BM09, Remark on page 153], shows that if $S \neq I$ then one can find a non-empty open set U of X such that $\{n \geq 0 : S^n x \in U\}$ has upper density zero for all $x \in X$; in particular, no vector in U is $\mathcal{U}$-frequently recurrent for S. Thus $S = I$, hence $T = \lambda I$. (b): This follows from (a) by the usual Riesz decomposition theorem and quasi-conjugacy arguments (see [BM09, Proposition 6.37] or [GEP11, Proposition 5.7]). □

It is not clear whether the result extends to reiterative recurrence. By a result of Salas [Sal95], see also [GEP11, Example 8.4], there exist hypercyclic compact perturbations $T = I + K$ of the identity with $\sigma(T) = \{1\}$.

Question 4.10. [H] Can the spectrum of a reiteratively hypercyclic operator be a singleton? Does there exist a reiteratively hypercyclic compact perturbation of the identity?

5 Recurrence of weighted backward shift operators

Backward shifts are the best understood class of operators in Linear Dynamics. In particular they will serve us here to distinguish five of the six types of recurrence considered in (1.1).

Apart from the proof of the latter fact, this section contains no proofs: the other results are special cases of stronger results proved either in Section 2, or in a forthcoming paper by the first two authors [BGE], or by other authors. We find it nonetheless instructive to highlight the recurrence behaviour of weighted shifts. We will just focus on Fréchet sequence spaces:

A *Fréchet sequence space* (over $\mathbb{N}$) is a Fréchet space that is a subspace of the space $\mathbb{K}^{\mathbb{N}}$ of all (real or complex) sequences and such that each coordinate functional $x = (x_n)_{n\geq1} \mapsto x_k$, $k \geq 1$, is continuous. The canonical unit sequences are denoted by $e_k = (\delta_{k,n})_{n\geq1}$. A weight sequence is a sequence $w = (w_n)_{n\geq1}$ of non-zero scalars. The (unilateral) weighted backward shift B_w is then defined by $B_w(x_n)_{n\geq1} = (w_{n+1}x_{n+1})_{n\geq1}$. Fréchet sequence spaces over $\mathbb{Z}$ and bilateral weighted backward shifts are defined analogously.

Now, Theorem 2.12 applies in particular to unilateral weighted backward shifts.

Corollary 5.1. *Let X be a Fréchet sequence space over $\mathbb{N}$ in which $(e_n)_{n\geq1}$ is a basis. Suppose that the weighted backward shift B_w is an operator on X. Then we have the following:*

(a) *If B_w is recurrent then it is hypercyclic.*

(b) *If B_w is reiteratively recurrent then it is reiteratively hypercyclic.*

(c) *If B_w is $\mathcal{U}$-frequently recurrent then it is $\mathcal{U}$-frequently hypercyclic.*

[H] The answer is negative; see [CM]. See also Section 2.1 of the *General discussion of the results.*

Each hypercyclic property in the corollary have been characterized in terms of the weights, at least if the basis is unconditional, see [GEP11, Theorem 4.8], [BGE18, Theorem 5.1].

There is a considerable strengthening of assertion (a). By a remarkable result of Chan and Seceleanu [CS12], if a weighted shift on $\ell^p(\mathbb{N})$, $1 \leq p < \infty$, admits an orbit with a non-zero limit point, then it is already a hypercyclic operator. Recently, this has been extended by He et al. [HHY18, Lemma 5] to all Fréchet sequence spaces over $\mathbb{N}$ in which $(e_n)_{n\geq 1}$ is an unconditional basis; see also [BGE].

For bilateral weighted shifts we have the analogue of (a).

Theorem 5.2. *Let X be a Fréchet sequence space over $\mathbb{Z}$ in which $(e_n)_{n\in\mathbb{Z}}$ is a basis. Suppose that the weighted backward shift B_w is an operator on X. If B_w is recurrent then it is hypercyclic.*

This was proved by Costakis and Parissis [CP12, Proposition 5.1] for the space $\ell^2(\mathbb{Z})$. For the general case, i.e. for $\ell^p(\mathbb{Z})$ spaces, this can be shown by combining the proof of these authors with the one of [GEP11, Theorem 4.12(a)], adding a standard conjugacy argument. But, again, Chan and Seceleanu [CS12] have the stronger result that if a weighted shift on $\ell^p(\mathbb{Z})$, $1 \leq p < \infty$, admits an orbit with a non-zero limit point, then it is hypercyclic. In [BGE], this is extended to all Fréchet sequence spaces over $\mathbb{Z}$ in which $(e_n)_{n\in\mathbb{Z}}$ is an unconditional basis.

Several questions remain (see also Question 2.13).

Question 5.3. Does the analogue of Corollary 5.1 hold for frequent recurrence? Does the analogue of Theorem 5.2 hold for reiterative ($\mathcal{U}$-frequent, frequent) recurrence?

The work by Chan and Seceleanu might suggest that the existence of a single non-zero vector with some recurrence property implies some type of hypercyclicity, and indeed:

Theorem 5.4. *Let X be a Fréchet sequence space (over $\mathbb{N}$ or $\mathbb{Z}$) in which the sequence $(e_n)_n$ is an unconditional basis. Suppose that the (unilateral or bilateral) weighted backward shift B_w is an operator on X. If B_w admits a non-zero uniformly recurrent vector, then it is chaotic and therefore frequently hypercyclic.*

For unilateral shifts this result is due to Galán et al. [GMJPO15, Theorem 2, Corollary 1 and the following Remark]; indeed, their statement is more restrictive but they actually prove the full result, which was also obtained in He et al. [HHY18, Corollary 4.2]. A special case is due to Grivaux et al. [GMM21, Remark 5.21]. For bilateral shifts the result was obtained by the first two authors of this paper [BGE].

The previous theorem can be considerably improved if the underlying space is ℓ^p. The following is a consequence of [BGE], using an idea of Bès et al. [BMPP16]. Note that it does not hold on $c_0(\mathbb{N})$ by [BR15, Theorem 5].

Theorem 5.5. *Let B_w be a weighted backward shift on $\ell^p(\mathbb{N})$ or $\ell^p(\mathbb{Z})$, $1 \leq p < \infty$. If B_w admits a non-zero reiteratively recurrent vector, then it is chaotic and therefore frequently hypercyclic.*

As we said above we do not know if, for general Fréchet sequence spaces, the existence of a single non-zero frequently recurrent vector, say, implies that the shift is frequently hypercyclic. For the main result of this section we only need the following weaker implication:

Lemma 5.6. *Let X be a Fréchet sequence space over $\mathbb{N}$ in which $(e_n)_n$ is an unconditional basis. Suppose that the weighted backward shift B_w is an operator on X. If there exists any non-zero vector that is frequently ($\mathcal{U}$-frequently or reiteratively) recurrent, then there is a set $A \subset \mathbb{N}$ of positive lower (upper or upper Banach) density such that*

$$\sum_{n\in A} \frac{1}{\prod_{\nu=1}^{n} w_\nu} e_n \quad \textit{converges in } X.$$

This follows from He et al. [HHY18, Lemma 5]; see also [BGE]. This allows us to prove:

Theorem 5.7. *The recurrence notions introduced in this paper can be strongly distinguished:*

(a) *There is a hypercyclic (mixing) operator without non-zero reiteratively recurrent vectors.*

(b) *There is a reiteratively hypercyclic operator without non-zero $\mathcal{U}$-frequently recurrent vectors.*

(c) *There is a $\mathcal{U}$-frequently hypercyclic operator without non-zero frequently recurrent vectors.*

(d) *There is a frequently hypercyclic operator without non-zero uniformly recurrent vectors.*

In view of Corollary 4.7 we may complete this list by the following assertion; note that, by Theorem 5.5, such an operator cannot be a weighted shift on $\ell^p(\mathbb{N})$ or $\ell^p(\mathbb{Z})$, $1 \le p < \infty$.

(e) *There is a hypercyclic uniformly recurrent operator without non-zero periodic points.*

Proof of Theorem 5.7. (a): By [BMPP16, Theorem 13], the weighted backward shift on $\ell^1(\mathbb{N})$ with weight sequence $w = (\frac{n+1}{n})_n$ is mixing but not reiteratively hypercyclic. By Theorem 5.5 it cannot have a non-zero reiteratively recurrent vector. (b): By [BMPP16, Theorem 7] there exists a reiteratively hypercyclic weighted backward shift B_w on $c_0(\mathbb{N})$ that is not $\mathcal{U}$-frequently hypercyclic (see also [BGE18, Theorem 7.1] for a simplified proof). In fact, the proofs show that the weight even satisfies that there is no set $A \subset \mathbb{N}$ of positive upper density such that

$$\prod_{\nu=1}^{n} w_\nu \to \infty \qquad \text{as } n \to \infty,\, n \in A.$$

Thus, by the previous lemma, B_w does not admit any non-zero $\mathcal{U}$-frequently recurrent vector. (c): This follows exactly as in (b), using [BR15, Theorem 5] or [BGE18, Theorem 7.2] and their proofs. (d): By [BG07, Corollary 5.2] (see also [BGE18, Theorem 7.3]) there exists a frequently hypercyclic weighted backward shift on $c_0(\mathbb{N})$ that is not chaotic. In view of Theorem 5.4, this operator cannot have non-zero uniformly recurrent vectors. □

Corollary 5.8. *The recurrence notions introduced in this paper can be distinguished:*

(a) *There is a recurrent operator without non-zero reiteratively recurrent vectors.*

(b) *There is a reiteratively recurrent operator without non-zero $\mathcal{U}$-frequently recurrent vectors.*

(c) *There is a $\mathcal{U}$-frequently recurrent operator without non-zero frequently recurrent vectors.*

(d) *There is a frequently recurrent operator without non-zero uniformly recurrent vectors.*

(e) *There is a uniformly recurrent operator without non-zero periodic points.*

6 $\mathcal{IP}^*$-recurrence

In the next section we will discuss recurrence properties of further operators. As we will see, a rich supply of eigenvectors to unimodular eigenvalues allows us not only to deduce uniform recurrence for many of these operators, but even a stronger notion that is defined in terms of the so-called IP*-sets. Before turning to these examples we will therefore study $\mathcal{IP}^*$-recurrence in this section.

The starting point is the family $\mathcal{IP}$ of IP-sets. As mentioned in [Fur81, page 52], this family arises naturally when one studies the structure of the sets of integers that can serve as the set of return times for some point in the system. We recall that $A \subset \mathbb{N}_0$ is an *IP-set* if there exists a strictly increasing sequence $(k_n)_{n\in\mathbb{N}}$ of positive integers such that $k_{j_1} + \cdots + k_{j_m} \in A$ whenever $j_1 < \cdots < j_m$ and $m \in \mathbb{N}$. Then a vector $x \in X$ is called *$\mathcal{IP}$-recurrent* for an operator $T \in \mathcal{L}(X)$ if, for any neighbourhood U of x, the return set $N(x, U)$ is an IP-set. However, it follows from [Fur81, Theorem 2.17] that every recurrent vector satisfies this property[I], so that the notions of recurrence and $\mathcal{IP}$-recurrence coincide.

It is more interesting to study the dual family $\mathcal{IP}^*$, that is, the family of all subsets of $\mathbb{N}_0$ that intersect every set in $\mathcal{IP}$ non-trivially. The elements of this family are called *IP*-sets*. A vector $x \in X$ is called *$\mathcal{IP}^*$-recurrent* for $T \in \mathcal{L}(X)$ if, for any neighbourhood U of x, the return set $N(x, U)$ is an IP*-set, see [Fur81, Chapter 9]. The corresponding set of vectors is denoted by $\mathcal{IP}^*\mathrm{Rec}(T)$. If this set is dense in X then the operator T is called *$\mathcal{IP}^*$-recurrent*. It is known that, for every $p \in \mathbb{N}$, the set $p\mathbb{N}_0 = \{kn : n \in \mathbb{N}_0\}$ is an IP*-set[J], and that every IP*-set is syndetic, see [Fur81, Lemma 9.2]. This implies that

$$\mathrm{Per}(T) \subset \mathcal{IP}^*\mathrm{Rec}(T) \subset \mathrm{URec}(T), \tag{1.4}$$

and every $\mathcal{IP}^*$-recurrent operator is uniformly recurrent. The notion of $\mathcal{IP}^*$-recurrence in a linear context has already been studied in [GMJPO15]. It was observed there that, thanks to the classical result [Fur81, Theorem 9.11], the $\mathcal{IP}^*$-recurrent vectors of an operator $T \in \mathcal{L}(X)$ coincide with *product recurrent vectors*, that is, those vectors $x \in X$ such that, for any operator $S \in \mathcal{L}(Y)$ on an F-space Y and for any recurrent vector y for S, the vector (x, y) is recurrent for $T \oplus S$. Note also that the vector x with unbounded orbit constructed in Example 3.3 or the one in [GMJPO15, Example 1] are $\mathcal{IP}^*$-recurrent since, for every neighbourhood U of x, we have that $k\mathbb{N}_0 \subset N(x, U)$ for some $k \in \mathbb{N}$.

An important property of the family $\mathcal{IP}$ is that it is partition regular, that is, if $A_1 \cup A_2 \in \mathcal{IP}$, then either $A_1 \in \mathcal{IP}$ or $A_2 \in \mathcal{IP}$; this implies have that $\mathcal{IP}^*$ is a filter, see [Fur81, Lemma 9.5].

Theorem 6.1. *Let $T \in \mathcal{L}(X)$. Then $\mathcal{IP}^*\mathrm{Rec}(T)$ is a vector subspace of X. In particular, if T is $\mathcal{IP}^*$-recurrent, then T admits a dense linear manifold of $\mathcal{IP}^*$-recurrent vectors.*

Proof. Let $x_1, x_2 \in \mathcal{IP}^*\mathrm{Rec}(T)$, and let λ_1, λ_2 be scalars. Given an arbitrary neighbourhood U of $x := \lambda_1 x_1 + \lambda_2 x_2$, we fix neighbourhoods U_j of x_j, $j = 1, 2$, such that $\lambda_1 U_1 + \lambda_2 U_2 \subset U$. Therefore we conclude, by the filter property of $\mathcal{IP}^*$, that

$$N(x, U) \supset N(x_1, U_1) \cap N(x_2, U_2) \in \mathcal{IP}^*,$$

the second part being a consequence of the definition of $\mathcal{IP}^*$-recurrence. □

[I]See Proposition 4.2 in Section 4 of the Appendix.

[J]See Lemma 4.3 in Section 4 of the Appendix.

We next obtain that, for power-bounded operators, uniform recurrence and $\mathcal{IP}^*$-recurrence coincide. To do this we need to recall the concept of *proximality*: given a dynamical system (X, T), where (X, d) is a metric space, we say that two points $x, y \in X$ are *proximal* for T if there exists an increasing sequence $(n_k)_{k\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that $d(T^{n_k}x, T^{n_k}y) \to 0$ as $k \to \infty$.

Theorem 6.2. *Let $T \in \mathcal{L}(X)$. If T is power-bounded, then*

$$\mathcal{IP}^*\mathrm{Rec}(T) = \mathrm{URec}(T).$$

Proof. We just need to show that every uniformly recurrent vector is $\mathcal{IP}^*$-recurrent. Thus let $x \in X$ be uniformly recurrent for T. By Theorem 3.4 the closure K of its orbit is a compact set, and it is T-invariant. Applying [Fur81, Theorem 9.11] to the dynamical system $(K, T|_K)$ we see that x is $\mathcal{IP}^*$-recurrent (for T) provided that no point $y \neq x$ in K is proximal to x.

Thus, suppose that there is some $y \in K$ with $y \neq x$ such that x and y are proximal for T. Let $(n_k)_{k\in\mathbb{N}}$ be an increasing sequence of positive integers such that $d(T^{n_k}x, T^{n_k}y) \to 0$ as $k \to \infty$. Since we may assume that the metric d on X is translation-invariant, we get that

$$T^{n_k}(x - y) \to 0 \quad \text{as } k \to \infty.$$

By equicontinuity of $(T^n)_{n\in\mathbb{N}_0}$ we then have that $T^n(x - y) \to 0$ as $n \to \infty$.

Now, since x is recurrent there is an increasing sequence $(m_k)_{k\in\mathbb{N}}$ of positive integers so that $T^{m_k}x \to x$ as $k \to \infty$. Thus, $T^{m_k}T^n x \to T^n x$ as $k \to \infty$, for each $n \in \mathbb{N}_0$. Again by equicontinuity of $(T^n)_{n\in\mathbb{N}_0}$ and the density of the orbit of x in K, we get that $T^{m_k}z \to z$ as $k \to \infty$, for every $z \in K$. In particular,

$$0 = \lim_{k\to\infty} T^{m_k}(x - y) = x - y \neq 0,$$

which is the desired contradiction. □

The above result suggests the following problem.

Question 6.3. [K] Is there an operator that is uniformly recurrent but not $\mathcal{IP}^*$-recurrent?

Let us comment on this problem.

Remark 6.4. (a) The León-Müller theorem holds for $\mathcal{IP}^*$-recurrence, that is, for any operator $T \in \mathcal{L}(X)$ and any scalar λ with $|\lambda| = 1$ we have that $\mathcal{IP}^*\mathrm{Rec}(\lambda T) = \mathcal{IP}^*\mathrm{Rec}(T)$. This is an easy consequence of the fact that a vector is $\mathcal{IP}^*$-recurrent if and only if it is product recurrent, and the fact that the León-Müller theorem holds for recurrence. It thus follows exactly as in the proof of Corollary 4.7 that there exists a hypercyclic operator on a Hilbert space that has a dense set of $\mathcal{IP}^*$-recurrent vectors but no non-zero periodic points. In particular, the first inclusion in (1.4) can be strict in a very strong sense, but we do not know the status of the second inclusion. Incidentally, $\mathcal{IP}^*$ does not have the CuSP, so that one cannot deduce the "Ansari-León-Müller" properties for $\mathcal{IP}^*$-recurrence as in the proofs of Theorems 4.2 and 4.4: the set $A = \mathbb{N}_0$ can be partitioned into the even and odd numbers, the set of even numbers is in $\mathcal{IP}^*$, but the set of odd numbers is not (see [Fur81, page 178])[L].

(b) For all of the operators considered in this paper, whenever we could show uniform recurrence, we even obtained $\mathcal{IP}^*$-recurrence. This will be a common pattern for the operators considered in the next section. For weighted backward shift operators see Theorem 5.4.

[K] This question has recently been solved in the negative for operator acting on Hilbert spaces; see Chapter 2.
[L] See Proposition 4.7 of Chapter 3 for the $\mathcal{IP}^*$ and Δ^*-recurrence "Ansari-León-Müller" properties.

7 Recurrence and unimodular eigenvectors

As promised we now study recurrence properties of various classes of operators. We limit ourselves to operators studied by Costakis et al. [CMP14]; our results strengthen several of their results. In order to keep the paper short we refer to that paper for the definition of the operators and the spaces involved. It turns out that for practically all of these operators their unimodular eigenvectors play a crucial role. In the sequel we will only consider Fréchet spaces over the complex field, and we recall that $\mathbb{T}$ denotes the unit circle in $\mathbb{C}$.

The following result is the key point in this section.

Lemma 7.1. *Let $\lambda_1, \ldots, \lambda_k \in \mathbb{T}$, $k \geq 1$. Then, for any $\varepsilon > 0$,*

$$\left\{ n \geq 0 : \sup_{j=1,\ldots,k} |\lambda_j^n - 1| < \varepsilon \right\} \in \mathcal{IP}^*;$$

in particular, it is a syndetic set.

Proof. Apply [Fur81, Proposition 9.8 with Lemma 9.2] to the Kronecker system consisting of the compact group $\mathbb{T}^k$ and the (left) multiplication $(z_1, \ldots, z_k) \mapsto (\lambda_1 z_1, \ldots, \lambda_k z_k)$. □

We mention that if one is only interested in proving that the sets are syndetic, then one finds a nice proof in [MT10, Lemma 3.1] based on Kronecker's theorem.

For $T \in \mathcal{L}(X)$ we denote the *set of unimodular eigenvectors* for T by

$$\mathcal{E}(T) = \{x \in X \setminus \{0\} : Tx = \lambda x \text{ for some } \lambda \in \mathbb{T}\}.$$

The following was obtained for uniform recurrence in [GMM21, Fact 5.6]:

Corollary 7.2. *Let $T \in \mathcal{L}(X)$. Then*

$$\operatorname{span}(\mathcal{E}(T)) \subset \mathcal{IP}^*\mathrm{Rec}(T)^{\mathrm{M}}.$$

Thus, if $\operatorname{span}(\mathcal{E}(T))$ is dense in X then T is $\mathcal{IP}^*$-recurrent, and hence uniformly recurrent.

Proof. Let $x \in \operatorname{span}(\mathcal{E}(T))$, and let W be a 0-neighbourhood. We can write $x = \sum_{j=1}^k a_j x_j$ with $a_j \in \mathbb{C}$ and $x_j \in X$ such that $Tx_j = \lambda_j x_j$, $\lambda_j \in \mathbb{T}$, for $j = 1, \ldots, k$. Then there is some $\varepsilon > 0$ such that $\sum_{j=1}^k \eta_j a_j x_j \in W$ whenever $|\eta_j| < \varepsilon$ for $j = 1, \ldots, k$. Now, by the lemma, there is a set $A \in \mathcal{IP}^*$ such that $|\lambda_j^n - 1| < \varepsilon$ for all $n \in A$ and $j = 1, \ldots, k$. Thus we have for any $n \in A$ that

$$T^n x - x = \sum_{j=1}^k (\lambda_j^n - 1) a_j x_j \in W,$$

which shows the claim. □

The corollary reminds the following well-known fact: the set of periodic points fulfills that

$$\operatorname{Per}(T) = \operatorname{span}\{x \in X : Tx = \lambda x \text{ for some root of unity } \lambda\},$$

see [GEP11, Proposition 2.33]. We will see in Remark 7.5 below that we do not necessarily have equality in Corollary 7.2.

[M] See Chapter 2, Section 4, Proposition 4.1 for a slight improvement in terms of the Δ^* family.

We start by looking at operators on finite-dimensional spaces.

Theorem 7.3. *Let $n \geq 1$. For a matrix $T : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, the following assertions are equivalent:*

(i) *T is recurrent;*

(ii) *T is uniformly recurrent;*

(iii) *T is $\mathcal{IP}^*$-recurrent;*

(iv) *T is similar to a diagonal matrix with unimodular diagonal entries.*

In that case, every vector in $\mathbb{C}^n$ is $\mathcal{IP}^*$-recurrent for T.

Proof. Costakis et al. [CMP14, Theorem 4.1] have shown that (i) and (iv) are equivalent. Thus it suffices to show that (iv) implies (iii): let S be an invertible matrix such that $S^{-1}TS$ is a diagonal matrix with unimodular diagonal entries. Then we have that $Se_k \in \mathcal{E}(T)$ for each $k = 1, \dots, n$. Thus $\operatorname{span}(\mathcal{E}(T)) = \mathbb{C}^n$, and we conclude with Corollary 7.2. □

This result suggests to consider general multiplication operators.

Theorem 7.4. *Let X be a complex Fréchet sequence space over $\mathbb{N}$ for which the vector space* $\operatorname{span}(\{e_n : n \in \mathbb{N}\})$ *is a dense subset. Let $(\lambda_n)_n$ be a sequence in $\mathbb{C}$, and let M_λ be the multiplication operator*

$$M_\lambda(x_n)_n = (\lambda_n x_n)_n,$$

which we suppose to be an operator on X.

(a) *The following assertions are equivalent:*

  (i) *M_λ is recurrent;*

  (ii) *M_λ is uniformly recurrent;*

  (iii) *M_λ is $\mathcal{IP}^*$-recurrent;*

  (iv) *$\lambda_n \in \mathbb{T}$ for all $n \geq 1$.*

(b) *If M_λ is a power-bounded operator and one of the conditions in* (a) *holds, then every vector in X is $\mathcal{IP}^*$-recurrent for M_λ.*

Proof. (a): It obviously suffices to show that (iv) implies (iii). For this, we need only observe that every sequence e_n, $n \geq 1$, belongs to $\mathcal{E}(M_\lambda)$ since $|\lambda_n| = 1$, so that $\operatorname{span}(\mathcal{E}(M_\lambda))$ is dense in the space X by hypothesis.

(b): This is a direct consequence of Theorems 3.1 and 6.2. □

We note that if $(e_n)_n$ is an unconditional basis of X and if and one of the conditions in (a) holds then M_λ is power-bounded, so that the conclusion of (b) holds in this case.

This result has an interesting consequence:

Remark 7.5. Let us consider a multiplication operator M_λ on $\ell^2(\mathbb{N})$, say, where the $\lambda_n \in \mathbb{T}$, $n \geq 1$, are pairwise distinct. Then the non-zero multiples of the e_n, $n \geq 1$, are the only unimodular eigenvectors, so that $\operatorname{span}(\mathcal{E}(T))$ contains exactly the finite sequences. On the other hand, by the previous result, every vector in $\ell^2(\mathbb{N})$ is $\mathcal{IP}^*$-recurrent. This shows that the inclusion in Corollary 7.2 may be strict.

We turn, more generally, to multiplication operators on spaces of measurable functions. If $(\Omega, \mathscr{A}, \mu)$ is a measure space, then we call a function $\phi : \Omega \longrightarrow \mathbb{C}$ essentially countably valued in $D \subset \mathbb{C}$ if there is a countable subset $C \subset D$ such that $\phi(t) \in C$ for μ-a.e. $t \in \Omega$.

Theorem 7.6. *Let $(\Omega, \mathscr{A}, \mu)$ be a measure space, ϕ a bounded measurable function on Ω, and let M_ϕ be the multiplication operator $M_\phi f = \phi f$ on $L^p(\Omega)$, $1 \leq p < \infty$.*

(a) ([CMP14]) *If M_ϕ is recurrent then $\phi(t) \in \mathbb{T}$ for μ-a.e. $t \in \Omega$.*

(b) *If ϕ is essentially countably valued in $\mathbb{T}$ then M_ϕ is $\mathcal{IP}^*$-recurrent; in fact, every vector in $L^p(\Omega)$ is $\mathcal{IP}^*$-recurrent.*

Proof. (a): was shown in the proof of [CMP14, Theorem 7.6]. (b): let $\phi(t) \in \{\lambda_1, \lambda_2, \ldots\} \subset \mathbb{T}$ for almost every $t \in \Omega$, and let $E_k = \{t \in \Omega : \phi(t) = \lambda_k\}$, $k \geq 1$. Then $\mu(\Omega \setminus \bigcup_{k\geq 1} E_k) = 0$. Now let $f \in L^p(\Omega)$, $f \neq 0$, and $\varepsilon > 0$. There exists some $N \geq 1$ such that

$$\int_{\bigcup_{k>N} E_k} |f|^p \, d\mu < \frac{\varepsilon}{2^{p+1}}.$$

By Lemma 7.1 there exists a set $A \in \mathcal{IP}^*$ such that $|\lambda_k^n - 1|^p < \frac{\varepsilon}{2\|f\|^p}$ for any $n \in A$ and $k = 1, \ldots, N$. Therefore we have for every $n \in A$ that

$$\begin{aligned} \|M_\phi^n f - f\|^p &= \sum_{k=1}^{N} \int_{E_k} |\phi^n - 1|^p \cdot |f|^p \, d\mu + \int_{\bigcup_{k>N} E_k} |\phi^n - 1|^p \cdot |f|^p \, d\mu \\ &\leq \frac{\varepsilon}{2\|f\|^p} \sum_{k=1}^{N} \int_{E_k} |f|^p \, d\mu + 2^p \int_{\bigcup_{k>N} E_k} |f|^p \, d\mu < \varepsilon. \end{aligned}$$

Thus, f is $\mathcal{IP}^*$-recurrent. □

We note that in order to only get the $\mathcal{IP}^*$-recurrence of the operator in (b) we could have used Corollary 7.2. Indeed, any indicator function $\mathbb{1}_E$ lies in $\mathcal{E}(T)$ when E is a measurable subset of some E_k, $k \geq 1$. Then clearly $\operatorname{span}(\mathcal{E}(T))$ is dense in $L^p(\Omega)$.

Example 7.7. In [CMP14, Example 7.9] it is considered the case when $\Omega = [0,1]$ with the Lebesgue measure and $\phi : [0,1] \longrightarrow \mathbb{T}$, $\phi(t) = e^{2\pi i f(t)}$, where $f : [0,1] \longrightarrow [0,1]$ is the famous Cantor-Lebesgue function. Then not only ϕ is essentially countably valued in $\mathbb{T}$, but almost all of its values are even roots of unity. Hence the operator M_ϕ has a dense set of periodic points by the previous argument.

We next turn to composition operators. We first look at operators on $H(\mathbb{C})$ and $H(\mathbb{D})$, the Fréchet spaces of entire functions and of holomorphic functions on $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, respectively, both endowed with the topology of uniform convergence on compact sets.

Theorem 7.8. *Let $\phi : \mathbb{C} \longrightarrow \mathbb{C}$ be a holomorphic function, and let C_ϕ be the composition operator on $H(\mathbb{C})$ given by $C_\phi f = f \circ \phi$. Then the following assertions are equivalent:*

(i) *C_ϕ is recurrent;*

(ii) *C_ϕ is uniformly recurrent;*

(iii) *C_ϕ is $\mathcal{IP}^*$-recurrent;*

(iv) *$\phi(z) = az + b$, $z \in \mathbb{C}$, with $a \in \mathbb{T}$ and $b \in \mathbb{C}$.*

Moreover, every vector is $\mathcal{IP}^*$-recurrent for C_ϕ if and only if $\phi(z) = az + b$, $z \in \mathbb{C}$, with $a = 1$ and $b = 0$, or $a \in \mathbb{T} \setminus \{1\}$ and $b \in \mathbb{C}$.

Proof. By [CMP14, Theorem 6.4] it suffices to show that (iv) implies (iii), and that the second claim holds. Thus, let $\phi(z) = az + b$, $z \in \mathbb{C}$, with $a \in \mathbb{T}$ and $b \in \mathbb{C}$. If $a = 1$ and $b \neq 0$, then C_ϕ is well known to be chaotic, see [GEP11, Example 2.35]; in that case, $\mathcal{IP}^*\mathrm{Rec}(C_\phi)$ is dense in but not all of $H(\mathbb{C})$. If $a = 1$ and $b = 0$ then clearly $\mathcal{IP}^*\mathrm{Rec}(C_\phi) = H(\mathbb{C})$. Finally let $a \in \mathbb{T} \setminus \{1\}$ and $b \in \mathbb{C}$. Let $f \in H(\mathbb{C})$, and fix $R > 0$ and $\varepsilon > 0$. It was shown in the proof of [CMP14, Theorem 6.4] that there is an $\eta > 0$ such that if $|a^n - 1| < \eta$, $n \geq 0$, then

$$\sup_{|z| \leq R} |C_\phi^n f(z) - f(z)| < \varepsilon.$$

Now it follows from Lemma 7.1 that $\{n \geq 0 : |a^n - 1| < \eta\} \in \mathcal{IP}^*$. This implies that f is $\mathcal{IP}^*$-recurrent. Thus we have again that $\mathcal{IP}^*\mathrm{Rec}(C_\phi) = H(\mathbb{C})$. $\square$

It is instructive to note that if, once more, one is only interested in obtaining $\mathcal{IP}^*$-recurrence of the operator then this can easily be done with Corollary 7.2. This is trivial if $a = 1$ and $b = 0$, and well known if $a = 1$ and $b \neq 0$, see [GEP11, Example 2.35]. Finally, if $a \in \mathbb{T} \setminus \{1\}$, $b \in \mathbb{C}$, then the functions $f_n(z) = (z + \frac{b}{a-1})^n$, $z \in \mathbb{C}$, $n \geq 0$, are eigenvectors for C_ϕ with eigenvalue $a^n \in \mathbb{T}$, and they span the set of polynomials, hence a dense subspace of $H(\mathbb{C})$.

For the unit disk we have the following.

Theorem 7.9. *Let $\phi : \mathbb{D} \longrightarrow \mathbb{D}$ be a holomorphic function, and let C_ϕ be the composition operator on $H(\mathbb{D})$ given by $C_\phi f = f \circ \phi$. Then the following assertions are equivalent:*

(i) *C_ϕ is recurrent;*

(ii) *C_ϕ is uniformly recurrent;*

(iii) *C_ϕ is $\mathcal{IP}^*$-recurrent;*

(iv) *either ϕ is univalent and has no fixed point, or ϕ is an elliptic automorphism.*

Moreover, every vector is $\mathcal{IP}^*$-recurrent for C_ϕ if and only if ϕ is an elliptic automorphism.

Proof. By [CMP14, Theorem 6.9] it suffices to show that (iv) implies (iii), and that the second claim holds. If ϕ is univalent and has no fixed point then C_ϕ is chaotic by [Sha01, Section 4]; hence $\mathcal{IP}^*\mathrm{Rec}(C_\phi)$ is dense in but not all of $H(\mathbb{D})$. If ϕ is an elliptic automorphism then, by the proof of [CMP14, Theorem 6.9], C_ϕ is conjugate to C_{ϕ_λ} for some $\lambda \in \mathbb{T}$, where $\phi_\lambda(z) = \lambda z$, $z \in \mathbb{D}$. It then follows easily from Lemma 7.1 that $\mathcal{IP}^*\mathrm{Rec}(C_\phi) = H(\mathbb{D})$. $\square$

We end the section by considering composition operators on the Hardy space $H^2(\mathbb{D})$.

Theorem 7.10. *Let $\phi : \mathbb{D} \longrightarrow \mathbb{D}$ be a linear fractional map $\phi(z) = \frac{az+b}{cz+d}$, $z \in \mathbb{D}$, with $ad-bc \neq 0$. For the composition operator C_ϕ on $H^2(\mathbb{D})$ given by $C_\phi f = f \circ \phi$, the following are equivalent:*

(i) *C_ϕ is recurrent;*

(ii) *C_ϕ is uniformly recurrent;*

(iii) *C_ϕ is $\mathcal{IP}^*$-recurrent;*

(iv) *ϕ is either hyperbolic with no fixed point in $\mathbb{D}$, or a parabolic automorphism, or an elliptic automorphism.*

Moreover, every vector is $\mathcal{IP}^*$-recurrent for C_ϕ if and only if ϕ is an elliptic automorphism.

Proof. By [CMP14, Theorem 6.12] it suffices to show that (iv) implies (iii), and that the second claim holds. If ϕ is hyperbolic with no fixed point in $\mathbb{D}$, or a parabolic automorphism, then it is chaotic by [Hos03, Corollary 7], hence $\mathcal{IP}^*\mathrm{Rec}(C_\phi)$ is dense in but not all of $H^2(\mathbb{D})$. If ϕ is an elliptic automorphism then we conclude as in the previous proof. □

Recurrence properties of further operators can be deduced from [CMP14, Sections 6 and 7].

8 $\mathcal{F}$-recurrence

In this paper we have concentrated on the most important types of recurrence in order to highlight their differing behaviour. In this section we will briefly study the general notion of *$\mathcal{F}$-recurrence*, and we consider operators on arbitrary topological vector spaces. The concept was introduced by Furstenberg [Fur81, Chapter 9] in a non-linear context.

Recall that a non-empty family $\mathcal{F}$ of subsets of $\mathbb{N}_0$ is called a *Furstenberg family* if $A \in \mathcal{F}$ and $B \supset A$ implies that $B \in \mathcal{F}$; we will assume throughout that $\mathcal{F}$ does not contain the empty set. A Furstenberg family is called *right-invariant* (respectively *left-invariant*) if $A \in \mathcal{F}$ and $n \geq 0$ implies that $A + n := \{k + n : k \in A\} \in \mathcal{F}$ (respectively $(A - n) \cap \mathbb{N}_0 \in \mathcal{F}$).

Definition 8.1. Let X be a topological vector space, $T \in \mathcal{L}(X)$, and let $\mathcal{F}$ be a Furstenberg family. A vector $x \in X$ is called *$\mathcal{F}$-recurrent* if, for any neighbourhood U of x, the return set $N(x, U)$ belongs to $\mathcal{F}$. The set of $\mathcal{F}$-recurrent vectors is denoted by $\mathcal{F}\mathrm{Rec}(T)$. If this set is dense in X then the operator is called *$\mathcal{F}$-recurrent*.

Remark 8.2. Costakis et al. [CMP14] have defined T to be recurrent if, for any non-empty open subset U of X, the set

$$N(U, U) = \{n \geq 0 : T^n(U) \cap U \neq \varnothing\} \text{ is non-empty,}$$

which amounts to demanding that it be in the family of infinite sets. This is equivalent to the definition used in this paper, by [CMP14, Proposition 2.1 with Remark 2.2], provided that the underlying space X is completely metrizable (as it happens for F-spaces).

More generally it might be interesting to study the operators T with the following property: for any non-empty open subset U of X,

$$N(U,U) = \{n \geq 0 : T^n(U) \cap U \neq \varnothing\} \in \mathcal{F}.$$

Motivated by [CP12] one might call these operators *topologically $\mathcal{F}$-recurrent*. This notion is naturally linked to the concept of *$\mathcal{F}$-transitive operators* as introduced by Bès et al. [BMPP19].

We have preferred the pointwise definition adopted in this paper in order to be close to the corresponding notion of $\mathcal{F}$-hypercyclicity. Recall that an operator $T \in \mathcal{L}(X)$ is *$\mathcal{F}$-hypercyclic* if there is some $x \in X$ such that, for any non-empty open set U in X, $N(x,U) \in \mathcal{F}$, see [BMPP16]. The vector x is then called *$\mathcal{F}$-hypercyclic*.

We have the following generalizations of results in the first part of the paper. The proofs follow as in the special cases, see Theorems 2.1, 2.5, 2.12, 3.1, 4.2 and 4.4, together with Remarks 4.3 and 4.5.

Theorem 8.3. *Let X be a topological vector space, $T \in \mathcal{L}(X)$, and let $\mathcal{F}$ be a right-invariant Furstenberg family. Then a vector is $\mathcal{F}$-hypercyclic if and only if it is both hypercyclic and also $\mathcal{F}$-recurrent at the same time. In particular, T is $\mathcal{F}$-hypercyclic if and only it admits a hypercyclic and $\mathcal{F}$-recurrent vector.*

For the following results we need the concept of an *(u.f.i.) upper Furstenberg family*[N]; we refer to [BGE18] and Theorem 3.1 there.

Theorem 8.4. *Let X be an F-space, $T \in \mathcal{L}(X)$, and $\mathcal{F}$ a right-invariant upper Furstenberg family. Then the following assertions are equivalent:*

(i) *T is $\mathcal{F}$-hypercyclic;*

(ii) *T is hypercyclic, and $\mathcal{F}\mathrm{Rec}(T)$ is a residual set;*

(iii) *T is hypercyclic, and $\mathcal{F}\mathrm{Rec}(T)$ is of second category;*

(iv) *T admits a hypercyclic $\mathcal{F}$-recurrent vector.*

In that case the set of hypercyclic and $\mathcal{F}$-recurrent vectors is residual.

Theorem 8.5. *Let X be an F-space, $T \in \mathcal{L}(X)$, and let $\mathcal{F}$ be a u.f.i. upper Furstenberg family. Suppose that there is a dense set of vectors $x \in X$ such that $T^n x \to 0$ as $n \to \infty$. Then T is $\mathcal{F}$-hypercyclic if and only if it is $\mathcal{F}$-recurrent.*

We can also generalize the paper of *power-boundedness* for general Furstenberg families:

Theorem 8.6. *Let X be a topological vector space, $T \in \mathcal{L}(X)$, and let $\mathcal{F}$ be any Furstenberg family. If T is power-bounded, then the set $\mathcal{F}\mathrm{Rec}(T)$ is closed.*

The CuSP property for a family of subsets of $\mathbb{N}_0$ was introduced in Section 4.

[N] See Example 3.2 in Section 3 of the Appendix for more on u.f.i. upper Furstenberg families.

Theorem 8.7. *Let X be a topological vector space, $T \in \mathcal{L}(X)$, and $\mathcal{F}$ a Furstenberg family with the CuSP property.*

(a) *Let $p \geq 1$. Assume that, for any $A \subset \mathbb{N}_0$, $A \in \mathcal{F}$ if and only if $pA \in \mathcal{F}$. Then T and T^p have the same $\mathcal{F}$-recurrent vectors. In particular, if T is $\mathcal{F}$-recurrent then so is T^p.*

(b) *Let λ be a scalar with $|\lambda| = 1$. Then the operators T and λT have the same $\mathcal{F}$-recurrent vectors. In particular, if T is $\mathcal{F}$-recurrent then so is λT.*

Theorems 8.3 and 8.7 have an interesting application to $\mathcal{F}$-hypercyclicity.

Theorem 8.8. *Let X be a topological vector space, $T \in \mathcal{L}(X)$, and let $\mathcal{F}$ be a right-invariant Furstenberg family with the CuSP property.*

(a) *Let $p \geq 1$. Assume that, for any $A \subset \mathbb{N}_0$, $A \in \mathcal{F}$ if and only if $pA \in \mathcal{F}$. Then T and T^p have the same $\mathcal{F}$-hypercyclic vectors. In particular, if T is $\mathcal{F}$-hypercyclic then so is T^p.*

(b) *Let λ be a scalar with $|\lambda| = 1$. Then the operators T and λT have the same $\mathcal{F}$-hypercyclic vectors. In particular, if T is $\mathcal{F}$-hypercyclic then so is λT.*

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Chapter 2

Recurrence properties for linear dynamical systems: An approach via invariant measures

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Adaptation: The notation has been slightly modified to use similar symbols in all chapters.

Abstract

We study different pointwise recurrence notions for linear dynamical systems from the Ergodic Theory point of view. We show that from any reiteratively recurrent vector x_0, for an adjoint operator T on a separable dual Banach space X, one can construct a T-invariant probability measure which contains x_0 in its support. This allows us to establish some equivalences, for these operators, between some strong pointwise recurrence notions which in general are completely distinguished. In particular, we show that (in our framework) reiterative recurrence coincides with frequent recurrence; for complex Hilbert spaces uniform recurrence coincides with the property of having a spanning family of unimodular eigenvectors; and the same happens for power-bounded operators on complex reflexive Banach spaces. These (surprising) properties are easily generalized to product and inverse dynamical systems, which implies some relations with the respective hypercyclicity notions. Finally, we study how typical is an operator with a non-zero reiteratively recurrent vector in the sense of Baire category.

1 Introduction and main results

1.1 General background

This paper focuses on some aspects of the interplay between the theories of Topological and Measurable Dynamics for *linear dynamical systems*, our main aim being to investigate the links in this context between various notions of *recurrence*.

A (*real* or *complex*) *linear dynamical system* (X, T) is given by the action of a bounded linear operator T on a (real or complex) separable infinite-dimensional Banach space X, and we will denote by $\mathcal{L}(X)$ the *set of bounded linear operators* acting on such a space X. A linear dynamical system is a particular case of a *Polish dynamical system* (i.e. a system given by the action of a continuous map on a separable completely metrizable space), and some of the results obtained in the paper hold in this more general context. Given a dynamical system $T : X \longrightarrow X$ and a point $x \in X$ we will denote by

$$\operatorname{Orb}(x, T) := \{T^n x : n \in \mathbb{N}_0\},$$

the *T-orbit of* x, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Examples of topological properties, for linear dynamical systems, which will be of interest to us in this work are:

(a) *recurrence*: the operator T is said to be *recurrent* if the set

$$\operatorname{Rec}(T) := \left\{x \in X : x \in \overline{\operatorname{Orb}(Tx, T)}\right\},$$

is dense in X, where each vector $x \in \operatorname{Rec}(T)$ is called a *recurrent vector* for T. By the (not too much known) Costakis-Manoussos-Parissis theorem (see [CMP14, Proposition 2.1]), this notion is equivalent to that of *topological recurrence*, i.e. for each non-empty open subset U of X one can find $n \in \mathbb{N}$ such that $T^n(U) \cap U \neq \varnothing$; and in this case, the set $\operatorname{Rec}(T)$ of recurrent vectors for T is a dense G_δ subset of X;

(b) *hypercyclicity*: we say that the operator T is *hypercyclic* if there exists a vector $x \in X$, called a *hypercyclic vector* for T, whose orbit $\operatorname{Orb}(x, T)$ is dense in X. By the Birkhoff transitivity theorem (see [GEP11, Theorem 1.16]), this notion equals *topological transitivity*, i.e. for each pair U, V of non-empty open subsets of X one can find $n \in \mathbb{N}_0$ such that $T^n(U) \cap V \neq \varnothing$; and in this case the set $\operatorname{HC}(T)$, of hypercyclic vectors for T, is a dense G_δ subset of X.

If given a point $x \in X$ and a set $U \subset X$ we denote the *return set from x to U* as

$$N_T(x, U) := \{n \in \mathbb{N}_0 : T^n x \in U\},$$

which will be denoted by $N(x, U)$ if no confusion with the map arises, we can reformulate the above notions in the following terms: a vector $x \in X$ is recurrent if and only if $N(x, U)$ is an infinite set for every neighbourhood U of x; and a vector $x \in X$ is hypercyclic if and only if $N(x, U)$ is an infinite set for every non-empty open subset U of X. Historically, hypercyclicity and its generalizations have been the most studied notions in Linear Dynamics while the study of the linear dynamical recurrence-kind properties (in a systematic way) started recently in 2014 with the work [CMP14], in spite of the great non-linear dynamical knowledge already existing in this area (see for instance [Fur81]).

Direct relations between the previous topological properties and Ergodic Theory arise when we are able to consider a *probability* (or a *positive finite*) *Borel measure* μ on the underlying space X (i.e. defined on the *σ-algebra of Borel sets* $\mathscr{B}(X)$ of X), which will sometimes be required to have *full support* (i.e. $\mu(U) > 0$ for every non-empty open subset U of X). We will only consider *Borel measures* in this work, and the word "Borel" will sometimes be omitted. If such a measure μ exists, we can study the dynamical system $(X, \mathscr{B}(X), \mu, T)$ from the point of view of Ergodic Theory and relevant properties are:

(a) *invariance*: the operator T is said to be *μ-invariant*, or equivalently, the measure μ is called *T-invariant*, if for each $A \in \mathscr{B}(X)$ the equality $\mu(T^{-1}(A)) = \mu(A)$ holds. By the Poincaré recurrence theorem (see [Wal82, Theorem 1.4]), this notion implies that for every $A \in \mathscr{B}(X)$ with $\mu(A) > 0$ there is $n \in \mathbb{N}$ such that $T^n(A) \cap A \neq \varnothing$. The *Dirac mass* δ_0 at 0 is always an invariant measure for any operator T, and we will say that a T-invariant probability measure μ is *non-trivial* if it is different from δ_0.

(b) *ergodicity*: the operator T is said to be *ergodic* with respect to μ, provided that the measure μ is T-invariant, and for each $A \in \mathscr{B}(X)$ with $T^{-1}(A) = A$ we have that $\mu(A) \in \{0, 1\}$. It is well known that the last statement is equivalent to the fact that, for each pair of sets $A, B \in \mathscr{B}(X)$ with $\mu(A), \mu(B) > 0$ there is $n \in \mathbb{N}_0$ such that $\mu\big(T^{-n}(A) \cap B\big) > 0$ (see [Wal82, Theorem 1.5]).

When T is ergodic with respect to a measure with full support, it follows from the Birkhoff pointwise ergodic theorem that T is not only hypercyclic, but even *frequently hypercyclic*: there exists a vector $x \in X$ such that for each non-empty open subset U of X the return set $N(x, U)$ has *positive lower density*; in other words:

$$\underline{\operatorname{dens}}(N(x, U)) = \liminf_{N \to \infty} \frac{\#(N(x, U) \cap [0, N])}{N + 1} > 0.$$

Such a vector x is said to be a *frequently hypercyclic vector* for T, and the set of all frequently hypercyclic vectors is denoted by $\operatorname{FHC}(T)$. See [BM09, Corollary 5.5] for the details of this argument, and for more on frequent hypercyclicity.

When T is only supposed to admit an invariant measure μ, it follows easily from the Poincaré recurrence theorem that μ-almost every $x \in X$ is recurrent for T (see [Fur81, Theorem 3.3]). Our main line of thought in this work will be to connect various (stronger) notions of recurrence via invariant measures, proceeding essentially in two steps:

– if T admits vectors with a certain (rather weak) recurrence property, prove that it admits a non-trivial invariant measure, perhaps with full support (see Theorem 2.3);

– if T admits a non-trivial invariant measure (perhaps with full support), prove that it admits vectors with a certain strong recurrence property (see Lemmas 3.1 and 4.4).

This approach in the context of linear dynamical systems comes from the paper [GM14], which extends to the linear setting some well-known results in the context of *compact dynamical systems* (see [Fur81, Chapter 3 and Lemma 3.17]). The various recurrence notions which we will consider were introduced and studied in the work [BGELMP22], but the initial study of recurrence in Linear Dynamics started in [CMP14]. In the next subsection, we recall the relevant definitions and present the first main result of this paper.

1.2 Furstenberg families: recurrence and hypercyclicity notions

The Banach spaces X considered in this subsection can be either real or complex. Let us first recall the following definitions from [BGELMP22]:

Definition 1.1. Given a non-empty collection of sets $\mathcal{F} \subset \mathscr{P}(\mathbb{N}_0)$ we say that it is a *Furstenberg family* (called *family* for short) if for each $A \in \mathcal{F}$ we have

(i) A is infinite;

(ii) if $A \subset B \subset \mathbb{N}_0$ then $B \in \mathcal{F}$ (i.e. $\mathcal{F}$ is hereditarily upward).

The *dual family* of $\mathcal{F}$ is the collection of sets $\mathcal{F}^* := \{A \subset \mathbb{N}_0 \text{ infinite} : A \cap B \neq \varnothing \text{ for all } B \in \mathcal{F}\}$.

Definition 1.2. Let (X, T) be a linear dynamical system and let $\mathcal{F}$ be a Furstenberg family. A point $x \in X$ is said to be *$\mathcal{F}$-recurrent* (resp. *$\mathcal{F}$-hypercyclic*) if $N(x, U) \in \mathcal{F}$ for every neighbourhood U of x (resp. for every non-empty open subset U of X). We denote by $\mathcal{F}\mathrm{Rec}(T)$ (resp. $\mathcal{F}\mathrm{HC}(T)$) the set of such points and we say that T is *$\mathcal{F}$-recurrent* (resp. *$\mathcal{F}$-hypercyclic*) if $\mathcal{F}\mathrm{Rec}(T)$ is dense in X (resp. if $\mathcal{F}\mathrm{HC}(T) \neq \varnothing$).

The families $\mathcal{F}$ for which there exist $\mathcal{F}$-hypercyclic operators are by far less common than those for which $\mathcal{F}$-recurrence exists since having an orbit distributed around the whole space is much more complicated than having it just around the initial point of the orbit. Furstenberg families associated just to recurrence will be used in the following subsection, but let us now focus on the most known cases for which both notions exist: a point $x \in X$ is said to be

(a) *frequently recurrent* (resp. *frequently hypercyclic*) if $\underline{\mathrm{dens}}(N(x, U)) > 0$ for every single neighbourhood U of x (resp. for every non-empty open subset U of X). We denote by $\mathrm{FRec}(T)$ (resp. $\mathrm{FHC}(T)$) the set of such points and we say that T is *frequently recurrent* (resp. *frequently hypercyclic*) if $\mathrm{FRec}(T)$ is dense in X (resp. if $\mathrm{FHC}(T) \neq \varnothing$);

(b) *$\mathcal{U}$-frequently recurrent* (resp. *$\mathcal{U}$-frequently hypercyclic*) if $\overline{\mathrm{dens}}(N(x, U)) > 0$ for every single neighbourhood U of x (resp. for every non-empty open subset U of X). We denote by $\mathrm{UFRec}(T)$ (resp. $\mathrm{UFHC}(T)$) the set of such points and T is called *$\mathcal{U}$-frequently recurrent* (resp. *$\mathcal{U}$-frequently hypercyclic*) if $\mathrm{UFRec}(T)$ is dense in X (resp. if $\mathrm{UFHC}(T) \neq \varnothing$);

(c) *reiteratively recurrent* (resp. *reiteratively hypercyclic*) if we have $\overline{\mathrm{Bd}}(N(x, U)) > 0$ for every neighbourhood U of x (resp. for every non-empty open subset U of X). We will denote by $\mathrm{RRec}(T)$ (resp. $\mathrm{RHC}(T)$) the set of such points and we say that T is *reiteratively recurrent* (resp. *reiteratively hypercyclic*) if $\mathrm{RRec}(T)$ is dense in X (resp. if $\mathrm{RHC}(T) \neq \varnothing$);

where for any $A \subset \mathbb{N}_0$ its:

(a) *lower density* is $\underline{\mathrm{dens}}(A) := \liminf_{N \to \infty} \dfrac{\#(A \cap [0, N])}{N+1}$;

(b) *upper density* is $\overline{\mathrm{dens}}(A) := \limsup_{N \to \infty} \dfrac{\#(A \cap [0, N])}{N+1}$;

(c) *upper Banach density* is $\overline{\mathrm{Bd}}(A) := \limsup_{N \to \infty} \left(\max_{m \geq 0} \dfrac{\#(A \cap [m, m+N])}{N+1} \right)$.

The introduced notions follow Definition 1.2 applied to the respective families of *positive* (*lower*, *upper* and *upper Banach*) *density sets*, and in fact, the inequalities between the respective densities imply the inclusions $\mathrm{FRec}(T) \subset \mathrm{UFRec}(T) \subset \mathrm{RRec}(T) \subset \mathrm{Rec}(T)$ and also their hypercyclicity version $\mathrm{FHC}(T) \subset \mathrm{UFHC}(T) \subset \mathrm{RHC}(T) \subset \mathrm{HC}(T)$. In particular, frequent, $\mathcal{U}$-frequent and reiterative recurrence are clearly stronger notions than "usual" recurrence as defined in Subsection 1.1, and frequent recurrence is a stronger notion than that of $\mathcal{U}$-frequent recurrence, which is in its turn stronger than reiterative recurrence.

We point out that all these notions are not specific to the linear setting; we will actually use them in the context of Polish dynamical systems in Sections 2, 3, 5 and 6. However, since we are focused on linear dynamical systems, our first main result connects all of them in the framework of *adjoint operators on separable dual Banach spaces*:

Theorem 1.3. *Let $T : X \longrightarrow X$ be an adjoint operator on a (real or complex) separable dual Banach space X. Then we have the equality*

$$\overline{\mathrm{FRec}(T)} = \overline{\mathrm{RRec}(T)}.$$

Moreover:

(a) *The following statements are equivalent:*

  (i) $\mathrm{FRec}(T) \setminus \{0\} \neq \varnothing$*;*

  (ii) $\mathrm{UFRec}(T) \setminus \{0\} \neq \varnothing$*;*

  (iii) $\mathrm{RRec}(T) \setminus \{0\} \neq \varnothing$*;*

  (iv) *T admits a non-trivial invariant probability measure.*

(b) *The following statements are equivalent:*

  (i) *T is frequently recurrent;*

  (ii) *T is $\mathcal{U}$-frequently recurrent;*

  (iii) *T is reiteratively recurrent;*

  (iv) *T admits an invariant probability measure with full support.*

In particular, these results hold whenever T is an operator on a (real or complex) separable reflexive Banach space X.

The above theorem is in spirit similar to [GM14, Theorem 1.3], where it is proved that every ($\mathcal{U}$-)frequently hypercyclic operator on a separable reflexive space admits an invariant measure with full support. It is observed in [GM14, Remark 3.5] that the arguments extend to every adjoint operator acting on a separable dual Banach space. In this same setting, it is also proved in [GM14, Proposition 2.11] that operators admitting an invariant measure with full support are exactly those which are frequently recurrent. However, the notion of frequent recurrence used in [GM14] is rather different from the one given in Definition 1.2: in [GM14] an operator $T \in \mathcal{L}(X)$ is called frequently recurrent if for every non-empty open subset U of X there exists a vector $x_U \in U$ for which just the positive lower density of the return set $N(x_U, U)$ is required. This notion is (at least formally) weaker than the one used here (see Remark 2.5).

The proof of Theorem 1.3 relies on some modifications of the arguments of [GM14, Section 2], which will be presented in Sections 2 and 3. We mention that it cannot be extended to all operators acting on separable (infinite-dimensional) Banach spaces. Indeed, it is shown in [BGELMP22, Theorem 5.7 and Corollary 5.8] that there even exist reiteratively hypercyclic operators on the space $c_0(\mathbb{N})$ which do not admit any non-zero $\mathcal{U}$-frequently recurrent vector.

1.3 Uniform, $\mathcal{IP}^*$, Δ^*-recurrence and unimodular eigenvectors

In this subsection the underlying Banach spaces X are assumed to be complex. A vector $x \in X$ is a *unimodular eigenvector* for T provided that $x \neq 0$ and $Tx = \lambda x$ for some unimodular complex number $\lambda \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. We will denote the set of such vectors by

$$\mathcal{E}(T) = \{x \in X \setminus \{0\} : Tx = \lambda x \text{ for some } \lambda \in \mathbb{T}\}.$$

Unimodular eigenvectors are clearly frequently recurrent vectors for T, but they enjoy some stronger recurrence properties like uniform, $\mathcal{IP}^*$ and even Δ^*-recurrence (see Definition 1.5). Our general aim in this paper is to investigate some contexts in which these strong forms of recurrence actually imply the existence of unimodular eigenvectors. We will see that it is indeed the case in (at least) the following two situations:

- when T is an operator acting on a complex Hilbert space (see Theorem 1.7 below);
- when T is a power-bounded operator on a complex reflexive Banach space (see Theorem 1.9).

Let us now introduce these stronger recurrence notions which are defined by considering Furstenberg families only relevant for the notion of recurrence, and, contrary to those used in Subsection 1.2, having no hypercyclicity analogue:

Definition 1.4. Let $A \subset \mathbb{N}_0$. We say that A is

(a) a *syndetic* set, if there is $m \in \mathbb{N}$ such that for every $x \in \mathbb{N}_0$ we have $[x, x+m] \cap A \neq \varnothing$. We will denote by $\mathcal{S} := \{A \subset \mathbb{N}_0 : A \text{ is syndetic}\}$ the *Furstenberg family of syndetic sets.*

(b) an *IP-set*, if there is a sequence $(x_n)_{n=1}^{\infty} \in \mathbb{N}_0^{\mathbb{N}}$ such that $\{\sum_{n\in F} x_n : F \subset \mathbb{N} \text{ finite}\} \subset A$. We will denote by $\mathcal{IP} := \{A \subset \mathbb{N}_0 : A \text{ is an IP-set}\}$, the *Furstenberg family of IP-sets.*

(c) a Δ*-set*, if there is an infinite set $B \subset \mathbb{N}_0$ such that $(B - B) \cap \mathbb{N} \subset A$. We will denote by $\Delta := \{A \subset \mathbb{N}_0 : A \text{ is a } \Delta\text{-set}\}$, the *Furstenberg family of* Δ*-sets.*

From Definition 1.2 and the dual families notation we have:

Definition 1.5. Let (X, T) be a linear dynamical system. A point $x \in X$ is said to be

(a) *uniformly recurrent* if $N(x, U) \in \mathcal{S}$ for every neighbourhood U of x. We will denote by $\mathrm{URec}(T)$ the set of such points and T is *uniformly recurrent* if $\mathrm{URec}(T)$ is dense in X;

(b) $\mathcal{IP}^*$*-recurrent* if $N(x, U) \in \mathcal{IP}^*$ for every neighbourhood U of x. We will denote by $\mathcal{IP}^*\mathrm{Rec}(T)$ the set of such points, and T is $\mathcal{IP}^*$*-recurrent* if $\mathcal{IP}^*\mathrm{Rec}(T)$ is dense in X;

(c) Δ^**-recurrent* if $N(x, U) \in \Delta^*$ for every neighbourhood U of x. We will denote by $\Delta^*\mathrm{Rec}(T)$ the set of such points, and T is Δ^**-recurrent* if $\Delta^*\mathrm{Rec}(T)$ is dense in X.

It is shown in [BMPP16, Proposition 2] that the above Furstenberg families do not admit a respective hypercyclicity counterpart. As in the previous subsection these recurrence notions could be defined for (non-linear) Polish dynamical systems, but since the eigenvectors will play a fundamental role in the connection between those concepts we will directly work with complex linear maps. The relation $\Delta^* \subset \mathcal{IP}^* \subset \mathcal{S}$ between the families (see [BD08]), Proposition 4.1 and [BGELMP22] imply the inclusions

$$\operatorname{span}(\mathcal{E}(T)) \subset \Delta^*\mathrm{Rec}(T) \subset \mathcal{IP}^*\mathrm{Rec}(T) \subset \mathrm{URec}(T) \subset \mathrm{FRec}(T) \subset \mathrm{UFRec}(T) \subset \mathrm{RRec}(T).$$

From there the following question was proposed in [BGELMP22]:

Question 1.6 ([BGELMP22, Question 6.3]). Does there exist any uniformly recurrent but not $\mathcal{IP}^*$-recurrent operator?

The uniformly recurrent operators considered in [BGELMP22] were also $\mathcal{IP}^*$-recurrent, and in fact a partial negative answer to Question 1.6 was already given in [BGELMP22, Theorem 6.2] for the particular case of power-bounded operators, condition which implies the equality of the two sets $\mathcal{IP}^*\mathrm{Rec}(T)$ and $\mathrm{URec}(T)$. The second main result of this paper provides a negative answer to Question 1.6 for operators acting on a complex separable *Hilbert space* H, by showing the following stronger statement: *any uniformly recurrent operator $T \in \mathcal{L}(H)$ has a spanning set of unimodular eigenvectors.* More precisely, define the sets

$$\begin{aligned}
\mathrm{FRec}^{bo}(T) &:= \mathrm{FRec}(T) \cap \{x \in H \text{ with bounded } T\text{-orbit}\};\\
\mathrm{UFRec}^{bo}(T) &:= \mathrm{UFRec}(T) \cap \{x \in H \text{ with bounded } T\text{-orbit}\};\\
\mathrm{RRec}^{bo}(T) &:= \mathrm{RRec}(T) \cap \{x \in H \text{ with bounded } T\text{-orbit}\}.
\end{aligned}$$

We always have that $\mathrm{URec}(T) \subset \mathrm{FRec}^{bo}(T) \subset \mathrm{UFRec}^{bo}(T) \subset \mathrm{RRec}^{bo}(T)$ because the uniformly recurrent vectors have bounded orbit. Furthermore:

Theorem 1.7. *Let $T \in \mathcal{L}(H)$ where H is a complex separable Hilbert space. Then we have the equalities*

$$\overline{\operatorname{span}(\mathcal{E}(T))} = \overline{\mathrm{URec}(T)} = \overline{\mathrm{RRec}^{bo}(T)}.$$

Moreover:

(a) *The following statements are equivalent:*

- (i) $\mathcal{E}(T) \neq \varnothing$;
- (ii) $\Delta^*\mathrm{Rec}(T) \setminus \{0\} \neq \varnothing$;
- (iii) $\mathcal{IP}^*\mathrm{Rec}(T) \setminus \{0\} \neq \varnothing$;
- (iv) $\mathrm{URec}(T) \setminus \{0\} \neq \varnothing$;
- (v) $\mathrm{FRec}^{bo}(T) \setminus \{0\} \neq \varnothing$;
- (vi) $\mathrm{UFRec}^{bo}(T) \setminus \{0\} \neq \varnothing$;
- (vii) $\mathrm{RRec}^{bo}(T) \setminus \{0\} \neq \varnothing$;
- (viii) *T admits a non-trivial invariant probability measure μ with $\int_H \|z\|^2 d\mu(z) < \infty$.*

(b) *The following statements are equivalent:*

(i) *the set* $\operatorname{span}(\mathcal{E}(T))$ *is dense in* H*;*

(ii) T *is* Δ^**-recurrent;*

(iii) T *is* $\mathcal{IP}^*$*-recurrent;*

(iv) T *is uniformly recurrent;*

(v) *the set* $\operatorname{FRec}^{bo}(T)$ *is dense in* H*;*

(vi) *the set* $\operatorname{UFRec}^{bo}(T)$ *is dense in* H*;*

(vii) *the set* $\operatorname{RRec}^{bo}(T)$ *is dense in* H*;*

(viii) T *admits an invariant probability measure* μ *with full support and* $\displaystyle\int_H \|z\|^2 d\mu(z) < \infty$.

The proof of Theorem 1.7 that we provide here is really specific to the Hilbertian setting in a somewhat roundabout way. It relies on the following three main arguments:

– the existence of a non-trivial invariant measure with a *finite second-order moment*, under the assumption of the existence of a reiteratively recurrent vector with *bounded orbit*; this argument is the same as the one employed in the proof of Theorem 1.3 above;

– the fact that any operator on a space of type 2, admitting an invariant measure with a finite second-order moment, admits in fact an invariant Gaussian measure whose support contains that of the initial measure (see Remark 4.5);

– and lastly, the fact that on spaces of cotype 2, the existence of an invariant Gaussian measure for an operator T implies that the unimodular eigenvectors of T span a dense subspace of the support of the measure (see *Step* 3 of Lemma 4.4).

These last two "facts" are far from being trivial, and we refer the reader to [BM09, Chapter 5] for a proof, as well as for an introduction to the role of Gaussian measures in Linear Dynamics. Since the only spaces which are both of type 2 and of cotype 2 are those which are isomorphic to a Hilbert space, our proof of Theorem 1.7 does not seem to admit any possible extension to a non-Hilbertian setting. The following question remains widely open:

Question 1.8. Let X be a complex Banach space and suppose that $T : X \longrightarrow X$ is a uniformly recurrent operator acting on X:

(a) Is $\operatorname{span}(\mathcal{E}(T))$ a dense set in X?

(b) What about the cases where T is an adjoint operator on a separable dual Banach space or where X is a reflexive Banach space?

A partial (but not completely satisfactory) answer is our third and last main result, which only concerns the *power-bounded* operators on complex reflexive Banach spaces. It extends the result [BGELMP22, Theorem 6.2] by showing that such an operator $T \in \mathcal{L}(X)$ is again uniformly recurrent if and only if it has a spanning set of unimodular eigenvectors. More precisely, we have that:

Theorem 1.9. *Let $T : X \longrightarrow X$ be a power-bounded operator on a complex reflexive Banach space X. Then we have the equality*

$$\overline{\text{span}(\mathcal{E}(T))} = \overline{\text{URec}(T)}.$$

In particular:

(a) *The following statements are equivalent:*

  (i) $\mathcal{E}(T) \neq \varnothing$;

  (ii) $\Delta^*\text{Rec}(T) \setminus \{0\} \neq \varnothing$;

  (iii) $\mathcal{IP}^*\text{Rec}(T) \setminus \{0\} \neq \varnothing$;

  (iv) $\text{URec}(T) \setminus \{0\} \neq \varnothing$.

(b) *The following statements are equivalent:*

  (i) *the set* $\text{span}(\mathcal{E}(T))$ *is dense in* X;

  (ii) T *is* Δ^**-recurrent;*

  (iii) T *is* $\mathcal{IP}^*$*-recurrent;*

  (iv) T *is uniformly recurrent.*

The proof of Theorem 1.9 relies on the classic splitting theorem of Jacobs-Deleeuw-Glicksberg (see [Kre85, Section 2.4]). Here the unimodular eigenvectors are obtained in a very different way than in the proof of Theorem 1.7 (via characters on a certain compact abelian group).

Although the arguments used in the proofs of the two theorems above still hold when the underlying space is complex and finite-dimensional, in this situation one can directly use the Jordan decomposition (see [CMP14, Theorem 4.1] and [BGELMP22, Theorem 7.3]) to obtain a spanning set of unimodular eigenvectors even from the notion of "usual" recurrence as defined in Subsection 1.1.

1.4 Organization of the paper

Section 2 is devoted to the statement and proof of a purely non-linear result (Theorem 2.3) which allows to construct invariant measures from reiteratively recurrent points for a rather general class of Polish dynamical systems (which includes the compact ones). Theorem 2.3 is a modest improvement of [GM14, Theorem 1.5, Remarks 2.6 and 2.12] and its proof is based on a modification of the construction given in [GM14, Section 2]. In Section 3, we prove some results where frequent recurrence is deduced from reiterative recurrence, in particular Theorem 1.3. Theorems 1.7 and 1.9, which provide links between strong forms of recurrence and the existence of unimodular eigenvectors, are proved in Section 4. Sections 5 and 6 present some applications of the above results in terms of product and inverse dynamical systems, while in Section 7 the "typicality", in the Baire category sense, of some recurrence properties is studied. We gather in Section 8 some possibly interesting open questions and a few comments related to them.

2 Invariant measures from reiterative recurrence

In this section, we present a modification of the construction of [GM14, Section 2] which allows to construct invariant measures from reiteratively recurrent points for a rather general class of Polish dynamical systems, including the compact ones (see Remark 2.4).

2.1 Topological assumptions and initial comments

We begin this section with some notation: whenever we consider a space of functions we will use the symbol $\mathbb{1}$ to denote the function constantly equal to 1, and given a subset A of the domain of the functions, we will write $\mathbb{1}_A$ for the indicator function of A, i.e. $\mathbb{1}_A(x) = 1$ if $x \in A$ and $\mathbb{1}_A(x) = 0$ if $x \notin A$. For instance, if we consider $\ell^\infty = \ell^\infty(\mathbb{N})$, the *space of all bounded sequences of real numbers*, $\mathbb{1} \in \ell^\infty$ is the sequence with all its terms equal to 1, and for every $A \subset \mathbb{N}$, $\mathbb{1}_A \in \ell^\infty$ will be the sequence in which the n-th coordinate is 1 if $n \in A$ and 0 otherwise.

A *Banach limit* is a continuous functional $\mathfrak{m} : \ell^\infty \longrightarrow \mathbb{R}$ such that for every pair of sequences $\phi = (\phi(n))_{n\geq 1}, \psi = (\psi(n))_{n\geq 1} \in \ell^\infty$, every $\alpha, \beta \in \mathbb{R}$ and every $a \in \mathbb{N}$:

(a) $\mathfrak{m}(\alpha\phi + \beta\psi) = \alpha\mathfrak{m}(\phi) + \beta\mathfrak{m}(\psi)$ (linearity);

(b) $\phi(n) \geq 0$ for every $n \in \mathbb{N}$ implies $\mathfrak{m}(\phi) \geq 0$ (positivity);

(c) $\mathfrak{m}((\phi(n + a)_{n\geq 1}) = \mathfrak{m}((\phi(n))_{n\geq 1})$ (shift-invariance);

(d) if ϕ is a convergent sequence then $\mathfrak{m}(\phi) = \lim_{n\to\infty} \phi(n)$ (which implies $\mathfrak{m}(\mathbb{1}) = 1$).

Following [GM14], each Banach limit $\mathfrak{m}$ should be viewed as a finitely additive measure on $\mathbb{N}$. In fact we will write the result of the action of $\mathfrak{m}$ on a "function" $\phi \in \ell^\infty$ as the integral:

$$\mathfrak{m}(\phi) = \int_{\mathbb{N}} \phi(i)\ d\mathfrak{m}(i).$$

Given a topological space (X, τ) we will denote by $\mathscr{B}(X, \tau)$ its *σ-algebra of Borel sets*. If there is no confusion with the topology we will simply write $\mathscr{B}(X)$. All the measures considered in this section will be *non-negative finite Borel measures*, i.e. they could be the null measure, and since they will be defined on Polish spaces the finiteness condition will imply their *regularity* (see for instance [Coh13, Proposition 8.1.12]). Given a (non-negative) finite Borel measure μ on a topological space (X, τ) we will denote its *support* by

$$\text{supp}(\mu) := X \setminus \bigcup_{U\in\tau, \mu(U)=0} U.$$

When μ is positive and regular it is easy to show that $\text{supp}(\mu)$ is non-empty, and the smallest τ-closed subset of X with full measure, i.e. $\mu(\text{supp}(\mu)) = \mu(X)$, the later being true even if μ is not regular but X is second-countable (see [Koz18, Proposition 2.3]). Moreover, a point x belongs to $\text{supp}(\mu)$ if and only if $\mu(U) > 0$ for every neighbourhood U of x.

Before presenting the "measures' constructing machine" that will be used in the rest of this work, we give name to some properties that a Polish dynamical system (X, T) may have. In particular, let (X, τ_X) be the underlying Polish space, τ a Hausdorff topology in X and let $\mathcal{K}_\tau$ be the set of τ-compact subsets of X. We will consider the following properties:

(I) T is a continuous self-map of (X, τ) (i.e. $T : X \longrightarrow X$ is τ-continuous);

(II) $\tau \subset \tau_X$ (i.e. τ is coarser than τ_X);

(III) $\mathscr{B}(X, \tau) = \mathscr{B}(X, \tau_X)$ (i.e. both topologies have the same Borel sets);

(IV) every τ-compact set is τ-metrizable (i.e. every $K \in \mathcal{K}_\tau$ is τ-metrizable);

(III*) every point of X has a neighbourhood basis for τ_X consisting of τ-compact sets.

In [GM14, Fact 2.1] it is shown easily how (II) and (III*) imply conditions (III) and (IV). For the concrete recurrence results that we obtain, it is necessary to assume conditions (I), (II) and (III*) in order to use the reiteratively recurrent points in a successful way. However, without property (III*) and assuming just conditions (I), (II), (III) and (IV) we can carry out the "construction" on which everything is based:

Lemma 2.1 (**Modification of [GM14, Remarks 2.6 and 2.12]**). *Let (X, T) be a Polish dynamical system. Assume that X is endowed with a Hausdorff topology τ which fulfills* (I), (II), (III) *and* (IV). *Then for each $x_0 \in X$ and each Banach limit $\mathfrak{m} : \ell^\infty \longrightarrow \mathbb{R}$ one can find a (non-negative) T-invariant finite Borel measure μ on X for which $\mu(X) \leq 1$ and such that*

$$\mu(K) \geq \mathfrak{m}(\mathbb{1}_{N(x_0,K)}) \quad \text{for every } K \in \mathcal{K}_\tau.$$

Moreover, we have the inclusion $\operatorname{supp}(\mu) \subset \overline{\operatorname{Orb}(x_0, T)}^\tau$.

Remark 2.2. Lemma 2.1 is a rather technical result which allows us to construct invariant measures. Note that:

(a) Assumptions (I), (II), (III) and (IV) are fulfilled by the initial topology τ_X. However, if the τ_X-compact sets are too small, given an arbitrary point $x_0 \in X$ (even with some kind of recurrence property) we could have $\mathfrak{m}(\mathbb{1}_{N(x_0,K)}) = 0$ for every τ_X-compact set $K \subset X$ and hence the measure μ obtained could be the null measure on X. We will consider a strictly coarser topology $\tau \subsetneq \tau_X$ in order to obtain "interesting measures" from Lemma 2.1.

(b) Following the previous comment, even if the τ-compact sets are big enough, the measure μ could be the null measure on X if we choose a point $x_0 \in X$ for which the return sets $N(x_0, K)$ are too small and hence $\mathfrak{m}(\mathbb{1}_{N(x_0,K)}) = 0$ for every $K \in \mathcal{K}_\tau$. We will get "interesting measures" whenever we combine Lemma 2.1 together with the existence of a point $x_0 \in X$ and a Banach limit $\mathfrak{m}$ for which $\mathfrak{m}(\mathbb{1}_{N(x_0,K)}) > 0$ for some τ-compact subsets K of X. Those conditions will come from property (III*) together with the existence of a reiteratively recurrent point $x_0 \in \operatorname{RRec}(T)$, see Theorem 2.3.

(c) In the proof of [GM14, Theorem 1.5] it is shown that under conditions (I), (II) and (III*), one can change the final statement of Lemma 2.1 into

– *then for each $x_0 \in X$ one can find a T-invariant finite Borel measure μ on X such that $\mu(K) \geq \underline{\operatorname{dens}}(N(x_0, K))$ for every $K \in \mathcal{K}_\tau$,*

simply by choosing a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and considering the Banach limit

$$\mathfrak{m}(\phi) := \lim_{\mathcal{U}} \frac{1}{n} \sum_{i=1}^{n} \phi(i) \quad \text{for every } \phi \in \ell^\infty.$$

Moreover, under the same assumptions it is also stated in [GM14, Remark 2.12] that

– *for each* $x_0 \in X$ *and each* $K \in \mathcal{K}_\tau$ *one can find a* T*-invariant finite Borel measure* μ *on* X *such that* $\mu(K) \geq \overline{\text{dens}}(N(x_0, K))$.

This just ensures that the above inequality holds true for only one fixed τ-compact subset K of X. We will encounter the same problem when working with the upper Banach density, and we will have to combine some more sophisticated Banach limits in order to cope with several τ-compact sets at the same time, see Subsection 2.3.

Here is the main result of this section:

Theorem 2.3. *Let* (X, T) *be a Polish dynamical system. Assume that* X *is endowed with a Hausdorff topology* τ *which fulfills* (I), (II), *and* (III*). *If* $x_0 \in X$ *is a reiteratively recurrent point for* T*, then one can find a* T*-invariant probability measure* μ_{x_0} *on* X *such that*

$$x_0 \in \text{supp}(\mu_{x_0}) \subset \overline{\text{Orb}(x_0, T)}^{\tau}.$$

Moreover, if T *is reiteratively recurrent then one can find a* T*-invariant probability measure* μ *on* X *with full support.*

Remark 2.4. If the Polish dynamical system $T : (X, \tau_X) \longrightarrow (X, \tau_X)$ is locally compact, its initial topology τ_X already fulfills properties (I), (II) and (III*), and hence (III) and (IV). In particular, the later is true whenever (X, τ_X) is a compact metrizable space.

2.2 Proof of Lemma 2.1

We modify the construction given in [GM14, Section 2.2]. Let (X, T) be a Polish dynamical system, denote by τ_X the initial topology of X and assume that it is also endowed with a Hausdorff topology τ which fulfills (I), (II), (III) and (IV). Fix $x_0 \in X$ and let $\mathfrak{m} : \ell^\infty \longrightarrow \mathbb{R}$ be a Banach limit. For each $K \in \mathcal{K}_\tau$ denote by $\mathscr{C}(K, \tau)$ the space of all τ-continuous real-valued functions on K.

Fact 2.2.1 (Modification of [GM14, Fact 2.2]). *For every* $K \in \mathcal{K}_\tau$ *there is a unique (non-negative) finite Borel regular measure* μ_K *on* K *such that*

$$\int_K f d\mu_K = \int_{\mathbb{N}} (\mathbb{1}_K f)(T^i x_0)\ d\mathfrak{m}(i) \quad \textit{for every } f \in \mathscr{C}(K, \tau).$$

The measure μ_K *satisfies* $0 \leq \mu_K(K) = \mathfrak{m}(\mathbb{1}_{N(x_0,K)}) \leq 1$.

Proof. The first part is obvious by the Riesz representation theorem since, as mentioned in [GM14, Fact 2.2], the formula

$$L(f) := \int_{\mathbb{N}} (\mathbb{1}_K f)(T^i x_0)\ d\mathfrak{m}(i) \quad \text{for every } f \in \mathscr{C}(K, \tau),$$

defines a (non-negative) linear functional on $\mathscr{C}(K, \tau)$. Moreover, the measure μ_K satisfies

$$0 \leq \mu_K(K) = \int_{\mathbb{N}} (\mathbb{1}_K)(T^i x_0)\ d\mathfrak{m}(i) = \mathfrak{m}(\mathbb{1}_{N(x_0,K)}) \leq \mathfrak{m}(\mathbb{1}) = 1. \qquad \square$$

By (III) we have the equality $\mathscr{B}(X,\tau) = \mathscr{B}(X,\tau_X)$ and hence for each $K \in \mathcal{K}_\tau$ we can extend the measure μ_K into a Borel measure on the whole space X (still denoted by μ_K) using the formula

$$\mu_K(A) := \mu_K(K \cap A) \quad \text{for every Borel set } A \in \mathscr{B}(X).$$

Clearly $\mu_K(X) \leq 1$, which implies the regularity of these measures. However, since the compact sets $K \in \mathcal{K}_\tau$ are not necessarily T-invariant and we could have $T^{-1}(K) \cap K = \varnothing$, the measures μ_K are not necessarily T-invariant. As in [GM14] we will define the T-invariant measure we are looking for by taking the supremum of the μ_K, which is possible due to the following fact:

Fact 2.2.2 (**[GM14, Fact 2.3]**). *If $K, F \in \mathcal{K}_\tau$ and if $K \subset F$, then $\mu_K \leq \mu_F$.*

Proof. The proof is exactly the same than that of [GM14, Fact 2.3] and it uses essentially conditions (II), (IV) and the positivity of $\mathfrak{m}$. □

Since a finite union of τ-compact subsets of X is still an element of $\mathcal{K}_\tau$, from Fact 2.2.2 we deduce that the family $(\mu_K)_{K\in\mathcal{K}_\tau}$ has the following property: for any pair $K_1, K_2 \in \mathcal{K}_\tau$ one can find $F \in \mathcal{K}_\tau$ such that $\mu_F \geq \max\{\mu_{K_1}, \mu_{K_2}\}$. We continue as in [GM14, Section 2.2]:

Fact 2.2.3 (**Modification of [GM14, Fact 2.4]**). *If we set*

$$\mu(A) := \sup_{K\in\mathcal{K}_\tau} \mu_K(A) \quad \textit{for every Borel set } A \in \mathscr{B}(X),$$

then μ is a (non-negative) Borel measure on X such that $\mu(X) \leq 1$.

Proof. The proof is exactly the same than that of [GM14, Fact 2.4] but with μ having the possibility of being the null measure. □

Fact 2.2.4 (**Modification of [GM14, Fact 2.5]**). *The measure μ is T-invariant and we have the inequality*

$$\mu(K) \geq \mathfrak{m}(\mathbb{1}_{N(x_0,K)}) \quad \textit{for every } K \in \mathcal{K}_\tau.$$

Proof. The first part of the proof is exactly the same than that of [GM14, Fact 2.5] and it uses essentially conditions (I), (IV), the positivity of $\mathfrak{m}$ and the fact that it is shift-invariant. By Fact 2.2.1, for each $K \in \mathcal{K}_\tau$ we have that $\mu(K) \geq \mu_K(K) = \mathfrak{m}(\mathbb{1}_{N(x_0,K)})$. □

To finish the proof of Lemma 2.1 we include a property, not shown in [GM14, Section 2.2], about the support of the measure constructed:

Fact 2.2.5. *We have the inclusion* $\mathrm{supp}(\mu) \subset \overline{\mathrm{Orb}(x_0,T)}^{\tau}$.

Proof. Write $O(x_0) := \overline{\mathrm{Orb}(x_0,T)}^{\tau}$. First we show that for each $K \in \mathcal{K}_\tau$ we have the inclusion $\mathrm{supp}(\mu_K) \subset K \cap O(x_0)$: indeed, for any $K \in \mathcal{K}_\tau$ and any point $x \in K \setminus O(x_0)$, by compactness, there exists a positive function $f \in \mathscr{C}(K,\tau)$ such that

$$f = 0 \text{ on } K \cap O(x_0) \quad \text{and} \quad f(x) = 1.$$

If we suppose that $x \in \mathrm{supp}(\mu_K)$ and if we take a τ-neighbourhood U of x in (K,τ) such that $f(U) \subset [1/2, \infty[$ then we have

$$0 < \frac{1}{2}\mu_K(U) \leq \int_K f d\mu_K = \int_{\mathbb{N}} (\mathbb{1}_K f)(T^i x_0)\ d\mathfrak{m}(i) = \mathfrak{m}(\mathbf{0}) = 0,$$

since $f(T^i x_0) = 0$ for every $i \in \mathbb{N}$, which is a contradiction. Hence $x \notin \mathrm{supp}(\mu_K)$.

Finally, given $x \notin O(x_0)$ there is an τ-open neighbourhood U of x in (X,τ), which by (II) is also a τ_X-neighbourhood of x in (X,τ_X), such that $U \cap O(x_0) = \varnothing$. Hence, since $\operatorname{supp}(\mu_K) \subset O(x_0)$ for every $K \in \mathcal{K}_\tau$, we deduce that $\mu_K(U) = \mu_K(K \cap U) = 0$ for every $K \in \mathcal{K}_\tau$. By the definition of μ we get $\mu(U) = 0$ and hence that $x \notin \operatorname{supp}(\mu)$. □

2.3 Proof of Theorem 2.3

Let (X,T) be a Polish dynamical system, denote by τ_X the initial topology of X and assume that X is endowed with a Hausdorff topology τ which fulfills (I), (II) and (III*).

Fact 2.3.1. *Given $x_0 \in X$ and $U \in \mathcal{K}_\tau$ with $\overline{\mathrm{Bd}}(N(x_0,U)) > 0$, there exists a T-invariant probability measure μ on X such that $\mu(U) > 0$. Moreover, we have the inclusion*

$$\operatorname{supp}(\mu) \subset \overline{\operatorname{Orb}(x_0,T)}^{\tau}.$$

Fact 2.3.1 allows us to (slightly) extend Theorem 2.3 in terms of the *recurrence* notion introduced in [GM14, Section 2.5] (see Remark 2.5 below for the explicit statement) which at the end turns out to be equivalent to the *recurrence* notion used here (see Theorem 3.3).

Proof of Fact 2.3.1. Since

$$\overline{\mathrm{Bd}}(N(x_0,U)) := \limsup_{N\to\infty} \left(\max_{m\geq 0} \frac{\#(N(x_0,U)\cap[m,m+N])}{N+1} \right) > 0,$$

there exists an increasing sequence of natural numbers $(N_k)_{k\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ and a sequence of intervals $I_k = [i_k+1, i_k+N_k] \subset \mathbb{N}$ such that

$$\overline{\mathrm{Bd}}(N(x_0,U)) = \lim_{k\to\infty} \frac{\#(N(x_0,U)\cap I_k)}{N_k}. \tag{2.1}$$

Then we fix the Banach limit $\mathfrak{m} : \ell^\infty \longrightarrow \mathbb{R}$ defined as

$$\mathfrak{m}(\phi) := \lim_{\mathcal{U}} \frac{1}{N_k} \sum_{n\in I_k} \phi(n) \quad \text{ for every } \phi \in \ell^\infty,$$

for some fixed non-principal ultrafilter $\mathcal{U} \subset \mathscr{P}(\mathbb{N})$ on $\mathbb{N}$. By (2.1) we have that

$$\mathfrak{m}(\mathbb{1}_{N(x_0,U)}) = \overline{\mathrm{Bd}}(N(x_0,U)) > 0.$$

Since τ fulfills (I), (II) and (III*), by [GM14, Fact 2.1] it also has properties (III) and (IV) so we can apply Lemma 2.1 to x_0 and $\mathfrak{m}$ obtaining a (non-negative) T-invariant finite Borel measure μ on X for which $\mu(K) \geq \mathfrak{m}(\mathbb{1}_{N(x_0,K)})$ for each $K \in \mathcal{K}_\tau$ and such that $\operatorname{supp}(\mu) \subset \overline{\operatorname{Orb}(x_0,T)}^{\tau}$. In particular we get $\mu(U) \geq \mathfrak{m}(\mathbb{1}_{N(x_0,U)}) > 0$ so μ is a positive T-invariant finite Borel measure. Normalizing μ we get the desired measure. □

Fact 2.3.2. *Given $x_0 \in \mathrm{RRec}(T)$, there exists a T-invariant probability measure μ_{x_0} on X such that*

$$x_0 \in \operatorname{supp}(\mu_{x_0}) \subset \overline{\operatorname{Orb}(x_0,T)}^{\tau}.$$

Proof. Set $O(x_0) := \overline{\mathrm{Orb}(x_0, T)}^{\tau}$. Using (III*), let $(U_n)_{n\in\mathbb{N}}$ be a basis of τ_X-neighbourhoods of x_0 consisting of τ-compact sets. Applying Fact 2.3.1 to each set U_n we obtain a sequence $(\mu_n)_{n\in\mathbb{N}}$ of T-invariant probability measures on X for which $\mu_n(U_n) > 0$ and such that $\mathrm{supp}(\mu_n) \subset O(x_0)$ for each $n \in \mathbb{N}$. Then the measure

$$\mu_{x_0} := \sum_{n\in\mathbb{N}} \frac{\mu_n}{2^n}$$

is a T-invariant probability measure on X. Moreover, for any τ_X-neighbourhood U of x_0 there is an integer $n \in \mathbb{N}$ with $U_n \subset U$ and hence

$$\mu_{x_0}(U) \geq \mu_{x_0}(U_n) \geq \frac{\mu_n(U_n)}{2^n} > 0.$$

This implies that $x_0 \in \mathrm{supp}(\mu_{x_0})$. Also, given $x \notin O(x_0)$ there is a τ-neighbourhood V of x, which by (II) is also a τ_X-neighbourhood of x, such that $V \cap O(x_0) = \varnothing$. Since $\mathrm{supp}(\mu_n) \subset O(x_0)$ for every $n \in \mathbb{N}$ we deduce that $\mu_n(V) = 0$ for every $n \in \mathbb{N}$ and by the definition of μ_{x_0} we get $\mu_{x_0}(V) = 0$. This implies that $x \notin \mathrm{supp}(\mu_{x_0})$ and hence $x_0 \in \mathrm{supp}(\mu_{x_0}) \subset O(x_0)$. □

To complete the proof of Theorem 2.3, let T be reiteratively recurrent. Since X is separable there is a countable set $\{x_n : n \in \mathbb{N}\} \subset \mathrm{RRec}(T)$ which is dense in X. Applying Fact 2.3.2 to each point x_n we obtain a sequence $(\mu_{x_n})_{n\in\mathbb{N}}$ of T-invariant probability measures on X such that $x_n \in \mathrm{supp}(\mu_{x_n})$ for each $n \in \mathbb{N}$. Finally, the measure $\mu := \sum_{n\in\mathbb{N}} \frac{\mu_{x_n}}{2^n}$ is a T-invariant probability measure on X with full support. □

Remark 2.5. Under the topological assumptions of Theorem 2.3, and in view of Fact 2.3.1, a generalization in terms of the *recurrence* notion introduced in [GM14, Section 2.5][A], and following the spirit of [GM14, Proposition 2.11], can be shown:

– *If for each open subset U of X there is a point $x_U \in X$ such that $\overline{\mathrm{Bd}}(N(x_U, U)) > 0$, then one can find a T-invariant probability measure μ on X with full support*[B].

Indeed, one just has to use (III*) to consider an appropriate countable family of τ-compact sets whose τ_X-interiors form a base of the initial topology τ_X, to apply Fact 2.3.1 to those τ-compact sets and then to take an infinite convex combination of the obtained measures.

3 From reiterative to frequent recurrence

Theorem 2.3 allows us to construct invariant measures starting from reiteratively recurrent points. In this section, we will exploit this result in order to show that reiterative recurrence for adjoint operators on separable dual Banach spaces actually implies the stronger notion of frequent recurrence (see Theorem 1.3).

3.1 A key lemma

An important tool for the proof of Theorem 1.3 is the following lemma:

[A] This has been called the $\mathcal{P}_{\mathcal{F}}$ *property*, see Section 2.2 of the *General discussion of the results.*
[B] This result has been independently proved in the recent work [CM].

Lemma 3.1 (**Frequent Recurrence from Invariant Measures**). *Let $T : X \longrightarrow X$ be a continuous map on a second-countable space X and let μ be a T-invariant probability measure on X. Then $\mu(\mathrm{FRec}(T)) = 1$ and in particular we have the inclusion*

$$\mathrm{supp}(\mu) \subset \overline{\mathrm{FRec}(T)}.$$

The above result is the recurrence version of [BM09, Corollary 5.5], and since recurrence is a local property the measure is not required to be with full support, condition under which the map T would clearly be frequently recurrent.

Proof of Lemma 3.1. Let $B \in \mathscr{B}(X)$ be an arbitrary but fixed Borel set with $\mu(B) > 0$. By the ergodic decomposition theorem (see [Gla03, Theorem 3.42]) there is a T-invariant probability measure m on X for which T is an ergodic map and such that $m(B) > 0$. Let $(U_n)_{n\in\mathbb{N}}$ be a countable basis of the topology and apply the Birkhoff pointwise ergodic theorem (see for instance [Gla03, Theorem 3.41]) to each of the indicator functions $\mathbb{1}_{U_n}$. This yields

$$\begin{aligned}
\mathrm{dens}(N(x,U_n)) &= \lim_{N\to\infty} \frac{\#(N(x,U_n)\cap[0,N])}{N+1} = \lim_{N\to\infty} \frac{1}{N+1}\sum_{k=0}^{N} \mathbb{1}_{U_n}(T^k x) \\
&= \int_X \mathbb{1}_{U_n}\, dm = m(U_n),
\end{aligned}$$

for m-a.e. point $x \in X$, that is, for each $n \in \mathbb{N}$ there is a set $A_n \subset X$ with $m(A_n) = 1$ such that $\mathrm{dens}(N(x,U_n)) = m(U_n)$ for every $x \in A_n$. Since a countable union of null sets is again null, the set

$$A := \mathrm{supp}(m) \cap \left(\bigcap_{n\in\mathbb{N}} A_n \right)$$

satisfies $m(A) = 1$. We claim that $A \subset \mathrm{FRec}(T)$: given $x \in A$ and a neighbourhood U of the point x there is an integer $n \in \mathbb{N}$ such that $x \in U_n \subset U$. Since $A \subset \mathrm{supp}(m)$ we have that $U_n \cap \mathrm{supp}(m) \neq \varnothing$ and hence $\underline{\mathrm{dens}}(N(x,U)) \geq \underline{\mathrm{dens}}(N(x,U_n)) = m(U_n) > 0$. The arbitrariness of the neighbourhood U of x implies that $x \in \mathrm{FRec}(T)$. Now, since $m(A) = 1$ and $m(B) > 0$ we obtain that $A \cap B \neq \varnothing$ and hence

$$\mathrm{FRec}(T) \cap B \neq \varnothing.$$

Since this is true for every set $B \in \mathscr{B}(X)$ with $\mu(B) > 0$ we deduce that $\mu(\mathrm{FRec}(T)) = 1$. Then $\mu(\overline{\mathrm{FRec}(T)}) = 1$ and in particular, since $\mathrm{supp}(\mu)$ is the smallest closed subset of X with full μ-measure, we get that $\mathrm{supp}(\mu) \subset \overline{\mathrm{FRec}(T)}$. □

Remark 3.2. Lemma 3.1 improves [Fur81, Theorem 3.3] in terms of frequent recurrence by using the Birkhoff pointwise ergodic theorem. Indeed, [Fur81, Theorem 3.3] shows that, under the assumptions of Lemma 3.1, μ-a.e. point is recurrent, i.e. $\mu(\mathrm{Rec}(T)) = 1$.

Combining Theorem 2.3 and Lemma 3.1 we deduce the following result:

Theorem 3.3 (**From Reiterative to Frequent Recurrence**). *Let the pair (X,T) be a Polish dynamical system, denote by τ_X the initial topology of X and assume that X is endowed with a Hausdorff topology τ which fulfills* (I), (II), *and* (III*). *Then we have the equality*

$$\overline{\mathrm{FRec}(T)}^{\tau_X} = \overline{\mathrm{RRec}(T)}^{\tau_X}.$$

Moreover:

(a) *The following statements are equivalent:*

  (i) $\mathrm{FRec}(T) \neq \varnothing$;
  (ii) $\mathrm{UFRec}(T) \neq \varnothing$;
  (iii) $\mathrm{RRec}(T) \neq \varnothing$;
  (iv) T *admits an invariant probability measure.*

(b) *The following statements are equivalent:*

  (i) T *is frequently recurrent;*
  (ii) T *is* $\mathcal{U}$*-frequently recurrent;*
  (iii) T *is reiteratively recurrent;*
  (iv) T *admits an invariant probability measure with full support.*

Proof. By definition we always have $\mathrm{FRec}(T) \subset \mathrm{UFRec}(T) \subset \mathrm{RRec}(T)$, so we just have to show that

$$\mathrm{RRec}(T) \subset \overline{\mathrm{FRec}(T)}^{\tau_X}.$$

Suppose that $\mathrm{RRec}(T) \neq \varnothing$. Given any $x_0 \in \mathrm{RRec}(T)$, by Theorem 2.3 one can find a T-invariant probability measure μ_{x_0} on X for which $x_0 \in \mathrm{supp}(\mu_{x_0})$. Since separable and metrizable spaces are second-countable, Lemma 3.1 implies that $x_0 \in \overline{\mathrm{FRec}(T)}^{\tau_X}$.

Moreover, in both cases (a) and (b) we have: (i) implies (ii) which implies (iii) by definition; (iii) implies (iv) by Theorem 2.3; and (iv) implies (i) by Lemma 3.1. □

As we already mentioned in the Introduction this result is false for general Polish dynamical systems: there even exist reiteratively hypercyclic operators acting on the $c_0(\mathbb{N})$ space without any non-zero $\mathcal{U}$-frequently recurrent vector (see [BGELMP22, Theorem 5.7 and Corollary 5.8]). Theorem 1.3 is the linear version of the previous result.

3.2 Proof of Theorem 1.3

Let $T : X \longrightarrow X$ be an adjoint operator on a separable dual Banach space X. Denote by $\tau_{\|\cdot\|}$ the norm topology, consider the weak-star topology w^* and note that:

(I) since T is an adjoint operator, it is a continuous self-map of (X, w^*);

(II) by the definition of the topologies, we have $w^* \subset \tau_{\|\cdot\|}$;

(III*) by the Alaoglu-Bourbaki theorem, the translation of the family of closed balls centred at 0 is a $\tau_{\|\cdot\|}$-neighbourhood basis consisting of w^*-compact sets.

If $T : X \longrightarrow X$ is an operator on a separable reflexive Banach space X the same conditions hold for the weak topology. From here one can apply the same arguments as those used in the proof of Theorem 3.3. In particular, if we consider a point $x_0 \in \mathrm{RRec}(T) \setminus \{0\}$ then the measure μ_{x_0} obtained by Theorem 2.3 is a non-trivial invariant probability measure. □

Remark 3.4. The equality $\overline{\mathrm{FRec}(T)} = \overline{\mathrm{RRec}(T)}$ and the equivalences (i) ⇔ (ii) ⇔ (iii) stated in Theorem 1.3 are still true when the underlying space X is a non-separable reflexive Banach space. Indeed, given an operator $T : X \longrightarrow X$ on a non-separable reflexive Banach space X, and given a point $x_0 \in \mathrm{RRec}(T)$ we can consider the separable closed T-invariant subspace

$$Z := \overline{\mathrm{span}(\mathrm{Orb}(x_0, T))},$$

which is again reflexive. Then $T|_Z : Z \longrightarrow Z$ is an operator on a separable reflexive Banach space. Moreover, recurrence is a local property, i.e. for each Furstenberg family $\mathcal{F}$ we have that

$$\mathcal{F}\mathrm{Rec}(T|_Z) = \mathcal{F}\mathrm{Rec}(T) \cap Z.$$

Applying Theorem 1.3 to $T|_Z$ we get that

$$x_0 \in \mathrm{RRec}(T|_Z) \text{ and hence } x_0 \in \overline{\mathrm{FRec}(T|_Z)} \subset \overline{\mathrm{FRec}(T)}.$$

However, we cannot say the same about statement (iv) of Theorem 1.3 since separability is essential to construct and extend the invariant measures onto the whole space. The above arguments are also restricted to the reflexive case because closed subspaces of a dual Banach space are not necessarily dual Banach spaces (consider $c_0(\mathbb{N}) \subset \ell^\infty(\mathbb{N})$).

4 From uniform recurrence to unimodular eigenvectors

Now we will connect some recurrence properties (stronger than those from Sections 2 and 3), for linear dynamical systems acting on *complex* Banach spaces, to the existence of unimodular eigenvectors. This comes motivated by the fact that, given any *complex linear map* $T : X \longrightarrow X$ on a *complex topological vector space* X, the linear span of its unimodular eigenvectors $\mathcal{E}(T)$ consists of Δ^*-recurrent vectors. It is shown in [BGELMP22, Lemma 7.1 and Corollary 7.2] that they are $\mathcal{IP}^*$-recurrent, and the same arguments hold by using [Fur81, Proposition 9.8] applied to the Kronecker system formed by the compact group $\mathbb{T}^k$ and the (left) multiplication $(z_1, ..., z_k) \mapsto (\lambda_1 z_1, ..., \lambda_k z_k)$ for a fixed k-tuple $(\lambda_1, ..., \lambda_k) \in \mathbb{T}^k$. We give an alternative proof via invariant measures:

Proposition 4.1. *Let $T : X \longrightarrow X$ be a complex linear map on a complex topological vector space X. Every linear combination of unimodular eigenvectors $\mathcal{E}(T)$ is a Δ^*-recurrent vector.*

Proof. Given $\lambda \in \mathbb{T}$ let $R_\lambda : \mathbb{T} \longrightarrow \mathbb{T}$ be the λ-rotation map where $z \mapsto \lambda z$. Then given $\varepsilon > 0$, since the Haar measure on $\mathbb{T}$ is a R_λ-invariant measure with full support, by the Poincaré recurrence theorem (see [Fur81, Theorem 3.2 and page 177]) there is $A \in \Delta^*$ such that for the ball centred at 1 and of radius $\varepsilon/2$, $B(1, \varepsilon/2) := \{z \in \mathbb{T} : |1 - z| < \varepsilon/2\}$, we have that

$$R_\lambda^n(B(1, \varepsilon/2)) \cap B(1, \varepsilon/2) \neq \varnothing \quad \text{for every } n \in A.$$

By the triangular inequality we get $|\lambda^n - 1| < \varepsilon$ for each $n \in A$ and hence

$$\Delta^* \ni A \subset \{n \in \mathbb{N} : |\lambda^n - 1| < \varepsilon\} \quad \text{so that} \quad \{n \in \mathbb{N} : |\lambda^n - 1| < \varepsilon\} \in \Delta^*.$$

Since the Furstenberg family Δ^* is a filter[C] (see [BD08]) and $\lambda \in \mathbb{T}$ and $\varepsilon > 0$ were chosen arbitrarily the proof is finished (see [BGELMP22, Corollary 7.2] for a detailed argument). □

[C] See Section 4 of the Appendix for more on the family Δ^*.

Hence, given a complex linear dynamical system $T : X \longrightarrow X$ we will always have that:

$$\operatorname{span}(\mathcal{E}(T)) \subset \Delta^*\mathrm{Rec}(T) \subset \mathcal{IP}^*\mathrm{Rec}(T) \subset \mathrm{URec}(T) \subset \mathrm{RRec}^{bo}(T).$$

Our goal in this section is to prove Theorem 1.7, which states that, for any operator acting on a complex Hilbert space, the existence of a non-zero reiteratively recurrent vector with bounded orbit (and in particular the existence of a uniformly recurrent vector) implies the existence of a unimodular eigenvector. The proof of Theorem 1.7 relies heavily on the *Gaussian measures* machinery. We begin by recalling some basic facts concerning these measures, as well as some deeper results pertaining to the Ergodic Theory of Gaussian linear dynamical systems. We refer the reader to one of the references [CTV87] or [DJT95] for more about Gaussian measures on Banach spaces, and to [BM09] and [BM16] for more on their role in Linear Dynamics.

4.1 Ergodic Theory and Gaussian measures in Linear Dynamics

The study of Ergodic Theory in the framework of Linear Dynamics started with the pioneering work of Flytzanis (see [Fly95a, Fly95b]), and was then further developed in the papers [BG06], [BG07] and [BM16], among others, focusing on the existence of invariant *Gaussian measures* satisfying some further dynamical properties such as weak/strong-mixing.

Definition 4.2. A Borel probability measure m on a complex Banach space X is said to be a *Gaussian measure* if every continuous linear functional $x^* \in X^*$ has a complex Gaussian distribution when considered as a random variable on $(X, \mathscr{B}(X), m)$.

It is now well understood that the dynamics of a linear dynamical system (X, T) are closely related to the properties of the unimodular eigenvectors of T. The situation is especially well understood in the Hilbertian setting, since the existence of an invariant Gaussian measure (with full support, or with respect to which T is ergodic or weakly/strongly-mixing) can be fully characterized in terms of the properties of the set $\mathcal{E}(T)$. See [BG06] and [BM09] for details. These characterizations do not hold true, in general, in the Banach space setting, but still many results are preserved allowing for a rather through understanding of Ergodic Theory of linear dynamical systems in this Gaussian framework. See [BG07], [BM09] and [BM16] for details.

Even though Gaussian measures are essential for our proof of Theorem 1.7 (see Lemma 4.4), the properties that such measures (may) have are properties that arbitrary probability measures can have too. We introduce these properties following [CTV87]:

Definition 4.3. Let μ be a probability measure on a Banach space X:

(a) suppose that there exists an element $x \in X$ such that

$$\int_X \langle x^*, z\rangle d\mu(z) = \langle x^*, x\rangle \quad \text{for every } x^* \in X^*,$$

then x is called the *expectation* of the measure μ and we will write $\int_X z d\mu(z) := x$;

(b) we say that μ is *centered* if its expectation exists and it is equal to $0 \in X$;

(c) we say that μ has a *finite second-order moment* if $\int_X \|z\|^2 d\mu(z) < \infty$.

If μ has a finite second-order moment then its expectation (called the *Pettis integral* of μ) exists (see [DU77, page 55]). Given a centered probability measure μ on X with a finite second-order moment, following [CTV87, page 169] and [BM09, Theorem 5.9], we can define the *covariance operator* of such a measure μ as the bounded linear operator

$$R : X^* \longrightarrow X \quad \text{satisfying} \quad \langle y^*, Rx^* \rangle = \int_X \langle y^*, z\rangle \langle x^*, z\rangle d\mu(z)$$

for every pair of elements x^* and y^* of X^*. In other words,

$$Rx^* := \int_X \langle x^*, z\rangle z d\mu(z) \quad \text{for every } x^* \in X^*. \tag{2.2}$$

Any Gaussian measure m on X has a finite second-order moment (see [BM09, Exercise 5.5]), and since we will consider in this work only centered Gaussian measures, we will always have an associated covariance operator for such a measure m.

When H is a complex separable Hilbert space, then the *covariance operator* of a centered probability measure μ on H with a finite second-order moment is usually defined, in a slightly different way, as the bounded linear operator $S : H \longrightarrow H$ for which

$$\langle Sx, y\rangle = \int_H \langle x, z\rangle \overline{\langle y, z\rangle} d\mu(z) \quad \text{for every } x, y \in H,$$

i.e.

$$Sx := \int_H \langle x, z\rangle z d\mu(z) \quad \text{for every } x \in H. \tag{2.3}$$

Observe that, contrary to (2.2), in this case $\langle Sx, \cdot\rangle : H \longrightarrow \mathbb{C}$ is an anti-linear functional acting on H. Also, S is a self-adjoint positive trace-class operator on H. It is a well-known and standard result (see for instance [BM09, Corollary 5.15]) that the Gaussian covariance operators on H are exactly the positive trace-class operators on H, i.e. for such an operator S there exists a Gaussian measure m on H for which we also have that

$$\langle Sx, y\rangle = \int_H \langle x, z\rangle \overline{\langle y, z\rangle} dm(z) \quad \text{for every } x, y \in H.$$

The possibility of constructing a Gaussian measure m with the same covariance operator as μ, together with the fact that the support of a Gaussian measure m is precisely the subspace $\overline{S(H)}$ (i.e. the closed linear span of its covariance operator range, see [BM09, Proposition 5.18]) is the key to prove the following lemma, inspired from the pioneering work [Fly95b] of Flytzanis:

Lemma 4.4 (**Unimodular Eigenvectors from Invariant Measures**). *Let $T \in \mathcal{L}(H)$, where H is a complex separable Hilbert space, and let μ be a (non-trivial) T-invariant probability measure on H such that $\int_H \|z\|^2 d\mu(z) < \infty$. Then we have the inclusions*

$$\operatorname{supp}(\mu) \subset \overline{\operatorname{span}(\operatorname{supp}(\mu))} \subset \overline{\operatorname{span}(\mathcal{E}(T))}.$$

Proof. Suppose first that μ is a centered measure on H. Then, since H is a Hilbert space, the covariance operator S of μ defined as in (2.3) satisfies

$$\langle Sx, y\rangle = \int_H \langle x, z\rangle \overline{\langle y, z\rangle} d\mu(z) = \int_{\operatorname{supp}(\mu)} \langle x, z\rangle \overline{\langle y, z\rangle} d\mu(z) \quad \text{for every } x, y \in H,$$

and by [BM09, Corollary 5.15] it is also the covariance operator of a certain Gaussian measure m on H. From now on we split the proof in three steps:

Step 1. *The Gaussian measure m is T-invariant*: Given $x, y \in H$ we have that

$$\begin{aligned} \langle TST^*x, y\rangle &= \langle ST^*x, T^*y\rangle = \int_H \langle T^*x, z\rangle \overline{\langle T^*y, z\rangle} d\mu(z) = \int_H \langle x, Tz\rangle \overline{\langle y, Tz\rangle} d\mu(z) \\ &= \int_H \langle x, z\rangle \overline{\langle y, z\rangle} d(\mu \circ T^{-1})(z) = \int_H \langle x, z\rangle \overline{\langle y, z\rangle} d\mu(z) = \langle Sx, y\rangle, \end{aligned}$$

since μ is T-invariant. By [BM09, Proposition 5.22] we deduce that m is T-invariant.

Step 2. *We have the equality* $\overline{\operatorname{span}(\operatorname{supp}(\mu))} = \operatorname{supp}(m)$: By [BM09, Proposition 5.18] we know that $\operatorname{supp}(m) = \operatorname{Ker}(S)^\perp = \overline{S(H)}$. Moreover, the subspace $\overline{\operatorname{span}(\operatorname{supp}(\mu))}^\perp$ is included in

$$\begin{aligned} &\left\{ y \in H : \langle Sx, y\rangle = \int_{\operatorname{supp}(\mu)} \langle x, z\rangle \overline{\langle y, z\rangle} d\mu(z) = 0 \text{ for every } x \in H \right\} = \\ &= \overline{S(H)}^\perp \subset \left\{ y \in H : \int_{\operatorname{supp}(\mu)} |\langle y, z\rangle|^2 d\mu(z) = 0 \right\} \\ &= \{ y \in H : \langle y, z\rangle = 0 \text{ for } \mu\text{-a.e. } z \in H \} \\ &\overset{(*)}{=} \{ y \in H : \langle y, z\rangle = 0 \text{ for every } z \in \operatorname{supp}(\mu) \} = \overline{\operatorname{span}(\operatorname{supp}(\mu))}^\perp, \end{aligned}$$

where the equality $(*)$ follows from the continuity of the maps $\langle y, \cdot\rangle : H \longrightarrow \mathbb{C}$.

Step 3. *We have the inclusion* $\operatorname{supp}(\mu) \subset \overline{\operatorname{span}(\mathcal{E}(T))}$: In [BM09, Theorem 5.46] it is stated that *if a Banach space X has cotype 2, then every operator in $\mathcal{L}(X)$ admitting an invariant Gaussian measure with full support has a spanning set of unimodular eigenvectors*. Since m is T-invariant by *Step* 1, we have that $\operatorname{supp}(m)$ is a T-invariant closed subspace of the Hilbert space H and then it has cotype 2. Hence, [BM09, Theorem 5.46] applied to $T|_{\operatorname{supp}(m)} : \operatorname{supp}(m) \longrightarrow \operatorname{supp}(m)$ and m, together with *Step* 2, implies that

$$\operatorname{supp}(\mu) \subset \overline{\operatorname{span}(\operatorname{supp}(\mu))} = \operatorname{supp}(m) \subset \overline{\operatorname{span}(\mathcal{E}(T))}.$$

Suppose now that μ is not centered and define the measure

$$\nu(A) := \int_{\mathbb{T}} \mu(\lambda A) d\lambda \quad \text{ for every Borel set } A \in \mathscr{B}(H).$$

Then ν is a (non-trivial) probability measure on H and it is T-invariant since

$$\nu(T^{-1}(A)) = \int_{\mathbb{T}} \mu(\lambda T^{-1}(A)) d\lambda = \int_{\mathbb{T}} \mu(T^{-1}(\lambda A)) d\lambda = \nu(A).$$

By the density of the simple functions in $L^1(H, \mathscr{B}(H), \nu)$ we have that ν is centered since

$$\int_H z d\nu(z) = \int_{\mathbb{T}} \left(\int_H \overline{\lambda} z d\mu(z) \right) d\lambda = \int_{\mathbb{T}} \overline{\lambda} \left(\int_H z d\mu(z) \right) d\lambda = 0,$$

and also that ν has a finite second-order moment since

$$\int_H \|z\|^2 d\nu(z) = \int_{\mathbb{T}} \left(\int_H \|\overline{\lambda} z\|^2 d\mu(z) \right) d\lambda = \int_H \|z\|^2 d\mu(z) < \infty.$$

The first part of the proof implies that $\text{supp}(\nu) \subset \overline{\text{span}(\text{supp}(\nu))} \subset \overline{\text{span}(\mathcal{E}(T))}$ so we just have to show that $\text{supp}(\mu) \subset \text{supp}(\nu)$. In order to check this pick $x_0 \in \text{supp}(\mu)$ and $\varepsilon > 0$. Then let $\delta := \mu(B(x_0, \varepsilon/2)) > 0$, where $B(x_0, \varepsilon/2)$ denotes the open ball of X centred at x_0 and of radius $\varepsilon/2$, and note that $B(x_0, \varepsilon/2) \subset \lambda B(x_0, \varepsilon)$ for any $\lambda \in \mathbb{T}$ with

$$|\lambda - 1| < \frac{\varepsilon}{2(\|x_0\| + 1)}.$$

Indeed, given $x \in B(x_0, \varepsilon/2)$ we have that $\|\lambda x_0 - x\| \leq \|(\lambda - 1)x_0\| + \|x_0 - x\| < \varepsilon$. Since for those $\lambda \in \mathbb{T}$ we have that $\mu(\lambda B(x_0, \varepsilon)) \geq \delta$ we deduce that $\nu(B(x_0, \varepsilon)) > 0$. The arbitrariness of $\varepsilon > 0$ implies that $x_0 \in \text{supp}(\nu)$. □

Remark 4.5. If we start the proof of Lemma 4.4 with the underlying space being a Banach space X which has type 2, then there exists a Gaussian measure m on X whose covariance operator is R as defined in (2.2). Indeed, since R is a symmetric and positive operator it admits a square root: there exist some separable Hilbert space H and an operator $K : H \longrightarrow X$ such that $R = KK^*$ (see [BM09, page 101]). Moreover, by the finite second-order moment condition of μ, the operator K^* is an absolutely 2-summing operator and hence such a Gaussian measure m on X exists by [BM09, Corollary 5.20]. However, in *Step* 3 of the proof above the underlying space needs to have cotype 2. The only spaces which are both of type 2 and of cotype 2 are isomorphic to Hilbert spaces, so the proof does not extend outside the Hilbertian setting.

We are now ready to prove Theorem 1.7.

4.2 Proof of Theorem 1.7

Let $T : H \longrightarrow H$ be an operator on a complex separable Hilbert space H. We already know that $\text{span}(\mathcal{E}(T)) \subset \text{URec}(T) \subset \text{RRec}^{bo}(T)$, so we just have to prove that

$$\text{RRec}^{bo}(T) \subset \overline{\text{span}(\mathcal{E}(T))}.$$

To see this pick $x_0 \in \text{RRec}^{bo}(T) \setminus \{0\}$ and let $M > 0$ be such that $\text{Orb}(x_0, T)$ is contained in MB_H, the $\|\cdot\|$-closed ball of radius M centred at 0. If we now denote by w the weak topology of the Hilbert space H, we have the inclusion

$$\overline{\text{Orb}(x_0, T)}^{w} \subset MB_H.$$

By Theorem 2.3 there is a (non-trivial, because $x_0 \neq 0$) T-invariant probability measure μ_{x_0} on X such that

$$x_0 \in \text{supp}(\mu_{x_0}) \subset \overline{\text{Orb}(x_0, T)}^{w},$$

and hence

$$\int_H \|z\|^2 d\mu_{x_0}(z) = \int_{\text{supp}(\mu_{x_0})} \|z\|^2 d\mu_{x_0}(z) \leq M^2 < \infty.$$

By Lemma 4.4 we get that $x_0 \in \text{supp}(\mu_{x_0}) \subset \overline{\text{span}(\mathcal{E}(T))}$ as we wanted to show.

Suppose now that there is a countable set $\{x_n : n \in \mathbb{N}\} \subset \text{RRec}^{bo}(T)$ which is dense in H. For each $n \in \mathbb{N}$ pick $M_n > 0$ and $k_n \in \mathbb{N}$ such that

$$\overline{\text{Orb}(x_n, T)}^{w} \subset M_n B_H \quad \text{and} \quad 2^n M_n^2 \leq 2^{k_n}.$$

Applying Theorem 2.3 to each x_n we obtain a sequence $(\mu_{x_n})_{n\in\mathbb{N}}$ of T-invariant probability measures on H such that $x_n \in \mathrm{supp}(\mu_{x_n}) \subset \overline{\mathrm{Orb}(x_n,T)}^w$ for each $n \in \mathbb{N}$. Consider the measure

$$\mu := \sum_{n\in\mathbb{N}} \frac{\mu_{x_n}}{2^{k_n}},$$

which is a (positive) T-invariant finite Borel measure on H with full support such that

$$\int_H \|z\|^2 d\mu(z) = \sum_{n\in\mathbb{N}} \frac{1}{2^{k_n}} \int_H \|z\|^2 d\mu_{x_n}(z) \leq \sum_{n\in\mathbb{N}} \frac{M_n^2}{2^{k_n}} \leq 1.$$

Normalizing the measure μ we get a T-invariant probability measure with full support and finite second-order moment.

Finally, in both cases (a) and (b) we have: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (vii) by definition; (vii) $\Rightarrow$ (viii) using Theorem 2.3 as in the above arguments; and (viii) $\Rightarrow$ (i) by the already used Lemma 4.4. □

Remark 4.6. The equalities $\overline{\mathrm{span}(\mathcal{E}(T))} = \overline{\mathrm{URec}(T)} = \overline{\mathrm{RRec}^{bo}(T)}$ and hence the equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (vii) established in Theorem 1.7 are still true when the underlying space H is a complex non-separable Hilbert space. Since the closed subspaces of a Hilbert space are again Hilbert spaces, the same arguments as those used in Remark 3.4 apply. We loose again the measures equivalences, i.e. statement (viii).

As mentioned in the Introduction and in Remark 4.5, the proof of Lemma 4.4 and hence that of Theorem 1.7 do not extend outside the Hilbertian setting.

We finish this section with the proof of Theorem 1.9, which concerns the power-bounded operators on complex reflexive Banach spaces X. The proof relies on the splitting theorem of Jacobs-Deleeuw-Glicksberg, and is really specific to the setting of power-bounded operators. We follow the presentation and notation of [Kre85, Section 2.4]: if $\mathscr{S}$ is a semigroup of $\mathcal{L}(X)$, we say that $\mathscr{S}$ is *weakly almost periodic* if for any $x \in X$ the set $\mathscr{S}x = \{Sx : S \in \mathscr{S}\}$ has a w-compact closure.

4.3 Proof of Theorem 1.9

Given a power-bounded operator $T : X \longrightarrow X$ on a complex reflexive Banach space X, we already know that $\mathrm{span}(\mathcal{E}(T)) \subset \mathrm{URec}(T)$ so we just have to show the inclusion

$$\mathrm{URec}(T) \subset \overline{\mathrm{span}(\mathcal{E}(T))}.$$

We set $O(x) := \overline{\mathrm{Orb}(x,T)}^w$ for each $x \in X$.

Since T is power-bounded, every T-orbit is bounded and has a w-compact closure. Hence, by the Jacobs-Deleeuw-Glicksberg theorem [Kre85, Section 2.4, Theorem 4.4] applied to the (weakly almost periodic) abelian semigroup of operators $\{T^n : n \in \mathbb{N}_0\} \subset \mathcal{L}(X)$, we obtain the direct sum decomposition

$$X = X_{rev} \oplus X_{fl},$$

where

$$X_{rev} := \{x \in X : y \in O(x) \Rightarrow x \in O(y)\} \quad \text{and} \quad X_{fl} := \{x \in X : 0 \in O(x)\}.$$

Moreover, by the second part of this same theorem [Kre85, Section 2.4, Theorem 4.5] we also get that

$$X_{rev} = \overline{\mathrm{span}(\mathcal{E}(T))}.$$

Let us now show that $\mathrm{URec}(T) \subset X_{rev}$. Indeed, given $x \in \mathrm{URec}(T) \setminus \{0\}$ we can consider the map $T|_{O(x)} : (O(x), w) \longrightarrow (O(x), w)$ which is a w-compact dynamical system. Since the weak topology is coarser than the norm topology we also have that $x \in \mathrm{URec}(T|_{O(x)})$ and hence by [Fur81, Theorem 1.17] the system $T|_{O(x)}$ is minimal so every $T|_{O(x)}$-orbit is dense in $O(x)$. Finally, given $y \in O(x)$ we have that

$$O(y) = \overline{\mathrm{Orb}(y,T)}^{w} = \overline{\mathrm{Orb}(y,T|_{O(x)})}^{w} = O(x)$$

which implies that $x \in O(y)$. The arbitrariness of $y \in O(x)$ shows that $x \in X_{rev}$. □

5 Product dynamical systems

Given a property of a dynamical system (X,T) it is usual to ask whether the *product dynamical system* $T \times T : X \times X \longrightarrow X \times X$ has the same property. Studied cases in Linear Dynamics are transitivity or hypercyclicity (which gives us the concept of topological weak-mixing), and in general $\mathcal{F}$-transitivity or $\mathcal{F}$-hypercyclicity (see [BMPP19] and [EEM21]). Here we show that the above theorems still work for the product systems.

Theorem 5.1 (From Reiterative to N-Dim. Frequent Recurrence). *Let $N \in \mathbb{N}$ and suppose that for each $1 \leq i \leq N$ there is a Polish dynamical system (X_i, T_i) such that (X_i, τ_{X_i}) can be endowed with a Hausdorff topology τ_i which fulfills* (I), (II), *and* (III*) *with respect to the map T_i and the topology τ_{X_i}. Then for the product dynamical system $T : (X, \tau_X) \longrightarrow (X, \tau_X)$, where τ_X is the product topology of the N-th τ_{X_i} topologies, we have the equality*

$$\overline{\mathrm{FRec}(T)}^{\tau_X} = \prod_{i=1}^{N} \overline{\mathrm{RRec}(T_i)}^{\tau_{X_i}}.$$

In particular:

(a) *The following statements are equivalent:*

- (i) $\mathrm{FRec}(T) \neq \varnothing$;
- (ii) $\mathrm{UFRec}(T) \neq \varnothing$;
- (iii) $\mathrm{RRec}(T) \neq \varnothing$;
- (iv) $\mathrm{RRec}(T_i) \neq \varnothing$ *for every* $1 \leq i \leq N$.

(b) *The following statements are equivalent:*

- (i) *T is frequently recurrent;*
- (ii) *T is $\mathcal{U}$-frequently recurrent;*
- (iii) *T is reiteratively recurrent;*
- (iv) *T_i is reiteratively recurrent for every $1 \leq i \leq N$.*

Proof. We clearly have the inclusion

$$\mathrm{FRec}(T) \subset \prod_{i=1}^{N} \mathrm{RRec}(T_i).$$

Now given $\boldsymbol{x}_0 = (x_1, ..., x_N) \in X$ such that $x_i \in \mathrm{RRec}(T_i)$ for each $1 \leq i \leq N$, let us show that $\boldsymbol{x}_0 \in \overline{\mathrm{FRec}(T)}^{\tau_X}$. Applying Theorem 2.3 we obtain a T_i-invariant measure μ_{x_i} on X_i such that $x_i \in \mathrm{supp}(\mu_{x_i})$ for each $1 \leq i \leq N$. Since

$$\mathscr{B}(X, \tau_X) = \prod_{i=1}^{N} \mathscr{B}(X, \tau_{X_i}),$$

we can consider the product measure $\mu_{\boldsymbol{x}_0} := \prod_{i=1}^{N} \mu_{x_i}$ on the product space X, which is a T-invariant measure (see [Wal82, Theorem 1.1 and Definition 1.2]) for which $\boldsymbol{x}_0 \in \mathrm{supp}(\mu_{\boldsymbol{x}_0})$. Applying now Lemma 3.1 we deduce that $\boldsymbol{x}_0 \in \overline{\mathrm{FRec}(T)}^{\tau_X}$. □

The following immediate corollaries yield a product version of Theorem 1.3:

Corollary 5.2. *Let $N \in \mathbb{N}$ and consider for each $1 \leq i \leq N$ an adjoint operator $T_i : X_i \longrightarrow X_i$ on a separable dual Banach space X_i. Then, for the direct sum operator $T = T_1 \oplus \cdots \oplus T_N$ acting on the direct sum space $X = X_1 \oplus \cdots \oplus X_N$, we have the equality*

$$\overline{\mathrm{FRec}(T)} = \prod_{i=1}^{N} \overline{\mathrm{RRec}(T_i)}.$$

In particular, the following statements are equivalent:

(i) *T is frequently recurrent;*

(ii) *T is $\mathcal{U}$-frequently recurrent;*

(iii) *T is reiteratively recurrent;*

(iv) *T_i is reiteratively recurrent for every $1 \leq i \leq N$.*

Moreover, the result holds whenever some of the T_i are operators acting on some reflexive Banach spaces X_i.

In the statement above, and whenever we consider a direct sum space $X_1 \oplus \cdots \oplus X_N$, one can use any norm defining the usual product topology on $X_1 \oplus \cdots \oplus X_N$ (see Theorem 5.8).

Definition 5.3. Let (X, T) be a linear dynamical system and let $N \in \mathbb{N}$. We will denote by $T_{(N)} : X^N \longrightarrow X^N$ the *N-fold direct sum* of T with itself, i.e. the dynamical system

$$T_{(N)} := \underbrace{T \oplus \cdots \oplus T}_{N} : \underbrace{X \oplus \cdots \oplus X}_{N} \longrightarrow \underbrace{X \oplus \cdots \oplus X}_{N},$$

where $X^N := \underbrace{X \oplus \cdots \oplus X}_{N}$ is the *N-fold direct sum* of X with itself.

Corollary 5.4. *Let $T : X \longrightarrow X$ be an adjoint operator on a separable dual Banach space X. Then the following statements are equivalent:*

(i) *$T_{(N)}$ is frequently recurrent for every $N \in \mathbb{N}$;*

(ii) *$T_{(N)}$ is $\mathcal{U}$-frequently recurrent for every $N \in \mathbb{N}$;*

(iii) *$T_{(N)}$ is reiteratively recurrent for every $N \in \mathbb{N}$;*

(iv) *T is reiteratively recurrent.*

In particular, the result holds whenever T is an operator on a reflexive Banach space X.

As a consequence of the above fact we can prove some results related with hypercyclicity. We start with an independent proof of [EEM21, Theorem 2.5 and Corollary 2.6] for the particular case of the reiteratively hypercyclic (adjoint) operators:

Theorem 5.5. *Let $T : X \longrightarrow X$ be a reiteratively hypercyclic adjoint operator on a separable dual Banach space X. Then $T_{(N)}$ is reiteratively hypercyclic and frequently recurrent for every $N \in \mathbb{N}$. The result holds if T is an operator on a separable reflexive Banach space X.*

Proof. Let $N \in \mathbb{N}$. Since T is reiteratively hypercyclic we know that:

(a) T is topologically weakly-mixing (see [BMPP16, page 548]), and hence $T_{(N)}$ is topologically transitive, and in particular hypercyclic;

(b) T is reiteratively recurrent, and by the above results $T_{(N)}$ is frequently recurrent, and in particular reiteratively recurrent.

By [BGELMP22, Theorem 2.1], reiterative recurrence plus hypercyclicity imply reiterative hypercyclicity. We deduce that $T_{(N)}$ is reiteratively hypercyclic and frequently recurrent. □

If we start just with reiterative recurrence, having a dense set of orbits converging to 0 implies a strong notion of hypercyclicity:

Theorem 5.6. *Let $T : X \longrightarrow X$ be an adjoint operator on a separable dual Banach space X. Suppose that there is a dense set $X_0 \subset X$ such that $T^k x \to 0$ as $k \to \infty$ for each $x \in X_0$. The following statements are equivalent:*

(i) *$T_{(N)}$ is $\mathcal{U}$-frequently hypercyclic and frequently recurrent for every $N \in \mathbb{N}$;*

(ii) *T is reiteratively recurrent.*

In particular, the result holds if T is an operator on a separable reflexive Banach space X.

Proof. Clearly (i) implies (ii) even if $T : X \longrightarrow X$ is not a linear map. If we suppose (ii) and we fix $N \in \mathbb{N}$, by the above results we get that $T_{(N)}$ is frequently recurrent and in particular $\mathcal{U}$-frequently recurrent. Let $Y_0 := X_0 \oplus \cdots \oplus X_0$ be the N-fold direct sum of the set X_0, which is a dense subset of the N-fold direct sum X^N whose orbits converge to $(0, ..., 0) \in X^N$. Using now [BGELMP22, Theorem 2.12], the existence of Y_0 and the $\mathcal{U}$-frequent recurrence imply that the operator $T_{(N)}$ is $\mathcal{U}$-frequently hypercyclic. □

It would be interesting to change the assumption of $\mathcal{U}$*-frequent hypercyclicity* in the above statement into that of *frequent hypercyclicity*, but as exposed in [BGELMP22, Question 2.13], the following is an open problem:

Question 5.7 ([**BGELMP22, Question 2.13**])**.** Let T be a frequently recurrent operator admitting a dense set of vectors whose orbits converge to 0. Is T frequently hypercyclic?

If now we focus on Theorems 1.7 and 1.9, their generalizations for product linear dynamical systems follow in a much easier way, since any N-tuple formed by unimodular eigenvectors is a linear combination of such vectors for the direct sum map:

Theorem 5.8. *Let $N \in \mathbb{N}$ and suppose that for each $1 \leq i \leq N$ we have:*

(a) *an operator $T_i : H_i \longrightarrow H_i$ on a complex Hilbert space H_i. Then, for the direct sum operator $T = T_1 \oplus \cdots \oplus T_N$ on the direct sum Hilbert space $H = H_1 \oplus \cdots \oplus H_N$, we have the equality*

$$\overline{\operatorname{span}(\mathcal{E}(T))} = \prod_{i=1}^{N} \overline{\operatorname{RRec}^{bo}(T_i)},$$

(b) *a power-bounded operator $T_i : X_i \longrightarrow X_i$ on a complex reflexive Banach space X_i. Then, for the direct sum operator $T = T_1 \oplus \cdots \oplus T_N$ on the direct sum space $X = X_1 \oplus \cdots \oplus X_N$, we have the equality*

$$\overline{\operatorname{span}(\mathcal{E}(T))} = \prod_{i=1}^{N} \overline{\operatorname{URec}(T_i)}.$$

In particular, in both cases $\operatorname{span}(\mathcal{E}(T))$ is dense in the (corresponding) direct sum space if and only if T_i is uniformly recurrent for every $1 \leq i \leq N$.

Proof. The vector $(0, ..., 0, x_i, 0, ..., 0)$ belongs to $\mathcal{E}(T)$ whenever $x_i \in \mathcal{E}(T_i)$ for each $1 \leq i \leq N$. It is enough to apply Theorems 1.7 and 1.9 to each operator T_i. □

Finally we get the desired generalization of Theorems 1.7 and 1.9:

Corollary 5.9. *Let $T \in \mathcal{L}(H)$ where H is a complex Hilbert space. The following are equivalent:*

(i) *the set $\operatorname{span}(\mathcal{E}(T_{(N)}))$ is dense in H^N for every $N \in \mathbb{N}$;*

(ii) *$T_{(N)}$ is Δ^*-recurrent for every $N \in \mathbb{N}$;*

(iii) *$T_{(N)}$ is $\mathcal{IP}^*$-recurrent for every $N \in \mathbb{N}$;*

(iv) *$T_{(N)}$ is uniformly recurrent for every $N \in \mathbb{N}$;*

(v) *the set $\operatorname{FRec}^{bo}(T_{(N)})$ is dense in H^N for every $N \in \mathbb{N}$;*

(vi) *the set $\operatorname{UFRec}^{bo}(T_{(N)})$ is dense in H^N for every $N \in \mathbb{N}$;*

(vii) *the set $\operatorname{RRec}^{bo}(T_{(N)})$ is dense in H^N for every $N \in \mathbb{N}$;*

(viii) *the set $\operatorname{RRec}^{bo}(T)$ is dense in H.*

Corollary 5.10. *Let $T : X \longrightarrow X$ be a power-bounded operator on a complex reflexive Banach space X. The following statements are equivalent:*

(i) *the set* $\operatorname{span}(\mathcal{E}(T_{(N)}))$ *is dense in* X^N *for every* $N \in \mathbb{N}$;

(ii) $T_{(N)}$ *is* Δ^**-recurrent for every* $N \in \mathbb{N}$;

(iii) $T_{(N)}$ *is* $\mathcal{IP}^*$*-recurrent for every* $N \in \mathbb{N}$;

(iv) $T_{(N)}$ *is uniformly recurrent for every* $N \in \mathbb{N}$;

(v) T *is uniformly recurrent.*

6 Inverse dynamical systems

As in the case of products, given a dynamical system $T : X \longrightarrow X$ with some property, it is natural to ask if the *inverse dynamical system* $T^{-1} : X \longrightarrow X$ (if it exists and is continuous) has the same property. This is true for hypercyclicity and reiterative hypercyclicity (see [BGE18]), but it fails for $\mathcal{U}$-frequent hypercyclicity (see [Men20]) and frequent hypercyclicity (see [Men22]). It is also known that the inverse of a frequently hypercyclic operator is $\mathcal{U}$-frequently hypercyclic (see [BR15, Proposition 20]).

If we focus on recurrence properties, the inverse of a recurrent operator is again recurrent as [CMP14, Proposition 2.6] shows. A simpler proof (in a transitive style) of that fact would be:

Proposition 6.1 ([**CMP14, Proposition 2.6**])**.** *Let $T : X \longrightarrow X$ be an invertible operator. Then T is recurrent if and only if so is T^{-1}.*

Proof. By [CMP14, Proposition 2.1] the result follows from the equivalence

$$T^n(U) \cap U \neq \varnothing \text{ if and only if } U \cap T^{-n}(U) \neq \varnothing,$$

valid for any non-empty open subset U of X. □

However, it is also shown in [CMP14, Remark 2.7] that the sets $\operatorname{Rec}(T)$ and $\operatorname{Rec}(T^{-1})$ may not be equal in spite of the fact that their closures coincide. For general $\mathcal{F}$-recurrence notions the following problem was proposed in [BGELMP22]:

Question 6.2 ([**BGELMP22, Question 2.14**])**.** Let T be an invertible operator which is reiteratively ($\mathcal{U}$-frequently, frequently, uniformly) recurrent, does T^{-1} have the same property?

We can ask the same question for $\mathcal{IP}^*$, Δ^*-recurrence and for the existence of unimodular eigenvectors. However, for the latest, linearity is enough to verify this property since

$$Tx = \lambda x \text{ for some } \lambda \in \mathbb{T} \implies T^{-1}x = \overline{\lambda}(T^{-1}\lambda x) = \overline{\lambda}x,$$

so clearly $\operatorname{span}(\mathcal{E}(T)) = \operatorname{span}(\mathcal{E}(T^{-1}))$. To answer Question 6.2 in our dual/reflexive setting we just have to recall the following trivial fact: given any homeomorphism $T : X \longrightarrow X$ of a Polish space X and any Borel measure μ on $(X, \mathscr{B}(X))$, then the measure μ is T-invariant if and only if it is T^{-1}-invariant.

Theorem 6.3 (**From Reiterative to Inverse Frequent Recurrence**). *Let $T : X \longrightarrow X$ be a homeomorphism of the Polish space (X, τ_X), and assume that X is endowed with a Hausdorff topology τ_1 which fulfills* (I) *for T,* (II)*, and* (III*)*. Then we have*

$$\overline{\mathrm{FRec}(T)}^{\tau_X} = \overline{\mathrm{RRec}(T)}^{\tau_X} \subset \overline{\mathrm{FRec}(T^{-1})}^{\tau_X} \subset \overline{\mathrm{RRec}(T^{-1})}^{\tau_X}.$$

Moreover:

(a) *If T is reiteratively recurrent then T^{-1} is frequently recurrent.*

(b) *If* $\mathrm{RRec}(T) \neq \varnothing$ *then* $\mathrm{FRec}(T) \cap \mathrm{FRec}(T^{-1}) \neq \varnothing$.

(c) *If X can be endowed with a Hausdorff topology τ_2 which fulfills* (I) *for T^{-1},* (II)*, and* (III*)*, then the above inclusions are equalities and T is reiteratively (and hence frequently) recurrent if and only if so is T^{-1}.*

Proof. The equality is shown in Theorem 3.3. The first inclusion follows from Lemma 3.1 applied to the measures constructed with Theorem 2.3 for each point of $\mathrm{RRec}(T)$, using the fact that they are T^{-1}-invariant. The second inclusion follows by definition. Moreover, if there exists a point $x_0 \in \mathrm{RRec}(T)$ and if we take the invariant probability measure μ_{x_0} on X constructed with Theorem 2.3, then by Lemma 3.1 we have $\mu_{x_0}(\mathrm{FRec}(T)) = 1 = \mu_{x_0}(\mathrm{FRec}(T^{-1}))$ which implies that $\mathrm{FRec}(T) \cap \mathrm{FRec}(T^{-1}) \neq \varnothing$. Finally, if such a topology τ_2 exists we can apply the first part of the result to T^{-1} obtaining $\overline{\mathrm{RRec}(T^{-1})}^{\tau_X} \subset \overline{\mathrm{FRec}(T)}^{\tau_X}$. □

As a corollary of the above theorem, and using the arguments from Theorem 1.3 we have:

Corollary 6.4. *Let $T : X \longrightarrow X$ be an invertible adjoint operator on a separable dual Banach space X. Then we have the equalities*

$$\overline{\mathrm{RRec}(T)} = \overline{\mathrm{FRec}(T)} = \overline{\mathrm{FRec}(T^{-1})} = \overline{\mathrm{RRec}(T^{-1})}.$$

Moreover:

(a) *T is reiteratively (and hence frequently) recurrent if and only if so is T^{-1}.*

(b) *If* $\mathrm{RRec}(T) \setminus \{0\} \neq \varnothing$ *then* $[\mathrm{FRec}(T) \cap \mathrm{FRec}(T^{-1})] \setminus \{0\} \neq \varnothing$.

In particular, the result holds whenever T is an operator on a reflexive Banach space X.

Proof. Let $S : Y \longrightarrow Y$ be an operator on a Banach space Y such that $Y^* = X$ and $S^* = T$. It is a known fact that T is invertible if and only if S is invertible, and in this case, $T^{-1} = (S^{-1})^*$, so T^{-1} is also an adjoint operator on the separable Banach space X and hence it is w^*-continuous. The above theorems can be now applied to $T : X \longrightarrow X$, $(X, \|\cdot\|)$ and the topology w^*. □

We can now give an alternative prove of [BGE18, Theorem 3.6] for adjoint operators:

Theorem 6.5. *Let $T : X \longrightarrow X$ be an invertible adjoint operator on a separable dual Banach space. If T is reiteratively hypercyclic (and hence frequently recurrent) then so is T^{-1}. In particular, the result holds whenever T is an operator on a separable reflexive Banach space X.*

Proof. By the above theorem T^{-1} is frequently recurrent and hence reiteratively recurrent. Since hypercyclicity is also preserved by taking the inverse system, [BGELMP22, Theorem 2.1] implies that T^{-1} is reiteratively hypercyclic. □

We cannot change the *reiterative hypercyclicity* in the statement of Theorem 6.5 above into the assumption of *$\mathcal{U}$-frequent hypercyclicity* since there are invertible $\mathcal{U}$-frequently hypercyclic operators on $\ell^p(\mathbb{N})$ $(1 \leq p < \infty)$ whose inverse is not $\mathcal{U}$-frequently hypercyclic (see [Men20]). However, it would be interesting to know whether it is possible to change the assumption of *reiterative hypercyclicity* into that of *frequent hypercyclicity*: even though it is known that there are invertible frequently hypercyclic operators on $\ell^1(\mathbb{N})$ whose inverse is not frequently hypercyclic (see [Men22]), one can check that these are not adjoint operators and, moreover, the inverse of a frequently hypercyclic operator is always $\mathcal{U}$-frequently hypercyclic as showed in [BR15, Proposition 20].

All the counterexamples mentioned here are C-type operators, which were introduced for the first time in [Men17] and further developed in [GMM21b, Men20, Men22], so a possible counterexample for the frequent hypercyclicity case could arise from those operators. If, on the other hand, one wishes to prove an analogue of Theorem 6.5 for the frequent hypercyclicity case in our dual/reflexive framework, one cannot take a similar approach since there are chaotic operators, which are in particular frequently recurrent and hypercyclic, but not $\mathcal{U}$-frequently hypercyclic (see [Men17] and [GMM21b]) and hence not frequently hypercyclic.

Theorems 1.7 and 1.9 and the equality $\operatorname{span}(\mathcal{E}(T)) = \operatorname{span}(\mathcal{E}(T^{-1}))$ give us the following:

Corollary 6.6. *Let $T : H \longrightarrow H$ be an invertible operator on a complex Hilbert space H. Then we have the equalities*

$$\overline{\mathrm{RRec}^{bo}(T)} = \overline{\operatorname{span}(\mathcal{E}(T))} = \overline{\operatorname{span}(\mathcal{E}(T^{-1}))} = \overline{\mathrm{RRec}^{bo}(T^{-1})}.$$

In particular, T is uniformly (and hence $\mathcal{IP}^*$ and Δ^*) recurrent if and only if so is T^{-1}.

Corollary 6.7. *Let $T : X \longrightarrow X$ be an invertible operator on a complex reflexive space X. If T is power-bounded, then we have*

$$\overline{\mathrm{URec}(T)} = \overline{\operatorname{span}(\mathcal{E}(T))} = \overline{\operatorname{span}(\mathcal{E}(T^{-1}))} \subset \overline{\mathrm{URec}(T^{-1})}.$$

In particular, if T is uniformly recurrent then $\operatorname{span}(\mathcal{E}(T^{-1}))$ is a dense set in X. Moreover, if T^{-1} is also power-bounded then the above inclusion is an equality and the operator T is uniformly (and hence $\mathcal{IP}^*$ and Δ^*) recurrent if and only if so is T^{-1}.

7 How typical is a reiteratively recurrent operator?

Let H be a complex separable Hilbert space. For any $M > 0$, denote by $\mathcal{L}_M(H)$ the set of bounded operators $T \in \mathcal{L}(H)$ such that $\|T\| \leq M$. Our aim in this short section is to present a result pertaining to the typicality of reiteratively recurrent operators of $\mathcal{L}_M(H)$, with $M > 1$, for one of the two (Polish) topologies SOT and SOT*. The framework that we use here is presented in detail in [GMM21b, Chapters 2 and 3], so we will be rather brief in our presentation and refer the readers to the works [GM22], [GMM21b] or [GMM21a] for more on typical properties of operators on Hilbert or Banach spaces.

We recall that the *Strong Operator Topology* (SOT) on $\mathcal{L}(H)$ is defined as follows: any $T_0 \in \mathcal{L}(H)$ has a SOT-neighbourhood basis consisting of sets of the form

$$U_{T_0,x_1,...,x_s,\varepsilon} := \{T \in \mathcal{L}(H) : \|(T - T_0)x_i\| < \varepsilon \text{ for } i = 1, ..., s\},$$

where $x_1, ..., x_s \in H$ and $\varepsilon > 0$. The *Strong* Operator Topology* (SOT*) is the "self-adjoint" version of the SOT: a basis of SOT*-neighbourhoods of $T_0 \in \mathcal{L}(H)$ is provided by the sets

$$V_{T_0,x_1,...,x_s,\varepsilon} := \{T \in \mathcal{L}(H) : \|(T - T_0)x_i\| < \varepsilon \text{ and } \|(T - T_0)^* x_i\| < \varepsilon \text{ for } i = 1, ..., s\},$$

where $x_1, ..., x_s \in H$ and $\varepsilon > 0$.

It is easily shown that $(\mathcal{L}_M(H), \text{SOT})$ and $(\mathcal{L}_M(H), \text{SOT}^*)$ are Polish spaces for any $M > 0$ (see [Ped89, Section 4.6.2]), and hence, a property of operators $T \in \mathcal{L}_M(H)$ will be called *typical* if the set of operators fulfilling it is co-meager (i.e. contains a dense G_δ-set), and *atypical* if its negation is typical. Following the notation used in [GMM21b] we can write

$$\begin{aligned} \text{HC}(H) &:= \{T \in \mathcal{L}(H) \text{ hypercyclic}\}; \\ \text{INV}(H) &:= \{T \in \mathcal{L}(H) \text{ admitting a non-trivial invariant measure}\}; \end{aligned}$$

and for each $M > 1$ the set $\text{HC}_M(H)$ is defined as $\text{HC}(H) \cap \mathcal{L}_M(H)$. Following the spirit of this study we introduce the following notation:

$$\begin{aligned} \text{RHC}(H) &:= \{T \in \mathcal{L}(H) \text{ reiteratively hypercyclic}\}; \\ \text{RRec}(H) &:= \{T \in \mathcal{L}(H) \text{ reiteratively recurrent}\}; \\ \text{RRec}^{\neq\varnothing}(H) &:= \{T \in \mathcal{L}(H) : \text{RRec}(T) \setminus \{0\} \neq \varnothing\}; \end{aligned}$$

and as it was done previously for the set $\text{HC}(H)$, for each $M > 1$, we will denote the respective bounded versions of these sets of operators by $\text{RHC}_M(H)$, $\text{RRec}_M(H)$ and $\text{RRec}_M^{\neq\varnothing}(H)$.

Let us first recall that a SOT-typical operator in $\mathcal{L}_M(H)$, for $M > 1$, has any form of recurrence one can wish for: by [EM13] a typical $T \in \mathcal{L}_M(H)$ is unitarily similar to MB_∞, where B_∞ denotes the backward shift of infinite multiplicity on $\ell^2(\mathbb{N}, \ell^2(\mathbb{N}))$, and MB_∞ is such that the linear span of its unimodular eigenvectors is dense, see [EM13, Theorem 5.2].

With respect to the topology SOT*, it is proved in [GMM21b, Theorem 2.29] that for every $M > 1$ the set $\text{HC}_M(H) \setminus \text{INV}(H)$ is co-meager in the space $(\text{HC}_M(H), \text{SOT}^*)$. In other words, a SOT*-typical hypercyclic operator on H admits no non-trivial invariant measure. Combining now Theorem 2.3 of the present work with [GMM21b, Theorem 2.29] we obtain:

Corollary 7.1. *For every $M > 1$, the set $\text{RRec}_M^{\neq\varnothing}(H)$ is meager in $(\mathcal{L}_M(H), \text{SOT}^*)$. In other words, a SOT*-typical operator on H has no non-zero reiteratively recurrent point.*

Since [BGELMP22, Theorem 2.1] shows that reiterative recurrence plus hypercyclicity equals reiterative hypercyclicity, we have that $\text{RHC}(H) = \text{RRec}(H) \cap \text{HC}(H)$. Using Corollary 7.1 we can improve [GMM21b, Corollary 2.36] in terms of reiterative recurrence:

Corollary 7.2. *For every $M > 1$, the set $\text{RRec}_M^{\neq\varnothing}(H) \cap \text{HC}_M(H)$ is meager in $(\text{HC}_M(H), \text{SOT}^*)$. In particular, the set $\text{RHC}_M(H)$ is meager in $(\text{HC}_M(H), \text{SOT}^*)$. In other words, a SOT*-typical hypercyclic operator on H does not admit any non-zero reiteratively recurrent point, and, in particular, is not reiteratively hypercyclic.*

8 Open problems

In this section we gather some possibly interesting open questions and a few comments related to them. We start by Questions 1.6 and 1.8, already stated in Subsection 1.3, which we recall here with some extra generality:

Question 8.1 (**Question 1.8**). Let T be a uniformly recurrent operator on a complex Fréchet space X. Is $\text{span}(\mathcal{E}(T))$ a dense set in X? What about the cases where T is an adjoint operator on a separable dual Banach space or where X is a reflexive Banach space?

Question 8.2 (**[BGELMP22, Question 6.3] and Question 1.6**). Does there exist an operator (possibly on a Fréchet space) which is uniformly recurrent but not Δ^*-recurrent? What about distinguishing uniform recurrence from $\mathcal{IP}^*$-recurrence?

Note that these two questions make sense in the more general context of *complex Fréchet spaces*, and in fact both questions are still unsolved for that rather general class of spaces. It is clear that, in every possible *complex* context, a positive answer to Question 8.1 implies a negative one to Question 8.2. Moreover, it would even imply a negative answer for the *real* case of Question 8.2: given any uniformly recurrent *real* linear dynamical system we could consider its complexification, and by the product-arguments used for Theorem 5.8 we would get unimodular eigenvectors and hence Δ^*-recurrence; the initial *real* dynamical system could possibly not contain the obtained unimodular eigenvectors, but the real and complex parts of such vectors would clearly be Δ^*-recurrent for the original *real* system.

It is worth mentioning that uniform and $\mathcal{IP}^*$-recurrence can be completely distinguished in the context of compact dynamical systems (see for instance the construction from [FL98], its properties in [CLW06] and then use [Fur81, Theorems 1.15 and 9.12]), so that the question here is if the linearity avoids that distinction.

The technique used in the proof of Theorem 1.7 (via Gaussian measures) is very different from the one used in Theorem 1.9 (via the Jacobs-Deleeuw-Glicksberg theorem). Indeed we loose the contact with measures and the unimodular eigenvectors are obtained from a totally different construction (see [Kre85, Section 2.4]). It seems to us that a more general "eigenvectors' constructing machine", not restricted to the measures or power-bounded assumptions, should be developed in order to provide a better answer to Question 8.1. What we know for the moment, leaving apart the power-bounded case which seems very specific, is the following:

Proposition 8.3. *Let $T \in \mathcal{L}(H)$ where H is a complex separable Hilbert space. Given a T-invariant w-compact subset K of H for which $0 \notin K$, we have*

$$\overline{\text{span}(\mathcal{E}(T))} \cap K \neq \varnothing \quad \textit{and in particular} \quad \mathcal{E}(T) \neq \varnothing.$$

Proof. We have that $T|_K : (K, w) \longrightarrow (K, w)$ is a w-compact dynamical system, so it admits a $T|_K$-invariant probability measure μ on K (see [Fur81, page 62]). Since the norm topology and the weak topology on H have the same Borel sets, we can extend the measure μ into a Borel probability measure on the whole space H (still denoted by μ) using the formula

$$\mu(A) := \mu(K \cap A) \quad \text{for every Borel set } A \in \mathscr{B}(H).$$

Note that μ is T-invariant. We deduce that: μ is non-trivial, since $0 \notin K$; and μ has a finite second-order moment, since $\text{supp}(\mu) \subset K$. Lemma 4.4 implies that $\overline{\text{span}(\mathcal{E}(T))} \cap K \neq \varnothing$ and in particular $\mathcal{E}(T) \neq \varnothing$. □

Remark 8.4. Let H be a complex separable Hilbert space. Since the set $\overline{\mathrm{Orb}(x,T)}^{w}$ is a T-invariant w-compact subset of H for any point $x \in H$ with bounded T-orbit, the arguments of the above proposition imply that *for any $M > 1$, a SOT^*-typical operator $T \in \mathcal{L}_M(H)$ has the property that every bounded orbit of T contains 0 in its weak closure.*

Proposition 8.5. *Let T be an adjoint operator on a complex separable dual Banach space X. Let $n \in \mathbb{N}$ and $\lambda \in \mathbb{T}$. Given a $[\lambda T]^n$-invariant w^*-compact and convex subset K of H for which $0 \notin K$, we have*

$$\mathcal{E}(T) \cap \mathrm{span}(\mathrm{Orb}(x,T)) \neq \varnothing \quad \textit{for some } x \in K,$$

and in particular $\mathcal{E}(T) \neq \varnothing$.

Proof. By the Schauder fixed-point theorem there is $x \in K$ for which the identity $[\lambda T]^n x = x$ holds. Taking $\alpha = \lambda^{-n} \in \mathbb{T}$ we get that $(\alpha - T^n)x = 0$. If we split the polynomial $(\alpha - z^n) \in \mathbb{C}[z]$ we have

$$(\alpha - z^n) = \prod_{i=1}^{n} (\alpha_i - z),$$

where the α_i are distinct n-th roots of α in $\mathbb{T}$. Considering the vectors

$$y_0 := x \quad \text{and} \quad y_j := (\alpha_j - T)y_{j-1} = \prod_{i=1}^{j} (\alpha_i - T)x \quad \text{for each } 1 \leq j \leq n,$$

we have $y_0 \neq 0$ since $0 \notin K$, but $y_n = (\alpha - T^n)x = 0$. Then for some $0 \leq k \leq n-1$ we have that $y_k \in \mathcal{E}(T) \cap \mathrm{span}(\mathrm{Orb}(x,T))$. In particular $\mathcal{E}(T) \neq \varnothing$. ☐

Another natural question concerning Theorem 1.7 is the relevance, in assertions (v) to (vii) of both parts (a) and (b), of the assumption that the vectors under consideration have *bounded* orbit. This fact is used in order to ensure that the invariant measures, which by Theorem 2.3 can be constructed from each reiteratively recurrent vector, have a finite second-order moment. To omit this boundedness assumption (or weak versions of it) seems to require new ideas. We recall here the following open problem from [GMM21b]:

Question 8.6 ([GMM21b, Question 8.3]). Is there any operator on a complex separable Hilbert space admitting a non-trivial invariant probability measure but no eigenvalues?

The following product and inverse questions also remain open:

Question 8.7. Let T be an linear operator acting on a Fréchet space X. If T is reiteratively ($\mathcal{U}$-frequently, frequently, uniformly) recurrent, does $T_{(N)}$ have the same property for $N \geq 2$?

Question 8.8 ([BGELMP22, Question 2.14] and Question 6.2). Let T be an invertible operator on a Fréchet space. If T is reiteratively ($\mathcal{U}$-frequently, frequently, uniformly, $\mathcal{IP}^*$, Δ^*) recurrent, does T^{-1} have the same property?

Question 8.7 is also open for usual recurrence, as defined in Subsection 1.1, and it seems to be a non-trivial question[D]. For the $\mathcal{IP}^*$ and Δ^* cases, the fact that such families have the filter property (see [BD08]) implies a positive answer.

[D] The usual recurrence version of this question has recently been solved in the negative; see Chapter 3.

As mentioned in [BGELMP22], the set of periodic points $\mathrm{Per}(T)$ of an operator T has the property that $\mathrm{Per}(T)$ is either equal to X or is a meager set (by the Baire category theorem, either $\mathrm{Per}(T)$ is of first category or else $T^n = I$ for some $n \in \mathbb{N}$). The same phenomenon happens (at least when X is a Banach space) with the set of uniformly recurrent vectors $\mathrm{URec}(T)$ since, by [BGELMP22, Corollary 3.2], if $\mathrm{URec}(T)$ is co-meager in X then T is a power-bounded operator and $\mathrm{URec}(T) = X$. This motivates the following question:

Question 8.9 ([BGELMP22, Question 2.9]). Let T be an operator on a Fréchet space X. Do we always have that either $\mathrm{FRec}(T) = X$ or $\mathrm{FRec}(T)$ is a meager set?

Even in the dual/reflexive setting we cannot say anything: the frequently recurrent points obtained in our construction form a "big" set with respect to a certain invariant measure, and usually this has nothing to do with the "bigness" from the Baire category point of view. In fact, any (Devaney) chaotic operator $T : X \longrightarrow X$ (i.e. hypercyclic with dense periodic vectors) admits an invariant probability measure μ on X with full support (see [GE06, Corollary 3.6]) and hence $\mu(\mathrm{FRec}(T)) = 1$ by Lemma 3.1. However, since T is hypercyclic we have that the set $\mathrm{FRec}(T)$ is meager, otherwise by [BGELMP22, Theorem 2.7] the set

$$\mathrm{FHC}(T) = \mathrm{FRec}(T) \cap \mathrm{HC}(T)$$

would be co-meager contradicting [BR15, Corollary 19].

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Chapter 3

Questions in linear recurrence: From the $T \oplus T$-problem to lineability

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Abstract

We study, for a continuous linear operator T on an F-space X, when the direct sum operator $T \oplus T$ is recurrent on $X \oplus X$. In particular: we establish the analogous notion, for recurrence, to that of (topological) *weak-mixing* for transitivity/hypercyclicity, namely *quasi-rigidity*; and we construct a recurrent but not quasi-rigid operator on each separable infinite-dimensional Banach space, solving the $T \oplus T$*-recurrence problem* in the negative way. The quasi-rigidity notion is closely related to the *dense lineability* of the set of recurrent vectors, and using similar conditions we study the lineability and dense lineability properties for the set of $\mathcal{F}$-recurrent vectors, under very weak assumptions on the Furstenberg family $\mathcal{F}$.

1 Introduction

This paper focuses on some aspects of *Linear Dynamics* and the general setting is the following: a (*real* or *complex*) *linear dynamical system* (X, T) is a pair formed by a continuous linear operator $T : X \longrightarrow X$ acting on a (real or complex) separable infinite-dimensional F-space (i.e. X is a completely metrizable topological vector space). We will denote by $\mathcal{L}(X)$ the *set of continuous linear operators* acting on such a space X, and the symbol $\mathbb{K}$ will stand for either the real or complex field, $\mathbb{R}$ or $\mathbb{C}$.

Given a linear dynamical system (X, T), the T*-orbit* of a vector $x \in X$ is the set

$$\mathrm{Orb}(x, T) := \{x, Tx, T^2x, T^3x, ...\} = \{T^n x : n \in \mathbb{N}_0\},$$

where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We say that a vector $x \in X$ is *hypercyclic* for T if $\mathrm{Orb}(x, T)$ is a dense set in X; and that the operator T is *hypercyclic* if it admits a hypercyclic vector. The *Birkhoff transitivity theorem* (see [BM09a, Theorem 1.2]) shows the equivalence between hypercyclicity and topological transitivity: a system (X, T) is called *topologically transitive* if for each pair of non-empty open subsets $U, V \subset X$ one can find some (and hence infinitely many) $n \in \mathbb{N}_0$ such that $T^n(U) \cap V \neq \varnothing$. It follows that the set $\mathrm{HC}(T)$ of hypercyclic vectors for T is a dense G_δ subset of X as soon as T is hypercyclic.

Hypercyclicity has been the main and most studied notion in Linear Dynamics. For instance, the following long-standing problem, which we call the $T \oplus T$*-hypercyclicity problem*, was posed in 1992 by D. Herrero [Her91]:

Question 1.1 (The $T \oplus T$-hypercyclicity problem). Let T be a hypercyclic operator on the F-space X. Is the operator $T \oplus T$ acting on the direct sum $X \oplus X$ hypercyclic?

Recall that an operator T is said to be (topologically) *weakly-mixing* if and only if $T \oplus T$ is transitive, so the question above asks if there exists any transitive but not weakly-mixing operator. The active research on hypercyclicity yielded many equivalent reformulations of Question 1.1, and the approach which will interest us most is the one taken in 1999 by Bès and the third author of this paper. Here is the main result they obtained in [BP99]:

– *A continuous linear operator $T \in \mathcal{L}(X)$ is (topologically) weakly-mixing if and only if it satisfies the so-called Hypercyclicity Criterion.*

In other words, $T \oplus T \in \mathcal{L}(X \oplus X)$ is a hypercyclic operator if and only if the following holds: *there exist two dense subsets $X_0, Y_0 \subset X$, an increasing sequence of positive integers $(n_k)_{k \in \mathbb{N}}$, and a family of (not necessarily continuous) mappings $S_{n_k} : Y_0 \longrightarrow X$ such that*

(i) $T^{n_k} x \to 0$ *for each* $x \in X_0$*;*

(ii) $S_{n_k} y \to 0$ *for each* $y \in Y_0$*;*

(iii) $T^{n_k} S_{n_k} y \to y$ *for each* $y \in Y_0$*.*

Question 1.1 was finally answered negatively in 2006 by De La Rosa and Read, when they exhibited a hypercyclic operator on a particular Banach space whose direct sum with itself is not hypercyclic (see [DLRR09]). The techniques used there were later refined by Bayart and Matheron in order to construct hypercyclic but not weakly-mixing operators on each Banach space admitting a normalized unconditional basis whose associated forward shift is continuous, as for instance on the classical $c_0(\mathbb{N})$ and $\ell^p(\mathbb{N})$ spaces, $1 \leq p < \infty$ (see [BM07, BM09b]).

The aim of this paper is to study the previous problem, and to develop the corresponding theory, for the notion of *recurrence*: we say that a vector $x \in X$ is *recurrent* for T if x belongs to the closure of its forward orbit $\overline{\text{Orb}(Tx,T)}$; and that the operator T is *recurrent* if its set $\text{Rec}(T)$ of recurrent vectors is *dense* in X. Note that, in order to say that an operator T has a *recurrent behaviour*, it is not enough to assume that $\text{Rec}(T) \neq \varnothing$ since the zero-vector is recurrent for every linear map.

It is shown in [CMP14, Proposition 2.1] that the notion of recurrence coincides with that of *topological recurrence*, i.e. the property that for each non-empty open subset $U \subset X$ one can find some (and hence infinitely many) $n \in \mathbb{N}$ such that $T^n(U) \cap U \neq \varnothing$. In addition, and as in the hypercyclicity case, when T is recurrent its set $\text{Rec}(T)$ of recurrent vectors is a dense G_δ subset of the underlying F-space X.

Recurrence is one of the oldest and most studied concepts in the *Topological Dynamics* area of knowledge (see [Fur81, Ban99, KLOY17]), but in Linear Dynamics its systematic study began with the 2014 paper of Costakis, Manoussos and Parissis [CMP14]. Many questions in *linear recurrence* remain open and here we will be particularly interested in the so-called *$T \oplus T$-recurrence problem*, which was stated in [CMP14, Question 9.6]:

Question 1.2 (**The $T \oplus T$-recurrence problem**)**.** Let T be a recurrent operator on the F-space X. Is the operator $T \oplus T$ acting on the direct sum $X \oplus X$ recurrent?

This question has been recently restated as an open problem in [CKV21] for C_0-semigroups of operators. In the first part of this work we identify the systems (X,T) for which the recurrence property of T implies that of $T \oplus T$. This characterization is given in terms of *quasi-rigidity* (see Definition 2.2), which will be for recurrence the analogous property to that of *weak-mixing* for hypercyclicity/transitivity (see Theorem 2.5). Then we answer Question 1.2 negatively, by constructing some recurrent but not quasi-rigid operators (see Section 3).

The second part of this paper is dedicated to the study of some properties related to the notion of lineability: recall that a subset Y of an F-space X is called (*dense*) *lineable* if $Y \cup \{0\}$ contains a (dense) infinite-dimensional vector space. A well-known result, due to Herrero and Bourdon, states that the set $\text{HC}(T)$ is always *dense lineable* for every hypercyclic operator T (see [GEP11, Theorem 2.55]); moreover, one easily observes that if T is a quasi-rigid operator, then $\text{Rec}(T)$ is also *dense lineable* (see Proposition 2.7). Here we generalize these results to a rather general class of Furstenberg families $\mathcal{F}$: first we show that it makes sense to study quasi-rigidity from the $\mathcal{F}$-recurrence point of view (see Proposition 4.5); and then we check that, for an $\mathcal{F}$-recurrent operator T, the set $\mathcal{F}\text{Rec}(T)$ of $\mathcal{F}$-recurrent vectors is always *lineable* and usually even *dense lineable* (see Theorems 5.5 and 5.8). As a consequence we obtain the *Herrero-Bourdon theorem* for $\mathcal{F}$-hypercyclicity (see Subsection 5.3).

The paper is organized as follows. Section 2 is devoted to define and study *quasi-rigidity*, which is the analogous property, for recurrence, to that of *weak-mixing* for hypercyclicity. In Section 3 we construct recurrent but not quasi-rigid operators, answering Question 1.2 in a negative way and solving the *$T \oplus T$-recurrence problem*. We recall the definition of $\mathcal{F}$-recurrence in Section 4, showing that quasi-rigidity is a particular case of such a concept, and studying both the weakest and strongest possible $\mathcal{F}$-recurrence notions. The *lineability* and *dense lineability* properties, for the set $\mathcal{F}\text{Rec}(T)$ of $\mathcal{F}$-recurrent vectors, are studied in Section 5. We finally gather, in Section 6, some left open problems.

We refer the reader to the textbooks [BM09a, GEP11] for any unexplained but standard notion about hypercyclicity, or more generally, about Linear Dynamics. Along the paper we use the environment "**Question**" to state the different problems that we solve, and we use "**Problem(s)**" to state the left open problems.

2 Quasi-rigid dynamical systems

In this section we introduce and study the concept of *quasi-rigidity*, which is naturally defined in the broader framework of *Topological Dynamics* (i.e. without assuming linearity): a pair (X, T) is called a *dynamical system* if T is a continuous self-map on a Hausdorff topological space X. An important class of dynamical systems, which we will repeatedly use, is the family of Polish systems: a pair (X, T) is said to be a *Polish dynamical system* whenever T is a continuous self-map of a separable completely metrizable space X.

Definition 2.1. For a dynamical system (X, T) and $N \in \mathbb{N}$ we denote by $T_{(N)} : X^N \longrightarrow X^N$ the *N-fold direct product* of T with itself, i.e. $(X^N, T_{(N)})$ is the dynamical system

$$T_{(N)} := \underbrace{T \times \cdots \times T}_{N} : \underbrace{X \times \cdots \times X}_{N} \longrightarrow \underbrace{X \times \cdots \times X}_{N}.$$

where $X^N := \underbrace{X \times \cdots \times X}_{N}$ is the *N-fold product* of X and $T_{(N)}(x_1, ..., x_N) := (Tx_1, ..., Tx_N)$.

If (X, T) is linear, then $(X^N, T_{(N)})$ will refer to the *N-fold direct sum* linear dynamical system

$$T_{(N)} := \underbrace{T \oplus \cdots \oplus T}_{N} : \underbrace{X \oplus \cdots \oplus X}_{N} \longrightarrow \underbrace{X \oplus \cdots \oplus X}_{N},$$

and in this case $X^N := \underbrace{X \oplus \cdots \oplus X}_{N}$ is the *N-fold direct sum* of X.

2.1 Quasi-rigidity: definition and equivalences

By a well-known theorem of Furstenberg, once a dynamical system (X, T) is weakly-mixing (that is, once $T \times T$ is topologically transitive) then so is every N-fold direct product system $(X^N, T_{(N)})$; see [GEP11, Theorem 1.51]. Hence, in order to completely answer Question 1.2 we should first study the recurrent dynamical systems (X, T) for which every N-fold direct product is again recurrent. A first attempt would be to rely on the notion of *rigidity*:

– A dynamical system (X, T) is said to be *rigid* if there exists an increasing sequence of natural numbers $(n_k)_{k \in \mathbb{N}}$ such that $T^{n_k} x \to x$ for every $x \in X$.

Rigidity has been studied in different contexts such as measure theoretic recurrence [FW77], dynamical systems on topological spaces [GM89] and also for linear systems [EG11, CMP14]. This is a really strong form of recurrence, as it implies that $X^N = \operatorname{Rec}(T_{(N)})$ for all $N \in \mathbb{N}$, so that it is not the exact property we are looking for. Nonetheless, the concept of rigidity motivates the definition of the following (as far as we know) *new* notion:

Definition 2.2 (Quasi-rigidity). Let (X, T) be a dynamical system and let $(n_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence of positive integers. We say that T is:

– *quasi-rigid* with respect to the sequence $(n_k)_{k\in\mathbb{N}}$, if there exists a dense subset Y of X such that $T^{n_k}x \to x$, as $k \to \infty$, for every $x \in Y$.

– *topologically quasi-rigid* with respect to $(n_k)_{k\in\mathbb{N}}$, if for every non-empty open subset $U \subset X$ there exists some $k_U \in \mathbb{N}$ such that $T^{n_k}(U) \cap U \neq \varnothing$ for every $k \geq k_U$.

Our aim in this section is proving that, given a dynamical system (X, T), the previous notions characterize the recurrent behaviour of every N-fold direct product system $(X^N, T_{(N)})$ under very weak assumptions (dynamically speaking) on the underlying space X. Let us start by showing that *topological quasi-rigidity* is for recurrence the analogous property to that of *weak-mixing* for transitivity, when the underlying space X is second-countable:

Lemma 2.3. *Let (X, T) be a dynamical system on a second-countable space X. Then the following statements are equivalent:*

(i) *T is topologically quasi-rigid;*

(ii) *$T_{(N)}$ is topologically recurrent for every $N \in \mathbb{N}$.*

Proof. (i) ⇒ (ii): suppose that T is topologically quasi-rigid with respect to $(n_k)_{k\in\mathbb{N}}$, let $N \in \mathbb{N}$ and consider any finite sequence of non-empty open sets $U_1, ..., U_N \subset X$. Using now the topological quasi-rigidity assumption we can find $k_0 \in \mathbb{N}$ such that

$$T^{n_k}(U_j) \cap U_j \neq \varnothing \quad \text{for every } 1 \leq j \leq N \text{ and all } k \geq k_0.$$

Then, for the open set $U = U_1 \times \cdots \times U_N \subset X^N$ we have that $T_{(N)}^{n_k}(U) \cap U \neq \varnothing$ for all $k \geq k_0$, so $(X^N, T_{(N)})$ is topologically recurrent.

(ii) ⇒ (i): assume that $T_{(N)}$ is topologically recurrent for every $N \in \mathbb{N}$ and let $(U_s)_{s\in\mathbb{N}}$ be a countable base of (non-empty) open sets for X. We can construct recursively an increasing sequence of positive integers $(n_k)_{k\in\mathbb{N}}$ such that $T^{n_k}(U_s) \cap U_s \neq \varnothing$ for every $s, k \in \mathbb{N}$ with $1 \leq s \leq k$: given any $k \in \mathbb{N}$, the set $U := U_1 \times \cdots \times U_k$ is a non-empty open subset of X^k and by the topological recurrence of the map $T_{(k)}$ we can pick $n_k \in \mathbb{N}$ sufficiently large with

$$T_{(k)}^{n_k}(U) \cap U \neq \varnothing \quad \text{and hence} \quad T^{n_k}(U_s) \cap U_s \neq \varnothing \quad \text{for all } 1 \leq s \leq k.$$

It is easy to check that T is topologically quasi-rigid with respect to $(n_k)_{k\in\mathbb{N}}$. □

As one would expect, if the underlying space is completely metrizable, we can identify the "pointwise" *quasi-rigidity* notion with that of *topological quasi-rigidity*:

Proposition 2.4. *Let (X, T) be a Polish dynamical system. The following are equivalent:*

(i) *T is quasi-rigid;*

(ii) *T is topologically quasi-rigid.*

Proof. The implication (i) $\Rightarrow$ (ii) is straightforward: assume that there is a dense subset $Y \subset X$ and an increasing sequence $(n_k)_{k\in\mathbb{N}}$ such that $T^{n_k}x \to x$ for every $x \in Y$. For each non-empty open subset $U \subset X$ we can select $x \in U \cap Y$, and then

$$T^{n_k}x \in T^{n_k}(U) \cap U \quad \text{for every sufficiently large } k \in \mathbb{N}.$$

Hence T is topologically quasi-rigid with respect to $(n_k)_{k\in\mathbb{N}}$. Let us prove (ii) $\Rightarrow$ (i): assume that $d(\cdot,\cdot)$ is a metric defining the complete topology of X, let $(U_s)_{s\in\mathbb{N}}$ be a countable base of open sets for X and set $V_{s,s} := U_s$ for each $s \in \mathbb{N}$. By (ii) we can find $n_1 \in \mathbb{N}$ such that

$$T^{n_1}(U_1) \cap U_1 = T^{n_1}(V_{1,1}) \cap V_{1,1} \neq \varnothing.$$

The continuity of T ensures the existence of a non-empty open subset $V_{1,2} \subset X$ of diameter (with respect to d) less than $\frac{1}{2^2}$ for which $\overline{V_{1,2}} \subset V_{1,1}$ and $T^{n_1}\left(\overline{V_{1,2}}\right) \subset V_{1,1}$. Suppose now that for some $k \in \mathbb{N}$ we have already constructed:

– finite sequences $(V_{s,j})_{s\leq j\leq k}$ of open subsets of X, for each $1 \leq s \leq k-1$;

– and a finite increasing sequence of positive integers $(n_j)_{1\leq j\leq k-1}$;

with the properties that $V_{s,j}$ has d-diameter less than $\frac{1}{2^j}$ when $s < j$, and also that

$$\overline{V_{s,j}} \subset V_{s,j-1} \quad \text{and} \quad T^{n_{j-1}}\left(\overline{V_{s,j}}\right) \subset V_{s,j-1} \quad \text{for all } 1 \leq s < j \leq k.$$

Then, considering the open subsets $V_{1,k}, V_{2,k}, ..., V_{k,k} \subset X$ and using again (ii) we can select a positive integer $n_k \in \mathbb{N}$ with $n_k > n_{k-1}$ for which $T^{n_k}(V_{s,k}) \cap V_{s,k} \neq \varnothing$ for all $1 \leq s \leq k$. Again the continuity ot T ensures the existence, for each $1 \leq s \leq k$, of a non-empty open subset $V_{s,k+1} \subset X$ of d-diameter less than $\frac{1}{2^{k+1}}$ and such that

$$\overline{V_{s,k+1}} \subset V_{s,k} \quad \text{and} \quad T^{n_k}\left(\overline{V_{s,k+1}}\right) \subset V_{s,k} \quad \text{for all } 1 \leq s \leq k.$$

A recursive argument gives us an increasing sequence of positive integers $(n_k)_{k\in\mathbb{N}}$ and, for each $s \in \mathbb{N}$, a sequence of non-empty open sets $(V_{s,k})_{k=s}^{\infty}$ such that each set $V_{s,k}$ has d-diameter less than $\frac{1}{2^k}$ when $s < k$, but also satisfying

$$\overline{V_{s,k+1}} \subset V_{s,k} \quad \text{and} \quad T^{n_k}\left(\overline{V_{s,k+1}}\right) \subset V_{s,k} \quad \text{for all } s, k \in \mathbb{N} \text{ with } s \leq k.$$

By the Cantor intersection theorem, for each $s \in \mathbb{N}$ there is a unique vector $y_s \in X$ such that

$$\{y_s\} = \bigcap_{k\geq s} \overline{V_{s,k}} \subset V_{s,s} = U_s.$$

We deduce that $Y := \{y_s : s \in \mathbb{N}\}$ is a dense subset of X. Since for each $s \in \mathbb{N}$ we have that $T^{n_k}y_s \in T^{n_k}\left(\overline{V_{s,k+1}}\right) \subset V_{s,k}$ for all $k > s$, we get that

$$\limsup_{k\to\infty} d(T^{n_k}y_s, y_s) \leq \limsup_{k\to\infty} (\operatorname{diam}_d(V_{s,k})) \leq \limsup_{k\to\infty} \frac{1}{2^k} = 0,$$

and hence T is quasi-rigid with respect to the sequence $(n_k)_{k\in\mathbb{N}}$. $\square$

The proof of the implication (i) $\Rightarrow$ (ii) in Proposition 2.4 shows that every quasi-rigid map is topologically quasi-rigid even if X is not a metrizable space. The converse fails in general (consider for instance [GEP11, Example 12.9]). Combining the previous results we get:

Theorem 2.5. *Let (X, T) be a Polish dynamical system. The following are equivalent:*

(i) *T is quasi-rigid;*

(ii) *T is topologically quasi-rigid;*

(iii) *$T_{(N)}$ is topologically recurrent for every $N \in \mathbb{N}$;*

(iv) *$T_{(N)}$ is recurrent for every $N \in \mathbb{N}$.*

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) (and also (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii)) are trivial even if the space X is not second-countable neither complete. When X is second-countable we get the equivalence (ii) $\Leftrightarrow$ (iii) by Lemma 2.3. If we just assume the completeness of X, then we have the equivalence between the last two statements (iii) $\Leftrightarrow$ (iv) by [CMP14, Proposition 2.1]. Finally, using the Polish hypothesis we get the equivalence (i) $\Leftrightarrow$ (ii) by Proposition 2.4. $\square$

Remark 2.6. Even though we have worked in the general setting of *Polish dynamical systems*, it is worth mentioning that Theorem 2.5 holds true for every:

– *compact dynamical system*, i.e. when T is a continuous self-map of a compact metrizable space X (many of references just study this class of systems, see [Fur81, Ban99, KLOY17]);

– continuous linear operator T acting on any separable F-space X. In other words, the previous result holds true for *linear dynamical systems* as defined in the Introduction.

In the linear setting we have got the following relations between the already exposed concepts:

$$\begin{array}{ccc} \textit{satisfying the HC} & \xrightarrow{\text{Theorem 2.5}} & \textit{quasi-rigidity} \\ \downarrow & & \downarrow \\ \textit{hypercyclicity} & \xrightarrow{\text{already known}} & \textit{recurrence} \end{array}$$

i.e. the *quasi-rigidity* notion is for recurrence, the analogous property to that of *satisfying the Hypercyclicity Criterion* (and hence to that of *weak-mixing*) for hypercyclicity; see [BP99].

2.2 Quasi-rigid operators

It is now time to discuss how the quasi-rigidity notion influences the structure of the set of recurrent vectors for a linear dynamical system (X, T). We start by showing that quasi-rigidity implies the dense lineability of the set of recurrent vectors:

Proposition 2.7. *If $T : X \longrightarrow X$ is a quasi-rigid operator, then* $\operatorname{Rec}(T)$ *is dense lineable.*

Proof. Assume that $Y \subset X$ is a dense subset such that there exists a sequence $(n_k)_{k\in\mathbb{N}}$ of integers with $T^{n_k}x \to x$ for all $x \in Y$. Then any $z \in Z := \operatorname{span}(Y)$ satisfies that $T^{n_k}z \to z$, so $Z \subset X$ is an infinite-dimensional dense vector space contained in $\operatorname{Rec}(T)$. $\square$

Remark 2.8. Let $T : X \longrightarrow X$ be a quasi-rigid operator with respect to a sequence $(n_k)_{k\in\mathbb{N}}$ and consider $Y := \{x \in X : T^{n_k}x \to x \text{ as } k \to \infty\}$. This is a dense subset of X which can be either of first or second Baire category, depending on the operator T:

(a) If T is a *rigid* operator, then $Y = X$. This is the case of the identity operator, but there also exist examples of rigid systems which are even (Devaney) chaotic (see [EG11]).

(b) If X is a Banach space and Y is a second category set, then $\sup_{k\in\mathbb{N}} \|T^{n_k}\| < \infty$ by the Banach-Steinhaus theorem. Consider the operator $T := \lambda B$ with $|\lambda| > 1$, where B is the (unilateral) *backward shift* on $c_0(\mathbb{N})$ or any $\ell^p(\mathbb{N})$ $(1 \leq p < \infty)$. This is a weakly-mixing and hence quasi-rigid operator, but the set Y as defined before has to be of first category since $x = (\frac{1}{n^2})_{n\in\mathbb{N}} \in X$ has the property that $\|T^n x\| \to \infty$ as $n \to \infty$.

Now we are going to give some (usually fulfilled) sufficient conditions for a dynamical system to be quasi-rigid. Recall that a vector $x \in X$ is called *cyclic* for an operator $T : X \longrightarrow X$ if

$$\operatorname{span}(\operatorname{Orb}(x, T)) = \operatorname{span}\{T^n x : n \in \mathbb{N}_0\} = \{p(T)x : p \text{ polynomial}\},$$

is a dense set in X; and an operator T is called *cyclic* as soon as it admits a cyclic vector. Moreover, a point $x \in X$ is called *periodic* for a map T whenever $T^p x = x$ for some positive integer $p \in \mathbb{N}$. See [BM09a, GEP11] for more on cyclicity and periodicity.

Proposition 2.9. *Let (X, T) be a dynamical system for which any of the following holds:*

(a) *the set* $\operatorname{Per}(T)$ *of periodic points for T is dense in X;*

(b) *T admits a recurrent vector $x \in \operatorname{Rec}(T)$ for which the following set is dense*

$$\{Sx : ST = TS, S : X \longrightarrow X \textit{ continuous mapping}\};$$

then T is quasi-rigid. In particular, a continuous linear operator T acting on a Hausdorff topological vector space X is quasi-rigid whenever any of the following holds:

– *T has dense periodic vectors;*

– *T admits a recurrent and cyclic vector;*

– *T admits a hypercyclic vector.*

Proof. To check (a) consider $Y = \operatorname{Per}(T)$ and $(n_k)_{k\in\mathbb{N}} = (k!)_{k\in\mathbb{N}}$. For (b) consider

$$Y = \{Sx : ST = TS, S : X \longrightarrow X \text{ continuous mapping}\}.$$

Indeed, since x is recurrent there exists $(n_k)_{k\in\mathbb{N}}$ such that $T^{n_k}x \to x$, but then, given any point $y = Sx \in Y$ we get that $T^{n_k}(Sx) = S(T^{n_k}x) \to Sx$. Finally, recall that every recurrent and cyclic (and hence every hypercyclic) vector fulfills condition (b). □

Proposition 2.9 together with [CMP14, Corollary 9.2] show that recurrence plus cyclicity imply quasi-rigidity when X is a complex Banach space. This extends to (real and complex) linear dynamical systems as defined in the Introduction via the following lemma, in which the concept of *complexification* is used (see Remark 5.2). We will write $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Lemma 2.10. *Let $T : X \longrightarrow X$ be a topologically recurrent continuous linear operator acting on a (real or complex) Hausdorff topological vector space X. The following statements hold:*

(a) *For every (real or complex) number $\lambda \in \mathbb{K} \setminus \mathbb{T}$ the operator $(T - \lambda)$ has dense range.*

(b) *If X is a real space, then for every $\lambda \in \mathbb{C} \setminus \mathbb{T}$ the complexified operator $(\widetilde{T} - \lambda)$, acting on the complexification $\widetilde{X}$ of X, has dense range.*

As a consequence, for any (real or complex) polynomial p without unimodular roots, the operator $p(T)$ has dense range on the (real or complex) space X.

The proof of this lemma is based on [GEP11, Lemma 12.13], which is a well-known result of Wengenroth for topologically transitive operators acting on Hausdorff topological vector spaces. Here we will just show statement (a), since (b) follows exactly as in [GEP11, Exercise 12.2.6] (*Hint*: when X is a real space define the *complexification* $\widetilde{T} : \widetilde{X} \longrightarrow \widetilde{X}$ as it is done in Remark 5.2 for operators on F-spaces, and use the fact that for each non-empty open subset $U \subset X$ there are infinitely many natural numbers $n \in \mathbb{N}$ fulfilling that $\widetilde{T}^n(U + iU) \cap (U + iU) \neq \varnothing$).

Proof of Lemma 2.10. Given $\lambda \in \mathbb{K} \setminus \mathbb{T}$ let $E := \overline{(T - \lambda)(X)}$, which is a closed subspace of X, and suppose that $E \neq X$. The quotient space X/E is then a (real or complex) Hausdorff topological vector space (since E is closed). If $q : X \longrightarrow X/E$ is the quotient map we have that $q((T - \lambda)x) = 0$ and hence $q(Tx) = \lambda q(x)$ for every $x \in X$. The operator S on X/E given by $S[x] = \lambda[x]$ is easily seen to be topologically recurrent since so is T.

However, if $|\lambda| < 1$ we can choose $[x] \in X/E \setminus \{0\}$ and a balanced 0-neighbourhood W such that $[x] \notin \overline{W}$. Since $\lambda^n[x] \to 0$ and $\lambda W \subset W$ there exists a neighbourhood U of $[x]$ and a natural number $N \in \mathbb{N}$ for which $U \cap W = \varnothing$ and $\lambda^n U \subset W$ for every $n \geq N$. Hence $S^n(U) \cap U = \varnothing$ for all $n \geq N$ so S is not topologically recurrent, which is a contradiction. If $|\lambda| > 1$ we can consider the map $S^{-1} : X/E \longrightarrow X/E$ for which $S^{-1}[x] = \lambda^{-1}[x]$. By the previous argument S^{-1} is not topologically recurrent, but then S is not topologically recurrent (see [GLM23, Section 6]), which is again a contradiction.

Let now p be any (real or complex) polynomial without unimodular roots: if X is a complex space then $p(T)$ has dense range since, splitting the polynomial, $p(T)$ can be written as a composition of dense range operators by (a); and if X is a real space then $p(\widetilde{T})$ also has dense range in $\widetilde{X}$ via the same argument as above but using (b). Finally, if p has real coefficients, then $p(\widetilde{T}) = p(T) + ip(T)$ and $p(T)$ has dense range in the real space X. □

The proof of [CMP14, Corollary 9.2] is highly based on the following result of D. Herrero:

– *Given an operator T on a complex Banach space X, if $\sigma_p(T^*)$ has empty interior then the set of cyclic vectors for T is dense, and hence it is a dense G_δ-set* (see [Her79]).

By using Lemma 2.10 we obtain this density result, for recurrent operators, in the more general setting of operators acting on arbitrary F-spaces:

Corollary 2.11 (Extension of [CMP14, Corollary 9.2]). *Let (X, T) be a (real or complex) linear dynamical system. If T is recurrent and cyclic then the set of cyclic vectors for T is a dense G_δ-set. In particular, if T is recurrent and cyclic, then it is quasi-rigid.*

Proof. The set of cyclic vectors is always a G_δ-set. To check its density let $x \in X$ be a cyclic vector for T and pick a non-empty open set $U \subset X$. Then there is a (real or complex) polynomial p such that $p(T)x \in U$. Splitting p we have that:

$$p(T) = \prod_{j=1}^{N} (T - \lambda_j) \quad \text{for some } (\lambda_j)_{j=1}^{N} \subset \mathbb{C}.$$

Choose a small perturbation $(\mu_j)_{j=1}^{N} \subset \mathbb{C} \setminus \mathbb{T}$ of $(\lambda_j)_{j=1}^{N} \subset \mathbb{C}$ with the property that the polynomial

$$\widehat{p}(T) := \prod_{j=1}^{N} (T - \mu_j) \quad \text{is still such that } \widehat{p}(T)x \in U.$$

Note that if p has real coefficients, then $(\mu_i)_{i=1}^{N} \subset \mathbb{C} \setminus \mathbb{T}$ can be chosen such that $\widehat{p}$ has still real coefficients. Finally, $\widehat{p}(T)x$ is a cyclic vector for T since

$$\widehat{p}(T) \left(\{ q(T)x : q \text{ polynomial} \} \right) = \{ q(T)(\widehat{p}(T)x) : q \text{ polynomial} \}$$

is a dense set by Lemma 2.10. Then $\mathrm{Rec}(T)$ and the set of cyclic vectors for T are both residual, so T admits a recurrent and cyclic vector. Proposition 2.9 ends the proof. □

3 Existence of recurrent but not quasi-rigid operators

After having studied in Section 2 the dynamical systems (X, T) for which all their N-fold direct sums $(X^N, T_{(N)})$ are recurrent, we are now ready to answer negatively Question 1.2 showing that there exist recurrent but not quasi-rigid operators. If for the moment we continue working in the (not necessarily linear) *dynamical systems* setting, the following are natural questions in view of the theory already developed here:

Question 3.1. Let (X, T) be a recurrent dynamical system:

(a) Is $(X \times X, T \times T)$ again a recurrent dynamical system?

(b) Assume that $(X \times X, T \times X)$ is recurrent. Is then (X, T) quasi-rigid?

If we change the "*recurrence*" assumption into that of "*topological transitivity*" the respective answers are:

(a) **No**, and in particular:

- for non-linear systems consider any irrational rotation, see [GEP11, Example 1.43];
- for linear maps we already mentioned the references [DLRR09, BM07, BM09b].

(b) **Yes**, in both linear and non-linear cases $(X^N, T_{(N)})$ is transitive for every $N \in \mathbb{N}$ if and only if $T \times T$ is transitive (see for instance [GEP11, Theorem 1.51]).

However, the answer to both questions is **no** in the recurrence setting, and indeed, it is shown in [Ban99, Lemma 9 and Example 4] that for each $N \in \mathbb{N}$ there exists a **non-linear dynamical system** $f : X \longrightarrow X$ on a **compact metric space** X for which

$$(X^N, f_{(N)}) \text{ is recurrent } \quad \text{while} \quad (X^{N+1}, f_{(N+1)}) \text{ is not recurrent.}$$

This construction is **highly non-linear** since each of these compact spaces X is the disjoint union of $N+1$ *sub-shifts* of the well-known *shift on two symbols*. It turns out that the answer to Question 3.1 in the framework of Linear Dynamics is also negative. Let us state the main result of this section:

Theorem 3.2. [A] *Let X be any (real or complex) separable infinite-dimensional Banach space. For each $N \in \mathbb{N}$ there exists an operator $T \in \mathcal{L}(X)$ such that*

$$T_{(N)} : X^N \longrightarrow X^N \textit{ is recurrent, and even } \mathrm{Rec}(T_{(N)}) = X^N,$$

but for which the operator $T_{(N+1)} : X^{N+1} \longrightarrow X^{N+1}$ is not recurrent any more.

Considering $N = 1$ we obtain examples of recurrent operators $T \in \mathcal{L}(X)$ for which $T \oplus T$ is not recurrent. Thus we get a negative answer to the *$T \oplus T$-recurrence problem*, i.e. we answer Question 1.2 and hence [CMP14, Question 9.6] in the negative.

Our proof of Theorem 3.2 **relies heavily** on a construction of Augé in [Aug12] of operators T on (infinite-dimensional, separable) Banach spaces X with **wild dynamics**: the two sets

$$A_T = \left\{ x \in X : \lim_{n\to\infty} \|T^n x\| = \infty \right\} \quad \text{and} \quad B_T = \left\{ x \in X : \liminf_{n\to\infty} \|T^n x - x\| = 0 \right\}$$

have non-empty interior and form a partition of X. Since $B_T = \mathrm{Rec}(T)$, these operators have plenty (but not a dense set) of recurrent vectors. The remainder of this section is devoted to prove the **complex** version of Theorem 3.2 by modifying the construction of [Aug12]. The **real** case follows in a really similar way by using the same arguments as in [Aug12, Section 3.2].

3.1 Necessary prerequisites

Assume that X is a complex separable infinite-dimensional Banach space. We will denote by X^* its *topological dual space* and by B_{X^*} the *unit ball* of X^*. Moreover, given $(x, x^*) \in X \times X^*$ we will denote by $\langle x^*, x \rangle = x^*(x)$ the dual evaluation. As in [Aug12], our operator will be built using a bounded biorthogonal system of X. The following is a really well-known and useful result (see [LT77, Vol I, Section 1.f] or [Oc75]):

– *Given a separable Banach space X one can find $(e_n, e_n^*)_{n\in\mathbb{N}} \subset X \times X^*$ such that:*

- $\mathrm{span}\{e_n : n \in \mathbb{N}\}$ *is dense in X;*
- $\langle e_n^*, e_m \rangle = \delta_{n,m}$ *where $\delta_{n,m} = 0$ if $n \neq m$ and 1 if $n = m$;*
- *for each $n \in \mathbb{N}$ we have that $\|e_n\| = 1$, and $K := \sup_{n\in\mathbb{N}} \|e_n^*\| < \infty$.*

[A] See Section 2.3 of the *General discussion of the results* for a slight improvement in terms of *$\mathcal{AP}$-recurrence*.

Once we have fixed a sequence $(e_n, e_n^*)_{n\in\mathbb{N}} \subset X \times X^*$ with the previous properties we set

$$c_{00} := \text{span}\{e_n : n \in \mathbb{N}\}.$$

Given $x \in X$ we will write $x_k := \langle e_k^*, x\rangle$ for each $k \in \mathbb{N}$, and we will repeatedly use that

$$\|x_k e_k\| \leq K\|x\| \quad \text{for each } k \in \mathbb{N}, \tag{3.1}$$

since

$$\|x\| := \sup_{x^*\in B_{X^*}} |\langle x^*, x\rangle| \geq \frac{|\langle e_k^*, x\rangle|}{\|e_k^*\|} = \frac{|x_k|}{\|e_k^*\|} = \frac{\|x_k e_k\|}{\|e_k^*\|} \geq \frac{\|x_k e_k\|}{K},$$

for each $k \in \mathbb{N}$, and hence that

$$\|x - y\| < \varepsilon \ \text{ implies } \ |x_k - y_k| < K\varepsilon \quad \text{for all } x, y \in X \text{ and } k \in \mathbb{N}. \tag{3.2}$$

Remark 3.3. If $(e_n)_{n\in\mathbb{N}}$ is not a Schauder basis of X, the above inequalities are still true but a vector $x \in X$ cannot in general be written as a convergent series $x = \sum_{k\in\mathbb{N}} x_k e_k$. However, for the vectors in $c_{00} = \text{span}\{e_n : n \in \mathbb{N}\}$ this equality will be true: for each $x \in c_{00}$ there is some $n \in \mathbb{N}$ such that

$$x = \sum_{k=1}^{n} \langle e_k^*, x\rangle e_k = \sum_{k=1}^{n} x_k e_k.$$

From now on we fix a natural number $N \in \mathbb{N}$. We are going to construct

– a *projection* P from X into itself;

– a *sequence of functionals* $(w_k^*)_{k\geq N+2} \subset X^*$;

– two *sequences* $(m_k)_{k\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ and $(\lambda_k)_{k\in\mathbb{N}} \in \mathbb{T}^{\mathbb{N}}$;

– and two *operators* R and T on X;

in such a way that $T_{(N)}$ is recurrent while $T_{(N+1)}$ is not.

3.2 The projection P and the sequences $(w_k^*)_{k\geq N+2}$, $(m_k)_{k\in\mathbb{N}}$ and $(\lambda_k)_{k\in\mathbb{N}}$

Denote by $P : X \longrightarrow \text{span}\{e_1, e_2, ..., e_{N+1}\}$ the projection of X onto the linear span of the first $N + 1$ basis vectors defined by

$$Px := \sum_{j=1}^{N+1} \langle e_j^*, x\rangle e_j \quad \text{for every } x \in X.$$

Note that P is continuous. In fact $\|P\| \leq (N + 1)K$. Let $E := \text{span}\{e_1^*, e_2^*, ..., e_{N+1}^*\}$ endowed with the norm $\|\cdot\|^*$ of X^*, and consider on E the equivalent norm $\|\cdot\|_\infty$ defined in the following way for each $\alpha_1, \alpha_2, ..., \alpha_{N+1} \in \mathbb{C}$,

$$\left\|\sum_{j=1}^{N+1} \alpha_j e_j^*\right\|_\infty := \max\{|\alpha_1|, |\alpha_2|, ..., |\alpha_{N+1}|\}.$$

This makes sense because the vectors $e_1^*, ..., e_{N+1}^*$ are linearly independent. There exist constants $M \geq m > 0$ such that $m\|w^*\|_\infty \leq \|w^*\|^* \leq M\|w^*\|_\infty$ for every $w^* \in E$.

Moreover, since the unit sphere $S_{E_\infty} := \{w^* \in E : \|w^*\|_\infty = 1\}$ is a compact metrizable space there exists a sequence $(w_k^*)_{k\geq N+2} \subset S_{E_\infty}$, which is dense in S_{E_∞}. Note that:

Fact 3.2.1. *For each $w^* \in S_{E_\infty}$ there exists an index $i \in \{1, 2, ..., N+1\}$ such that the following holds: for every $\varepsilon > 0$ and every $x \in X$ with $\|x - e_i\| < \varepsilon$ we have that*

$$|\langle w^*, Px\rangle| > 1 - (N+1)K\varepsilon.$$

Proof. Since $w^* \in S_{E_\infty}$ there are coefficients $(\alpha_j)_{j=1}^{N+1} \in \mathbb{C}^{N+1}$ such that $w^* = \sum_{j=1}^{N+1} \alpha_j e_j^*$, and there exists at least one index $i \in \{1, 2, ..., N+1\}$ with $|\alpha_i| = 1$. Given $\varepsilon > 0$ and $x \in X$ with $\|x - e_i\| < \varepsilon$, the inequality (3.2) implies that $|x_i - 1| < K\varepsilon$ while $|x_j| < K\varepsilon$ for every $1 \leq j \leq N+1$ with $j \neq i$. Since $0 \leq |\alpha_j| \leq 1$ for every $1 \leq j \leq N+1$ we get that

$$|\langle w^*, Px\rangle| = \left|\sum_{j=1}^{N+1} \alpha_j x_j\right| \geq |x_i| - \sum_{j=1, j\neq i}^{N+1} |\alpha_j|\,|x_j| > (1 - K\varepsilon) - NK\varepsilon = 1 - (N+1)K\varepsilon. \qquad \square$$

Let $(m_k)_{k\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ be a sequence of positive integers with the following properties:

(a) $m_k \mid m_{k+1}$ for each $k \geq 1$;

(b) $m_1 = m_2 = \cdots = m_{N+1} = 1$;

and starting from $k = N+2$, the sequence $(m_k)_{k\geq N+2}$ grows fast enough to satisfy:

(c) $\lim_{j\to\infty} \left(m_j \cdot \sum_{k=j+1}^{\infty} \frac{1}{m_k}\right) = 0$.

The sequence $(\lambda_k)_{k\in\mathbb{N}} \in \mathbb{T}^{\mathbb{N}}$ is then defined by setting $\lambda_k := e^{2\pi i \frac{1}{m_k}}$ for each $k \in \mathbb{N}$. Using the inequality $|e^{i\theta} - 1| \leq |\theta|$, which is true for every $\theta \in \mathbb{R}$, and also that $\lambda_1 = \lambda_2 = \cdots = \lambda_{N+1} = 1$, we deduce the inequality

$$\sum_{k=1}^{\infty} |\lambda_k - 1| \leq 2\pi \sum_{k=N+2}^{\infty} \frac{1}{m_k} < \infty, \tag{3.3}$$

since condition (c) on the sequence $(m_k)_{k\in\mathbb{N}}$ implies that the series $\sum_{k=1}^{\infty} \frac{1}{m_k}$ is convergent.

3.3 The operators R and T

For each $x \in c_{00} = \text{span}\{e_n : n \in \mathbb{N}\}$, which we write as $x = \sum_{k=1}^{n} x_k e_k$, we set

$$Rx := \sum_{k=1}^{n} \lambda_k x_k e_k.$$

Note that for each $x = \sum_{k=1}^{n} x_k e_k \in c_{00}$, and using (3.1), we have that:

$$\|Rx\| \leq \|Rx - x\| + \|x\| \leq \sum_{k=1}^{n} |\lambda_k - 1| \cdot \|x_k e_k\| + \|x\| \leq \left(K \sum_{k=1}^{\infty} |\lambda_k - 1| + 1\right) \|x\|.$$

By (3.3) the map R extends to a bounded operator on X still denoted by R.

We now define the operator T on X by setting

$$Tx \ := \ Rx \ + \sum_{k=N+2}^{\infty} \frac{1}{m_{k-1}} \langle w_k^*, Px \rangle e_k \quad \text{for every } x \in X. \tag{3.4}$$

The second sum in the expression (3.4) defines a bounded operator by the assumption (c) on the sequence $(m_k)_{k\in\mathbb{N}}$, the fact that $\|P\| \leq (N+1)K$ and also that $\|w_k^*\|^* \leq M$ for each $k \geq N+2$. Indeed,

$$\left\| \sum_{k=N+2}^{\infty} \frac{1}{m_{k-1}} \langle w_k^*, Px \rangle e_k \right\| \leq \sum_{k=N+2}^{\infty} \frac{\|w_k^*\|^* \cdot \|Px\|}{m_{k-1}} \leq \left(M(N+1)K \sum_{k=N+2}^{\infty} \frac{1}{m_{k-1}} \right) \|x\|,$$

for every $x \in X$, where the last parenthesis has finite value by assumption (c) on $(m_k)_{k\in\mathbb{N}}$. It follows that T is a continuous operator on X. Let us now compute its n-th power:

Fact 3.3.1 (Modification of [Aug12, Lemma 3.5]). *For every $x \in X$ and $n \geq 1$ we have that*

$$T^n x \ = \ R^n x \ + \sum_{k=N+2}^{\infty} \frac{\lambda_{k,n}}{m_{k-1}} \langle w_k^*, Px \rangle e_k,$$

where $\lambda_{k,n} := \sum_{l=0}^{n-1} \lambda_k^l = \frac{\lambda_k^n - 1}{\lambda_k - 1}$ for each $k \geq N+2$.

Proof. Suppose that the formula holds for some $n \geq 1$. Then

$$T^{n+1}x = TR^n x \ + \sum_{k=N+2}^{\infty} \frac{\lambda_{k,n}}{m_{k-1}} \langle w_k^*, Px \rangle Te_k = R^{n+1}x \ + \sum_{k=N+2}^{\infty} \frac{1}{m_{k-1}} \langle w_k^*, PR^n x \rangle e_k$$

$$+ \sum_{k=N+2}^{\infty} \frac{\lambda_{k,n}}{m_{k-1}} \langle w_k^*, Px \rangle \left(Re_k \ + \sum_{j=N+2}^{\infty} \frac{1}{m_{j-1}} \langle w_j^*, Pe_k \rangle e_j \right)$$

$$= R^{n+1}x \ + \sum_{k=N+2}^{\infty} \frac{1 + \lambda_{k,n} \cdot \lambda_k}{m_{k-1}} \langle w_k^*, Px \rangle e_k = R^{n+1}x \ + \sum_{k=N+2}^{\infty} \frac{\lambda_{k,n+1}}{m_{k-1}} \langle w_k^*, Px \rangle e_k.$$

In order to obtain these equalities we have used the fact that $PR^n x = Px$, which follows from the equality $\lambda_1 = \lambda_2 = \cdots = \lambda_{N+1} = 1$, and also the fact that $Pe_k = 0$ for $k \geq N+2$. □

We will also need the following properties regarding the numbers $\lambda_{k,n}$:

Fact 3.3.2 (Modification of [Aug12, Fact 3.6]). *Let $n \geq 1$. Then:*

(i) $|\lambda_{k,n}| \leq n$ *for all* $k \geq N+2$*;*

(ii) $\lambda_{k,m_n} = 0$ *whenever* $n \geq k \geq N+2$*;*

(iii) $|\lambda_{k,n}| \geq \frac{2}{\pi} n > \frac{m_{k-1}}{\pi}$ *whenever* $k = \min\{j \geq N+2 : 2n \leq m_j\}$*.*

Proof. Each $\lambda_{k,n}$, for $k \geq N+2$, is the geometric sum

$$\lambda_{k,n} = \textstyle\sum_{l=0}^{n-1} \lambda_k^l = \frac{\lambda_k^n - 1}{\lambda_k - 1} \quad \text{with } \lambda_k = e^{2\pi i \frac{1}{m_k}};$$

then (i) follows from the triangle inequality; (ii) since $n \geq k$ implies that $\lambda_k^{m_n} = e^{2\pi i \frac{m_n}{m_k}} = 1$ by condition (a) on the sequence $(m_k)_{k\in\mathbb{N}}$; and (iii) because

$$|\lambda_{k,n}| = \left| \frac{e^{2\pi i \frac{n}{m_k}} - 1}{e^{2\pi i \frac{1}{m_k}} - 1} \right| = \frac{\left|\sin(\pi \frac{n}{m_k})\right|}{\left|\sin(\pi \frac{1}{m_k})\right|} \geq \frac{2}{\pi} n > \frac{m_{k-1}}{\pi},$$

using that $\sin(\theta) \geq \frac{2}{\pi}\theta$ for each $\theta \in [0, \frac{\pi}{2}]$ and that $|\sin(\theta)| \leq |\theta|$ for every $\theta \in \mathbb{R}$. □

Up to now, we have defined the operator T and computed its powers T^n. It remains to check that $T_{(N)}$ is recurrent while $T_{(N+1)}$ is not. We start by showing that $R : X \longrightarrow X$ is a rigid operator (see Subsection 2.1 or [CMP14, Definition 1.2]) with respect to the (eventually) increasing sequence of positive integers $(m_k)_{k\in\mathbb{N}}$:

Fact 3.3.3 (Modification of [Aug12, Lemma 3.4]). *We have* $\lim_{k\to\infty} R^{m_k} x = x$ *for all* $x \in X$.

Proof. Following [Aug12, Lemma 3.4] we first show that the sequence $(\|R^{m_j}\|)_{j\in\mathbb{N}}$ is bounded. In fact, if we fix $x = \sum_{k=1}^n x_k e_k \in c_{00}$ such that $\|x\| = 1$, for all $j \geq 1$ we have that

$$\begin{aligned}
\|R^{m_j} x - x\| &= \left\| \sum_{k=1}^n (\lambda_k^{m_j} - 1) x_k e_k \right\| \\
&= \left\| \sum_{k=j+1}^n (\lambda_k^{m_j} - 1) x_k e_k \right\| \qquad \text{since } \lambda^{m_j} = 1 \text{ for all } k \leq j, \\
&\leq K\|x\| \sum_{k=j+1}^n |\lambda_k^{m_j} - 1| = K \sum_{k=j+1}^n \left| e^{2\pi i \frac{m_j}{m_k}} - 1 \right| \qquad \text{by (3.1)}, \\
&\leq 2\pi K \sum_{k=j+1}^\infty \frac{m_j}{m_k} \qquad \text{since } |e^{i\theta} - 1| \leq |\theta| \text{ for every } \theta \in \mathbb{R},
\end{aligned}$$

which is less than a constant independent of j by condition (c) on $(m_k)_{k\in\mathbb{N}}$. The density of c_{00} in X implies that $(\|R^{m_j}\|)_{j\in\mathbb{N}}$ is a bounded sequence. Now if given $x \in X$ and $\varepsilon > 0$ we find $y \in c_{00}$ with $\|y - x\| < \varepsilon$, then

$$\|R^{m_j} x - x\| \leq \|R^{m_j}(x-y)\| + \|R^{m_j} y - y\| + \|y - x\| < \sup_{j\in\mathbb{N}} \|R^{m_j}\|\varepsilon + \|R^{m_j} y - y\| + \varepsilon.$$

The claim follows since $R^{m_j} y = y$ for every $j \in \mathbb{N}$ large enough, and ε was arbitrary. □

Now we can prove the *recurrence* property of $T_{(N)}$:

Proposition 3.4. *The operator* $T_{(N)} : X^N \longrightarrow X^N$ *is recurrent. Moreover,* $\mathrm{Rec}(T_{(N)}) = X^N$.

Proof. Fix $x^{(1)}, x^{(2)}, ..., x^{(N)} \in X$ and set $x_j^{(i)} := \langle e_j^*, x^{(i)} \rangle$ for each $1 \le i \le N$ and $j \ge 1$. Choose $(\alpha_j)_{j=1}^{N+1} \in \mathbb{C}^{N+1}$ such that

$$\begin{pmatrix} x_1^{(1)} & x_2^{(1)} & \cdots & x_{N+1}^{(1)} \\ x_1^{(2)} & x_2^{(2)} & \cdots & x_{N+1}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ x_1^{(N)} & x_2^{(N)} & \cdots & x_{N+1}^{(N)} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{N+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{and} \quad \max_{1 \le j \le N+1} |\alpha_j| = 1.$$

This is indeed possible since the system considered has N equations and $N+1$ unknowns. Set $w^* := \sum_{j=1}^{N+1} \alpha_j e_j^* \in E$. Then $w^* \in S_{E_\infty}$ and clearly $\langle w^*, Px^{(i)} \rangle = 0$ for all $1 \le i \le N$. Let $(k_n)_{n \in \mathbb{N}}$ be an increasing sequence of positive integers (all greater than $N+2$) such that $\lim_{n \to \infty} w_{k_n}^* = w^*$ in E. By Fact 3.3.1, for each $1 \le i \le N$ we have that

$$\begin{aligned} \left(T^{m_{k_n-1}} - R^{m_{k_n-1}}\right) x^{(i)} &= \sum_{k=N+2}^{\infty} \frac{\lambda_{k,m_{k_n-1}}}{m_{k-1}} \langle w_k^*, Px^{(i)} \rangle e_k = \underbrace{\sum_{N+2 \le k < k_n} \frac{\lambda_{k,m_{k_n-1}}}{m_{k-1}} \langle w_k^*, Px^{(i)} \rangle e_k}_{(*)} \\ &+ \underbrace{\frac{\lambda_{k_n,m_{k_n-1}}}{m_{k_n-1}} \langle w_{k_n}^*, Px^{(i)} \rangle e_{k_n}}_{(**)} + \underbrace{\sum_{k_n < k} \frac{\lambda_{k,m_{k_n-1}}}{m_{k-1}} \langle w_k^*, Px^{(i)} \rangle e_k}_{(***)} . \end{aligned}$$

Note that for each $1 \le i \le N$,

$$(*) = \sum_{N+2 \le k < k_n} \frac{\lambda_{k,m_{k_n-1}}}{m_{k-1}} \langle w_k^*, Px^{(i)} \rangle e_k = 0,$$

since by (ii) of Fact 3.3.2 we have that $\lambda_{k,m_{k_n-1}} = 0$ for $k_n - 1 \ge k$. By (i) of Fact 3.3.2 we have that $|\lambda_{k_n,m_{k_n-1}}| \le m_{k_n-1}$ and hence

$$\|(**)\| = \left| \frac{\lambda_{k_n,m_{k_n-1}}}{m_{k_n-1}} \langle w_{k_n}^*, Px^{(i)} \rangle \right| \le |\langle w_{k_n}^*, Px^{(i)} \rangle| \underset{n \to \infty}{\longrightarrow} 0,$$

since $\lim_{n \to \infty} w_{k_n}^* = w^*$ and $\langle w^*, Px^{(i)} \rangle = 0$ for every $1 \le i \le N$. Moreover, by condition (c) on the sequence $(m_k)_{k \in \mathbb{N}}$ we get that, for each $1 \le i \le N$,

$$\|(***)\| \le \sum_{k_n < k} \frac{\left|\lambda_{k,m_{k_n-1}}\right|}{m_{k-1}} \|w_k^*\|^* \cdot \|P\| \cdot \|x^{(i)}\| \le M \cdot (N+1)K \cdot \|x^{(i)}\| \cdot \sum_{k_n < k} \frac{m_{k_n-1}}{m_{k-1}} \underset{n \to \infty}{\longrightarrow} 0.$$

Finally, for each $1 \le i \le N$ we have that

$$\begin{aligned} \|T^{m_{k_n-1}} x^{(i)} - x^{(i)}\| &\le \|\left(T^{m_{k_n-1}} - R^{m_{k_n-1}}\right) x^{(i)}\| + \|R^{m_{k_n-1}} x^{(i)} - x^{(i)}\| \\ &\le \|(*)\| + \|(**)\| + \|(***)\| + \|R^{m_{k_n-1}} x^{(i)} - x^{(i)}\| \underset{n \to \infty}{\longrightarrow} 0, \end{aligned}$$

where $\|R^{m_{k_n-1}} x^{(i)} - x^{(i)}\| \to 0$ by Fact 3.3.3. That is, $(x^{(1)}, x^{(2)}, ..., x^{(N)}) \in \mathrm{Rec}(T_{(N)})$ and the arbitrariness of the vectors $x^{(1)}$, $x^{(2)}$, ..., $x^{(N)}$ implies that $\mathrm{Rec}(T_{(N)}) = X^N$. □

To end the proof of Theorem 3.2 we just have to show the following:

Proposition 3.5. *The operator $T_{(N+1)} : X^{N+1} \longrightarrow X^{N+1}$ is not recurrent.*

Proof. By contradiction suppose that $T_{(N+1)}$ is recurrent. Fix any positive value

$$0 < \varepsilon < \frac{1}{K(N+1+2\pi)}, \tag{3.5}$$

and consider the open balls $B(e_1, \varepsilon), B(e_2, \varepsilon), ..., B(e_{N+1}, \varepsilon) \subset X$ centred at e_i, $1 \leq i \leq N+1$, with radius ε. Then there exists some $n \in \mathbb{N}$ such that

$$T^n(B(e_i, \varepsilon)) \cap B(e_i, \varepsilon) \neq \varnothing \quad \text{for every } i = 1, 2, ..., N+1.$$

Let $k_n := \min\{j \geq N+2 : 2n \leq m_j\} \geq N+2$. By (iii) of Fact 3.3.2 we have that

$$|\lambda_{k_n,n}| > \tfrac{m_{k_n-1}}{\pi}. \tag{3.6}$$

Moreover, Fact 3.2.1 applied to the functional $w^*_{k_n}$ yields the existence of (at least) one index $i_n \in \{1, 2, ..., N+1\}$ for which

$$\left|\langle w^*_{k_n}, Px\rangle\right| > 1 - (N+1)K\varepsilon \quad \text{whenever } \|x - e_{i_n}\| < \varepsilon. \tag{3.7}$$

Picking any $x \in B(e_{i_n}, \varepsilon) \cap T^{-n}(B(e_{i_n}, \varepsilon)) \cap c_{00}$ we get the following inequalities

$$\begin{aligned} \varepsilon &> \|T^n x - e_{i_n}\| \geq \frac{1}{K}\left|\langle e^*_{k_n}, T^n x - e_{i_n}\rangle\right| = \frac{1}{K}\left|\langle e^*_{k_n}, T^n x\rangle\right| && \text{by (3.1)}, \\ &= \frac{1}{K}\left|\lambda^n_{k_n} x_{k_n} + \frac{\lambda_{k_n,n}}{m_{k_n-1}}\langle w^*_{k_n}, Px\rangle\right| && \text{by Fact 3.3.1}. \end{aligned}$$

Hence

$$\begin{aligned} \varepsilon &> \frac{1}{K}\left(\left|\frac{\lambda_{k_n,n}}{m_{k_n-1}}\right| \cdot \left|\langle w^*_{k_n}, Px\rangle\right| - K\varepsilon\right) > \frac{1}{K\pi} \cdot \left|\langle w^*_{k_n}, Px\rangle\right| - \varepsilon && \text{by (3.2) and (3.6)}, \\ &> \frac{(1-(N+1)K\varepsilon)}{K\pi} - \varepsilon > \frac{1}{K(N+1+2\pi)} && \text{by (3.7) and (3.5)}. \end{aligned}$$

This is a contradiction with (3.5). ☐

The **complex** version of Theorem 3.2 is now proved. The construction can be adapted to the **real** case using the same arguments as in [Aug12], so every separable infinite-dimensional Banach space supports a recurrent operator which is not quasi-rigid. Moreover, there are operators whose N-fold direct sum is recurrent while its $N+1$-fold direct sum is not, and taking $N = 1$ we solve negatively the *$T \oplus T$-recurrence problem*. We recall here that the respective problem for *C_0-semigroups of operators*, stated in [CKV21], remains open.

4 Furstenberg families for pointwise recurrence

In the last two decades hypercyclicity has been studied from the *frequency of visits* point of view: one investigates "how often" the orbit of a vector returns to every open subset of the space. The recent notions of *frequent*, *$\mathcal{U}$-frequent* or *reiterative hypercyclicity* have emerged following this line of thought (see [BG06, Shk09, BMPP16, BMPP19]). As showed in [BMPP16, BGELMP22], under some natural assumptions this behaviour can be decomposed in two factors:

(1) *usual hypercyclicity* (we have to require that the operator under study has a dense orbit);

(2) some kind of *strong recurrence* property (we have to require that each non-empty open subset of the space contains an orbit returning to it with some frequency).

Even though the study of *linear recurrence* began in 2014 with [CMP14], *stronger recurrence* notions have only recently been orderly investigated in the 2022 paper [BGELMP22], and then in the following works [GLM23, CM], where the concepts of *$\mathcal{F}$-recurrence* and *$\mathcal{F}$-hypercyclicity* with respect to a Furstenberg family $\mathcal{F}$ are deeply studied. In this section we show that quasi-rigidity can be studied from an $\mathcal{F}$-recurrence perspective for a very particular kind of Furstenberg families called *free filters* (see Proposition 4.5). Then we study both the weakest and strongest possible $\mathcal{F}$-recurrence notions together with the Furstenberg families associated to them (see Subsection 4.2), and we finish the section recalling that $\mathcal{F}$-recurrence behaves well under homomorphisms, quasi-conjugacies and commutants (see Subsection 4.3).

4.1 Definitions, examples and $\mathcal{F}(A)$-recurrence

A collection of sets $\mathcal{F} \subset \mathscr{P}(\mathbb{N}_0)$ is said to be a *Furstenberg family* (a *family* for short) provided that: each set $A \in \mathcal{F}$ is infinite; $\mathcal{F}$ is hereditarily upward (i.e. $B \in \mathcal{F}$ when $A \in \mathcal{F}$ and $A \subset B$); and also $A \cap [n, \infty[\in \mathcal{F}$ for all $A \in \mathcal{F}$ and $n \in \mathbb{N}$. The *dual family* $\mathcal{F}^*$ for a given family $\mathcal{F}$ is the collection of sets $A \subset \mathbb{N}_0$ such that $A \cap B \neq \varnothing$ for every $B \in \mathcal{F}$. The rather vague notion of *frequency* mentioned above will be defined in terms of families: given a dynamical system (X, T), a point $x \in X$ and a non-empty (open) subset $U \subset X$ we denote by

$$N_T(x, U) := \{n \in \mathbb{N}_0 : T^n x \in U\},$$

the *return set from x to U*. It will be denoted by $N(x, U)$ if no confusion with the map arises.

Definition 4.1. Let (X, T) be a dynamical system and let $\mathcal{F}$ be a Furstenberg family. A point $x \in X$ is said to be

– *$\mathcal{F}$-recurrent* if $N(x, U) \in \mathcal{F}$ for every neighbourhood U of x;

– *$\mathcal{F}$-hypercyclic* if $N(x, U) \in \mathcal{F}$ for every non-empty open subset $U \subset X$.

Moreover, we will denote by

– $\mathcal{F}\mathrm{Rec}(T)$ the *set of $\mathcal{F}$-recurrent points*, and we will say that T is *$\mathcal{F}$-recurrent* whenever the set $\mathcal{F}\mathrm{Rec}(T)$ is dense in X;

– $\mathcal{F}\mathrm{HC}(T)$ the *set of $\mathcal{F}$-hypercyclic points*, and we will say that T is *$\mathcal{F}$-hypercyclic* whenever the set $\mathcal{F}\mathrm{HC}(T)$ is non-empty.

Example 4.2. The previous Linear Dynamics works [BM09a, GEP11, CMP14, BMPP16, BGE18, BGELMP22, CM22a, CM22b, CM, GLM23] have considered the families formed by:

(a) the *infinite sets*

$$\mathcal{I} = \{A \subset \mathbb{N}_0 : A \text{ is infinite}\}.$$

The notion of $\mathcal{I}$-recurrence (resp. $\mathcal{I}$-hypercyclicity) coincides with the concept of recurrence (resp. hypercyclicity) as defined in the Introduction. We denote by $\mathrm{Rec}(T)$ (resp. $\mathrm{HC}(T)$) the set of recurrent (resp. hypercyclic) vectors associated to this family.

(b) the *sets containing arbitrarily long arithmetic progressions* or *AP-sets*

$$\mathcal{AP} := \{A \subset \mathbb{N}_0 : A \text{ contains arbitrarily long arithmetic progressions}\}.$$

In other words, $A \in \mathcal{AP}$ if $\forall l \in \mathbb{N}$, $\exists a, m \in \mathbb{N}$ such that $\{a + km : 0 \leq k \leq l\} \subset A$, or equivalently A contains the *arithmetic progression* of length $l+1$, common difference $m \in \mathbb{N}$ and initial term $a \in \mathbb{N}_0$. The concept of $\mathcal{AP}$-recurrence is equivalent to the notion of (topological) *multiple recurrence*, which requires that for each non-empty open subset $U \subset X$ and each length $l \in \mathbb{N}$ there exists some $n \in \mathbb{N}$ fulfilling that

$$U \cap T^{-n}(U) \cap \cdots \cap T^{-ln}(U) \neq \varnothing.$$

This concept was introduced for linear dynamical systems in [CP12] and further developed in [KLOY17] and [CM22b], where $\mathcal{AP}$-hypercyclicity is also considered. We will denote by $\mathcal{AP}\mathrm{Rec}(T)$ (resp. $\mathcal{AP}\mathrm{HC}(T)$) the set of $\mathcal{AP}$-recurrent (resp. $\mathcal{AP}$-hypercyclic) vectors.

(c) the *sets with positive upper Banach density* $\overline{\mathcal{BD}} := \{A \subset \mathbb{N}_0 : \overline{\mathrm{Bd}}(A) > 0\}$, where for each $A \subset \mathbb{N}_0$ its *upper Banach density* is defined as

$$\overline{\mathrm{Bd}}(A) := \lim_{N\to\infty} \left(\max_{n\geq 0} \frac{\#(A \cap [n+1, n+N])}{N} \right). \quad \text{See [GTT10] for alternative definitions.}$$

The $\overline{\mathcal{BD}}$-recurrence (resp. $\overline{\mathcal{BD}}$-hypercyclicity) has been called *reiterative recurrence* (resp. *reiterative hypercyclicity*), and $\mathrm{RRec}(T)$ (resp. $\mathrm{RHC}(T)$) is the set of reiteratively recurrent (resp. reiteratively hypercyclic) vectors, see [BMPP16, BMPP19, BGELMP22, GLM23].

(d) the *sets with positive upper density* $\overline{\mathcal{D}} := \{A \subset \mathbb{N}_0 : \overline{\mathrm{dens}}(A) > 0\}$, where for each $A \subset \mathbb{N}_0$ its *upper density* is defined as

$$\overline{\mathrm{dens}}(A) := \limsup_{N\to\infty} \frac{\#(A \cap [1, N])}{N}.$$

The $\overline{\mathcal{D}}$-recurrence (resp. $\overline{\mathcal{D}}$-hypercyclicity) notion is commonly called *$\mathcal{U}$-frequent recurrence* (resp. *$\mathcal{U}$-frequent hypercyclicity*) and $\mathrm{UFRec}(T)$ (resp. $\mathrm{UFHC}(T)$) is the set of $\mathcal{U}$-frequent recurrent (resp. $\mathcal{U}$-frequent hypercyclic) vectors, see [Shk09, CP12, BGELMP22, GLM23].

(e) the *sets with positive lower density* $\underline{\mathcal{D}} := \{A \subset \mathbb{N}_0 : \underline{\mathrm{dens}}(A) > 0\}$, where for each $A \subset \mathbb{N}_0$ its *lower density* is defined as

$$\underline{\mathrm{dens}}(A) := \limsup_{N\to\infty} \frac{\#(A \cap [1, N])}{N}.$$

The notion of $\underline{\mathcal{D}}$-recurrence (resp. $\underline{\mathcal{D}}$-hypercyclicity) is usually called *frequent recurrence* (resp. *frequent hypercyclicity*), and we will denote by $\mathrm{FRec}(T)$ (resp. $\mathrm{FHC}(T)$) the set of frequently recurrent (resp. frequently hypercyclic) vectors, see [BG06, BMPP16, GLM23].

(f) the *syndetic sets*

$$\mathcal{S} := \{A \subset \mathbb{N}_0 : A \text{ is syndetic}\},$$

i.e. $A \in \mathcal{S}$ whenever $\exists m \in \mathbb{N}$ such that $\forall a \in \mathbb{N}_0$, $[a, a+m] \cap A \neq \varnothing$. The notion of $\mathcal{S}$-recurrence has been called *uniform recurrence* and we denote by $\mathrm{URec}(T)$ the set of uniformly recurrent vectors, see [Fur81, BGELMP22, GLM23]. The family $\mathcal{S}$ coincides with the family of *sets with positive lower Banach density* $\underline{\mathcal{BD}} := \{A \subset \mathbb{N}_0 : \underline{\mathrm{Bd}}(A) > 0\}$, where for each $A \subset \mathbb{N}_0$ its *lower Banach density* is defined as

$$\underline{\mathrm{Bd}}(A) := \lim_{N\to\infty} \left(\inf_{n\geq 0} \frac{\#(A \cap [n+1, n+N])}{N} \right).$$ See [GTT10] for alternative definitions.

(g) the *IP-sets*

$$\mathcal{IP} := \{A \subset \mathbb{N}_0 : A \text{ is an IP-set}\},$$

i.e. $A \in \mathcal{IP}$ whenever $\{\sum_{k\in F} n_k : F \subset \mathbb{N} \text{ finite}\} \subset A$ for some sequence $(n_k)_{k\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$. In this case we use the dual family $\mathcal{IP}^*$, and for the $\mathcal{IP}^*$-recurrence notion we denote by $\mathcal{IP}^*\mathrm{Rec}(T)$ the set $\mathcal{IP}^*$-recurrent vectors, see [Fur81, GMJPO15, BGELMP22, GLM23].

(h) the Δ-*sets*

$$\Delta := \{A \subset \mathbb{N}_0 : A \text{ is an } \Delta\text{-set}\},$$

i.e. $A \in \Delta$ whenever $\exists B \in \mathcal{I}$ with $(B - B) \cap \mathbb{N} \subset A$. In this case we again use the dual family Δ^*, and for the Δ^*-recurrence notion we denote by $\Delta^*\mathrm{Rec}(T)$ the set of Δ^*-recurrent vectors, see [Fur81, GLM23].

We have the following relations between the families listed above:

$$\Delta^* \subset \mathcal{IP}^* \subset \mathcal{S} = \underline{\mathcal{BD}} \subset \underline{\mathcal{D}} \subset \overline{\mathcal{D}} \subset \overline{\mathcal{BD}} \subset \mathcal{AP} \subset \mathcal{I}^{\mathrm{B}}.$$

Among these inclusions, some of them follow easily from the definitions while others depend on deep theorems like the celebrated *Szemerédi theorem*, see [BD08, CM22b, HS98, Sze75]. They imply the respective inclusions between the introduced sets of recurrent vectors

$$\Delta^*\mathrm{Rec}(T) \subset \mathcal{IP}^*\mathrm{Rec}(T) \subset \mathrm{URec}(T) \subset \dots$$

$$\dots \subset \mathrm{FRec}(T) \subset \mathrm{UFRec}(T) \subset \mathrm{RRec}(T) \subset \mathcal{AP}\mathrm{Rec}(T) \subset \mathrm{Rec}(T),$$

and also between the sets of hypercyclic vectors

$$\mathrm{FHC}(T) \subset \mathrm{UFHC}(T) \subset \mathrm{RHC}(T) \subset \mathcal{AP}\mathrm{HC}(T) \subset \mathrm{HC}(T).$$

It is worth mentioning that the families $\mathcal{F}$ for which there exist $\mathcal{F}$-hypercyclic operators are by far less common than those for which $\mathcal{F}$-recurrence exists, which is not surprising since having an orbit distributed around the whole space is much more complicated than having it just coming back around the initial point of the orbit (it is known that there does not exist any Δ^*, $\mathcal{IP}^*$ neither $\mathcal{S}$-hypercyclic operator, see [BMPP16, BGELMP22]). Other families have been considered from the $\mathcal{F}$-hypercyclicity perspective in the works [BGE18, EEM21]. Our aim now is to show that a dynamical system is *quasi-rigid* if and only if it is $\mathcal{F}$-recurrent with respect to a *free filter* $\mathcal{F}$. Let us recall the following classical definitions (see [Bou89]):

[B] See Sections 1, 2 and 3 of the Appendix for more details on the stated inclusions.

Definition 4.3. Let $\mathcal{F} \subset \mathscr{P}(\mathbb{N}_0)$ be a collection of sets of natural numbers (note that $\mathcal{F}$ is not necessarily a Furstenberg family here). We say that $\mathcal{F}$:

– has the *finite intersection property*, if $\bigcap_{A\in\mathcal{A}} A \neq \varnothing$ for every $\mathcal{A} \subset \mathcal{F}$ with $\#\mathcal{A} < \infty$;

– is a *filter*, if $\mathcal{F}$ is hereditarily upward and for every $A, B \in \mathcal{F}$ we have that $A \cap B \in \mathcal{F}$;

– is a *free filter*, if it is a filter and $\bigcap_{A\in\mathcal{F}} A = \varnothing$.

Finally, given any infinite subset $A \subset \mathbb{N}_0$ we denote by $\mathcal{F}(A)$ the *free filter generated by* A, which is the Furstenberg family

$$\mathcal{F}(A) := \{B \subset \mathbb{N}_0 : \#(A \setminus B) < \infty\}.$$

In the literature, $\mathcal{F}(A)$ has been called the *Fréchet filter* on A, or also the *eventuality filter* generated by the increasing sequence of integers $(n_k)_{k\in\mathbb{N}}$ forming the set $A \subset \mathbb{N}_0$ (see [Bou89]).

Remark 4.4. It is clear that every *free filter* is a *filter*, and that every *filter* has the *finite intersection property*. However, when $\mathcal{F}$ is a *Furstenberg family* as defined in this paper the previous definitions have some extra (and immediate) **consequences**:

(a) *A Furstenberg family* $\mathcal{F}$ *has the finite intersection property if and only if* $\bigcap_{A\in\mathcal{A}} A \in \mathcal{I}$ *for every* $\mathcal{A} \subset \mathcal{F}$ *with* $\#\mathcal{A} < \infty$: indeed, if $\mathcal{F}$ has the finite intersection property but for some $\mathcal{A} \subset \mathcal{F}$ with $\#\mathcal{A} < \infty$ we had that $\bigcap_{A\in\mathcal{A}} A$ is finite, then for any fixed $n > \max\left(\bigcap_{A\in\mathcal{A}} A\right)$ we would arrive to the contradiction:

$$\mathcal{A}' := \{A \cap [n, \infty[: A \in \mathcal{A}\} \subset \mathcal{F} \quad \text{with} \quad \#\mathcal{A}' < \infty \quad \text{but} \quad \bigcap_{A\in\mathcal{A}'} A = \varnothing.$$

(b) *Let* $\mathcal{F}$ *be a Furstenberg family. Then,* $\mathcal{F}$ *is a filter if and only if it is a free filter*: note that given any B from a filter $\mathcal{F}$ which is also a Furstenberg family we have that

$$\bigcap_{A\in\mathcal{F}} A \subset \bigcap_{n\in\mathbb{N}} B \cap [n, \infty[= \varnothing,$$

by the definition of Furstenberg family used in this paper.

We are now ready to characterize quasi-rigidity in terms of $\mathcal{F}$-recurrence and free filters:

Proposition 4.5. *Let* (X, T) *be a dynamical system. The following are equivalent:*

(i) T *is quasi-rigid;*

(ii) T *is* $\mathcal{F}(A)$*-recurrent for some infinite subset* $A \subset \mathbb{N}_0$*;*

(iii) T *is* $\mathcal{F}$*-recurrent with respect to a free filter* $\mathcal{F}$ *with a countable base.*

Moreover, if X *is a second-countable space, the previous statements are equivalent to:*

(iv) T *is* $\mathcal{F}$*-recurrent for a family* $\mathcal{F}$ *with the finite intersection property.*

Proof. For (i) $\Rightarrow$ (ii), if T is quasi-rigid with respect to $(n_k)_{k\in\mathbb{N}}$ consider $A := \{n_k : k \in \mathbb{N}\}$. The implication (ii) $\Rightarrow$ (iii) is obvious since $\mathcal{F}(A)$ has a countable base. To see (iii) $\Rightarrow$ (i) assume that the map T is $\mathcal{F}$-recurrent and let $(A_k)_{k\in\mathbb{N}}$ be a decreasing countable base of the free filter $\mathcal{F}$. Setting $n_k := \min(A_k)$ for each $k \in \mathbb{N}$, and taking a subsequence if necessary, we get an increasing sequence $(n_k)_{k\in\mathbb{N}}$ with respect to which T is quasi-rigid.

We always have that (i), (ii) and (iii) $\Rightarrow$ (iv) even if the space X is not second-countable. Assume now that X is second-countable and that T is $\mathcal{F}$-recurrent for a family $\mathcal{F}$ with the finite intersection property. Let $\{x_s : s \in \mathbb{N}\} \subset \mathcal{F}\mathrm{Rec}(T)$ be a dense set in X and, for each $s \in \mathbb{N}$, let $(U_{s,k})_{k\in\mathbb{N}}$ be a decreasing neighbourhood basis of x_s. We can now recursively construct an increasing sequence of positive integers $(n_k)_{k\in\mathbb{N}}$ such that

$$T^{n_k}x_s \in U_{s,k} \quad \text{for every } s,k \in \mathbb{N} \text{ with } 1 \leq s \leq k,$$

by taking a sufficiently large integer $n_k \in \bigcap_{s=1}^{k} N_T(x_s, U_{s,k}) \in \mathcal{I}$. It is easily seen that the map T is quasi-rigid with respect to $(n_k)_{k\in\mathbb{N}}$, which shows that (iv) $\Rightarrow$ (i), (ii) and (iii). □

4.2 Appropriate Furstenberg families for $\mathcal{F}$-recurrence

We have just shown that quasi-rigidity is indeed a particular case of $\mathcal{F}$-recurrence for a kind of family not listed in Example 4.2. It is then natural to ask the following:

Question 4.6. Are the families of Example 4.2, i.e. $\mathcal{F} = \mathcal{I}, \mathcal{AP}, \overline{\mathcal{BD}}, \overline{\mathcal{D}}, \underline{\mathcal{D}}, \mathcal{S}, \mathcal{IP}^*, \Delta^*$, and the free filters $\mathcal{F}(A)$ the only ones for which $\mathcal{F}$-recurrence should be considered?

This may seem a very open query. Our objective in this part of the paper is to look in detail into two very different classes of Furstenberg families: those for which $\mathcal{F}$-recurrence is the *weakest* possible recurrence notion, i.e. families $\mathcal{F} \subsetneq \mathcal{I}$ such that $\mathcal{F}\mathrm{Rec}(T) = \mathrm{Rec}(T)$; and those for which $\mathcal{F}$-recurrence is the *strongest* possible recurrence notion, i.e. periodicity.

Starting with the first of these two classes (i.e. the weaker ones):

– *For any dynamical system (X, T) every recurrent point $x \in \mathrm{Rec}(T)$ is an $\mathcal{IP}$-recurrent point.*

This is shown in [Fur81, Theorem 2.17] when X is a *metric space*, and that proof easily extends for arbitrary dynamical systems[C]. Hence

$$\mathrm{Rec}(T) \subset \mathcal{IP}\mathrm{Rec}(T) \subset \Delta\mathrm{Rec}(T) \subset \mathcal{I}\mathrm{Rec}(T).$$

Since $\mathcal{I}\mathrm{Rec}(T) = \mathrm{Rec}(T)$ we always have that

$$\mathcal{IP}\mathrm{Rec}(T) = \Delta\mathrm{Rec}(T) = \mathrm{Rec}(T).$$

This apparently basic relation allows us to give an alternative (and much simpler) proof to the so-called *Ansari* and *León-Müller recurrence theorems*, which assert that powers and unimodular multiples of an operator share the same set of recurrent vectors (see [CMP14, Proposition 2.3] for the original, different for each of the T^p and λT cases, and rather long proof):

[C]See Proposition 4.2 in Section 4 of the Appendix.

Proposition 4.7 (**[CMP14, Proposition 2.3]**). *Let (X,T) be a dynamical system. Then:*

(a) *For every positive integer $p \in \mathbb{N}$ we have that* $\operatorname{Rec}(T) = \operatorname{Rec}(T^p)$.

(b) *If (X,T) is linear, then for every $\lambda \in \mathbb{T}$ we have that* $\operatorname{Rec}(T) = \operatorname{Rec}(\lambda T)$.

Proof. [D] Suppose that $x \in \operatorname{Rec}(T)$, i.e. $N_T(x,V)$ belongs to Δ for every neighbourhood V of x. Fixed any $p \in \mathbb{N}$ and any neighbourhood U of x, since the set $(p \cdot \mathbb{N}_0)$ belongs to the Δ^* family[E] we have that

$$\varnothing \neq N_T(x,U) \cap (p \cdot \mathbb{N}_0) \subset N_{T^p}(x,U),$$

which implies that $x \in \operatorname{Rec}(T^p)$, proving (a). To check (b) assume moreover that (X,T) is a linear dynamical system and fix any $\lambda \in \mathbb{T}$. Let $\varepsilon > 0$ and let $U_0 \subset U$ be a neighbourhood of x such that $\mu \cdot U_0 \subset U$ for all $|\mu - 1| < \varepsilon$. Since the set $\{n \in \mathbb{N} : |\lambda^n - 1| < \varepsilon\}$ belongs to Δ^* (see [GLM23, Proposition 4.1]) we have that

$$\varnothing \neq N_T(x,U_0) \cap \{n \in \mathbb{N} : |\lambda^n - 1| < \varepsilon\} \subset N_{\lambda T}(x,U),$$

which implies that $x \in \operatorname{Rec}(\lambda T)$. □

Let us now look at the *strongest* possible recurrence notion by using families of big subsets of natural numbers, which have been also historically considered in (linear) dynamics:

Example 4.8. Consider the Furstenberg families formed by:

(a) the *cofinite sets*

$$\mathcal{I}^* := \{A \subset \mathbb{N}_0 : A \text{ is cofinite}\},$$

which is the dual family of that formed by the infinite sets $\mathcal{I}$, and which is used to define the notion of (topological) *mixing*: a dynamical system (X,T) is called *mixing* if the set $\{n \in \mathbb{N}_0 : T^n(U) \cap V \neq \varnothing\}$ belongs to $\mathcal{I}^*$ for every pair of non-empty open subset $U, V \subset X$ (see [GEP11, Definition 1.38]).

(b) the *thick sets*

$$\mathcal{T} := \{A \subset \mathbb{N}_0 : A \text{ is thick}\},$$

i.e. $A \in \mathcal{T}$ if $\forall m \in \mathbb{N}$, $\exists a_m \in A$ with $[a_m, a_m + m] \subset A$. This family is known to characterize the *weak-mixing* property (see [GEP10, Theorem 3] or [GEP11, Theorem 1.54]):

– *For every dynamical system (X,T) the following statements are equivalent:*
  (i) *(X,T) is weakly mixing;*
  (ii) *the set $\{n \in \mathbb{N}_0 : T^n(U) \cap V \neq \varnothing\}$ belongs to $\mathcal{T}$ for every pair of non-empty open subsets $U, V \subset X$.*

(c) the *thickly syndetic sets*

$$\mathcal{TS} := \{A \subset \mathbb{N}_0 : A \text{ is thickly syndetic}\},$$

i.e. $A \in \mathcal{TS}$ if $\forall m \in \mathbb{N}$, $\exists A_m$ syndetic such that $A_m + [0,m] \subset A$. This family characterizes the *topological ergodicity* (i.e. the property that $\{n \in \mathbb{N}_0 : T^n(U) \cap V \neq \varnothing\}$ is syndetic for every pair of non-empty open subsets $U, V \subset X$) for **linear** dynamical systems, as it is shown in [GEP11, Exercise 2.5.4] and [BMPP19]:

[D] The same arguments show the "Ansari-León-Müller" properties for the families $\mathcal{IP}^*$ and Δ^*.
[E] See Lemma 4.3 in Section 4 of the Appendix.

– *For every linear dynamical system* (X, T) *the following are equivalent:*
  (i) (X, T) *is topologically ergodic;*
  (ii) *the set* $\{n \in \mathbb{N}_0 : T^n(U) \cap V \neq \varnothing\}$ *belongs to* $\mathcal{TS}$ *for every pair of non-empty open subsets* $U, V \subset X$.

(d) the *sets of density greater or equal to* $\delta > 0$, which for each $0 < \delta \leq 1$ are the families

$$\underline{\mathcal{BD}}_\delta := \{A \subset \mathbb{N}_0 : \underline{\mathrm{Bd}}(A) \geq \delta\}, \qquad \overline{\mathcal{BD}}_\delta := \{A \subset \mathbb{N}_0 : \overline{\mathrm{Bd}}(A) \geq \delta\},$$
$$\underline{\mathcal{D}}_\delta := \{A \subset \mathbb{N}_0 : \underline{\mathrm{dens}}(A) \geq \delta\}, \qquad \overline{\mathcal{D}}_\delta := \{A \subset \mathbb{N}_0 : \overline{\mathrm{dens}}(A) \geq \delta\}.$$

The density families have also been studied in Linear Dynamics, and we refer the reader to [BMPP16, BMPP19] for more about them.

We are about to show that the recurrence notions associated to the Furstenberg families introduced in Example 4.8 above imply different periodicity notions (see Proposition 4.11). We replicate the arguments from [BMPP16, Proposition 3], which have been used in an independent way in [CM, Lemma 2.14 and Corollary 2.15]. Our contribution here is to rewrite these results in their "*pointwise* $\mathcal{F}$*-recurrence*" version. We start by proving two key lemmas:

Lemma 4.9. *Let* (X, T) *be a dynamical system,* $x \in X$ *and* $N \in \mathbb{N}$. *The following statements are equivalent:*

(i) $x \in \mathrm{Per}(T)$ *and its period is strictly lower than* N;

(ii) *for every neighbourhood* U *of* x *there exists* $n_U \in \mathbb{N}_0$ *such that*
$$\#(N(x, U) \cap [n_U + 1, n_U + N]) \geq 2;$$

(iii) *for every neighbourhood* U *of* x *we have that* $T^p(U) \cap U \neq \varnothing$ *for some* $1 \leq p < N$.

Proof. For (i) $\Rightarrow$ (ii) let n_U be the period of x minus one. For (ii) $\Rightarrow$ (iii) recall that

$$\text{given } n_1 < n_2 \in N(x, U) \quad \text{we have that} \quad T^{n_2 - n_1}(U) \cap U \neq \varnothing.$$

Finally, if we suppose that $T^p x \neq x$ for all $1 \leq p < N$, by continuity we can find an open neighbourhood U of x such that $T^p(U) \cap U = \varnothing$ for every $1 \leq p < N$, so (iii) $\Rightarrow$ (i). □

Lemma 4.10. *Let* $0 < \delta \leq 1$ *and* $N \in \mathbb{N}$ *with* $\frac{1}{N} < \delta$. *Then, for every* $A \in \overline{\mathcal{BD}}_\delta$ *there is* $n_A \in \mathbb{N}_0$ *such that* $\#(A \cap [n_A + 1, n_A + N]) \geq 2$.

Proof. Otherwise we would have that $\#(A \cap [n+1, n+N]) \leq 1$ for every $n \in \mathbb{N}_0$ obtaining the contradiction

$$\overline{\mathrm{Bd}}(A) = \lim_{K \to \infty} \left(\max_{n \geq 0} \frac{\#(A \cap [n+1, n+(N \cdot K)])}{N \cdot K} \right) \leq \lim_{K \to \infty} \frac{K}{N \cdot K} = \frac{1}{N} < \delta. \qquad \square$$

Finally we get the desired result, in which we classify the periodic points of a dynamical system in terms of the density of the return sets:

Proposition 4.11. *Let (X,T) be a dynamical system and let $x \in X$. Then:*

(a) *Given $0 < \delta \leq 1$, the following statements are equivalent:*

(i) *$x \in \mathrm{Per}(T)$ and its period is lower or equal to $\left\lfloor \frac{1}{\delta} \right\rfloor$;*

(ii) *$x \in \underline{\mathcal{BD}}_\delta\mathrm{Rec}(T)$;*

(iii) *$x \in \underline{\mathcal{D}}_\delta\mathrm{Rec}(T)$;*

(iv) *$x \in \overline{\mathcal{D}}_\delta\mathrm{Rec}(T)$;*

(v) *$x \in \overline{\mathcal{BD}}_\delta\mathrm{Rec}(T)$.*

In particular, if T is $\overline{\mathcal{BD}}_\delta$-recurrent then $T^N = I$ for $N = 1 \cdot 2 \cdots \left(\left\lfloor \frac{1}{\delta} \right\rfloor - 1\right) \cdot \left\lfloor \frac{1}{\delta} \right\rfloor$.

(b) *The following statements are equivalent:*

(i) *x is a fixed point, i.e. $Tx = x$;*

(ii) *$x \in \mathcal{I}^*\mathrm{Rec}(T)$;*

(iii) *$x \in \mathcal{TS}\mathrm{Rec}(T)$;*

(iv) *$x \in \mathcal{T}\mathrm{Rec}(T)$;*

(v) *$x \in \overline{\mathcal{BD}}_\delta\mathrm{Rec}(T)$ for some $\delta > \frac{1}{2}$;*

(vi) *for every neighbourhood U of x the set $N(x,U)$ contains two consecutive integers.*

Proof. (a): Let us first show that (i) $\Rightarrow$ (ii): let $p \in \left\{1, 2, ..., \left\lfloor \frac{1}{\delta} \right\rfloor\right\}$ be such that $T^p x = x$. Then

$$\underline{\mathrm{Bd}}(N(x,U)) \geq \underline{\mathrm{Bd}}(p \cdot \mathbb{N}_0) = \frac{1}{p} \geq \delta$$

for every neighbourhood U of x, so $N(x,U) \in \overline{\mathcal{BD}}_\delta$. For (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) recall that $\underline{\mathcal{BD}}_\delta \subset \underline{\mathcal{D}}_\delta \subset \overline{\mathcal{D}}_\delta \subset \overline{\mathcal{BD}}_\delta$. For (v) $\Rightarrow$ (i) let $N := \left\lfloor \frac{1}{\delta} \right\rfloor + 1$. By Lemma 4.10, for every neighbourhood U of x there is $n_U \in \mathbb{N}$ such that

$$\#(N(x,U) \cap [n_U + 1, n_U + N]) \geq 2,$$

so Lemma 4.9 finishes the work.

(b): The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) follow from the following well-known relations $\mathbb{N}_0 \in \mathcal{I}^* \subset \mathcal{TS} \subset \mathcal{T} = \overline{\mathcal{BD}}_1 \subset \overline{\mathcal{BD}}_\delta$ for every $0 < \delta \leq 1$ (see [BD08, HS98]). In order to prove (v) $\Rightarrow$ (vi) use Lemma 4.10 applied to $N = 2$. Finally, the equivalence (i) $\Leftrightarrow$ (vi) follows from Lemma 4.9. □

The characterization obtained for periodic points in part (a) of Proposition 4.11 tells us which are the Furstenberg families $\mathcal{F}$ whose respective $\mathcal{F}$-recurrence notion is trivial, in the sense that it coincides with periodicity. However, it is still interesting to study all the recurrence notions introduced in Example 4.2: recall that for each positive integer $p \in \mathbb{N}$ the set $p \cdot \mathbb{N}_0$ belongs to the Δ^* family[F], so that $\mathrm{Per}(T) \subset \Delta^*\mathrm{Rec}(T)$ for every dynamical system (X,T) and hence Δ^*-recurrence is weaker than periodicity. We have also reproved [BMPP16, Proposition 3]:

Corollary 4.12. *There is no $\overline{\mathcal{BD}}_\delta$-hypercyclic operator for any $0 < \delta \leq 1$.*

[F]See Lemma 4.3 in Section 4 of the Appendix.

4.3 Quasi-conjugacies and commutants in $\mathcal{F}$-recurrence

We end Section 4 with some basic but useful tools about *quasi-conjugacies* and (non-linear) *commutants* which will be used in Section 5 to study the *lineability* and *dense lineability* of the set of $\mathcal{F}$-recurrent vectors. Following [Fur81, GEP11] we define:

Definition 4.13. Given two dynamical systems (X, T) and (Y, S), a map $\phi : Y \longrightarrow X$ is called an *homomorphism of dynamical systems* if ϕ is continuous and the diagram

$$\begin{array}{ccc} Y & \xrightarrow{S} & Y \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ X & \xrightarrow{T} & X \end{array}$$

commutes, i.e. $\phi \circ S = T \circ \phi$. Moreover, we say that the map ϕ is a:

– *quasi-conjugacy*, if ϕ has dense range, and hence that (X, T) is *quasi-conjugate* to (Y, S);

– *conjugacy*, if ϕ is an homeomorphism between X and Y (see Remark 5.2).

Many dynamical properties are preserved by quasi-conjugacy (see [GEP11, Chapter 1]): if (Y, S) admits a dense orbit, is topologically transitive, weakly-mixing or even (Devaney) chaotic, then so is (X, T). On the other hand, to preserve the *recurrence* of a point it is enough to have an homomorphism. This is a really well-known fact (see [Fur81, Proposition 1.3]), and indeed similar arguments were already used in [Fur81, Proposition 9.9] for $\mathcal{F}$-recurrence with respect to the Furstenberg families Δ^* and $\mathcal{IP}^*$, or in [CM, Proposition 2.7] for general Furstenberg families. We include here the general argument regarding return sets:

Lemma 4.14. *Let (X, T) and (Y, S) be dynamical systems and suppose that $\phi : Y \longrightarrow X$ is an homomorphism between them. Given $y \in Y$ and any neighbourhood $U \subset X$ of $\phi(y)$, the set $V := \phi^{-1}(U) \subset Y$ is a neighbourhood of y for which $N_S(y, V) = N_T(\phi(y), U)$. In particular, for any Furstenberg family $\mathcal{F}$ we have that:*

(a) *If $y \in Y$ is $\mathcal{F}$-recurrent for S then $\phi(y)$ is $\mathcal{F}$-recurrent for T.*

(b) *If ϕ is a quasi-conjugacy and S is $\mathcal{F}$-recurrent then so is T.*

Proof. By continuity of ϕ the set $V = \phi^{-1}(U)$ is a neighbourhood of y. Since $\phi \circ S = T \circ \phi$ we have that $\phi(S^n y) = T^n \phi(y)$ for every $n \in \mathbb{N}_0$. It follows that

$$N_T(\phi(y), U) = \{n \in \mathbb{N}_0 : T^n \phi(y) \in U\} = \{n \in \mathbb{N}_0 : S^n y \in \phi^{-1}(U)\} = N_S(y, V).$$

In particular, if y is $\mathcal{F}$-recurrent then the return set $N_T(\phi(y), U) = N_S(y, \phi^{-1}(U))$ belongs to $\mathcal{F}$ for every neighbourhood U of $\phi(y)$. Statement (b) follows immediately from (a). □

Considering $Y := X$, $S := T$ and $\phi := S$ we obtain the following:

Corollary 4.15. *Let (X, T) be a dynamical system and let $S : X \longrightarrow X$ be a continuous map commuting with T, that is $S \circ T = T \circ S$. Given $x \in X$ and any neighbourhood U of Sx, the set $V := S^{-1}(U)$ is a neighbourhood of x for which $N_T(x, V) = N_T(Sx, U)$. In particular, for any Furstenberg family $\mathcal{F}$ we have that*

$$S(\mathcal{F}\mathrm{Rec}(T)) \subset \mathcal{F}\mathrm{Rec}(T).$$

Corollary 4.15 unifies some of the well-known arguments used in [AB, Proposition 2.5], [CMP14, Proof of Theorem 9.1] and [Fur81, Propositions 1.3 and 9.9], generalizing them to arbitrary Furstenberg families and (not necessarily linear) maps acting on arbitrary topological spaces. From now on we will write

$$\mathscr{C}(X) := \{S : X \longrightarrow X \text{ continuous map}\},$$

i.e. the *set of continuous* (and not necessarily linear) *self-maps* of the Hausdorff topological space X. Following the notation of [Jun76] we define:

Definition 4.16. Let (X, T) be a dynamical system. We will denote by

$$\mathcal{C}_T := \{S \in \mathscr{C}(X) : S \circ T = T \circ S\},$$

the (non-linear) *commutant* of T. Given $x \in X$ we will denote by $\mathcal{C}_T(x) := \{Sx : S \in \mathcal{C}_T\}$ the $\mathcal{C}_T$*-orbit* of x. We say that a subset $Y \subset X$ is $\mathcal{C}_T$*-invariant* if $S(Y) \subset Y$ for every $S \in \mathcal{C}_T$.

Remark 4.17. Since $T \in \mathcal{C}_T$ we always have that:

– every $\mathcal{C}_T$-invariant set is also T-invariant;

– if X is a linear space, then $\{p(T) : p \text{ polynomial}\} \subset \mathcal{C}_T$ and for every $x \in X$ we get that

$$\text{span}(\text{Orb}(x, T)) = \text{span}\{T^n x : n \in \mathbb{N}_0\} = \{p(T)x : p \text{ polynomial}\} \subset \mathcal{C}_T(x).$$

In particular, every *cyclic* vector for an operator $T \in \mathcal{L}(X)$ has a *dense* $\mathcal{C}_T$*-orbit*.

Moreover, in the linear setting every $\mathcal{C}_T$-orbit is a vector subspace of X:

Lemma 4.18. *Let (X, T) be a linear dynamical system. Then $(\mathcal{C}_T, +, \circ)$ is a subring of the ring of continuous maps $(\mathscr{C}(X), +, \circ)$. In particular, given any $x \in X$ the set $\mathcal{C}_T(x)$ is a vector subspace of X and the smallest $\mathcal{C}_T$-invariant subset of X containing the point x.*

Proof. Given $S, R \in \mathcal{C}_T$ and $\alpha, \beta \in \mathbb{K}$ we have that $(\alpha S + \beta R) \in \mathcal{C}_T$ and also that $(S \circ R) \in \mathcal{C}_T$. In particular, given $x \in X$ the set $\mathcal{C}_T(x)$ is a vector subspace of X. Moreover, if $Y \subset X$ is any $\mathcal{C}_T$-invariant subset of X and $x \in Y$ then $\mathcal{C}_T(x) = \{Sx : S \in \mathcal{C}_T\} \subset \bigcup_{S \in \mathcal{C}_T} S(Y) \subset Y$. □

When (X, T) is a non-linear system it is still true that $\mathcal{C}_T(x)$ is the *smallest* $\mathcal{C}_T$*-invariant subset of* X *containing the point* $x \in X$. Let us now generalize Corollary 4.15 to direct products by using the following notation: given a system (X, T) and a subset $Y \subset X$ we will denote by

$$\mathcal{N}_{pr}(Y) := \{A \subset \mathbb{N}_0 : N(x, U) \subset A \text{ for some } x \in Y \text{ and some neighbourhood } U \text{ of } x\},$$

the *family of pointwise-recurrent return sets* of the points of Y. Note that $\mathcal{N}_{pr}(\{x\})$ is a filter, for a point $x \in X$, if and only if $x \in \text{Rec}(T)$: indeed, if $x \notin \text{Rec}(T)$ then $\mathcal{N}_{pr}(\{x\}) = \mathscr{P}(\mathbb{N}_0)$, which is not a filter; conversely, if we choose a pair U and V of neighbourhoods for a recurrent vector $x \in \text{Rec}(T)$ then we have that $N(x, U \cap V) \subset N(x, U) \cap N(x, V)$. Note also that a dynamical system (X, T) is $\mathcal{F}$-recurrent if there exits a dense set $Y \subset X$ such that $\mathcal{N}_{pr}(Y) \subset \mathcal{F}$.

Theorem 4.19. *Let (X,T) be a (linear) dynamical system and let $\mathcal{F}$ be a Furstenberg family. Given $x \in \mathcal{F}\mathrm{Rec}(T)$ we have that*

$$\mathcal{N}_{pr}(\mathcal{C}_T(x)) \ = \ \mathcal{N}_{pr}(\{x\}),$$

and hence $\mathcal{C}_T(x)$ is a $\mathcal{C}_T$-invariant (vector) subspace of X with the property that

$$\mathcal{C}_T(x)^N \ \subset \ \mathcal{F}\mathrm{Rec}(T_{(N)}) \quad \textit{for every } N \in \mathbb{N}.$$

Proof. Obviously $\mathcal{N}_{pr}(\{x\}) \subset \mathcal{N}_{pr}(\mathcal{C}_T(x))$. For $\mathcal{N}_{pr}(\mathcal{C}_T(x)) \subset \mathcal{N}_{pr}(\{x\})$ we use Corollary 4.15: given $y \in \mathcal{C}_T(x)$ and any neighbourhood U of y there is a neighbourhood V of x such that

$$N_T(y,U) = N_T(x,V) \in \mathcal{N}_{pr}(\{x\}).$$

The rest of the result follows from the filter condition of the family $\mathcal{N}_{pr}(\{x\})$: for any $N \in \mathbb{N}$ and any set $\{x_1,x_2,...,x_N\} \subset \mathcal{C}_T(x)$ let $z := (x_1,x_2,...,x_N) \in X^N$. Then, given any neighbourhood $U \subset X^N$ of z we can find neighbourhoods $U_i \subset X$ of x_i, for $1 \leq i \leq N$, such that

$$U_1 \times \cdots \times U_N \subset U \qquad \text{and hence} \qquad N_{T_{(N)}}(z,U) \supset \textstyle\bigcap_{i=1}^N N_T(x_i,U_i) \in \mathcal{N}_{pr}(\{x\}) \subset \mathcal{F}.$$

The arbitrariness of U implies that $z \in \mathcal{F}\mathrm{Rec}(T_{(N)})$. Finally, if (X,T) is a linear dynamical system, then the set $\mathcal{C}_T(x)$ is a $\mathcal{C}_T$-invariant vector subspace of X by Lemma 4.18. □

Remark 4.20. Theorem 4.19 is an extension of Corollary 4.15 to N-fold direct products/sums. As we have already mentioned, these kind of arguments have been used many times: for example in [CMP14, Theorem 9.1] they were used for usual recurrence, and in [CM, Proposition 2.7] for arbitrary Furstenberg families.

5 Infinite-dimensional vector spaces in $\mathcal{F}\mathrm{Rec}(T)$

The objective of this section is to study, for a linear dynamical system (X,T) and arbitrary Furstenberg families $\mathcal{F}$, when the set of $\mathcal{F}$-recurrent vectors is *lineable* or *dense lineable*, i.e. to establish if it contains a (possibly dense) infinite-dimensional vector subspace. Theorem 4.19 will be our main tool for this study, which motivation stems from the following two already showed facts:

– quasi-rigidity coincides with $\mathcal{F}(A)$-recurrence (see Proposition 4.5);

– quasi-rigidity for an operator T implies that $\mathrm{Rec}(T)$ is dense lineable (see Proposition 2.7).

It is then natural to ask whether the set $\mathcal{F}\mathrm{Rec}(T)$ contains an infinite-dimensional vector space for other Furstenberg families $\mathcal{F}$. We are about to show that $\mathcal{F}\mathrm{Rec}(T)$ is lineable as soon as T is $\mathcal{F}$-recurrent (see Theorem 5.5 and Corollary 5.6 below), and we obtain some (natural) sufficient conditions implying that $\mathcal{F}\mathrm{Rec}(T)$ is dense lineable (see Theorem 5.8). As a consequence we obtain the *Herrero-Bourdon theorem* for $\mathcal{F}$-hypercyclicity (see Subsection 5.3).

5.1 Lineability

Given a vector space X and a subset of vectors $Y \subset X$ with some property, we say that Y is *lineable* if there exists an infinite-dimensional vector subspace $Z \subset X$ such that $Z \setminus \{0\} \subset Y$. In our case, given a linear dynamical system (X, T) we always have that $0 \in \mathcal{F}\mathrm{Rec}(T)$ for every Furstenberg family $\mathcal{F}$, so the set of $\mathcal{F}$-recurrent vectors will be lineable if it admits an infinite-dimensional vector space (which includes the zero-vector).

In order to prove that $\mathcal{F}\mathrm{Rec}(T)$ is lineable as soon as (X, T) is $\mathcal{F}$-recurrent, we will observe that the (span of the) *unimodular eigenvectors* are the only recurrent vectors whose orbits have a finite-dimensional linear span (see Lemma 5.3 below). Recall that:

Definition 5.1. Given a **complex-linear** dynamical system $T : X \longrightarrow X$, a vector $x \in X$ is called a *unimodular eigenvector* for T if $x \neq 0$ and $Tx = \lambda x$ for some unimodular complex number $\lambda \in \mathbb{T}$. We denote by $\mathcal{E}(T)$ the *set of unimodular eigenvectors* for T, i.e.

$$\mathcal{E}(T) = \{x \in X \setminus \{0\} : Tx = \lambda x \text{ for some } \lambda \in \mathbb{T}\}.$$

Every finite linear combination of unimodular eigenvectors is a Δ^*-recurrent vector (see for instance [GLM23, Proposition 4.1]) and the following holds (see [GEP11, Proposition 2.33]):

$$\mathrm{Per}(T) = \mathrm{span}\{x \in X : Tx = e^{\alpha\pi i}x \text{ for some } \alpha \in \mathbb{Q}\} \subset \mathrm{span}(\mathcal{E}(T)) \subset \Delta^*\mathrm{Rec}(T).$$

In order to treat both real and complex cases at the same time, we need a set of (**real**) vectors having an analogous recurrent-behaviour to that of unimodular eigenvectors:

Remark 5.2. The *complexification* $(\widetilde{X}, \widetilde{T})$ of a **real-linear** system $T : X \longrightarrow X$ is defined in the following way (see [MST99, MMFPSS22] and [GEP11, Exercise 2.2.7]):

- the space $\widetilde{X} := \{x + iy : x, y \in X\}$, which is topologically identified with $X \oplus X$ and becomes a **complex** F-space endowed with the multiplication $(\alpha + i\beta)(x + iy) = (\alpha x - \beta y) + i(\alpha y + \beta x)$ for every $\alpha, \beta \in \mathbb{R}$ and every $x, y \in X$;
- and the operator $\widetilde{T} : \widetilde{X} \longrightarrow \widetilde{X}$ is defined as $\widetilde{T}(x + iy) = Tx + iTy$ for every $x, y \in X$. Note that this is a continuous **complex-linear** operator acting on $\widetilde{X}$.

Defining the map $J : X \oplus X \longrightarrow \widetilde{X}$ as $J(x, y) := x + iy \in \widetilde{X}$ for every $(x, y) \in X \oplus X$, the diagram

$$\begin{array}{ccc} X \oplus X & \xrightarrow{T \oplus T} & X \oplus X \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle J} \\ \widetilde{X} & \xrightarrow{\widetilde{T}} & \widetilde{X} \end{array}$$

commutes so J is a conjugacy (see Definition 4.13). In this setting we define the (**real**) *set of unimodular eigenvectors* for T as

$$\mathcal{E}(T) := \left\{x \in X : \text{ there exists } y \in X \text{ such that } x + iy \in \mathcal{E}(\widetilde{T})\right\}.$$

These **real** unimodular eigenvectors have the same recurrent-behaviour than the **complex** unimodular eigenvectors. In fact, the following properties are easily checked:

$$\mathrm{span}(\mathcal{E}(T)) = \left\{x \in X : \text{ there exists } y \in X \text{ such that } x + iy \in \mathrm{span}(\mathcal{E}(\widetilde{T}))\right\},$$

$$\mathrm{Per}(T) \subset \mathrm{span}(\mathcal{E}(T)) \subset \Delta^*\mathrm{Rec}(T).$$

Lemma 5.3. *Let (X, T) be a (real or complex) linear dynamical system. For each recurrent vector $x \in \mathrm{Rec}(T)$ the following statements are equivalent:*

(i) $x \in \mathrm{span}(\mathcal{E}(T))$;

(ii) $\dim(\mathrm{span}\{T^n x : n \in \mathbb{N}_0\}) < \infty$.

In particular, given any Furstenberg family $\mathcal{F}$ for which $\mathcal{F}\mathrm{Rec}(T) \setminus \mathrm{span}(\mathcal{E}(T)) \neq \varnothing$, there exists an infinite-dimensional T-invariant vector subspace $Z \subset X$ with the property that

$$Z^N \subset \mathcal{F}\mathrm{Rec}(T_{(N)}) \quad \textit{for every } N \in \mathbb{N}.$$

Proof. For (i) $\Rightarrow$ (ii) compute the orbit of a vector from $\mathrm{span}(\mathcal{E}(T))$ in both real and complex cases, and observe that its span is finite-dimensional. For (ii) $\Rightarrow$ (i) let $x \in \mathrm{Rec}(T)$ and suppose that the T-invariant subspace $E = \mathrm{span}\{T^n x : n \in \mathbb{N}_0\}$ is finite-dimensional. We have two cases:

(1) If (X, T) is complex, then $T|_E : E \longrightarrow E$ is a recurrent complex-linear operator on a finite-dimensional space. By [CMP14, Theorem 4.1] there exists a basis of E formed by unimodular eigenvectors for $T|_E$ (and hence for T) so $x \in E \subset \mathrm{span}(\mathcal{E}(T))$.

(2) If (X, T) is real, then we can identify the complexification $\widetilde{T|_E} : \widetilde{E} \longrightarrow \widetilde{E}$ with the direct sum $T|_E \oplus T|_E$ (see Remark 5.2). Theorem 4.19 implies that

$$E \oplus E \subset \mathrm{Rec}(T|_E \oplus T|_E),$$

so $\widetilde{T|_E}$ is a recurrent complex-linear operator on a finite-dimensional space. As in case (1) above, by [CMP14, Theorem 4.1] there exists a basis of $\widetilde{E}$ formed by unimodular eigenvectors for $\widetilde{T}$, so $x + i0 \in \widetilde{E} \subset \mathrm{span}(\mathcal{E}(\widetilde{T}))$ and hence $x \in \mathrm{span}(\mathcal{E}(T))$.

Finally, if $\mathcal{F}$ is a Furstenberg family for which there exists some $x \in \mathcal{F}\mathrm{Rec}(T) \setminus \mathrm{span}(\mathcal{E}(T))$, the equivalence (i) $\Leftrightarrow$ (ii) implies that $Z := \mathrm{span}\{T^n x : n \in \mathbb{N}_0\}$ is an infinite-dimensional T-invariant vector subspace of X. Theorem 4.19 then shows

$$Z^N \subset \mathcal{C}_T(x)^N \subset \mathcal{F}\mathrm{Rec}(T_{(N)}) \quad \text{for every } N \in \mathbb{N}. \qquad \square$$

The strongest $\mathcal{F}$-recurrence notion considered in this paper and fulfilling the condition that

$$\mathrm{span}(\mathcal{E}(T)) \subset \mathcal{F}\mathrm{Rec}(T) \quad \textit{for every } T \in \mathcal{L}(X),$$

is that of Δ^*-recurrence. The properties of "*having a spanning set of unimodular eigenvectors*" and that of "*being Δ^*-recurrent*" have been deeply related in the recent work [GLM23] and they are specially near when one considers power-bounded operators (see [GLM23, Theorem 1.9]). It is then natural to ask if the equality $\mathrm{span}(\mathcal{E}(T)) = \Delta^*\mathrm{Rec}(T)$ holds for some general class of linear dynamical systems (X, T), a natural candidate being that of power-bounded operators.

The answer is negative as we show in the following trivial example by using Lemma 5.3:

Example 5.4. *There exists a linear dynamical system admitting a* Δ^**-recurrent vector such that the linear span of its orbit is infinite-dimensional*: consider $X = c_0(\mathbb{N})$ or $\ell^p(\mathbb{N})$ for some $1 \leq p < \infty$ with their usual norms. Define a linear map $T : c_{00}(\mathbb{N}) \longrightarrow c_{00}(\mathbb{N})$ in the following way: $Te_1 = e_1$ and for every $k \geq 2$

$$Te_k = \begin{cases} e_{k+1}, & \text{if } 2^m < k < 2^{m+1}, \\ e_{2^m+1}, & \text{if } k = 2^{m+1}, \end{cases}$$

for each $m \geq 0$, where $e_k = (\delta_{k,n})_{n=1}^{\infty}$ is the k-th vector of the canonical basis of X. Since $\|Tx\| = \|x\|$ for each $x \in c_{00}(\mathbb{N})$, T extends to a *linear isometry* on the whole space X. We now show that the vector $x = \sum_{m\geq 0} \frac{1}{2^m} e_{2^m+1} \in X$ has the required properties:

(1) $x \in \Delta^*\mathrm{Rec}(T)$: given any $\varepsilon > 0$ there is $m_\varepsilon \in \mathbb{N}$ such that $\left\|\sum_{m>m_\varepsilon} \frac{1}{2^m} e_{2^m+1}\right\| < \frac{\varepsilon}{2}$. Then, given $n \in 2^{m_\varepsilon} \cdot \mathbb{N}_0$ we have that

$$T^n\left(\sum_{m\leq m_\varepsilon} \frac{1}{2^m} e_{2^m+1}\right) = \sum_{m\leq m_\varepsilon} \frac{1}{2^m} e_{2^m+1} \quad \text{so} \quad \|T^n x - x\| \leq 2\left\|\sum_{m>m_\varepsilon} \frac{1}{2^m} e_{2^m+1}\right\| < \varepsilon.$$

Hence $2^{m_\varepsilon} \cdot \mathbb{N}_0 \subset N_T(x, B(x,\varepsilon))$, so $x \in \Delta^*\mathrm{Rec}(T)$ since $2^m \cdot \mathbb{N}_0 \in \Delta^*$ for all $m \in \mathbb{N}$.

(2) $\mathrm{span}\{T^n x : n \in \mathbb{N}_0\}$ is infinite-dimensional: otherwise there would exist a polynomial

$$p(z) = \sum_{i=1}^{N} a_i z^i \text{ with } a_i \in \mathbb{K} \text{ and } a_N \neq 0 \quad \text{such that} \quad p(T)x = 0,$$

and taking $m_0 \in \mathbb{N}$ such that $N < 2^{m_0}$ we would arrive to the contradiction

$$0 = [p(T)x]_{2^{m_0}+1+N} = \left[a_N \cdot T^N x\right]_{2^{m_0}+1+N} = \frac{a_N}{2^{m_0}} \neq 0.$$

We finally state the desired *lineability* results:

Theorem 5.5 (**Lineability**). *Let* $\mathcal{F}$ *be a Furstenberg family with the property that the inclusion*

$$\mathrm{span}(\mathcal{E}(S)) \subset \mathcal{F}\mathrm{Rec}(S),$$

holds for every linear dynamical system (Y, S)*. Then, for each* $\mathcal{F}$*-recurrent linear system* (X, T) *there exists an infinite-dimensional* T*-invariant vector subspace* $Z \subset X$ *with the property that*

$$Z^N \subset \mathcal{F}\mathrm{Rec}(T_{(N)}) \quad \text{for every } N \in \mathbb{N}.$$

Proof. By assumption $\mathcal{F}\mathrm{Rec}(T)$ is dense. We distinguish two cases:

(1) If $\mathrm{span}(\mathcal{E}(T))$ is finite-dimensional (or simply if it is not dense in X), then we have that $\mathcal{F}\mathrm{Rec}(T) \setminus \mathrm{span}(\mathcal{E}(T)) \neq \varnothing$ and Lemma 5.3 yields the desired vector subspace.

(2) If $\mathrm{span}(\mathcal{E}(T))$ is infinite-dimensional, then $(\mathrm{span}(\mathcal{E}(T)))^N = \mathrm{span}(\mathcal{E}(T_{(N)})) \subset \mathcal{F}\mathrm{Rec}(T_{(N)})$ for every $N \in \mathbb{N}$, and $Z := \mathrm{span}(\mathcal{E}(T))$ is the required vector subspace. □

The previous theorem is "optimal" for the families considered in this paper: we always have the inclusions

$$\mathrm{Per}(T) \subset \mathrm{span}(\mathcal{E}(T)) \subset \Delta^*\mathrm{Rec}(T),$$

and we observed in Section 4 that the families $\mathcal{F}$ for which $\mathcal{F}\mathrm{Rec}(T) \subset \Delta^*\mathrm{Rec}(T)$ are such that $\mathcal{F}\mathrm{Rec}(T) \subset \mathrm{Per}(T)$, for every system (X, T). We deduce (at least) the following result:

Corollary 5.6. *Let (X, T) be a linear dynamical system. If the linear operator T is recurrent (resp. $\mathcal{AP}$, reiterative, $\mathcal{U}$-frequent, frequent, uniformly, $\mathcal{IP}^*$ or Δ^*-recurrent), then the set* $\mathrm{Rec}(T)$ *(resp.* $\mathcal{AP}\mathrm{Rec}(T)$, $\mathrm{RRec}(T)$, $\mathrm{UFRec}(T)$, $\mathrm{FRec}(T)$, $\mathrm{URec}(T)$, $\mathcal{IP}^*\mathrm{Rec}(T)$ *or* $\Delta^*\mathrm{Rec}(T)$*) admits an infinite-dimensional T-invariant vector space, and in particular it is lineable.*

Theorem 5.5 (and hence Corollary 5.6) is still true with the same proof if we replace the original assumption that "*T is $\mathcal{F}$-recurrent*" by the very much less restrictive hypothesis that "*$\mathcal{F}\mathrm{Rec}(T)$ is dense in some infinite-dimensional closed subspace $Y \subset X$*", or even by the formally weaker assumption that "*$\mathcal{F}\mathrm{Rec}(T)$ spans an infinite-dimensional vector subspace*". These are also necessary conditions for the lineability property as the following example shows:

Example 5.7. Let $X = c_0(\mathbb{N})$ or $\ell^p(\mathbb{N})$ for some $1 \leq p < \infty$ with their usual norm. Let $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ be a bounded sequence and take the multiplication operator defined as

$$T\left((x_n)_{n\in\mathbb{N}}\right) := (\lambda_n x_n)_{n\in\mathbb{N}} \quad \text{for each } (x_n)_{n\in\mathbb{N}} \in X.$$

For any fixed any $N \in \mathbb{N}$, consider a sequence $(\lambda_n)_{n\in\mathbb{N}}$ with the properties that $\lambda_n = 1$ for $n \leq N$ and $\lambda_n \notin \mathbb{T}$ for $n > N$. Then it is trivial to check that, for any Furstenberg family $\mathcal{F}$, the set $\mathcal{F}\mathrm{Rec}(T) = \mathrm{span}(\mathcal{F}\mathrm{Rec}(T))$ is exactly a finite-dimensional subspace of dimension N.

5.2 Dense Lineability

Given a topological vector space X and a subset of vectors $Y \subset X$ with some property, we say that Y is *dense lineable* if there exists a dense infinite-dimensional vector subspace $Z \subset X$ such that $Z \setminus \{0\} \subset Y$. As it happens for *lineability*, given a linear system (X, T) we always have that $0 \in \mathcal{F}\mathrm{Rec}(T)$ for every family $\mathcal{F}$, so the set of $\mathcal{F}$-recurrent vectors will be dense lineable if it admits a dense infinite-dimensional vector space (including the zero-vector).

Given a linear dynamical system (X, T) we provide some sufficient conditions for the set $\mathcal{F}\mathrm{Rec}(T)$ to be dense lineable. Filters will play a fundamental role in our study. The motivation for our main result (see Theorem 5.8 below) is again the notion of quasi-rigidity:

– quasi-rigidity implies that $\mathrm{Rec}(T) = \mathcal{I}\mathrm{Rec}(T)$ is dense lineable (see Proposition 2.7);

– quasi-rigidity coincides with $\mathcal{F}(A)$-recurrence, and $\mathcal{F}(A)$ is a filter (see Proposition 4.5);

– i.e. *there is a dense set $Y \subset X$ and a filter $\mathcal{F}' = \mathcal{F}(A)$ such that $\mathcal{N}_{pr}(Y) \subset \mathcal{F}' \subset \mathcal{I}$.*

We can generalize these ideas to arbitrary Furstenberg families $\mathcal{F}$, finding some filter $\mathcal{F}'$ contained in $\mathcal{F}$. In particular, we extend [BGELMP22, Theorem 6.1] where it is shown that the set of $\mathcal{IP}^*$-recurrent vectors is always a vector subspace:

Theorem 5.8 (**Dense Lineability**). *Let (X,T) be a linear dynamical system and let $\mathcal{F}$ be a Furstenberg family. If there exist a dense subset $Y \subset X$ and a filter $\mathcal{F}'$ such that*

$$\mathcal{N}_{pr}(Y) \subset \mathcal{F}' \subset \mathcal{F},$$

then $Z := \mathrm{span}(\bigcup_{S\in\mathcal{C}_T} S(Y)) \subset X$ is a dense infinite-dimensional T-invariant vector subspace with the property that $Z^N \subset \mathcal{F}\mathrm{Rec}(T_{(N)})$ for every $N \in \mathbb{N}$. In particular, if $\mathcal{F}$ is a filter itself, then for every $N \in \mathbb{N}$ we have the equality $\mathcal{F}\mathrm{Rec}(T)^N = \mathcal{F}\mathrm{Rec}(T_{(N)})$, which is a T-invariant vector subspace of X^N, and the following statements are equivalent:

(i) *T is $\mathcal{F}$-recurrent;*

(ii) *$\mathcal{F}\mathrm{Rec}(T_{(N)})$ is a dense infinite-dimensional vector subspace of X^N for every $N \in \mathbb{N}$.*

Proof. It is clear that $Z \subset X$ is a dense infinite-dimensional T-invariant vector subspace. Let us show that $\mathcal{N}_{pr}(Z) \subset \mathcal{F}'$: given $y_1, y_2, ..., y_N \in \bigcup_{S\in\mathcal{C}_T} S(Y)$ and any neighbourhood $U \subset X$ of $x := \sum_{i=1}^N y_i \in Z$ we can find neighbourhoods U_i of y_i, for $1 \leq i \leq N$, such that

$$\sum_{i=1}^N U_i \subset U \subset X \quad \text{and hence} \quad N_T(x,U) \supset \bigcap_{i=1}^N N_T(y_i,U_i) \in \mathcal{F}',$$

since $\mathcal{N}_{pr}(\{y_i\}) \subset \mathcal{F}'$ for all $1 \leq i \leq N$ by Theorem 4.19. Finally, given any $N \in \mathbb{N}$, $x_1, x_2, ..., x_N \in Z$ and any neighbourhood $V \subset X^N$ of the N-tuple $z := (x_1, ..., x_N) \in X^N$, we can find neighbourhoods V_i of z_i, for $1 \leq i \leq N$, such that

$$V_1 \oplus \cdots \oplus V_N \subset V \subset X^N \quad \text{and hence} \quad N_{T_{(N)}}(z,V) \supset \bigcap_{i=1}^N N_T(z_i,V_i) \in \mathcal{F}' \subset \mathcal{F}.$$

The arbitrariness of $V \subset X^N$ implies that $z \in \mathcal{F}\mathrm{Rec}(T_{(N)})$. □

Note that Theorem 5.8 extends the result obtained in Proposition 2.7 to every N-fold direct sum operator. Moreover, in view of Theorems 4.19 and 5.8 we can generalize Proposition 2.9 establishing, for dense lineability, the following (natural) sufficient conditions:

Proposition 5.9. *Let (X,T) be a linear dynamical system and let $\mathcal{F}$ be a Furstenberg family. If T admits an $\mathcal{F}$-recurrent vector $x \in \mathcal{F}\mathrm{Rec}(T)$ with a dense $\mathcal{C}_T$-orbit*

$$\mathcal{C}_T(x) = \{Sx : S \in \mathcal{C}_T\},$$

then $\mathcal{F}\mathrm{Rec}(T_{(N)})$ is dense lineable in X^N for every $N \in \mathbb{N}$. In particular, the later is true whenever any of the following holds:

– *T admits an $\mathcal{F}$-recurrent and cyclic vector;*

– *T admits an $\mathcal{F}$-hypercyclic vector.*

Proof. Given a vector $x \in \mathcal{F}\mathrm{Rec}(T)$ for which $\mathcal{C}_T(x)$ is dense in X, Theorem 4.19 implies that $\mathcal{N}_{pr}(\mathcal{C}_T(x)) \subset \mathcal{N}_{pr}(\{x\}) \subset \mathcal{F}$ and the result follows from Theorem 5.8. □

The dense lineability of the set $\mathcal{F}\text{Rec}(T_{(N)})$ implies that $T_{(N)}$ is $\mathcal{F}$-recurrent. A slightly weaker hypothesis allows us to keep the $\mathcal{F}$-recurrence of every N-fold direct product:

Proposition 5.10 (**$\mathcal{F}$-recurrence for all $T_{(N)}$**)**.** *Let (X,T) be a linear dynamical system and let $\mathcal{F}$ be a Furstenberg family. If T is $\mathcal{F}$-recurrent and if there exists a vector $x \in X$ with a dense $\mathcal{C}_T$-orbit*

$$\mathcal{C}_T(x) = \{Sx : S \in \mathcal{C}_T\},$$

then $T_{(N)} : X^N \longrightarrow X^N$ is $\mathcal{F}$-recurrent for every $N \in \mathbb{N}$. In particular, the later is true whenever any of the following holds:

– *T is $\mathcal{F}$-recurrent and cyclic;*

– *T is $\mathcal{F}$-recurrent and hypercyclic.*

This has been recently proved in [CM, Proposition 2.7], but we repeat here the argument for the sake of completeness:

Proof. Fix $N \in \mathbb{N}$. By Theorem 4.19 we have the inclusion

$$Y := \bigcup_{y \in \mathcal{F}\text{Rec}(T)} \mathcal{C}_T(y)^N \subset \mathcal{F}\text{Rec}(T_{(N)}),$$

so it is enough to show that Y is dense in X^N. By assumption the point $x \in X$ has a dense $\mathcal{C}_T$-orbit, so given any non-empty open subset $U \subset X^N$ there are N continuous maps $S_1, S_2, ..., S_N \in \mathcal{C}_T$ such that $(S_1x, S_2x, ..., S_Nx) \in U$. By the continuity of $S_1, S_2, ..., S_N$, and since T is $\mathcal{F}$-recurrent, we can find $y \in \mathcal{F}\text{Rec}(T)$ near enough to x such that

$$(S_1y, S_2y, ..., S_Ny) \in \ \mathcal{C}_T(y)^N \cap U \ \subset \ Y \cap U. \qquad \square$$

Remark 5.11. We include here some comments about the previous results:

(a) Proposition 5.10 remains valid for non-linear dynamical systems. It is a slight improvement of [CMP14, Theorem 9.1] in terms of Furstenberg families and non-linear commuting maps, and it has been independently proved in the recent work [CM, Proposition 2.7].

(b) Given a Furstenberg family $\mathcal{F}$, the following is a natural open question:

Problem 5.12. Suppose that a linear dynamical system (X,T) does not admit any dense $\mathcal{C}_T$-orbit. Can every N-fold direct sum $(X^N, T_{(N)})$ be $\mathcal{F}$-recurrent?

(c) The assumptions of Proposition 5.9 imply those of Proposition 5.10. Conversely:

Problem 5.13. Let $\mathcal{F}$ be a Furstenberg family. Can an $\mathcal{F}$-recurrent linear system (X,T) admit dense $\mathcal{C}_T$-orbits but fulfill that $\mathcal{F}\text{Rec}(T) \cap \{x \in X : \overline{\mathcal{C}_T(x)} = X\} = \varnothing$?

This seems to be a tough question since the commutator of an operator is usually difficult to describe. If instead of dense $\mathcal{C}_T$-orbits we just consider the set of *hypercyclic* vectors, then the answer is yes: Menet constructed a *chaotic* (and hence with dense periodic vectors) operator T, which is not $\mathcal{U}$-frequently hypercyclic (see [Men17]), i.e.

– *Such an operator T fulfills the assumptions of Proposition 5.10 for every Furstenberg family $\mathcal{F}$ satisfying that $\Delta^* \subset \mathcal{F} \subset \overline{\mathcal{D}}$, but it has no $\mathcal{F}$-recurrent and hypercyclic vector.*

Considering now the set of *cyclic* (instead of the dense $\mathcal{C}_T$-orbit or hypercyclic) vectors, then Corollary 2.11 and Proposition 5.9 imply that:

– *If* (X, T) *is a recurrent and cyclic linear dynamical system, then the set* $\mathrm{Rec}(T_{(N)})$ *is dense lineable for every* $N \in \mathbb{N}$.

In general we cannot change $\mathrm{Rec}(T_{(N)})$ into $\mathcal{F}\mathrm{Rec}(T_{(N)})$ unless we could ensure that $\mathcal{F}\mathrm{Rec}(T)$ is a co-meager set. This is the case for $\mathcal{AP}$-recurrence (see [KLOY17, CM22b]):

– *If* (X, T) *is an* $\mathcal{AP}$*-recurrent (i.e. multiple recurrent) and cyclic linear dynamical system, then the set* $\mathcal{AP}\mathrm{Rec}(T_{(N)})$ *is dense lineable for every* $N \in \mathbb{N}$.

In [BGELMP22, Example 2.4] it is exhibited a $\overline{\mathcal{BD}}$-recurrent (i.e. reiteratively recurrent) operator $T \in \mathcal{L}(X)$ for which the set $\overline{\mathcal{BD}}\mathrm{Rec}(T) = \mathrm{RRec}(T)$ is meager. Such an operator T is not cyclic, and in view of [BGELMP22, Theorem 2.1] (result which states that: *the set of* $\overline{\mathcal{BD}}$*-recurrent vectors is co-meager for every* $\overline{\mathcal{BD}}$*-recurrent and hypercyclic operator*) the following question is then a natural open problem:

Problem 5.14. Let (X, T) be a $\overline{\mathcal{BD}}$-recurrent (i.e. reiteratively recurrent) and cyclic linear dynamical system. Is $\mathrm{RRec}(T)$ a co-meager set?

5.3 Dense Lineability for $\mathcal{F}$-hypercyclicity

Under some natural conditions on the Furstenberg family $\mathcal{F}$ we can apply the $\mathcal{F}$-recurrence theory developed above in order to obtain some results regarding $\mathcal{F}$-hypercyclicity. We will consider *right-invariant* and *upper* Furstenberg families. Let us recall the definitions:

A family $\mathcal{F}$ is said to be *right-invariant* if for every $A \in \mathcal{F}$ and every $n \in \mathbb{N}_0$ the set $A + n = \{k + n : k \in A\}$ also belongs to $\mathcal{F}$. For example, the families $\mathcal{I}, \mathcal{AP}, \mathcal{S}, \mathcal{T}, \mathcal{TS}, \mathcal{I}^*$ and the density ones are easily seen to be right-invariant. However, the families $\mathcal{IP}$ and Δ together with their dual families $\mathcal{IP}^*$ and Δ^* are not right-invariant (see [BMPP19, Proposition 5.6]).

It is shown in [BGELMP22] that given any right-invariant Furstenberg family $\mathcal{F}$, then every single $\mathcal{F}$-recurrent and hypercyclic vector is indeed $\mathcal{F}$-hypercyclic. As a consequence we obtain a kind of *Herrero-Bourdon theorem* (see [GEP11, Theorem 2.55]) for $\mathcal{F}$-hypercyclicity:

Corollary 5.15. *Let* (X, T) *be a linear dynamical system and assume that* $\mathcal{F}$ *is a right-invariant Furstenberg family. If* x *is an* $\mathcal{F}$*-hypercyclic vector for* T*, then*

$$\{p(T)x : p \text{ polynomial}\} \setminus \{0\},$$

is a dense set of $\mathcal{F}$*-hypercyclic vectors. In particular, any* $\mathcal{F}$*-hypercyclic operator admits a dense invariant vector space consisting, except for zero, of* $\mathcal{F}$*-hypercyclic vectors.*

Proof. Follows from the original Herrero-Bourdon theorem, together with Theorem 4.19 and the equality

$$\mathcal{F}\mathrm{HC}(T) = \mathcal{F}\mathrm{Rec}(T) \cap \mathrm{HC}(T),$$

showed in [BGELMP22] for every right-invariant Furstenberg family $\mathcal{F}$. □

Apart from *right-invariance* we may assume the *u.f.i. upper* condition (defined in [BGE18]) to obtain even stronger results: a family $\mathcal{F}$ is said to be *upper* if it can be written as

$$\mathcal{F} = \bigcup_{\delta \in D} \mathcal{F}_\delta \quad \text{with} \quad \mathcal{F}_\delta = \bigcap_{m \in M} \mathcal{F}_{\delta,m}$$

for sets $\mathcal{F}_{\delta,m} \subset \mathscr{P}(\mathbb{N}_0)$ such that $\delta \in D$ and $m \in M$, where D is arbitrary, M is countable and they have the following properties:

(i) for any $\mathcal{F}_{\delta,m}$ and any $A \in \mathcal{F}_{\delta,m}$ there exists a finite set $F \subset \mathbb{N}_0$ such that

$$A \cap F \subset B \text{ implies } B \in \mathcal{F}_{\delta,m};$$

(ii) for any $A \in \mathcal{F}$ there exists some $\delta \in D$ such that for all $n \in \mathbb{N}_0$ we have

$$(A - n) \cap \mathbb{N}_0 \in \mathcal{F}_\delta.$$

We say that an upper family $\mathcal{F}$ as above is *uniformly finitely invariant* (called *u.f.i.* for short), if for any $A \in \mathcal{F}$, there is some $\delta \in D$ such that, for all $n \in \mathbb{N}$, $A \cap [n, \infty[\in \mathcal{F}_\delta$.

The families $\mathcal{I}, \mathcal{AP}, \overline{\mathcal{BD}}$ and $\overline{\mathcal{D}}$ are easily checked to be *u.f.i. upper*, while $\underline{\mathcal{D}}$ is not even *upper* (see [BGE18, CM22b])[G]. As it is shown in [BGE18], there exists a kind of *Birkhoff transitivity theorem* for these *u.f.i. upper* families, and in particular, the following holds:

Corollary 5.16. *Let (X, T) be an $\mathcal{F}$-recurrent linear dynamical system where $\mathcal{F}$ is a u.f.i. upper Furstenberg family. If there is a dense set $X_0 \subset X$ such that $T^n x \to 0$ for each $x \in X_0$, then the set $\mathcal{F}\mathrm{Rec}(T_{(N)})$ is dense lineable for every $N \in \mathbb{N}$. In particular:*

(a) *If T is recurrent then $\mathrm{Rec}(T_{(N)})$ is dense lineable for every $N \in \mathbb{N}$;*

(b) *If T is $\mathcal{AP}$-recurrent then $\mathcal{AP}\mathrm{Rec}(T_{(N)})$ is dense lineable for every $N \in \mathbb{N}$;*

(c) *If T is reiteratively recurrent then $\mathrm{RRec}(T_{(N)})$ is dense lineable for every $N \in \mathbb{N}$;*

(d) *If T is $\mathcal{U}$-frequently recurrent then $\mathrm{UFRec}(T_{(N)})$ is dense lineable for every $N \in \mathbb{N}$.*

Proof. By [BGELMP22, Theorem 8.5], if T is $\mathcal{F}$-recurrent for a u.f.i. upper family $\mathcal{F}$ and if there exists a dense set of vectors $x \in X$ such that $T^n x \to 0$ as $n \to \infty$, then T is $\mathcal{F}$-hypercyclic. By Proposition 5.9 the set $\mathcal{F}\mathrm{Rec}(T_{(N)})$ is then dense lineable for every $N \in \mathbb{N}$. The particular cases follow from the fact that $\mathcal{I}$, $\mathcal{AP}$, $\overline{\mathcal{BD}}$ and $\overline{\mathcal{D}}$ are u.f.i. upper Furstenberg families. □

Moreover, if we consider the Furstenberg family $\overline{\mathcal{BD}}$ of *positive upper Banach density sets* we can prove a particular case of [EEM21, Theorem 2.5 and Corollary 2.8] in a much simpler way, which is also a particular case of [CM, Proposition 2.8]:

Corollary 5.17 ([EEM21, Theorem 2.5 and Corollary 2.8]). *Let (X, T) be a reiteratively hypercyclic linear system. Then $T_{(N)} : X^N \longrightarrow X^N$ is reiteratively hypercyclic for every $N \in \mathbb{N}$.*

Proof. By [BMPP16, Proposition 4] we have that $T_{(N)}$ is hypercyclic. By Corollary 5.10 we have that $T_{(N)}$ is reiteratively recurrent. By [BGELMP22, Theorem 2.1] every hypercyclic vector for $T_{(N)}$ is also reiteratively hypercyclic, so that $T_{(N)}$ is reiteratively hypercyclic. □

[G] See Example 3.2 in Section 3 of the Appendix for more on u.f.i. upper Furstenberg families.

6 Open problems

In this section we include three open questions and a few comments related to them. They essentially ask "what happens if we cannot apply Theorem 5.8":

Problems 6.1. Let $\mathcal{F}$ be a Furstenberg family and assume that it is not a filter. Suppose that $T \in \mathcal{L}(X)$ is a continuous linear $\mathcal{F}$-recurrent operator:

(a) Is the set $\mathcal{F}\mathrm{Rec}(T_{(N)})$ necessarily dense lineable for every $N \in \mathbb{N}$?

(b) Is the set $\mathcal{F}\mathrm{Rec}(T)$ necessarily dense lineable?

(c) Is $T \oplus T$ necessarily an $\mathcal{F}$-recurrent operator?

Note that **Problems (b)** and **(c)** are weaker questions than **Problem (a)**, i.e. a positive answer to **(a)** would imply a positive answer to both **(b)** and **(c)**. However, for usual recurrence (i.e. for $\mathcal{I}$-recurrence) we already have a negative answer to **Problems (a)** and **(c)**:

– *in Section 3 we construct recurrent operators $T \in \mathcal{L}(X)$ for which $T \oplus T$ is not recurrent.*

Nevertheless, that example is not solving **Problem (b)** in the negative since the mentioned operator fulfills that $\mathrm{Rec}(T) = X$. Thus, finding a possible recurrent operator T for which the set $\mathrm{Rec}(T)$ is not dense lineable demands to solve again the *$T \oplus T$-recurrence problem*. It is not hard to check that all the operators constructed in Section 3 have a dense lineable set of $\mathcal{AP}$-recurrent vectors[H]. In particular:

– *there exist multiple recurrent operators $T \in \mathcal{L}(X)$ for which $T \oplus T$ is not even recurrent*;

and the same comments done for usual recurrence hold for $\mathcal{AP}$-recurrence. These arguments are not valid for the other recurrence notions considered in this document:

Proposition 6.2. *Let (X,T) be a reiteratively recurrent linear dynamical system. Then T is a quasi-rigid operator, and hence $\mathrm{Rec}(T_{(N)})$ is dense lineable for every $N \in \mathbb{N}$.*

Proof. Following [BMPP16, Proposition 4] and [BMPP19, Proposition 5.6]: given a non-empty subset $U \subset X$ we can pick a vector $x \in \mathrm{RRec}(T) \cap U$ and hence $\overline{\mathrm{Bd}}(N_T(x,U)) > 0$. Recall that

$$\big(N_T(x,U) - N_T(x,U)\big) := \{s_2 - s_1 : s_1 \leq s_2 \in N_T(x,U)\} \subset \{n \geq 0 : T^n(U) \cap U \neq \varnothing\}.$$

Since given $A \subset \mathbb{N}_0$ with $A \in \overline{\mathcal{BD}}$ we have that $A - A$ belongs to Δ^* by [Fur81, Theorem 3.18][I], then: for every non-empty open subset $U \subset X$ we have that

$$\{n \geq 0 : T^n(U) \cap U \neq \varnothing\} \in \Delta^*.$$

Since Δ^* is a filter (see [BD08, HS98]) it is easily checked that every N-fold direct sum $(X^N, T_{(N)})$ is topologically recurrent. By Theorem 2.5 we have that T is quasi-rigid and finally Proposition 2.7 (or Theorem 5.8) implies that $\mathrm{Rec}(T_{(N)})$ is dense lineable for every $N \in \mathbb{N}$. □

Proposition 6.2 together with the very recent results [GLM23, Theorems 5.1 and 5.8] show that **Problems (a)**, **(b)** and **(c)** may have a positive answer for Furstenberg families implying stronger recurrence notions than those of usual and $\mathcal{AP}$-recurrence.

[H]The proof of this fact has been included in Section 2.3 of the *General discussion of the results*.

[I]See also statement (e) of Proposition 2.10 in Section 2 of the Appendix.

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Chapter 4

Recurrent subspaces in Banach spaces

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Adaptation: The notation has been slightly modified to use similar symbols in all chapters. Moreover, in comparison to the previous version available on arXiv, some extra comments have been added and some spelling errors have been rectified.

Abstract

We study the *spaceability* of the set of recurrent vectors $\mathrm{Rec}(T)$ for an operator $T : X \longrightarrow X$ on a Banach space X. In particular: we find sufficient conditions for a quasi-rigid operator to have a recurrent subspace; when X is a complex Banach space we show that having a recurrent subspace is equivalent to the fact that the essential spectrum of the operator intersects the closed unit disc; and we extend the previous result to the real case through some complexification techniques. As a consequence we obtain that: *a weakly-mixing operator on a real or complex separable Banach space has a hypercyclic subspace if and only if it has a recurrent subspace*. The results exposed exhibit a symmetry between the hypercyclic and recurrent spaceability theories showing that, at least for the *spaceable* property, hypercyclicity and recurrence can be treated as equals.

1 Introduction

In *Linear Dynamics* the interest in *hypercyclicity* comes from the *invariant subset problem*, which remains open for operators acting on Hilbert spaces: an operator has no non-trivial invariant closed subset if and only if each non-zero vector is hypercyclic for it. This motivated the study of the structure of the set of hypercyclic vectors, and it is well-known due to Herrero and Bourdon that such a set is always *dense lineable*, i.e. every hypercyclic operator admits a (not necessarily closed) dense invariant subspace that consists (except for the zero-vector) entirely of hypercyclic vectors; see [GEP11, Theorem 2.55]. The *spaceability* (i.e. the property of admitting an infinite-dimensional closed subspace) of such a set has also been deeply studied and, for the case of operators acting on Banach spaces, we will follow the basic references [MR96, LSMR97, LSMR01, GLSMR00], [BM09, Chapter 8] and [GEP11, Chapter 10].

In its turn, *recurrence* is one of the fundamental concepts in dynamics since the beginning of the theory at the end of the 19th century when Poincaré introduced his *recurrence theorem*, even though most of the literature in the context of Linear Dynamics is built around the central concept of hypercyclicity. In fact, we refer to the 2014 paper of Costakis, Manoussos and Parissis [CMP14] as the beginning of the systematic study of *linear recurrence*, despite the great non-linear dynamical knowledge already existing in this area; see [Fur81]. The linear structure of the set of recurrent vectors has been recently studied, and results about *lineability* and *dense lineability* have been obtained; see [GLMP]. The aim of this paper is rewriting the already existent *hypercyclic spaceability theory* on Banach spaces for recurrence and, moreover, linking both concepts in a deep way when we consider weakly-mixing operators.

The paper is organized as follows. In Section 2 we introduce the notation, basic concepts and the historical development of the *hypercyclic spaceability theory* on Banach spaces. In Section 3 we present the obtained results about the spaceability of the recurrent vectors by establishing a symmetry with the hypercyclic case. Section 4 is devoted to prove sufficient conditions for a quasi-rigid operator to have a *recurrent subspace*. In Section 5 we study the essential spectrum for recurrent operators and its consequences on the existence of recurrent subspaces when the underlying space is complex. Finally, Section 6 exhibits the real case of the previous results while in Section 7 we investigate their applications and we show that every *C-type operator*, as defined in the recent works [Men17] and [GMM21], has a *hypercyclic subspace*.

2 Notation and basic concepts

2.1 General background

From now on let $T : X \longrightarrow X$ be a *bounded linear operator* on a (*real* or *complex*) *separable infinite-dimensional Banach space* X, denote by $\mathcal{L}(X)$ the *set of bounded linear operators* acting on such a space X, and let $\mathbb{K}$ be the real or complex field, $\mathbb{R}$ or $\mathbb{C}$. We say that a subset of vectors $Y \subset X$ is *spaceable* if there exists an infinite-dimensional closed subspace $Z \subset X$ such that $Z \setminus \{0\} \subset Y$. In this paper we will study the *spaceability* property for subsets of vectors with certain dynamical attributes. Given a vector $x \in X$ we will denote its T-*orbit* by

$$\mathrm{Orb}(x, T) := \{T^n x : n \in \mathbb{N}_0\},$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and we are interested in the dynamical properties:

(a) *hypercyclicity*: a vector $x \in X$ is called *hypercyclic* for T if its T-orbit is dense in X. We denote by $\mathrm{HC}(T)$ the set of such vectors, and T is called *hypercyclic* if $\mathrm{HC}(T)$ is non-empty;

(b) *recurrence*: a vector $x \in X$ is called *recurrent* for T if there exists an increasing sequence $(k_n)_{n\in\mathbb{N}}$ of natural numbers for which $T^{k_n}x \to x$. We denote by $\mathrm{Rec}(T)$ the set of such vectors, and we say that T is *recurrent* if $\mathrm{Rec}(T)$ is dense in X.

Following the notation of [LSMR01, GLSMR00], [BM09, Chapter 8], [GEP11, Chapter 10], and having present that the vector $0 \in X$ is always a recurrent vector for any operator:

Definition 2.1. Let X be a (real or complex) Banach space and $T \in \mathcal{L}(X)$:

(a) a *hypercyclic subspace* for T is an infinite-dimensional closed subspace $Z \subset X$ with the property that $Z \setminus \{0\} \subset \mathrm{HC}(T)$;

(b) a *recurrent subspace* for T is an infinite-dimensional closed subspace $Z \subset X$ with the property that $Z \subset \mathrm{Rec}(T)$.

In the next subsection we recall the existent *hypercyclic spaceability theory* for operators acting on Banach spaces, pointing out the similarities that we will find in the recurrent case.

2.2 Hypercyclic subspaces

The first known result, which established sufficient conditions for a general operator to have a hypercyclic subspace, is due to A. Montes-Rodríguez [MR96]:

Theorem 2.2 ([**MR96**]). *Let X be a (real or complex) separable Banach space and $T \in \mathcal{L}(X)$. Assume that there is an increasing sequence of integers $(k_n)_{n\in\mathbb{N}}$ such that:*

(a) *T satisfies the Hypercyclicity Criterion with respect to $(k_n)_{n\in\mathbb{N}}$;*

(b) *there is an infinite-dimensional closed subspace $E \subset X$ such that $T^{k_n}x \to 0$ for all $x \in E$.*

Then T has a hypercyclic subspace.

One may note various things when looking at Theorem 2.2:

– the original statement used the whole sequence $(k_n)_{n\in\mathbb{N}} = (n)_{n\in\mathbb{N}}$, i.e. T was required to satisfy the Kitai's Criterion instead of the more refined Hypercyclicity Criterion (see [LSM06]);

– it is now well-known (due to J. Bès and A. Peris [BP99]) that an operator $T \in \mathcal{L}(X)$ satisfies the Hypercyclicity Criterion if and only if it is *weakly-mixing*, i.e. if and only if the N-fold direct sum operator $T \oplus \cdots \oplus T$ is hypercyclic for every $N \in \mathbb{N}$;

– however, the sequence with respect to which conditions (a) and (b) are satisfied must coincide, so a priori one cannot exchange the hypothesis that "*T satisfies the Hypercyclicity Criterion with respect to $(k_n)_{n\in\mathbb{N}}$*" by the apparently equivalent "*T is weakly-mixing*";

– there exists a nice alternative proof for Theorem 2.2 given by K. Chan (see [Cha99, CT01]). We will not use that approach here, but it is worth mentioning that it also needs to assume both (a) and (b) conditions with respect to the same sequence $(k_n)_{n\in\mathbb{N}}$.

The next result that appeared about the existence of hypercyclic subspaces for operators acting on Banach spaces is due to F. León-Saavedra and A. Montes-Rodríguez [LSMR97]. They showed that Theorem 2.2 applies for weakly-mixing compact perturbations of the identity:

Theorem 2.3 ([**LSMR97**])**.** *Let X be a (real or complex) separable Banach space and consider any $T \in \mathcal{L}(X)$. Assume that T is weakly-mixing and that there exists some compact operator K such that $\|T - K\| \leq 1$. Then T has a hypercyclic subspace.*

Note that, contrary to Theorem 2.2, no assumption on the sequence with respect to which T satisfies the Hypercyclicity Criterion is necessary for Theorem 2.3 and, indeed, we have stated the result with the *weak-mixing* assumption. The development of the theory followed by giving a complete characterization of the operators that admit a hypercyclic subspace, provided they are weakly-mixing. This was first established by F. León-Saavedra and A. Montes-Rodríguez for Hilbert spaces [LSMR01], and later by M. González and the previous two authors for the general Banach space case [GLSMR00].

To state the announced characterization we move into the *complex* setting, and in fact, we need the concept of *essential spectrum* for an operator acting on a complex Banach space (see [Con89]). We also introduce the concept of *left-essential spectrum*, since it turns out to be an important tool for the result and we also use it along the paper (see Section 5):

Definition 2.4. Let X be a complex Banach space and let $\mathcal{K}(X)$ be the two-sided ideal of $\mathcal{L}(X)$ consisting of all compact operators on X. Given $T \in \mathcal{L}(X)$ we denote by $[T]_{\mathcal{L}/\mathcal{K}}$ the image of the operator T under the standard projection $\mathcal{L}(X) \hookrightarrow \mathcal{L}(X)/\mathcal{K}(X)$, where the Banach algebra $\mathcal{L}(X)/\mathcal{K}(X)$ is the known *Calkin algebra*. Then:

(a) the *left-essential spectrum* of T is the compact set of complex numbers

$$\sigma_{\ell e}(T) = \left\{ \lambda \in \mathbb{C} : [T - \lambda]_{\mathcal{L}/\mathcal{K}} \text{ is not left-invertible in } \mathcal{L}(X)/\mathcal{K}(X) \right\};$$

(b) the *essential spectrum* of T is the compact set of complex numbers

$$\sigma_{e}(T) = \left\{ \lambda \in \mathbb{C} : [T - \lambda]_{\mathcal{L}/\mathcal{K}} \text{ is not invertible in } \mathcal{L}(X)/\mathcal{K}(X) \right\}.$$

In Section 5 we recall the relation between the previous concepts and the so-called *Fredholm* and *semi-Fredholm operators*. For the moment, and setting $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, we are able to state the mentioned characterization:

Theorem 2.5 ([**GLSMR00**])**.** *Let X be a complex separable Banach space and let $T \in \mathcal{L}(X)$. If T is weakly-mixing, then the following statements are equivalent:*

(i) *T has a hypercyclic subspace;*

(ii) *there exists an infinite-dimensional closed subspace $E \subset X$ and an increasing sequence of integers $(l_n)_{n\in\mathbb{N}}$ such that $T^{l_n} x \to 0$ for all $x \in E$;*

(iii) *there exists an infinite-dimensional closed subspace $E \subset X$ and an increasing sequence of integers $(l_n)_{n\in\mathbb{N}}$ such that $\sup_{n\in\mathbb{N}} \|T^{l_n}|_E\| < \infty$;*

(iv) *the essential spectrum of T intersects the closed unit disk, i.e. $\sigma_e(T) \cap \overline{\mathbb{D}} \neq \varnothing$.*

Theorem 2.5 contains the *complex* cases of Theorems 2.2 and 2.3, the second being true since the compact perturbation of any operator preserves its essential spectrum. As we have already mentioned the *weakly-mixing* property is a really important tool in the three stated results, but in order to conclude the existence of a hypercyclic subspace there is always a second assumption about the "regularity" of the operator T on some subspace of X.

3 Recurrent subspaces

If we want to rewrite the previous theory for recurrence, then we should introduce the analogous ingredients to those observed in Theorems 2.2, 2.3 and 2.5. First of all we need the operator $T \in \mathcal{L}(X)$ to have a recurrence property similar to the *weak-mixing* condition:

– *the N-fold direct sum operator $T \oplus \cdots \oplus T$ has to be recurrent for every $N \in \mathbb{N}$.*

It has been recently shown that, for separable Banach spaces, this property is equivalent to the notion of *quasi-rigidity* (see [GLMP]):

Definition 3.1 ([**GLMP**])**.** We say that $T \in \mathcal{L}(X)$ is *quasi-rigid with respect to* an increasing sequence $(k_n)_{n\in\mathbb{N}}$ if there exists a dense subset $Y \subset X$ such that $T^{k_n}x \to x$ for all $x \in Y$.

Secondly, we need an assumption about the "regularity" of the operator T. In Theorem 2.2 this assumption was condition (b). However, we would like the identity operator $I : X \longrightarrow X$ to fulfill the main theorems of the *recurrent spaceability theory*, so we will use a more realistic condition than that of having, for a recurrent (and not necessarily hypercyclic) operator, an infinite-dimensional closed subspace of vectors with 0-convergent sub-orbits:

Theorem 3.2. *Let X be a (real or complex) separable Banach space and let $T \in \mathcal{L}(X)$. Assume that there is an increasing sequence of integers $(k_n)_{n\in\mathbb{N}}$ such that:*

(a) *T is quasi-rigid with respect to $(k_n)_{n\in\mathbb{N}}$;*

(b) *there exists a non-increasing sequence $(E_n)_{n\in\mathbb{N}}$ of infinite-dimensional closed subspaces of the space X such that $\sup_{n\in\mathbb{N}} \left\| T^{k_n}|_{E_n} \right\| < \infty$.*

Then T has a recurrent subspace. In particular, there exists an infinite-dimensional closed subspace $F \subset X$ and a subsequence $(l_n)_{n\in\mathbb{N}}$ of $(k_n)_{n\in\mathbb{N}}$ such that $T^{l_n}x \to x$ for all $x \in F$, so

$$F \oplus \cdots \oplus F \subset \operatorname{Rec}(T \oplus \cdots \oplus T)$$

for every N-fold direct sum operator $T \oplus \cdots \oplus T : X \oplus \cdots \oplus X \longrightarrow X \oplus \cdots \oplus X$.

Theorem 3.2 can be compared with Theorem 2.2. Condition (b) of Theorem 3.2 is satisfied by the identity operator and, for recurrent (and not necessarily hypercyclic) operators, it is a more suitable assumption than the original condition (b) of Theorem 2.2. The proof of Theorem 3.2 is highly based on the arguments and *basic sequence*'s techniques used in the proof of Theorem 2.2 (see Section 4). In particular, our proof can be compared with the very well-known results [LSM06, Theorem 20] or [GEP11, Theorem 10.29], where condition (b) of Theorem 3.2 was already used to obtain the existence of hypercyclic subspaces.

An application of the Banach-Steinhaus theorem together with Theorem 3.2 yields that, *given a (real or complex) Banach space X and given $T \in \mathcal{L}(X)$ a quasi-rigid operator with respect to the sequence $(k_n)_{n\in\mathbb{N}}$, the following conditions are equivalent:*

- *there exists an infinite-dimensional closed subspace $E \subset X$ and a subsequence $(l_n)_{n\in\mathbb{N}}$ of $(k_n)_{n\in\mathbb{N}}$ such that $T^{l_n}x \to x$ for all $x \in E$;*

- *there exists an infinite-dimensional closed subspace $E \subset X$ and a subsequence $(l_n)_{n\in\mathbb{N}}$ of $(k_n)_{n\in\mathbb{N}}$ such that $(T^{l_n}x)_{n\in\mathbb{N}}$ converges in X for all $x \in E$;*

- *there exists an infinite-dimensional closed subspace $E \subset X$ and a subsequence $(l_n)_{n\in\mathbb{N}}$ of $(k_n)_{n\in\mathbb{N}}$ such that $(T^{l_n}x)_{n\in\mathbb{N}}$ is bounded in X for all $x \in E$;*

- *there exists an infinite-dimensional closed subspace $E \subset X$ and a subsequence $(l_n)_{n\in\mathbb{N}}$ of $(k_n)_{n\in\mathbb{N}}$ such that $\sup_{n\in\mathbb{N}} \|T^{l_n}|_E\| < \infty$;*

- *there exists a non-increasing sequence $(E_n)_{n\in\mathbb{N}}$ of infinite-dimensional closed subspaces of X and a subsequence $(l_n)_{n\in\mathbb{N}}$ of $(k_n)_{n\in\mathbb{N}}$ such that $\sup_{n\in\mathbb{N}} \left\|T^{l_n}|_{E_n}\right\| < \infty$;*

and if any of them holds, then T has a recurrent subspace.

A stronger result is true: following [LSMR01, GLSMR00] and [BM09, Chapter 8] and the philosophy of Theorem 2.5 we can establish the equivalence between the previous statements and the (apparently weaker) condition of admitting a recurrent subspace:

Theorem 3.3. *Let X be a complex separable Banach space and $T \in \mathcal{L}(X)$. If T is quasi-rigid, then the following statements are equivalent:*

(i) *T has a recurrent subspace;*

(ii) *there exists an infinite-dimensional closed subspace $E \subset X$ and an increasing sequence of integers $(l_n)_{n\in\mathbb{N}}$ such that $T^{l_n}x \to x$ for all $x \in E$;*

(iii) *there exists an infinite-dimensional closed subspace $E \subset X$ and an increasing sequence of integers $(l_n)_{n\in\mathbb{N}}$ such that $\sup_{n\in\mathbb{N}} \|T^{l_n}|_E\| < \infty$;*

(iv) *the essential spectrum of T intersects the closed unit disk, i.e. $\sigma_e(T) \cap \overline{\mathbb{D}} \neq \varnothing$.*

Theorem 3.3 reaffirms the fact that *quasi-rigidity* is, for recurrence, the analogous property to that of *weak-mixing* for hypercyclicity. The proof of Theorem 3.3 (see Section 5) is highly based on the proof of Theorem 2.5, but we shall study the structure of the essential spectrum for recurrent operators in order to complete the different implications required.

In view of Theorems 2.5 and 3.3 we deduce that:

- *A weakly-mixing operator on a complex separable Banach space has a hypercyclic subspace if and only if it has a recurrent subspace.*

However, we can also prove that equivalence for operators acting on *real* Banach spaces by using some *complexification*'s techniques (see [MST99, MMFPSS22]): for a real Banach space $(X, \|\cdot\|)$ the *complexification* $\widetilde{X}$ of X is defined formally as the vector space

$$\widetilde{X} := \{x + iy : x, y \in X\},$$

which can be (algebraically and topologically) identified with $X \oplus X$. Indeed, if we define the multiplication by complex scalars as $(a + ib)(x + iy) = (ax - by) + i(ay + bx)$, for any $a, b \in \mathbb{R}$ and $x, y \in X$, then $\widetilde{X}$ becomes a complex Banach space with the norm

$$\|x + iy\|_c := \sup_{t\in[0,2\pi]} \|\cos(t)x - \sin(t)y\|,$$

for $x, y \in X$. It is easy to check that $\|\cdot\|_c$ is a $\mathbb{C}$-homogeneous norm that endows $\widetilde{X} = X + iX$ with an homeomorphic topology to that of the usual direct sum space $X \oplus X$, and the map

$$J : \widetilde{X} \longrightarrow X \oplus X \quad \text{ with } \quad J(x + iy) = (x, y) \in X \oplus X,$$

is an $\mathbb{R}$-isomorphism. Given a (real-linear) operator $T : X \longrightarrow X$ on the real Banach space X, the *complexified* operator (or simply its *complexification*) $\widetilde{T} : \widetilde{X} \longrightarrow \widetilde{X}$ is defined by

$$\widetilde{T}(x + iy) = Tx + iTy \quad \text{ for every } x, y \in X,$$

which is a (complex-linear) operator on the (complex) Banach space $\widetilde{X}$. It is easily seen that $\widetilde{T}$ is conjugate to $T \oplus T$ via J, i.e. $J \circ \widetilde{T} = T \oplus T \circ J$, and also that $\|T\| = \|\widetilde{T}\|$.

By using this complex structure we can state the real version of Theorems 2.5 and 3.3:

Theorem 3.4. *Let X be a real separable Banach space and let $T \in \mathcal{L}(X)$:*

(a) *If T is weakly-mixing, then the following statements are equivalent:*

 (i) *T has a hypercyclic subspace;*

 (ii) *there exists an infinite-dimensional closed subspace $E \subset X$ and an increasing sequence of integers $(l_n)_{n\in\mathbb{N}}$ such that $T^{l_n}x \to 0$ for all $x \in E$;*

 (iii) *there exists an infinite-dimensional closed subspace $E \subset X$ and an increasing sequence of integers $(l_n)_{n\in\mathbb{N}}$ such that $\sup_{n\in\mathbb{N}} \|T^{l_n}|_E\| < \infty$;*

 (iv) *the essential spectrum of the complexification $\widetilde{T} : \widetilde{X} \longrightarrow \widetilde{X}$ intersects the closed unit disk, i.e. $\sigma_e(\widetilde{T}) \cap \overline{\mathbb{D}} \neq \varnothing$.*

(b) *If T is quasi-rigid, then the following statements are equivalent:*

 (i) *T has a recurrent subspace;*

 (ii) *there exists an infinite-dimensional closed subspace $E \subset X$ and an increasing sequence of integers $(l_n)_{n\in\mathbb{N}}$ such that $T^{l_n}x \to x$ for all $x \in E$;*

 (iii) *there exists an infinite-dimensional closed subspace $E \subset X$ and an increasing sequence of integers $(l_n)_{n\in\mathbb{N}}$ such that $\sup_{n\in\mathbb{N}} \|T^{l_n}|_E\| < \infty$;*

 (iv) *the essential spectrum of the complexification $\widetilde{T} : \widetilde{X} \longrightarrow \widetilde{X}$ intersects the closed unit disk, i.e. $\sigma_e(\widetilde{T}) \cap \overline{\mathbb{D}} \neq \varnothing$.*

Theorem 3.4 is a modest extension of the previous theory to the real setting. Its proof is highly based on the proofs of Theorems 2.5 and 3.3, but we require some basic lemmas in order to pass from (hypercyclic or recurrent) subspaces for the real-linear system to subspaces for its complexification and vice versa, see Section 6.

Since the weak-mixing property implies quasi-rigidity for every operator on a separable Banach space (see [GLMP, Theorem 2.5]), from Theorems 2.5, 3.3 and 3.4 we obtain that:

Corollary 3.5. *Let $T : X \longrightarrow X$ be a weakly-mixing operator on a (real or complex) separable Banach space X. Then the following statements are equivalent:*

(i) *T has a hypercyclic subspace;*

(ii) *T has a recurrent subspace.*

4 Sufficient conditions for recurrent subspaces

In this section we prove Theorem 3.2. As in the proof of Theorem 2.2:

– we can extract a *basic sequence* from the family $(E_n)_{n\in\mathbb{N}}$ of infinite-dimensional subspaces, by using the Mazur theorem (see [LT77, Vol I, page 4] or [BM09, Lemma C.1.1]);

– we will "perturb" that basic sequence to obtain an equivalent one formed by vectors with a strong recurrent property (see [BM09, Lemmas 8.4 and C.1.2] or [GEP11, Lemma 10.6]);

We refer to the textbooks [Die84, LT77] for any unexplained but standard notion about Schauder basis and basic sequences.

4.1 Proof of Theorem 3.2

By the Mazur theorem (see [LT77, Vol I, page 4] or [BM09, Lemma C.1.1]) there exists a normalized basic sequence $(e_n)_{n\in\mathbb{N}}$ such that $e_n \in E_n$ for each $n \in \mathbb{N}$. This sequence is then a Schauder basis of

$$E := \overline{\operatorname{span}\{e_n : n \in \mathbb{N}\}}.$$

Moreover, for every strictly increasing sequence of integers $(l_n)_{n\in\mathbb{N}}$, the sequence $(e_{l_n})_{n\in\mathbb{N}}$ is a normalized Schauder basis of the closed subspace

$$\overline{\operatorname{span}\{e_{l_n} : n \in \mathbb{N}\}} \subset E \subset X.$$

For each $n \in \mathbb{N}$ consider the *coefficient functional* $e_n^* : E \longrightarrow \mathbb{K}$ such that

$$\langle e_n^*, x\rangle = \langle e_n^*, \sum_{k\in\mathbb{N}} a_k e_k\rangle = a_n \qquad \text{for each } x \in E \text{ with } x = \sum_{n\in\mathbb{N}} a_n e_n.$$

The family $(e_n^*)_{n\in\mathbb{N}}$ is uniformly continuous since the we have constructed a **normalized** basic sequence $(e_n)_{n\in\mathbb{N}}$ (see for instance [LT77, Vol. I, 1.b] or [BM09, Appendix C.1]). Denote by $\|e_n^*\|$ the norm of e_n^* as a functional in E and write $K := 1 + \max_{n\in\mathbb{N}} \|e_n^*\|$.

By assumption (a) there is a dense subset Y of X such that $T^{k_n}x \to x$ for every $x \in Y$.

Claim. *There exists:*

(1) *an increasing sequence of integers $(l_n)_{n\in\mathbb{N}}$, subsequence of $(k_n)_{n\in\mathbb{N}}$;*

(2) *a sequence of vectors $(f_{l_n})_{n\in\mathbb{N}} \subset Y$, and the related sequence $(g_{l_n})_{n\in\mathbb{N}} := (f_{l_n} - e_{l_n})_{n\in\mathbb{N}}$;*

with the properties:

(i) $\|f_{l_n} - e_{l_n}\| = \|g_{l_n}\| < \frac{1}{2^{n+1}K}$ *for every* $n \in \mathbb{N}$*;*

(ii) $\|T^{l_j} g_{l_n}\| < \frac{1}{2^{j+n}}$ *for each* $j \in \mathbb{N}$ *and* $n > j$*;*

(iii) $\|T^{l_j} f_{l_n} - f_{l_n}\| < \frac{1}{2^{j+n}}$ *for each* $j \in \mathbb{N}$ *and* $1 \leq n \leq j$*;*

(iv) $k_{l_{n+1}} > l_n \in \{k_j : j \in \mathbb{N}\}$ *for every* $n \in \mathbb{N}$*.*

Proof of the Claim. Suppose that we have constructed $(f_{l_n})_{n=1}^{i}$ and $(l_n)_{n=1}^{i}$ with:

(i) $\|f_{l_n} - e_{l_n}\| = \|g_n\| < \frac{1}{2^{n+1}K}$ for each $n \leq i$;

(ii) $\|T^{l_j} g_{l_n}\| < \frac{1}{2^{j+n}}$ for each $1 \leq j < i$ and $i \geq n > j$;

(iii) $\|T^{l_j} f_{l_n} - f_{l_n}\| < \frac{1}{2^{j+n}}$ for each $1 \leq j \leq i$ and $1 \leq n \leq j$;

(iv) $k_{l_{n+1}} > l_n \in \{k_j : j \in \mathbb{N}\}$ for every $n < i$.

By the continuity of T there is $\varepsilon > 0$ such that

$$\|T^{l_j} y\| < \frac{1}{2^{j+(i+1)}} \qquad \text{for every } 1 \leq j \leq i \text{ and } y \in X \text{ with } \|y\| < \varepsilon.$$

Therefore, taking $f_{l_{i+1}} \in Y$ such that $\|f_{l_{i+1}} - e_{l_{i+1}}\| < \max\left\{\frac{1}{2^{i+2}K}, \varepsilon_i\right\}$ we get

(i) $\|f_{l_{i+1}} - e_{l_{i+1}}\| = \|g_{l_{i+1}}\| < \frac{1}{2^{(i+1)+1}K}$, which is (i) for $i+1$;

(ii) $\|T^{l_j} g_{l_{i+1}}\| < \frac{1}{2^{j+(i+1)}}$ for each $1 \leq j < i+1$, which is (ii) for $i+1$.

Choose $l_{i+1} \in \{k_j : j \in \mathbb{N}\}$ large enough such that $k_{l_{i+1}} > l_i$ and $\|T^{l_{i+1}} f_{l_n} - f_{l_n}\| < \frac{1}{2^{(i+1)+n}}$ for every $1 \leq n \leq i+1$. This is possible since $f_{l_n} \in Y$, and it implies (iii) and (iv) for $i+1$. □

Once the *Claim* is proved, by condition (i) we get that

$$\sum_{n\in\mathbb{N}} \|e^*_{l_n}\| \cdot \|f_{l_n} - e_{l_n}\| = \sum_{n\in\mathbb{N}} \|e^*_{l_n}\| \cdot \|g_{l_n}\| \overset{\text{(i)}}{<} \sum_{n\in\mathbb{N}} \frac{1}{2^{n+1}} = \frac{1}{2} < 1,$$

so [BM09, Lemmas 8.4 and C.1.2] or [GEP11, Lemma 10.6] imply that $(f_{l_n})_{n\in\mathbb{N}}$ is a basic sequence equivalent to $(e_{l_n})_{n\in\mathbb{N}}$. It follows that $(f_{l_n})_{n\in\mathbb{N}}$ is a Schauder basis of

$$F := \overline{\operatorname{span}\{f_{l_n} : n \in \mathbb{N}\}} \subset X,$$

which is an infinite-dimensional closed subspace of X.

We claim that $T^{l_j}x \to x$ for all $x \in F$: indeed, given $x \in F$ there is a 0-convergent sequence $a = (a_n)_{n\in\mathbb{N}} \in c_0(\mathbb{N})$ such that

$$x = \sum_{n\in\mathbb{N}} a_n f_{l_n} = \sum_{n\in\mathbb{N}} a_n(e_{l_n} + g_{l_n}) = \sum_{n\in\mathbb{N}} a_n e_{l_n} + \sum_{n\in\mathbb{N}} a_n g_{l_n},$$

where $\sum_{n\in\mathbb{N}} a_n e_{l_n}$ is convergent since $(f_{l_n})_{n\in\mathbb{N}}$ and $(e_{l_n})_{n\in\mathbb{N}}$ are equivalent basic sequences, and $\sum_{n\in\mathbb{N}} a_n g_{l_n}$ is an absolutely convergent series by (i). Hence

$$\begin{aligned}
\|T^{l_j}x - x\| &= \left\| \left(\sum_{n\leq j} a_n T^{l_j} f_{l_n}\right) + T^{l_j}\left(\sum_{n>j} a_n g_{l_n}\right) + T^{l_j}\left(\sum_{n>j} a_n e_{l_n}\right) - \sum_{n\in\mathbb{N}} a_n f_{l_n} \right\| \\
&\leq \left\|\sum_{n\leq j} a_n (T^{l_j} f_{l_n} - f_{l_n})\right\| + \left\|\sum_{n>j} a_n T^{l_j} g_{l_n}\right\| + \left\|T^{l_j}\left(\sum_{n>j} a_n e_{l_n}\right)\right\| + \left\|\sum_{n>j} a_n f_{l_n}\right\| \\
&\leq \|a\|_\infty \left(\sum_{n\leq j} \|T^{l_j} f_{l_n} - f_{l_n}\| + \sum_{n>j} \|T^{l_j} g_{l_n}\|\right) \\
&\quad + \left\|T^{l_j}|_{E_{l_{j+1}}}\right\| \cdot \left\|\sum_{n>j} a_n e_{l_n}\right\| + \left\|\sum_{n>j} a_n f_{l_n}\right\| \\
&\overset{\text{(ii),(iii)}}{\underset{\text{(iv),(b)}}{<}} \|a\|_\infty \sum_{i>j} \frac{1}{2^i} + \sup_{n\in\mathbb{N}} \left\|T^{k_n}|_{E_n}\right\| \cdot \left\|\sum_{n>j} a_n e_{l_n}\right\| + \left\|\sum_{n>j} a_n f_{l_n}\right\| \longrightarrow 0
\end{aligned}$$

when $j \to \infty$ since $(e_{l_n})_{n\in\mathbb{N}}$ and $(f_{l_n})_{n\in\mathbb{N}}$ are basic sequences. □

4.2 Comments on Theorem 3.2

The previous proof allows us to relax the hypothesis of Theorem 3.2 in two different ways: we can delete the *separability* hypothesis; and we can assume just *local quasi-rigidity*:

Remark 4.1 (Non-Separability). The *separability* of the underlying space is crucial to study hypercyclicity: Theorem 2.2 applies to bounded linear operators on *separable* Banach spaces. However, as it is mentioned in [CMP14], the *separability* is not a necessary assumption for recurrence: Theorem 3.2 is still true, with the same proof, for non-separable spaces.

Remark 4.2 (Local Quasi-Rigidity is allowed in Theorem 3.2). Suppose that $(E_n)_{n\in\mathbb{N}}$ is the non-increasing sequence of infinite-dimensional closed subspaces that comes from the assumption (b) of Theorem 3.2. The proof of Theorem 3.2 still holds if we replace (a) by the following much weaker condition:

(a*) *there is a set of vectors* $Y \subset X$ *with* $E_1 \subset \overline{Y}$ *such that* $T^{k_n}x \to x$ *for all* $x \in Y$.

This last condition is a kind of *local quasi-rigidity*. Note that, contrary to the hypercyclicity case, an operator T does not need to be recurrent in order to have a recurrent subspace. For instance, given $\lambda \in \mathbb{R}$ with $|\lambda| \neq 1$ we can consider the operator

$$T := I \oplus \lambda I : X \oplus X \longrightarrow X \oplus X,$$

where $I : X \longrightarrow X$ is the identity operator on a Banach space X. Clearly T is not a recurrent operator (i.e. the set of recurrent vectors $\text{Rec}(T)$ is not dense) but it contains a recurrent subspace. Note that T fulfills property (a*) for any sequence $(k_n)_{n\in\mathbb{N}}$ whenever $E_1 \subset X \oplus \{0\}$.

5 The complex case

In this section the underlying Banach spaces X are assumed to be complex. Theorem 3.3 is proved, characterising the quasi-rigid operators admitting a recurrent subspace in a similar way than the already well-known Theorem 2.5 does for weakly-mixing operators admitting a hypercyclic subspace. We start by studying the essential spectrum for recurrent operators.

5.1 The essential spectrum for recurrent operators

In the proof of Theorem 2.5 the *left-essential spectrum* of the operator $T : X \longrightarrow X$ plays a fundamental role even though in the statement just appears the *essential spectrum*. This happens because both sets coincide when T is hypercyclic (see [GLSMR00]). Here we prove that the same holds for recurrent operators and we will use this fact to prove Theorem 3.3.

Let us recall why the latter is true for hypercyclic operators. Following the general theory of *Fredholm* operators, see for instance [LT77, Vol. I, 2.c], we have that:

(a) $\lambda \in \sigma_{\ell e}(T)$ if and only if $T - \lambda$ is not a *left-Fredholm* operator. Recall that the operator $T - \lambda$ is *left-Fredholm* if the following conditions hold:

$$\text{Ran}(T - \lambda) \text{ is closed} \qquad \text{and} \qquad \dim \text{Ker}(T - \lambda) < \infty.$$

(b) $\lambda \in \sigma_e(T)$ if and only if $T - \lambda$ is not a *Fredholm* operator. Recall that the operator $T - \lambda$ is *Fredholm* if the following conditions hold:

$$\text{Ran}(T - \lambda) \text{ is closed}, \qquad \dim \text{Ker}(T - \lambda) < \infty \qquad \text{and} \qquad \text{codim}\, \text{Ran}(T - \lambda) < \infty.$$

Clearly $\sigma_{\ell e}(T) \subset \sigma_e(T)$. Conversely, given $\lambda \in \sigma_e(T)$ for which $\text{Ran}(T - \lambda)$ is dense we can deduce that $\lambda \in \sigma_{\ell e}(T)$. This is why we have the equality

$$\sigma_{\ell e}(T) = \sigma_e(T),$$

for every hypercyclic operator T, since hypercyclicity implies that $\text{Ran}(T - \lambda)$ is dense for every $\lambda \in \mathbb{C}$ (see for instance [GEP11, Lemma 2.53]). For a recurrent operator T, and setting $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, the previous argument just gives us that

$$\sigma_{\ell e}(T) \setminus \mathbb{T} = \sigma_e(T) \setminus \mathbb{T},$$

since in this case $\text{Ran}(T-\lambda)$ is not necessarily dense when $\lambda \in \mathbb{T}$, see [CMP14, Proposition 2.14].

However, if given any $T \in \mathcal{L}(X)$ we denote by T^* its *adjoint operator* acting on the respective *dual Banach space* X^*, then the following argument will allow us to show the complete equality between both spectrums for every recurrent operator:

Lemma 5.1. *Let X be a complex Banach space and let $T \in \mathcal{L}(X)$. Suppose that $\Omega \subset \mathbb{C}$ is a non-empty connected open set with the property that $\Omega \setminus \sigma_p(T^*) \neq \varnothing$. Then we have that*

$$\sigma_{\ell e}(T) \cap \overline{\Omega} \neq \varnothing \quad \text{if and only if} \quad \sigma_e(T) \cap \overline{\Omega} \neq \varnothing.$$

Proof. Since $\sigma_{\ell e}(T) \subset \sigma_e(T)$ we just argue the case in which $\sigma_e(T) \cap \overline{\Omega} \neq \varnothing$. Suppose by contradiction that $\sigma_{\ell e}(T) \cap \overline{\Omega} = \varnothing$, then

$$\overline{\Omega} \subset \{\lambda \in \mathbb{C} : \operatorname{Ran}(T-\lambda) \text{ is closed and } \dim \operatorname{Ker}(T-\lambda) < \infty\},$$

so the index

$$\operatorname{ind}(T-\lambda) := \dim \operatorname{Ker}(T-\lambda) - \operatorname{codim} \operatorname{Ran}(T-\lambda),$$

is well defined as an element of $\mathbb{Z} \cup \{\pm\infty\}$ for every $\lambda \in \overline{\Omega}$ (see [LT77, Vol I, 2.c]). Since the set of semi-Fredholm operators is a norm-open subset of $\mathcal{L}(X)$ and the index function is norm-discrete-continuous on it (see [O'S92, Theorem 2.2]), we deduce that there is a connected open set $U \subset \mathbb{C}$ with $\overline{\Omega} \subset U$ and such that the index function

$$\lambda \mapsto \operatorname{ind}(T-\lambda) \in \mathbb{Z} \cup \{\pm\infty\},$$

is constant on U. Moreover, this value is finite and non-negative since by assumption we can find some $\lambda \in \Omega \setminus \sigma_p(T^*)$, which implies that $\operatorname{Ran}(T-\lambda)$ is closed, $T-\lambda$ has dense range, $\operatorname{codim} \operatorname{Ran}(T-\lambda) = 0$, and then we have the equality $\operatorname{ind}(T-\lambda) = \dim \operatorname{Ker}(T-\lambda) \in \mathbb{N}_0$. Finally, given $\mu \in \sigma_e(T) \cap \overline{\Omega} \subset U$ we were assuming that $\mu \notin \sigma_{\ell e}(T)$, but then

$$\operatorname{Ran}(T-\mu) \text{ is closed} \qquad \text{and} \qquad \dim \operatorname{Ker}(T-\mu) < \infty,$$

so we necessarily have that $\operatorname{codim} \operatorname{Ran}(T-\mu) = \infty$, and hence $\operatorname{ind}(T-\mu) = -\infty$, which yields a contradiction since we have proved that $\operatorname{ind}(T-\mu) \in \mathbb{N}_0$. □

Proposition 5.2. *Let X be a complex Banach space and let $T \in \mathcal{L}(X)$. If $\sigma_p(T^*)$ has empty interior, then we have the equality $\sigma_{\ell e}(T) = \sigma_e(T)$.*

Proof. We already know that $\sigma_{\ell e}(T) \subset \sigma_e(T)$. Suppose by contradiction that there exists some element $\lambda \in \sigma_e(T) \setminus \sigma_{\ell e}(T)$. Using that the set $\sigma_{\ell e}(T)$ is compact (and hence closed) we can find a connected open neighbourhood Ω of λ such that $\sigma_{\ell e}(T) \cap \overline{\Omega} = \varnothing$. This is a contradiction with Lemma 5.1 since $\Omega \setminus \sigma_p(T^*) \neq \varnothing$, by the empty interior assumption, and also $\lambda \in \sigma_e(T) \cap \overline{\Omega}$. □

The previous result applies to recurrent operators:

Corollary 5.3. *Let X be a complex Banach space and let $T \in \mathcal{L}(X)$. If T is recurrent, then we have the equality $\sigma_{\ell e}(T) = \sigma_e(T)$.*

Proof. If $T \in \mathcal{L}(X)$ is recurrent we have that $\sigma_p(T^*) \subset \mathbb{T}$ by [CMP14, Proposition 2.14]. Proposition 5.2 yields the result. □

We are now ready to prove Theorem 3.3.

5.2 Proof of Theorem 3.3

Clearly (ii) $\Rightarrow$ (i). It is also direct that (ii) $\Rightarrow$ (iii) by the Banach-Steinhaus theorem applied to the family of operators $\left\{T^{l_n}|_E : n \in \mathbb{N}\right\}$.

To see the implications (i) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (iv) we use the following fact originally proved in [GLSMR00, Proof of Theorem 4.1] for operators on complex separable Banach spaces:

Lemma 5.4. *If $\sigma_{\ell e}(T) \cap \overline{\mathbb{D}} = \varnothing$, then every infinite-dimensional closed subspace $Z \subset X$ admits a vector $x \in Z$ such that $\lim_{n\to\infty} \|T^n x\| = \infty$.*

Since T is recurrent we have the equality $\sigma_e(T) = \sigma_{\ell e}(T)$ by Corollary 5.3. Then, if any of the statements (i) or (iii) holds but $\sigma_e(T) \cap \overline{\mathbb{D}} = \varnothing$ we arrive to a contradiction: the vector with divergent orbit obtained by Lemma 5.4 cannot be in a recurrent subspace, neither in the subspace described in statement (iii).

To finish the proof we show that (iv) $\Rightarrow$ (ii): suppose that (iv) holds and let $\lambda \in \sigma_e(T) \cap \overline{\mathbb{D}}$ which by Corollary 5.3 has the property $\lambda \in \sigma_{\ell e}(T)$. Then $T - \lambda$ is not left-Fredholm so we can apply [BM09, Proposition D.3.4] obtaining an infinite-dimensional closed subspace $E \subset X$ and a compact operator $R \in \mathcal{L}(X)$ such that $(T - R)|_E = \lambda I|_E$, which implies that

$$\|(T - R)^n|_E\| \leq 1 \quad \text{for every } n \in \mathbb{N}.$$

From now we modify the proof of [BM09, Lemma 8.16]: by assumption T is quasi-rigid with respect to some sequence $(k_n)_{n\in\mathbb{N}}$. For each $n \in \mathbb{N}$ we can write

$$T^{k_n} = (T - R)^{k_n} + R_n,$$

where R_n is a compact operator. By [BM09, Lemma 8.13] there exists a non-increasing sequence $(E_n)_{n\in\mathbb{N}}$ of finite-codimensional (and hence infinite-dimensional) closed subspaces of E with

$$\|R_n|_{E_n}\| \leq 1 \quad \text{for every } n \in \mathbb{N}.$$

Then

$$\left\|T^{k_n}|_{E_n}\right\| = \left\|[(T - R)^{k_n} + R_n]|_{E_n}\right\| \leq \left\|(T - R)^{k_n}|_{E_n}\right\| + \|R_n|_{E_n}\| \leq 2 < \infty.$$

By Theorem 3.2 we get (ii) for some subsequence $(l_n)_{n\in\mathbb{N}}$ of $(k_n)_{n\in\mathbb{N}}$. □

Remark 5.5. The **complex case** of Corollary 3.5 is now proved as a direct consequence of Theorems 2.5 and 3.3. However, in view of Lemma 5.4 there is an alternative proof:

- It is clear that having a hypercyclic subspace implies having a recurrent subspace since every hypercyclic vector is recurrent.
- On the other hand Lemma 5.4 implies that: a necessary condition for a weakly-mixing operator to admit a recurrent subspace is that the left-essential spectrum must intersect the closed unit disc, since a vector with a divergent orbit is not recurrent. Hence, Theorem 2.5 implies that the operator must have a hypercyclic subspace.

In Section 6 we obtain the **real case** of Corollary 3.5.

5.3 Comments on Theorem 3.3

Before the comments on Theorem 3.3 let us introduce the following standard notation:

Definition 5.6. Let X and Y be a Banach spaces:

– We will denote by $\mathcal{L}(X,Y)$ the *set of bounded linear operators* from X to Y.

– Let $(Y_j)_{j\in J}$ be a collection of Banach spaces and $\mathcal{K} \subset \bigcup_{j\in J} \mathcal{L}(X,Y_j)$ a family of compact operators acting on X. We will say that a sequence of vectors $(x_n)_{n\in\mathbb{N}} \subset X$ is *$\mathcal{K}$-null* if $\lim_{n\to\infty} Kx_n = 0$ for every operator $K \in \mathcal{K}$.

As we did for Theorem 3.2 in Subsection 4.2, we can relax the hypothesis of Theorem 3.3:

Remark 5.7 (Non-Separability). Theorem 3.3 is still true if X is a *non-separable* space. This is not completely direct since the implications (i) ⇒ (vii) and (vi) ⇒ (vii) are deduced from Lemma 5.4, which with the proof of [GLSMR00, Proof of Theorem 4.1] is only valid for complex *separable* Banach spaces. Nevertheless, if one reads carefully the proof given for those implications in [BM09, Section 8.3], it can be deduced the following more general fact that uses directly the *essential spectrum*:

Proposition 5.8. *Let X be a complex (and not necessarily separable) Banach space and let $T \in \mathcal{L}(X)$. If $\sigma_e(T) \cap \overline{\mathbb{D}} = \varnothing$, then every infinite-dimensional closed subspace $Z \subset X$ admits a vector $x \in Z$ such that $\lim_{n\to\infty} \|T^n x\| = \infty$.*

If we let X and $(Y_n)_{n\in\mathbb{N}}$ be complex Banach spaces, $T \in \mathcal{L}(X)$, and we consider any $Z \subset X$ infinite-dimensional closed subspace, then Proposition 5.8 can be proved in three steps:

– [BM09, Lemma 8.15 (b)] *If $\sigma_e(T) \cap \overline{\mathbb{D}} = \varnothing$, then one can find $\lambda > 1$, $n_0 \in \mathbb{N}$ and a countable family of compact operators $\mathcal{K}_0 \subset \mathcal{L}(X)$ such that the following holds for any normalized $\mathcal{K}_0$-null sequence $(e_j)_{j\in\mathbb{N}} \subset X$:*

$$\liminf_{j\to\infty} \|T^n e_j\| \geq \lambda^n \quad \textit{for each } n \geq n_0.$$

– [BM09, Lemma 8.17] *If Z is separable, then there exist a complex separable Banach space $\hat{Y}$ and a countable family of compact operators $\mathcal{K}_1 \subset \mathcal{L}(Z,\hat{Y})$ such that the following holds for any normalized $\mathcal{K}_1$-null sequence $(e_j)_{j\in\mathbb{N}} \subset Z$: given any summable sequence of positive numbers $(\alpha_n)_{n\in\mathbb{N}}$, there exists some vector $x \in \overline{\operatorname{span}_{\mathbb{R}}\{e_j : j \in \mathbb{N}\}} \subset Z$ such that*

$$\|T^n x\| \geq \alpha_n \limsup_{j\to\infty} \|T^n e_j\| \quad \textit{for each } n \in \mathbb{N}.$$

– [BM09, Corollary 8.14] *If $\mathcal{K} = (K_n : Z \longrightarrow Y_n)_{n\in\mathbb{N}}$ is a countable family of compact operators, then Z contains a normalized $\mathcal{K}$-null basic sequence.*

One just needs to apply [BM09, Corollary 8.14] to the countable family of compact operators $\mathcal{K} = \mathcal{K}_0 \cup \mathcal{K}_1$ and choose a summable sequence $(\alpha_n)_{n\in\mathbb{N}}$ for which $\alpha_n\lambda^n \to \infty$ when $n \to \infty$. The separability of X is replaced by the separability of Z. It is worth mentioning that the vector $x \in Z$ constructed lies in the closure of the **real-linear** span of the sequence $(e_j)_{j\in\mathbb{N}}$ selected. This fact will be important in the real case, see Section 6.

Remark 5.9 (Local Quasi-Rigidity is not allowed in Theorem 3.3). We cannot repeat the exchange of assumption done for Theorem 3.2 in Remark 4.2: the infinite-dimensional closed subspace $E \subset X$ obtained in the proof by the non-left-Fredholm condition of $T - \lambda$ cannot be controlled to be included in the closure of a set $Y \subset X$ such that $T^{k_n}x \to x$ for all $x \in Y$. In fact, let X be any of the complex spaces $\ell^p(\mathbb{N})$, $1 \leq p < \infty$, or $c_0(\mathbb{N})$, and let $B : X \longrightarrow X$ be the well-known *backward shift* operator on such a space X. Then for any fixed $\lambda \in \mathbb{D} \setminus \{0\}$ consider the operator

$$T := \lambda^{-1}B \oplus \lambda I : X \oplus X \longrightarrow X \oplus X,$$

where $I : X \longrightarrow X$ is the identity operator on X. It is easy to verify that:

(1) T is not quasi-rigid neither recurrent since $(x, y) \in \mathrm{Rec}(T)$ implies that $y = 0$;

(2) $\lambda^{-1}B$ is quasi-rigid since the Rolewicz operator is known to be (weakly-)mixing;

(3) $\lambda \in \sigma_e(T) \cap \overline{\mathbb{D}}$ since $\mathrm{Ker}(T - \lambda)$ is infinite-dimensional;

(4) T has no recurrent subspace: otherwise $\lambda^{-1}B$ would have a recurrent subspace, but it is really well-known that $\sigma_e(\lambda^{-1}B) = \sigma_{\ell e}(\lambda^{-1}B) = \lambda^{-1}\mathbb{T} = \{\mu \in \mathbb{C} : |\mu| = \frac{1}{|\lambda|}\}$, contradiction.

6 The real case

In this section we extend Theorems 2.5 and 3.3 to the real case via Theorem 3.4. Since given a real-linear operator $T : X \longrightarrow X$ one of the equivalences included in Theorem 3.4 is:

– *the essential spectrum of the complexification* $\widetilde{T} : \widetilde{X} \longrightarrow \widetilde{X}$ *intersects the closed unit disk*;

we need some previous results allowing us to pass from subspaces for the real-linear system to subspaces for its complexification and vice versa.

The first of such results is the following: given a complex infinite-dimensional closed subspace Z of the complexification $\widetilde{X}$ we want to conclude that its projection

$$P(Z) := \{x \in X : x + iy \in Z \text{ for some } y \in X\}, \tag{4.1}$$

contains an infinite-dimensional closed subspace. Note that $P(Z)$, expressed in (4.1) as the *real part projection* of Z, coincides with the *imaginary part projection* of Z, namely

$$Q(Z) := \{y \in X : x + iy \in Z \text{ for some } x \in X\}.$$

Indeed, $P(Z) = Q(Z)$ because a vector $x + iy$ belongs to Z if and only if $i \cdot (x + iy) = -y + ix$ belongs to Z and also if and only if $(-i) \cdot (x + iy) = y - ix$ belongs to Z. Since $\widetilde{X}$ can be identified with the direct sum space $X \oplus X$ we will prove the following more general fact:

Lemma 6.1. *Let* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be (real or complex) Banach spaces and consider an infinite-dimensional closed subspace* $Z \subset X \oplus Y$ *of its direct sum. If we denote by*

$$P_X : X \oplus Y \longrightarrow X \qquad \textit{and} \qquad P_Y : X \oplus Y \longrightarrow Y,$$

the standard projections on the corresponding subspace, then at least one of the subspaces $P_X(Z)$ *or* $P_Y(Z)$ *contains an infinite-dimensional closed subspace.*

Proof. Suppose that $P_Y(Z) \subset Y$ does not admit any infinite-dimensional closed subspace:

Claim. *For every $\delta > 0$ and every infinite-dimensional closed subspace $V \subset Z$ there exist a vector $(x, y) \in V$ such that $\|x\|_X = 1$ and $\|y\|_Y < \delta$.*

Proof of the Claim. By the initial assumption on $P_Y(Z) \subset Y$ we just have two possibilities: the subspace $\overline{P_Y(V)}$ is closed (and hence finite-dimensional) or $P_Y(V)$ is not closed. In both cases $P_Y|_V : V \longrightarrow \overline{P_Y(V)}$ is not a Banach isomorphism so it is not bounded from below and considering the norm $\|(x, y)\| := \max\{\|x\|_X, \|y\|_Y\}$ in the space $X \oplus Y$ we get the *Claim*. □

Now one can modify the Mazur theorem (see [LT77, Vol I, Theorem 1.a.5 and Lemma 1.a.6]) and easily construct a basic sequence $(x_n, y_n)_{n\in\mathbb{N}} \subset Z$ with the properties:

(a) $\|x_n\|_X = 1$ and $\|y_n\|_Y < \frac{1}{2^n}$ for every $n \in \mathbb{N}$ (by using the *Claim*);

(b) $(x_n)_{n\in\mathbb{N}}$ is a basic sequence for $(X, \|\cdot\|_X)$ (adding to each step of [LT77, Vol I, Lemma 1.a.6] the corresponding functionals of the type $(x^*, 0) \in X^* \oplus Y^* = (X \oplus Y)^*$).

We claim that $P_X(Z)$ contains the subspace $\overline{\operatorname{span}\{x_n : n \in \mathbb{N}\}}$: given any convergent series $x = \sum_{n\in\mathbb{N}} a_n x_n$ we have that $a = (a_n)_{n\in\mathbb{N}} \in c_0(\mathbb{N})$ since $(x_n)_{n\in\mathbb{N}}$ is a normalized sequence, and hence $\sum_{n\in\mathbb{N}} a_n y_n$ is an absolutely convergent series to some vector $y \in Y$, i.e.

$$y = \sum_{n\in\mathbb{N}} a_n y_n \in Y, \quad \text{so} \quad (x, y) = \sum_{n\in\mathbb{N}} a_n (x_n, y_n) \in Z,$$

and finally $x = P_X(x, y) \in P_X(Z)$. □

Remark 6.2 (W. B. Johnson's Proof). Lemma 6.1 admits an equivalent shorter proof in terms of *strictly singular operators*: if $P_X(Z)$ and $P_Y(Z)$ do not contain any infinite-dimensional closed subspace, then the operators $P_X : Z \longrightarrow X \oplus Y$ and $P_Y : Z \longrightarrow X \oplus Y$ are strictly singular. By [LT77, Vol I, Theorem 2.c.5] we would have that $P_X + P_Y = I : Z \longrightarrow Z$, which is an isomorphism, has to be a strictly singular operator yielding a contradiction.

Remark 6.3. It follows from Lemma 6.1 that for any real Banach space X and any $T \in \mathcal{L}(X)$: *if there exists an infinite-dimensional closed subspace $Z \subset \widetilde{X}$ fulfilling any dynamical property (among those described in Theorems 2.5 or 3.3) with respect to the complexification $\widetilde{T}$, then $P(Z) \subset X$ as defined in (4.1) admits an infinite-dimensional closed subspace fulfilling the same dynamical property with respect to the real-linear operator T.*

6.1 Divergent orbits for real-linear operators

Once we know how to pass from a "*recurrent or hypercyclic subspace for $\widetilde{T}$*" to one for T, we need to study the converse implication. We do it by proving a real version of Proposition 5.8:

Proposition 6.4. *Let X be a real (and not necessarily separable) Banach space and $T \in \mathcal{L}(X)$. If $\sigma_e(\widetilde{T}) \cap \overline{\mathbb{D}} = \varnothing$, then every infinite-dimensional closed subspace $E \subset X$ admits a vector $x \in E$ such that $\lim_{n\to\infty} \|T^n x\| = \infty$.*

In order to prove Proposition 6.4 we will rewrite the proof of Proposition 5.8 but using the following real version of [BM09, Corollary 8.14]:

Lemma 6.5. *Let X be a real Banach space, and let E be an infinite-dimensional closed subspace of X. If $\mathcal{K} = (K_n : \widetilde{E} \longrightarrow Y_n)_{n\in\mathbb{N}}$ is a countable family of complex-linear compact operators, where each Y_n is a complex Banach space, then there exist a normalized basic sequence $(e_j)_{j\in\mathbb{N}}$ on E such that $(\widetilde{e_j})_{j\in\mathbb{N}} = (e_j + i0)_{j\in\mathbb{N}} \subset \widetilde{E}$ is a $\mathcal{K}$-null sequence.*

This is just the *complexification* version of the well-known (real and complex) Banach spaces results [BM09, Lemma 8.13 and Corollary 8.14]:

Proof of Lemma 6.5. We start by claiming that there exists a decreasing sequence $(E_j)_{j\in\mathbb{N}}$ of finite-codimensional closed subspaces of E such that $\|K_n|_{\widetilde{E_j}}\| \leq \frac{1}{2^j}$ whenever $n \leq j$:

Since K_1 is compact, the adjoint operator K_1^* is also compact, so one can find a finite number of functionals $z_1^*, ..., z_N^* \in \widetilde{E}^*$ such that

$$K_1^*(B_{Y_1^*}) \subset \bigcup_{1\leq k\leq N} B(z_k^*, \tfrac{1}{2}),$$

where $B_{Y_1^*}$ is the closed unit ball of Y_1^* and $B(z_k^*, \frac{1}{2})$ is the ball on $\widetilde{E}^*$ centred at each vector z_k^* and of radius $\frac{1}{2}$. Then we have that

$$\|K_1(z)\| = \sup\{|z^*(z)| : z^* \in K_1^*(B_{Y_1^*})\} \leq \max_{1\leq k\leq N} |z_k^*(z)| + \tfrac{1}{2}\|z\| \quad \text{for all } z \in \widetilde{E}. \tag{4.2}$$

By [MMFPSS22, Section 4.10] we can identify $(\widetilde{E})^*$ with $(\widetilde{E^*})$, and in particular, for each z_k^* there are $x_k^*, y_k^* \in E^*$ such that

$$z_k^*(x + iy) = [x_k^*(x) - y_k^*(y)] + i\,[y_k^*(x) + x_k^*(y)]\,,$$

for every $x + iy \in \widetilde{E}$ and $1 \leq k \leq N$. Hence we have the inclusion

$$\widetilde{\mathrm{Ker}(x_k^*) \cap \mathrm{Ker}(y_k^*)} \subset \mathrm{Ker}(z_k^*) \subset \widetilde{E} \quad \text{for every } 1 \leq k \leq N.$$

Therefore, by (4.2), the finite-codimensional closed subspace

$$E_1 := E \quad \cap \bigcap_{1\leq k\leq N} \mathrm{Ker}(x_k^*) \cap \mathrm{Ker}(y_k^*) \subset E, \text{ has the property } \|K_1|_{\widetilde{E_1}}\| < \tfrac{1}{2}.$$

Recursively, if we have $E_1, ..., E_j$ already constructed we obtain E_{j+1} in the same way: consider a finite covering of $\bigcup_{n=1}^{j+1} K_n^*(B_{Y_n^*})$ with balls of diameter lower than $\frac{1}{2^{j+1}}$ and intersect E_j with the kernels of the corresponding finite sequence of functionals.

The already mentioned Mazur theorem provides now a normalized basic sequence $(e_j)_{j\in\mathbb{N}}$ on E such that $e_j \in E_j$ for all $j \in \mathbb{N}$. This sequence $(e_j)_{j\in\mathbb{N}}$ has the required properties. □

Proof of Proposition 6.4. Assume that E is separable. Let $\mathcal{K}_0 \subset \mathcal{L}(\widetilde{X})$ and $\mathcal{K}_1 \subset \mathcal{L}(\widetilde{E}, \hat{Y})$ be the countable family of compact operators obtained from the results [BM09, Lemma 8.15 (b)] and [BM09, Lemma 8.17] respectively. Apply Lemma 6.5 to the countable family of compact operators $\mathcal{K} = \mathcal{K}_0 \cup \mathcal{K}_1$ obtaining a a normalized basic sequence $(e_j)_{j\in\mathbb{N}} \subset E$ such that its complexification $(\widetilde{e_j})_{j\in\mathbb{N}} = (e_j + i0)_{j\in\mathbb{N}} \subset \widetilde{E}$ is a $\mathcal{K}$-null sequence. Choose a summable sequence $(\alpha_n)_{n\in\mathbb{N}}$ for which $\alpha_n \lambda^n \to \infty$ and apply [BM09, Lemma 8.17] to $(\widetilde{e_j})_{j\in\mathbb{N}}$ obtaining a vector

$$z = x + iy \in \overline{\mathrm{span}_{\mathbb{R}}\{\widetilde{e_j} : j \in \mathbb{N}\}} \quad \text{for which} \quad \lim_{n\to\infty} \|\widetilde{T}^n z\| = \infty.$$

Note that $x \in E$, $y = 0$ and hence $\lim_{n\to\infty} \|T^n x\| = \lim_{n\to\infty} \|\widetilde{T}^n z\| = \infty$. □

We are now ready to prove Theorem 3.4.

6.2 Proof of Theorem 3.4

We show (a) and (b) at the same time following the proofs of Theorems 2.5 and 3.3: firstly, we observe that (ii) $\Rightarrow$ (iii) by the Banach-Steinhaus theorem.

To see (i) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (iv) note that if (i) or (iii) hold but $\sigma_e(\widetilde{T}) \cap \overline{\mathbb{D}} = \varnothing$ we arrive to a contradiction: the vector with divergent orbit obtained by Proposition 6.4 cannot be in a recurrent subspace, neither in the subspace described by statement (iii).

Finally, since T is weakly-mixing or quasi-rigid if and only if so is $\widetilde{T}$, we have that statement (iv) implies (i), (ii) and (iii) for the complexification operator $\widetilde{T}$ by Theorems 2.5 and 3.3. By Lemma 6.1 we deduce (i), (ii) and (iii) for T itself (see Remark 6.3). □

6.3 Comments on Theorem 3.4

Remark 6.6 (Non-Separability). As it happens in Theorems 3.2 and 3.3, the quasi-rigid part of Theorem 3.4 is still true for non-separable Banach spaces.

Remark 6.7 (Operators without common hypercyclic subspaces). It was first shown in [ABLSP05, Example 2.1] that there are two operators admitting hypercyclic subspaces which do not have a common hypercyclic subspace. The example was extended in [BM09, Exercise 8.2]. In particular, given any $\lambda \in \mathbb{C} \setminus \overline{\mathbb{D}}$:

– *Let $T \in \mathcal{L}(\ell^2(\mathbb{N}))$ be a weakly-mixing operator, on the complex space $\ell^2(\mathbb{N})$, with a hypercyclic subspace. Then*

$$T_1 := T \oplus \lambda B : \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N}) \longrightarrow \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N}),$$
$$T_2 := \lambda B \oplus T : \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N}) \longrightarrow \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N}),$$

are weakly-mixing operators with a hypercyclic subspace, but they do not share any common hypercyclic subspace (where $B : \ell^2(\mathbb{N}) \longrightarrow \ell^2(\mathbb{N})$ is the known backward shift).

The proof is based on the following four facts:

(1) T_1 and T_2 are weakly-mixing because λB is mixing (it is the Rolewicz operator);

(2) $\sigma_e(T_1) \cap \overline{\mathbb{D}} \neq \varnothing$ and $\sigma_e(T_2) \cap \overline{\mathbb{D}} \neq \varnothing$ because $\sigma_e(T) \cap \overline{\mathbb{D}} \neq \varnothing$;

(3) for a Hilbert space H there is a simple proof, using orthogonality, of the fact that: *for any infinite-dimensional closed subspace $Z \subset H \oplus H$, at least one of the projections $P_1(Z)$ or $P_2(Z)$ admits an infinite-dimensional closed subspace* (see [ABLSP05, Example 2.1]);

(4) the Rolewicz operator λB has no hypercyclic subspace (see Remark 5.9).

By Theorem 3.4 and Lemma 6.1 we get that, for any (real or complex) number $\lambda \in \mathbb{K} \setminus \overline{\mathbb{D}}$:

– *Let $T \in \mathcal{L}(X)$ be a weakly-mixing operator with a hypercyclic subspace on a (real or complex) Banach space X. Let Y be the (real or complex) $\ell^p(\mathbb{N})$ $(1 \leq p < \infty)$ or $c_0(\mathbb{N})$ space and denote by $B : Y \longrightarrow Y$ the backward shift on Y. Then*

$$T_1 := T \oplus \lambda B : X \oplus Y \longrightarrow X \oplus Y \quad \text{and} \quad T_2 := \lambda B \oplus T : Y \oplus X \longrightarrow Y \oplus X,$$

are weakly-mixing operators with a hypercyclic subspace, but they do not share any common hypercyclic subspace.

7 Further results, applications and open problems

One of the objectives of the *hypercyclic spaceability theory* is establishing sufficient conditions for an operator to admit a hypercyclic subspace. This is the case of Theorem 2.3 which can be easily reproved with the theory developed here: for any operator $S = T - K \in \mathcal{L}(X)$ with $\|S\| \leq 1$ we have that its essential spectrum (which is a subset of the spectrum of S) is included in the closed unit disk, so any weakly-mixing compact perturbation $T = S + K$ of S admits a hypercyclic subspace (use Theorem 3.4 for the real case). We can also recover the following well-known result (see [GEP11, Corollary 10.11]):

Corollary 7.1. *Let $T \in \mathcal{L}(X)$ be a weakly-mixing (resp. quasi-rigid) operator acting on complex separable Banach space X. If $\operatorname{Ker}(T - \lambda)$ is infinite-dimensional for some $\lambda \in \overline{\mathbb{D}}$, then T has a hypercyclic (resp. recurrent) subspace.*

Proof. In that case $\lambda \in \sigma_e(T) \cap \overline{\mathbb{D}}$ and Theorem 3.3 applies. □

Nonetheless, Theorem 2.3 and the previous corollary are somehow restrictive: the first needs that the perturbed operator has norm bounded by 1, and the second requires the existence of plenty of eigenvectors associated to the same eigenvalue. We propose the following alternative "sufficient condition" in order to obtain the existence of hypercyclic subspaces:

Corollary 7.2. *Let X be a (real or complex) Banach space and let $T, S, K \in \mathcal{L}(X)$ where the operator T is weakly-mixing, S is quasi-rigid, K is compact and $T = S + K$. Then T has a hypercyclic subspace if and only if S has a recurrent subspace.*

Proof. Theorems 2.5 and 3.3 yield the result in the complex case since $\sigma_e(T) = \sigma_e(S)$. Use Theorem 3.4 for the real case. □

The idea behind Corollary 7.2 is to apply it to *C-type operators*.

7.1 Application to C-type operators

In the last years lots of dynamical properties have been distinguished in Linear Dynamics by the so-called *C-type operators*. Using them one can construct operators which are (Devaney) *chaotic* but not *$\mathcal{U}$-frequently hypercyclic*, see [Men17]; *chaotic* and *frequently hypercyclic* but do not admit an *ergodic probability measure with full support*; *chaotic* and *$\mathcal{U}$-frequently hypercyclic* but not *frequently hypercyclic*; *chaotic* and *mixing* but not *$\mathcal{U}$-frequently hypercyclic*, see [GMM21]; *invertible* and *frequently hypercyclic* (resp. *$\mathcal{U}$-frequently hypercyclic*) but whose inverse is not *frequently hypercyclic* (resp. *$\mathcal{U}$-frequently hypercyclic*), see [Men20, Men22].

In this section we show that every *C-type* operator as defined in [Men17, GMM21] admits a hypercyclic subspace. We do it in two different ways:

- first by using the essential spectrum and applying Theorem 3.3;

- and secondly by constructing explicitly a sequence of subspaces to apply Theorem 3.2.

However, in both cases we use Corollary 7.2 to simplify the problem.

As defined in [GMM21], the *C-type operators* are chaotic (and hence weakly-mixing) compact perturbations of operators with a dense set of periodic points (and hence quasi-rigid). More precisely, each *C-type operator* is associated to four parameters v, w, φ and b, where:

- $w = (w_j)_{j\in\mathbb{N}}$ is a sequence of complex numbers which is both bounded and bounded from below, i.e. $0 < \inf_{j\in\mathbb{N}} |w_j| \leq \sup_{j\in\mathbb{N}} |w_j| < \infty$;
- $\varphi : \mathbb{N}_0 \longrightarrow \mathbb{N}_0$ is a map such that $\varphi(0) = 0$, $\varphi(n) < n$ for every $n \in \mathbb{N}_0$, and the set $\varphi^{-1}(l) = \{n \in \mathbb{N}_0 : \varphi(n) = l\}$ is infinite for every $l \in \mathbb{N}_0$;
- $b = (b_n)_{n\in\mathbb{N}_0}$ is a strictly increasing sequence of positive integers such that $b_0 = 0$ and $b_{n+1} - b_n$ is a multiple of $2(b_{\varphi(n)+1} - b_{\varphi(n)})$ for every $n \in \mathbb{N}$;
- $v = (v_n)_{n\in\mathbb{N}}$ is a sequence of non-zero complex numbers such that $\sum_{n\in\mathbb{N}} |v_n| < \infty$.

Definition 7.3 ([GMM21])**.** Let $(e_k)_{k\in\mathbb{N}_0}$ be the canonical basis of the linear space

$$c_{00}(\mathbb{N}_0) := \left\{ (x_j)_{j\in\mathbb{N}_0} \in \mathbb{C}^{\mathbb{N}_0} : \text{ there exists } j_0 \in \mathbb{N} \text{ with } x_j = 0 \text{ for all } j \geq j_0 \right\},$$

i.e. $e_k = (\delta_{k,j})_{j\in\mathbb{N}_0}$. The linear map $T_{w,b}$ on $c_{00}(\mathbb{N}_0)$ associated to the data w and b is defined by:

$$T_{w,b}\ e_k := \begin{cases} w_{k+1} e_{k+1}, & \text{if } k \in [b_n, b_{n+1} - 1[, \\ \\ -\left(\prod_{j=b_n+1}^{b_{n+1}-1} w_j\right)^{-1} e_{b_n}, & \text{if } k = b_{n+1} - 1, \text{ for } n \geq 0. \end{cases}$$

Moreover, the linear map $T_{w,\varphi,b,v}$ on $c_{00}(\mathbb{N}_0)$ associated to the data w, φ, b and v given as above is defined by:

$$T_{w,\varphi,b,v}\ e_k := \begin{cases} w_{k+1} e_{k+1}, & \text{if } k \in [b_n, b_{n+1} - 1[, n \geq 0, \\ \\ v_n e_{b_{\varphi(n)}} - \left(\prod_{j=b_n+1}^{b_{n+1}-1} w_j\right)^{-1} e_{b_n}, & \text{if } k = b_{n+1} - 1, n \geq 1, \\ \\ -\left(\prod_{j=b_0+1}^{b_1-1} w_j\right)^{-1} e_0, & \text{if } k = b_1 - 1. \end{cases}$$

When they extend continuously to an $\ell^p(\mathbb{N}_0)$ space (with $1 \leq p < \infty$) the resulting continuous operators are still denoted by $T_{w,b}$ and $T_{w,\varphi,b,v}$ respectively, and this last operator $T_{w,\varphi,b,v}$ is called the *operator of C-type* on $\ell^p(\mathbb{N}_0)$ associated to the data w, φ, b and v.

As showed in [GMM21, Lemma 6.2 and Proposition 6.5] we have that $T_{w,\varphi,b,v} = T_{w,b} + K_{\varphi,v}$ where

$$K_{\varphi,v}\ x := \sum_{n\in\mathbb{N}} v_n x_{b_{n+1}-1} \cdot e_{b_{\varphi(n)}} \qquad \text{for each } x = (x_j)_{j\in\mathbb{N}_0} \in \ell^p(\mathbb{N}_0)$$

is a compact operator. The *continuity* of $T_{w,b}$ (and hence that of $T_{w,\varphi,b,v}$) depends on the following condition

$$\inf_{n\in\mathbb{N}_0} \prod_{j=b_n+1}^{b_{n+1}-1} |w_j| > 0,$$

and the *chaotic* behaviour of $T_{w,\varphi,b,v}$ can be deduced whenever the following holds:

$$\limsup_{\substack{N\to\infty\\ N\in\varphi^{-1}(n)}} |v_N| \prod_{j=b_N+1}^{b_{N+1}-1} |w_j| = \infty \quad \text{for every } n \geq 0. \tag{4.3}$$

Since $T_{w,b}$ has a dense set of periodic points it is quasi-rigid (see [GLMP, Proposition 2.9]), so that Corollary 7.2 can be applied to every *C-type operator* $T_{w,\varphi,b,v}$ defined as above, that is, studying if it admits a hypercyclic subspace is equivalent to study if the respective operator $T_{w,b}$ (which has much easier dynamics than $T_{w,\varphi,b,v}$) admits a recurrent subspace. As we advanced, we will show that $T_{w,b}$ always has a recurrent subspace in two different ways:

First Option ($T_{w,b}$ has a recurrent subspace): via the essential spectrum. In view of Theorem 3.3 and Corollary 5.3, the operator $T_{w,b}$ has a recurrent subspace if and only if there exists some $\lambda \in \overline{\mathbb{D}}$ fulfilling at least one of the following properties:

(a) $\operatorname{Ker}(T_{w,b} - \lambda)$ is infinite-dimensional;

(b) $\operatorname{Ran}(T_{w,b} - \lambda)$ is not closed.

With respect to (a), it can be checked that every element in the *point spectrum* of $T_{w,b}$ has to be an appropriated root of the unity, so in particular $\sigma_p(T_{w,b}) \subset \mathbb{T} \subset \overline{\mathbb{D}}$. Moreover, it is not hard to check that:

– *The subspace* $\operatorname{Ker}(T_{w,b} - \lambda)$ *is infinite-dimensional for some* $\lambda \in \mathbb{T}$ ***if and only if*** *there are infinitely many blocks* $[b_n, b_{n+1}[$ *for which* $\lambda^{b_{n+1}-b_n} = -1$.

This is a problem for our proposes since the condition that $b_{n+1} - b_n$ has to be a multiple of $2(b_{\varphi(n)+1} - b_{\varphi(n)})$, which is an assumption on the parameter $b = (b_n)_{n\in\mathbb{N}_0}$, fights against having infinitely many blocks with an appropriated length for λ: the equality $\lambda^{b_{\varphi(n)+1}-b_{\varphi(n)}} = -1$ implies that $\lambda^{b_{n+1}-b_n} = 1$. Indeed, for the most interesting examples exhibited in [Men17, GMM21], there is no $\lambda \in \mathbb{T}$ with $\operatorname{Ker}(T_{w,b} - \lambda)$ being infinite-dimensional.

Regarding property (b), we can show that $\operatorname{Ran}(T_{w,b}) = \operatorname{Ran}(T_{w,b} - 0)$ is not closed whenever the operator fulfills (4.3), and hence that $0 \in \sigma_e(T_{w,b}) \cap \overline{\mathbb{D}}$. We first claim that $T_{w,b}$ is not invertible: otherwise $T_{w,b}^{-1}$ would act on $c_{00}(\mathbb{N}_0)$ as:

$$T_{w,b}^{-1}\, e_k := \begin{cases} \frac{1}{w_{k-1}} e_{k-1}, & \text{if } k \in]b_n, b_{n+1}-1], \\[2ex] -\left(\prod_{j=b_n+1}^{b_{n+1}-1} w_j\right) e_{b_{n+1}-1}, & \text{if } k = b_n, \text{ for } n \geq 0. \end{cases}$$

However, since $T_{w,b}$ is assumed to fulfill (4.3) and $(v_N)_{N\in\mathbb{N}}$ is a summable sequence we get that

$$\sup_{n\in\mathbb{N}_0} \prod_{j=b_n+1}^{b_{n+1}-1} |w_j| = \infty, \tag{4.4}$$

which implies that $T_{w,b}^{-1} : c_{00}(\mathbb{N}_0) \longrightarrow c_{00}(\mathbb{N}_0)$ does not extend continuously to $\ell^p(\mathbb{N}_0)$. Since $T_{w,b}$ is quasi-rigid it has dense range. Moreover, $T_{w,b}$ is one-to-one so the open mapping theorem implies that $T_{w,b}$ is not surjective (otherwise it would be invertible). We deduce that $\operatorname{Ran}(T_{w,b})$ is not closed, that $0 \in \sigma_e(T_{w,b}) \cap \overline{\mathbb{D}}$ and that $T_{w,b}$ has a *recurrent subspace*. By Corollary 7.2 the C-type operator $T_{w,\varphi,b,v}$ has a *hypercyclic subspace*.

The previous argument is a particular case of the following result, which was already used in [LSMR01] for the particular case of the *bilateral weighted shifts* (the spectrum of such operators is either an annulus or a disk, depending on whether T is invertible or not):

Corollary 7.4. *Let X be a (real or complex) separable Banach space and suppose that $T \in \mathcal{L}(X)$ is a weakly-mixing (resp. quasi-rigid) operator. If T is one-to-one but not invertible, then T has a hypercyclic (resp. recurrent) subspace.*

Proof. We have that T is a recurrent operator (weak-mixing implies quasi-rigidity, which implies recurrence in its turn) so T has dense range. Then $\mathrm{Ran}(T)$ is not closed (otherwise T would be invertible by the open mapping theorem and the one-to-one assumption). The previous implies that $0 \in \sigma_e(T) \cap \overline{\mathbb{D}}$ (and $0 \in \sigma_e(\tilde{T}) \cap \overline{\mathbb{D}}$ for the real case). □

Second Option ($T_{w,b}$ has a recurrent subspace): by finding explicit subspaces. In view of Theorem 3.2 and since $T_{w,b}$ is quasi-rigid with respect to some increasing sequence $(k_n)_{n\in\mathbb{N}}$, it is enough to find a decreasing sequence $(E_n)_{n\in\mathbb{N}}$ of infinite-dimensional and closed subspaces of $\ell^p(\mathbb{N}_0)$ for which

$$\sup_{n\in\mathbb{N}} \left\| T_{w,b}^{k_n}|_{E_n} \right\| < \infty. \tag{4.5}$$

We use (4.4), which again comes from (4.3) and the fact that $(v_N)_{N\in\mathbb{N}}$ is a summable sequence (and hence convergent to 0). Fix $M \geq \sup_{j\in\mathbb{N}} |w_j| > 0$ and construct recursively a strictly increasing sequence of natural numbers $(l_n)_{n\in\mathbb{N}}$ such that

$$(k_n - 1) < (b_{l_n+1} - b_{l_n}) \quad \text{ and } \quad M^{k_n-1} \leq \prod_{j=b_{l_n}+1}^{b_{l_n+1}-1} |w_j| \quad \text{ for all } n \in \mathbb{N}.$$

This election can be done since we can always choose a big enough $l_n \in \mathbb{N}$ by (4.4). Now, for each $n \in \mathbb{N}$ consider the subspace

$$E_n := \overline{\mathrm{span}\{e_{b_{l_m+1}-1} : m \geq n\}},$$

and note that the sequence $(E_n)_{n\in\mathbb{N}}$ is a decreasing family of infinite-dimensional and closed subspaces of $\ell^p(\mathbb{N}_0)$. Hence, fixed $n \in \mathbb{N}$ and given $x = \sum_{m\geq n} a_m \cdot e_{b_{l_m+1}-1} \in E_n$ we have that

$$T_{w,b}^{k_n}(x) = \sum_{m\geq n} a_m \cdot \frac{\prod_{j=b_{l_m}+1}^{b_{l_m}+k_n-1} w_j}{\prod_{j=b_{l_m}+1}^{b_{l_m+1}-1} w_j} \cdot e_{b_{l_m}+k_n-1},$$

which implies that

$$\left\| T_{w,b}^{k_n}(x) \right\|_p^p \leq \sum_{m\geq n} \left(|a_m| \cdot \frac{M^{k_n-1}}{\prod_{j=b_{l_m}+1}^{b_{l_m+1}-1} |w_j|} \right)^p \leq \sum_{m\geq n} |a_m|^p = \|x\|_p^p,$$

so $\|T_{w,b}^{k_n}|_{E_n}\| \leq 1$ and (4.5) holds. The constructive proof of Theorem 3.2 yields now to the existence of a *recurrent subspace* for the operator $T_{w,b}$. Finally, and again by Corollary 7.2, we deduce that the C-type operator $T_{w,\varphi,b,v}$ has a *hypercyclic subspace*.

Example 7.5. The previous arguments, together with the work and examples exhibited in [Men17, GMM21], allow us to construct (Devaney) *chaotic* operators on every $\ell^p(\mathbb{N}_0)$-space having a *hypercyclic subspace*, which (among others) can be chosen to be:

– not $\mathcal{U}$*-frequently hypercyclic*, see [Men17];

– *frequently hypercyclic* but not *ergodic*, see [GMM21, Examples 7.7 and 7.19];

– $\mathcal{U}$*-frequently hypercyclic* but not *frequently hypercyclic*, see [GMM21, Example 7.11];

– *topologically mixing* but not $\mathcal{U}$*-frequently hypercyclic*, see [GMM21, Example 7.16].

Example 7.5 exhibits how the theory developed in this paper can be used to simplify the study of the dynamics for certain weakly-mixing operators. In particular, there could be plenty of interesting weakly-mixing operators whose spaceability may be studied via Corollary 7.2.

7.2 Open problems

The following open problems could be interesting:

Question 7.6. Can Theorem 3.2 be proved via the K. Chan approach (see [Cha99, CT01])?

There also exists a *hypercyclic spaceability theory* for operators acting on *Fréchet spaces*. However, for Fréchet spaces we loose the tool of the essential spectrum, so that this theory is much weaker than the one presented here for Banach spaces. Indeed, and as far as we know, there does not exist a general characterization of the weakly-mixing operators acting on Fréchet spaces that admit a hypercyclic subspace, even though some sufficient conditions about the existence of hypercyclic subspaces, such as Theorem 2.2, are still true in the Fréchet setting (see [GEP11, Chapter 10, Section 10.5]). In view of that we propose the following problem:

Question 7.7. Is (a proper variation of) Theorem 3.2 still true for Fréchet spaces?

When we use the term "*proper variation*" we refer to using, in the Fréchet setting, the proper notion of "*equicontinuity*" instead of the original "*equiboundedness*" used in this paper. In particular, we really believe that the works [Men11, Men13, Men14] have developed a sufficiently powerful *theory of basic sequences*, on Fréchet spaces with a continuous norm, in order to extend the proof of Theorem 3.2 to this more general class of spaces.

Note also that an affirmative answer for Question 7.6 could lead to a positive solution for Question 7.7, simpler than the constructive proof showed in Section 4, by using the techniques related to left-multiplication operators (acting on the algebra of linear operators) and tensor products (see [BMGP04, BGE12] for the hypercyclic and frequently hypercyclic cases).

The following and last problem seems to be more intriguing:

Question 7.8. Is Corollary 3.5 still true for operators acting on Fréchet spaces?

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General discussion of the results

This chapter presents two independent sections. First we discuss the nature of the different results achieved, and then we include some remarks and further results related to each of the chapters/articles forming this memoir.

1 Nature of the results

This brief section is more philosophical than mathematical in character since we will focus on the spirit of the results obtained more than on the results themselves. Our main idea has been to compare *recurrence* with the central Linear Dynamics notion of *hypercyclicity*.

Along the work we can find three different kind of results:

(1) **Results following the existing general evolution of hypercyclicity**. After the 2014 Costakis, Manoussos and Parissis paper [30], which presents the basis of linear recurrence, the natural evolution of the theory seemed to be adding the "frequency of visits" point of view that appeared in hypercyclicity with the works of Bayart and Grivaux [6] in 2006, Shkarin [84] in 2009, and Bès, Menet, Peris and Puig [16] in 2016 with the concepts of frequent, $\mathcal{U}$-frequent and reiterative hypercyclicity.

Article [21] (i.e. Chapter 1) provides a first approach in this direction by defining the notions of frequent, $\mathcal{U}$-frequent and reiterative recurrence, and exhibiting a symmetry between the theory of "frequent recurrence" and the last advances in hypercyclicity.

This train of thought is mainly concentrated in Chapter 1, Sections 2, 4, 5 and 8, but also in Chapter 3, Sections 4 and 5. We are referring to all those results where the *$\mathcal{F}$-hypercyclicity* is decomposed in *$\mathcal{F}$-recurrence + hypercyclicity*, which usually allows us to prove strong theorems for $\mathcal{F}$-hypercyclicity by proving them first for "usual hypercyclicity" and then for $\mathcal{F}$-recurrence (see for instance the Ansari-León-Müller-type theorems).

We also include in this category the results that exhibit a symmetry between the notions of "usual" hypercyclicity and recurrence: the *$T \oplus T$-recurrence problem* presents a negative answer as it happens for the *$T \oplus T$-hypercyclicity problem* (see Section 3 of Chapter 3); and the conditions used to show the existence of recurrent subspaces are very similar to those already developed in the hypercyclicity literature to obtain hypercyclic subspaces (see Chapter 4 and compare Theorem 2.2 with Theorem 3.2 stated there).

This last pair of "usual recurrence" results appear naturally when one is able to define properly the notion of *quasi-rigidity*, that is, the analogous property for recurrence, to that of *weak-mixing/satisfying the Hypercyclicity Criterion* for transitivity/hypercyclicity.

(2) **Results linking hypercyclicity and recurrence in a deep way**. In this category we include the situations in which hypercyclicity and recurrence behave exactly in the same manner. This happens when some restrictions are assumed on the underlying space, on the operator studied or on the Furstenberg family used:

- *$\mathcal{F}$-hypercyclicity is equivalent to $\mathcal{F}$-recurrence*, when $\mathcal{F}$ is an u.f.i. upper Furstenberg family and there exists a dense set of 0-convergent orbits, see Theorems 2.12 and 8.5 stated in Chapter 1, but also its consequences in Theorem 5.6 included in Chapter 2 or Corollary 5.16 included in Chapter 3. These results apply for every unilateral backward shift, which are a very important type of operators in Linear Dynamics.

- *Having a hypercyclic subspace is equivalent to having a recurrent subspace*, at least when the operator is weakly-mixing and acts on a Banach space, see Theorems 2.5, 3.3, 3.4 and Corollary 3.5 stated in Chapter 4. These results show that, for some properties such as *spaceability*, the hypercyclic behaviour of an operator can be fully studied by looking only at its recurrence characteristics.

(3) **Results strictly related to recurrence**. This are the type of results that do not have sense for, are not related to, or differ widely from what happens for hypercyclicity:

- There exist, and we can study, power-bounded and recurrent operators (see Chapter 1, Section 3; and Chapter 2, Theorem 1.9), but linear recurrence can be also considered on finite-dimensional spaces (see Chapter 1, Section 7). It is well-known that hypercyclicity is an infinite-dimensional phenomenon requiring the existence of unbounded orbits.

- We can study $\mathcal{F}$-recurrence for Furstenberg families $\mathcal{F}$ that do not admit $\mathcal{F}$-hypercyclic operators as it happens for the Δ^* and $\mathcal{IP}^*$ families, and for the uniform recurrence notion (see Chapter 1, Sections 6, 7; and Chapter 2, Section 4). This is also the case of periodicity, which can be characterized in terms of families (see Chapter 3, Section 4).

- *The notions of frequent and reiterative recurrence coincide*, for adjoint operators on dual Banach spaces (see Theorem 1.3 stated in Chapter 2). Also, *having a spanning set of unimodular eigenvectors is equivalent to uniform recurrence*, on Hilbert spaces and for power-bounded operators on reflexive spaces (see Chapter 2, Theorems 1.7 and 1.9). These results appear when we are able to use Ergodic Theory, and they are included in this category because the respective hypercyclicity notions are not equivalent even in the dual/reflexive/Hilbertian setting (see [73, 53, 74]), although some implications for the respective $\mathcal{F}$-hypercyclicity notions can be obtained (see Chapter 2, Sections 5 and 6).

- The *$T \oplus T$-recurrence problem* has a negative answer as the *$T \oplus T$-hypercyclicity problem*, but a huge difference between recurrence and hypercyclicity appears when we assume that the operator $T \oplus T$ is recurrent/hypercyclic and we look at the recurrence/hypercyclicity property of the N-fold direct sum operator $T_{(N)}$ for $N \geq 3$. In this sense Theorem 3.2 stated in Chapter 3 shows a behaviour for recurrence, which differs in a strong way to what happens for hypercyclicity.

The results exhibited along the work can be classified using these three categories, showing how other dynamical properties may interact with hypercyclicity in very different ways.

2 Remarks and further results

Before focusing on each chapter let us dedicate a few words to the *separability* and *linearity* assumptions: from the beginning of this work we have assumed that the operator under study $T \in \mathcal{L}(X)$ was acting on a *separable* F-space X (see page 2), but recurrence can be also considered when X is non-separable (the identity map is recurrent). However, one of the main ideas of this work was to connect recurrence with hypercyclicity, and since this last property needs separability we preferred to assume it in order to avoid unnecessary difficulties. Along the chapters we have briefly pointed out when the *separability* assumption can be dropped, see Chapter 2, Remarks 3.4 and 4.6, but also Chapter 4, Remarks 4.1, 5.7 and 6.6. This usually happens for the type of **results** that are **strictly related to recurrence**.

Historically, the underlying spaces considered in dynamics have been separable. This is the case of Linear Dynamics because of hypercyclicity (even though there are at least two remarkable exceptions, see [15] and [71]), but also for classical non-linear systems since the most usual set up is that of *continuous maps* acting on *compact metrizable spaces*, which are separable. Many references framed on the topic of *compact dynamical systems* have been used here (see [1, 4, 27, 38, 41, 45, 48, 61, 65]). Among them we must highlight the 1981 Furstenberg's book [41] entitled "*Recurrence in Ergodic Theory and Combinatorial Number Theory*", which can be considered the theoretical basis of this thesis.

This last comment links to our second remark: the necessity of *linearity*. It is clear that some of the results and ideas used in this work are only valid in the linear setting. This is the case when we consider power-bounded operators, unimodular eigenvectors, or the (dense) lineability and spaceability properties. However, the rest of the results do still work in the broader context of *Polish dynamical systems*, that is, when $T : X \longrightarrow X$ is a continuous map acting on a separable completely metrizable space X (see Chapter 2, Section 2, 5 and 6 but also Chapter 3, Sections 2 and 4). For better readability, and exactly as we have done in the very first **Introduction** chapter, we will mainly treat and focus on Linear Dynamics in both the **General discussion of the results** and the final **Conclusions** chapters.

Let us now include some remarks and further results:

2.1 On Chapter 1

The original version of article [21] (i.e. Chapter 1) did not include the notion of $\mathcal{AP}$*-recurrence*. Indeed, the point of the paper was to observe, for each operator $T \in \mathcal{L}(X)$ and each set $A \subset \mathbb{N}_0$, the following nice symmetry:

$$\operatorname{Per}(T) \subset \operatorname{URec}(T) \subset \operatorname{FRec}(T) \subset \operatorname{UFRec}(T) \subset \operatorname{RRec}(T) \subset \operatorname{Rec}(T),$$

$$0 \ \leq \ \underline{\operatorname{Bd}}(A) \ \leq \ \underline{\operatorname{dens}}(A) \ \leq \ \overline{\operatorname{dens}}(A) \ \leq \ \overline{\operatorname{Bd}}(A) \ \leq \ 1.$$

The syndetic sets that define uniform recurrence are exactly those sets with positive lower Banach density[A], the sets with positive lower and upper density define frequent and $\mathcal{U}$-frequent recurrence, while the sets of positive upper Banach density define reiterative recurrence.

[A] See Proposition 2.10 in Section 2 of the Appendix.

The referee of article [21] suggested the inclusion of *$\mathcal{AP}$-recurrence* recently studied in [24], which is equivalent to the notion of (topological) *multiple recurrence* introduced for linear operators by Costakis and Parissis in the 2012 paper [31]. We finally incorporated this notion in the Introduction of article [21] but, in order to maintain a good readability, we did not elaborate further on it. Recall that $\mathcal{AP}$ is the Furstenberg family formed by sets containing arbitrarily long arithmetic progressions, and that $\mathcal{AP}\mathrm{Rec}(T)$ denotes the set of $\mathcal{AP}$-recurrent vectors for an operator $T \in \mathcal{L}(X)$. We have the following inclusions:

$$\mathrm{RRec}(T) \subset \mathcal{AP}\mathrm{Rec}(T) \subset \mathrm{Rec}(T).$$

We can show that these inclusions are strict, in a rather strong sense, obtaining an analogous result to that of Theorem 5.7 stated in Chapter 1:

Proposition 2.1. *Usual, $\mathcal{AP}$ and reiterative recurrence can be strongly distinguished:*

(a) *There is a hypercyclic operator $T \in \mathcal{L}(X)$ for which $\mathcal{AP}\mathrm{Rec}(T)$ is nowhere dense.*

(b) *There is an $\mathcal{AP}$-hypercyclic operator $T \in \mathcal{L}(X)$ for which $\mathrm{RRec}(T)$ is nowhere dense.*

Proof. (a): By [31, Proposition 5.8] there exists a hypercyclic (even weakly-mixing) bilateral weighted shift $T \in \mathcal{L}(\ell^2(\mathbb{Z}))$, which is not topologically multiply recurrent. If the set $\mathcal{AP}\mathrm{Rec}(T)$ was somewhere dense, then a similar argument to that used in [51, Proposition 5.10] would show that T is $\mathcal{AP}$-recurrent arriving to a contradiction.

(b): By using [8] and [9], in [24, Section 4] it is shown the existence of an $\mathcal{AP}$-hypercyclic operator $T \in \mathcal{L}(\ell^1(\mathbb{N}))$, which is not weakly-mixing. If $\mathrm{RRec}(T)$ was somewhere dense, again by a similar argument to that used in [51, Proposition 5.10] we would obtain that T is reiteratively recurrent and then reiteratively hypercyclic by [21, Theorem 2.1]. This yields a contradiction since every reiteratively hypercyclic operator is weakly-mixing by [16, Proposition 4]. □

An analogous result to that of Corollary 5.8 stated in Chapter 1 follows:

Corollary 2.2. *Usual, $\mathcal{AP}$ and reiterative recurrence can be distinguished:*

(a) *There is a recurrent operator $T \in \mathcal{L}(X)$ for which $\mathcal{AP}\mathrm{Rec}(T)$ is nowhere dense.*

(b) *There is an $\mathcal{AP}$-recurrent operator $T \in \mathcal{L}(X)$ for which $\mathrm{RRec}(T)$ is nowhere dense.*

These results are based on the interchange of ideas between the chapters/articles forming this memoir and the recent investigations done by Rodrigo Cardeccia and Santiago Muro in the works [23, 24, 25], which have significantly contributed to the development of $\mathcal{F}$-hypercyclicity and $\mathcal{F}$-recurrence. A good example of this fact is the following result from [25], which follows from the negative answer obtained there to Question 4.10 stated in Chapter 1:

Cardeccia and Muro, [25, Corollary 3.7]: *There are separable infinite-dimensional Banach spaces without reiteratively hypercyclic operators.*

The approach used in [25] considers recurrence and Furstenberg families via the so-called *$\mathcal{P}_{\mathcal{F}}$ property*, which we define minutely in the following section.

It is also shown in [25] that a general $\mathcal{F}$-hypercyclicity version of Theorem 2.1 stated in Chapter 1 holds (see [25, Proposition 2.8]). Indeed, the same six equivalences are still true for u.f.i. upper *block families*. Given a Furstenberg family $\mathcal{F} \subset \mathscr{P}(\mathbb{N}_0)$:

– we define its *block family* $b\mathcal{F}$ as the collection of sets $A \subset \mathbb{N}_0$ for which there exists $B \in \mathcal{F}$ fulfilling that for every finite subset $F \subset B$ there is $n \in \mathbb{N}_0$ such that $F + n \subset A$.

– we say that $\mathcal{F}$ is a *block family* whenever $b\mathcal{F} = \mathcal{F}$.

Among other relations, it is checked in [25] that $b\overline{\mathcal{D}} = b\underline{\mathcal{D}} = b\overline{\mathcal{BD}} = \overline{\mathcal{BD}}$, so that $\overline{\mathcal{BD}}$ is a block family[B]. It is also clear that $\mathcal{AP}$ is a block family. These type of Furstenberg families have been also used in previous works such as [45], [59] and [68], and they enjoy the following nice properties: first, *an operator $T \in \mathcal{L}(X)$ is $\mathcal{F}$-hypercyclic if and only if it is $\mathcal{F}$-recurrent and hypercyclic*; and *once T is $\mathcal{F}$-hypercyclic, then every hypercyclic vector is $\mathcal{F}$-hypercyclic.*

General Furstenberg families have been also used in the context of *topological transitivity.* Indeed, if for every pair U, V of non-empty open subsets of X one demands that the *return set*

$$N_T(U, V) := \{n \in \mathbb{N}_0 : T^n(U) \cap V \neq \varnothing\}$$

belongs to $\mathcal{F}$, then the concept of *topological $\mathcal{F}$-transitivity* appears. This has been deeply studied in [17] and concepts such as *mixing*, *weak-mixing* and *topological ergodicity* can be seen as *topological $\mathcal{I}^*$-transitivity*, *$\mathcal{T}$-transitivity* and *$\mathcal{TS}$-transitivity*[C]. It might be interesting to study the concept of *topological $\mathcal{F}$-recurrence*: require that $N_T(U, U)$ belongs to $\mathcal{F}$ for every non-empty open subset U of X. As far as we know this is an unexplored field, but it can be used to characterize *topological quasi-rigidity* in terms of filters (see Section 2.3 of this chapter).

See Section 2.1 of the *Conclusions* for more on Chapter 1.

2.2 On Chapter 2

In article [50] (i.e. Chapter 2) the results exhibit some strong equivalences between (a priori) different recurrence notions, in the context of adjoint operators acting on separable dual Banach spaces. These results are based on the w^*-w^*-continuity of the adjoint operators, and under this hypothesis Cardeccia and Muro have been able to characterize the *linear chaos* in terms of $\mathcal{F}$-hypercyclicity (see [23]).

Another notion that they have studied minutely in [25] and deeply related with $\mathcal{F}$-recurrence is the *$\mathcal{P}_{\mathcal{F}}$ property*, which was formally defined in 2018 by Puig [81], but previously considered for particular Furstenberg families by several authors such as Costakis and Parissis [31], Badea and Grivaux [3, Proposition 4.6] and Grivaux and Matheron [52, Section 2.5]:

Definition 2.3. Let $\mathcal{F}$ be a Furstenberg family. We say that $T \in \mathcal{L}(X)$ has the *$\mathcal{P}_{\mathcal{F}}$ property* if for every non-empty open subset U of X there exists $x_U \in U$ such that $N_T(x_U, U) \in \mathcal{F}$.

Note that, if $\mathcal{F}$ is a left-invariant family, then $T \in \mathcal{L}(X)$ has the $\mathcal{P}_{\mathcal{F}}$ property if and only if for every non-empty open subset U of X there exists $x_U \in X$ such that $N_T(x_U, U) \in \mathcal{F}$.

[B] See Lemma 2.12 of the Appendix for an argument showing that $\overline{\mathcal{BD}}$ is a block family.
[C] See Sections 1 and 3 of the Appendix for the definition of the families $\mathcal{I}^*$, $\mathcal{T}$ and $\mathcal{TS}$.

Recall that the notion of $\mathcal{F}$-recurrence requires the density of the set $\mathcal{F}\mathrm{Rec}(T)$, and that the vectors of this set return to each of their neighbourhoods with frequency $\mathcal{F}$. It is then clear that the $\mathcal{P}_{\mathcal{F}}$ property is, at least formally, weaker than $\mathcal{F}$-recurrence. In [25, Section 5] it is asked, and left as an open problem, if the $\mathcal{P}_{\mathcal{F}}$ property is equivalent to $\mathcal{F}$-recurrence. The results that we have obtained in Chapter 2, and in particular Lemma 3.1 there, solve positively this question under the already mentioned "adjoint operator's assumption".

As we argue in Section 2 of Chapter 2, these strong results, that appear when we are able to use Ergodic Theory, are still true for many natural classes of *Polish dynamical systems*. In particular, they hold for *compact dynamical systems*, which have been the main context in classical non-linear dynamics. This is probably the reason why the notions of frequent and reiterative recurrence have not been considered for classical systems (there they are equivalent to the existence of an invariant measure with full support), while stronger recurrence notions such as uniform, $\mathcal{IP}^*$ or Δ^*-recurrence have been deeply studied (see [41]).

Returning to the linear setting, we have also shown in Chapter 2 that having a spanning set of unimodular eigenvectors is equivalent to be uniformly recurrent, at least for operators acting on Hilbert spaces (Theorem 1.3) and for power-bounded operators on reflexive Banach spaces (Theorem 1.9). Along Section 4 of Chapter 2 we have discussed the difficulties that the "Hilbert space's result" presents when one tries to extend it for other spaces. However, for the case of "power-bounded operators on reflexive Banach spaces", it seems that a much stronger result (which will appear in a forthcoming work) can be achieved.

See Section 2.2 of the *Conclusions* for more on Chapter 2.

2.3 On Chapter 3

In article [51] (i.e. Chapter 3) we define the notion of *quasi-rigidity* characterizing the operators $T \in \mathcal{L}(X)$ for which every N-fold direct sum $T_{(N)}$ is again recurrent. Then we construct in Theorem 3.2 some recurrent but not quasi-rigid operators, which are not reiteratively recurrent by Proposition 6.2 of Chapter 3. However, these operators are $\mathcal{AP}$-recurrent:

Corollary 2.4. *Let X be any (real or complex) separable infinite-dimensional Banach space. For each $N \in \mathbb{N}$ there exists an operator $T \in \mathcal{L}(X)$ such that*

$$T_{(N)} : X^N \longrightarrow X^N \textit{ is } \mathcal{AP}\textit{-recurrent,}$$

but for which $T_{(N+1)} : X^{N+1} \longrightarrow X^{N+1}$ (and hence $T_{(J)}$ for all $J > N$) is not recurrent.

Proof. We just show the "$N = 1$ *and complex*" case. Let X be a complex Banach space and suppose that $(e_n, e_n^*)_{n\in\mathbb{N}} \subset X \times X^*$ is the biorthogonal sequence; P the projection; $M > 0$ the constant; $(w_k^*)_{k\geq 3}$, $(m_k)_{k\in\mathbb{N}}$ and $(\lambda_k)_{k\in\mathbb{N}}$ the sequences of functionals, integers and complex numbers; and R, T the operators considered in Section 3 of Chapter 3.

It is enough to check that $c_{00} := \mathrm{span}\{e_n : n \in \mathbb{N}\} \subset \mathcal{AP}\mathrm{Rec}(T)$. Indeed, let us prove that given an arbitrary but fixed vector $x \in c_{00}$ the following holds:

– *For each $l \in \mathbb{N}$ there exist some $n \in \mathbb{N}$ such that $\|T^{jm_{n-1}}x - x\| < \frac{1}{l}$ for all $1 \leq j \leq l$.*

First of all note that, if $x = \sum_{k=1}^{n_0} x_k e_k$, then $R^{jm_{n-1}}x = x$ for all $n > n_0$ and all $j \in \mathbb{N}_0$. Now, by the assumptions on the sequence $(m_k)_{k\in\mathbb{N}}$ we can fix $n_1 \in \mathbb{N}$ such that

$$M \cdot 2K \cdot \|x\| \cdot \sum_{n<k} \frac{m_{n-1}}{m_{k-1}} < \frac{1}{2l^2} \quad \text{for all } n > n_1.$$

By the construction of $(w_k^*)_{k\geq 3}$ we can now choose $n > \max(n_0, n_1)$ such that $|\langle w_n^*, Px\rangle| < \frac{1}{2l^2}$. Finally, for each $1 \leq j \leq l$ we have that

$$T^{jm_{n-1}}x - x = \Big((T^{jm_{n-1}} - R^{jm_{n-1}})x\Big) + \Big(R^{jm_{n-1}}x - x\Big) = (T^{jm_{n-1}} - R^{jm_{n-1}})x,$$

since $n > n_0$. Hence, and since $n > n_1$, we have that

$$\begin{aligned}
\|T^{jm_{n-1}}x - x\| &= \|(T^{jm_{n-1}} - R^{jm_{n-1}})x\| \overset{\text{Fact 3.3.1}}{\underset{\text{Chapter 3}}{\leq}} \left\| \sum_{3\leq k<n} \frac{\lambda_{k,jm_{n-1}}}{m_{k-1}} \langle w_k^*, Px\rangle e_k \right\| \\
&+ \left\| \frac{\lambda_{n,jm_{n-1}}}{m_{n-1}} \langle w_n^*, Px\rangle e_n \right\| + \left\| \sum_{n<k} \frac{\lambda_{k,jm_{n-1}}}{m_{k-1}} \langle w_k^*, Px\rangle e_k \right\| \\
&\overset{\text{Fact 3.3.2}}{\underset{\text{Chapter 3}}{\leq}} j \cdot |\langle w_n^*, Px\rangle| + j \cdot \left| M \cdot 2K \cdot \|x\| \cdot \sum_{n<k} \frac{m_{n-1}}{m_{k-1}} \right| < 2j \cdot \frac{1}{2l^2} \leq \frac{1}{l},
\end{aligned}$$

for all $1 \leq j \leq l$ as we wanted to show. The later easily implies that x belongs to $\mathcal{AP}\mathrm{Rec}(T)$ because for each $l \in \mathbb{N}$, and for the ball $B(x, \frac{1}{l})$ centred at x and of radius $\frac{1}{l}$, we have that the return set $N_T(x, B(x, \frac{1}{l}))$ contains an arithmetic progression of length $l + 1$. ☐

Quasi-rigidity seems to be a new notion even for classical non-linear dynamical systems, but we have also defined *topological quasi-rigidity* (see Chapter 3, Definition 2.2). It is worth mentioning that this can be also characterized in terms of free filters. Indeed, if we consider the definition of *topological $\mathcal{F}$-recurrence* given at the end of Section 2.1 of this chapter then:

Proposition 2.5. *Let (X, T) be a dynamical system. The following are equivalent:*

(i) *T is topologically quasi-rigid;*

(ii) *T is topologically $\mathcal{F}(A)$-recurrent for some infinite subset $A \subset \mathbb{N}_0$;*

(iii) *T is topologically $\mathcal{F}$-recurrent with respect to a free filter $\mathcal{F}$ with a countable base.*

Moreover, if X is a second-countable space, the previous statements are equivalent to:

(iv) *T is topologically $\mathcal{F}$-recurrent for a family $\mathcal{F}$ with the finite intersection property.*

Proof. The proof is completely analogous to that of Proposition 4.5 stated in Chapter 3. ☐

As a final remark on Chapter 3 we would like to include here a short argument, which shows (the really well-known fact) that hypercyclicity is an infinite-dimensional phenomenon, but using the recurrence theory developed:

Corollary 2.6. *There are no hypercyclic operators on a finite-dimensional space $X \neq \{0\}$.*

Proof. Suppose that $T : X \longrightarrow X$ was a complex hypercyclic matrix. Then T would be a recurrent matrix. By [30, Theorem 4.1] there would exist a base of X formed by unimodular eigenvectors for T so that every orbit would be bounded and no vector could be hypercyclic. The real case follows similarly by the arguments used in Section 5.1 of Chapter 3. □

See [10, Proposition 1.1] or [55, Proposition 2.57 or Exercises 2.7.1 and 2.7.2] for some of the standard proofs of this last fact.

See Section 2.3 of the *Conclusions* for more on Chapter 3.

2.4 On Chapter 4

In article [69] (i.e. Chapter 4) we identify those quasi-rigid operators (acting on Banach spaces) that admit a *recurrent subspace*, just as in the case of weakly-mixing operators admitting a *hypercyclic subspace*. The theory developed could simplify the study of those weakly-mixing operators that can be written as a compact perturbation of a quasi-rigid operator with a much easier dynamical behaviour (see Chapter 4, Section 7, Corollary 7.2). In view of these results it could be interesting to find other properties for which recurrence and hypercyclicity behave exactly in the same way, as it happens for spaceability in the "*weakly-mixing*" context.

For non-weakly-mixing operators, it is an open problem to characterize when they admit hypercyclic subspaces (see [43, Question 7]). We can at least show the following:

Corollary 2.7. *Let X be a complex (resp. real) separable Banach space and let $T \in \mathcal{L}(X)$ be a hypercyclic but not weakly-mixing operator. The following statements are equivalent:*

(i) *T has a recurrent subspace;*

(ii) *there exists an infinite-dimensional closed subspace $E \subset X$ and an increasing sequence of integers $(l_n)_{n\in\mathbb{N}}$ such that $T^{l_n}x \to x$ for all $x \in E$;*

(ii') *there exists an infinite-dimensional closed subspace $E \subset X$ and an increasing sequence of integers $(l_n)_{n\in\mathbb{N}}$ such that $T^{l_n}x \to 0$ for all $x \in E$;*

(iii) *there exists an infinite-dimensional closed subspace $E \subset X$ and an increasing sequence of integers $(l_n)_{n\in\mathbb{N}}$ such that $\sup_{n\in\mathbb{N}} \|T^{l_n}|_E\| < \infty$;*

(iv) *the essential spectrum of T (resp. $\widetilde{T}$) intersects the closed unit disk $\overline{\mathbb{D}}$.*

Proof. Recall that every hypercyclic operator is quasi-rigid by Proposition 2.9 of Chapter 3. Then (i) ⇔ (ii) ⇔ (iii) ⇔ (iv) follow from Theorem 3.3 and 3.4 of Chapter 4; (ii') ⇒ (iii) by the Banach-Steinhaus theorem; and (iv) ⇒ (ii') by the proof of (iv) ⇒ (ii) showed in Theorem 3.3 of Chapter 4 combined with [55, Proof of Theorem 10.29], since by hypercyclicity there exist a dense set $X_0 \subset X$ and an increasing sequence $(k_n)_n \in \mathbb{N}^{\mathbb{N}}$ such that $T^{k_n}x \to 0$ for all $x \in X_0$. □

See Section 2.4 of the *Conclusions* for more on Chapter 4.

We refer to the ***Conclusions*** chapter for the final concluding remarks.

Conclusions

In this chapter we summarize the conclusions we got in this work: first we state the general conclusions; and then we collect the lines of research and problems that remain open.

1 General conclusions

In this thesis we have studied *linear recurrence*, which provides a wide range of different research possibilities: one can focus on the original notion of "usual recurrence", but it is also possible to use Combinatorial Number Theory and work with the generalized and stronger recurrence properties of "frequent recurrence" called $\mathcal{F}$-recurrence along this memoir.

From the beginning of this document we have insisted on the novelty of recurrence for linear dynamical systems: it has been just deeply studied from 2014 while the notion of hypercyclicity has more than 30 years of development. This contrast comes to show a fact that has recently appeared in Linear Dynamics: the problems that remain open in (also "frequent") hypercyclicity are, by far, much more difficult than those arising right now for other more modern dynamical properties (see [43] for a very recent compendium of open problems in hypercyclicity).

Among these newer properties, it seems that linear recurrence is particularly important because of the natural links that appear with hypercyclicity, as we have shown along this work. However, it is more or less clear that hypercyclicity is and will continue to be the quintessential property in Linear Dynamics, in part due to the *invariant subspace* and *subset problems* since these represent the basis on which the theory of linear systems has been developed.

From our point of view it is at least nice (and hopeful) that the lines of though used in the chapters/articles forming this memoir have aroused the interest of other researchers such as the already mentioned Rodrigo Cardeccia and Santiago Muro, whose work has complemented ours in a really deep way. This recent interchange of ideas shows that Linear Dynamics can still be a really active research area, and that other dynamical properties (such as recurrence) may be exploited in order to give alternative approaches to hypercyclicity.

Plenty of open problems and research lines remain open in linear recurrence as one could expect due to its newness. These questions are not only connected to hypercyclicity since plenty of queries are strictly related to recurrence. Indeed, recall that there are many Furstenberg families $\mathcal{F}$ for which there does not exist any $\mathcal{F}$-hypercyclic operator and just $\mathcal{F}$-recurrence can be considered. This is the case of uniform, $\mathcal{IP}^*$ and Δ^*-recurrence, and studying the structure of the sets $\mathcal{F}\mathrm{Rec}(T)$ of $\mathcal{F}$-recurrent vectors for these, but also for other strong recurrence properties such as frequent recurrence, seems to be a challenging problem.

2 Open problems and concluding remarks

Let us now focus on each chapter to include the main left open problems:

2.1 On Chapter 1

In article [21] (i.e. Chapter 1) we have studied $\mathcal{F}$-recurrence for many natural Furstenberg families $\mathcal{F}$. For the strong recurrence notions of frequent, uniform, $\mathcal{IP}^*$ and Δ^*-recurrence it is still unknown if the respective sets of $\mathcal{F}\mathrm{Rec}(T)$ can be of second category without taking up the whole space (see Questions 2.9 and 2.11 of Chapter 1):

Open Problem 1. *For any operator $T \in \mathcal{L}(X)$ acting on a (real or complex) F-space X:*

(a) *Do we always have that either* $\mathrm{FRec}(T) = X$ *or* $\mathrm{FRec}(T)$ *is of first category?*

(b) *Do we always have that either* $\mathrm{URec}(T) = X$ *or* $\mathrm{URec}(T)$ *is of first category?*

(c) *Do we always have that either* $\mathcal{IP}^*\mathrm{Rec}(T) = X$ *or* $\mathcal{IP}^*\mathrm{Rec}(T)$ *is of first category?*

(d) *Do we always have that either* $\Delta^*\mathrm{Rec}(T) = X$ *or* $\Delta^*\mathrm{Rec}(T)$ *is of first category?*

The raise of these questions is based on the following two facts (already stated in Chapter 1):

(1) the sets considered can be of second category since the identity operator $I : X \longrightarrow X$ fulfills that $\Delta^*\mathrm{Rec}(T) = \mathcal{IP}^*\mathrm{Rec}(T) = \mathrm{URec}(T) = \mathrm{FRec}(T) = X$;

(2) and also for the set of periodic vectors $\mathrm{Per}(T)$ the corresponding property holds: if we let $\mathrm{Per}_n(T) := \{x \in X : T^n x = x\}$ then

$$\mathrm{Per}(T) = \bigcup_{n\in\mathbb{N}} \mathrm{Per}_n(T),$$

and once $\mathrm{Per}(T)$ is of second category we have that $\mathrm{Per}_n(T)$ is somewhere dense for some $n \in \mathbb{N}$, which implies $T^n = I$ (i.e. either $\mathrm{Per}(T)$ is of first category or else $\mathrm{Per}(T) = X$).

Let us include some comments regarding **Problem 1**:

– We know that (a) is true when T is a hypercyclic operator: in that case $\mathrm{FRec}(T)$ is necessarily of first category (see Chapter 1, Theorem 2.8). When T is not hypercyclic we do not know anything even in the dual/reflexive setting used in Chapter 2: the set of frequently recurrent vectors in that case is “big” with respect to a certain invariant measure, but this has usually nothing to do with the “bigness” in the Baire category sense.

– If X is a Banach space then (b), (c) and (d) are true (see Chapter 1, Corollary 3.2). In contrast, note that Example 3.3 given in Chapter 1 also shows a Δ^*-recurrent vector with an unbounded orbit for an operator acting on a Fréchet space.

– The next possible stronger recurrence notion of those studied in this memoir is that of having a dense set of unimodular eigenvectors. Note that, if $\mathrm{span}(\mathcal{E}(T))$ is a second category set, then T is a power-bounded operator on the F-space X so that $\mathcal{F}\mathrm{Rec}(T) = X$, for every Furstenberg family $\mathcal{F}$, whenever T is $\mathcal{F}$-recurrent (see Chapter 1, Theorem 8.6).

Regarding weaker recurrence notions we have shown in Example 2.4 of Chapter 1 that there exist a reiteratively recurrent operator with a first category set of reiteratively recurrent vectors. Since this operator is not cyclic and in view of Theorem 2.1 of Chapter 1, which shows that

– *the set of reiteratively recurrent vectors is always of second category for every reiteratively recurrent and hypercyclic operator,*

the following is then a natural question already stated in Problem 5.14 of Chapter 3:

Open Problem 2. *Let $T \in \mathcal{L}(X)$ be a reiteratively recurrent and cyclic operator. Is then the set of reiteratively recurrent vectors* $\mathrm{RRec}(T)$ *necessarily a second category set?*

Let us turn to finding sufficient conditions to show the equivalence between $\mathcal{F}$-hypercyclicity and $\mathcal{F}$-recurrence. Question 2.13 of Chapter 1 is still unanswered:

Open Problem 3. *Let $T \in \mathcal{L}(X)$ be a frequently recurrent operator (or even Devaney chaotic) acting on an F-space X and such that $T^n x \to 0$ as $n \to \infty$ for all x from a dense subset of X. Does it follow that the operator T is frequently hypercyclic?*

It seems to be even open whether every chaotic operator with a dense generalized kernel (that is, $\overline{\bigcup_{n\geq 0} \mathrm{Ker}(T^n)} = X$) is frequently hypercyclic.

Another kind of question is that related to the T^{-1}*-type* and $T \oplus T$*-type problems*, which have been asked in Question 2.14 of Chapter 1 and Questions 8.7 and 8.8 of Chapter 2:

Open Problem 4. *Let $T \in \mathcal{L}(X)$ be an operator acting on a (real or complex) F-space X:*

(a) *If T is an invertible reiteratively ($\mathcal{U}$-frequently, frequently, uniformly, $\mathcal{IP}^*$, Δ^*) recurrent operator, does T^{-1} have the same property?*

(b) *If T is a reiteratively ($\mathcal{U}$-frequently, frequently, uniformly) recurrent operator, does $T \oplus T$ (or more generally $T_{(N)}$ for all $N \geq 2$) have the same property?*

All these questions present positive answers in the very particular dual/reflexive/Hilbertian setting used in Chapter 2, so the question is if these properties still hold for operators on general F-spaces. In this broader context we prove, in Proposition 5.10 of Chapter 3, that the answer to question (b) is positive whenever T admits a dense $\mathcal{C}_T$-orbit (in particular if T is cyclic).

We could also include here Question 6.3 stated in Chapter 1, which asks the equivalence between uniform and $\mathcal{IP}^*$-recurrence. However, we treat this problem in a much deeper way (involving unimodular eigenvectors) in the following section.

2.2 On Chapter 2

In article [50] (i.e. Chapter 2) we have used Ergodic Theory and measure preserving systems to obtain strong recurrence notions from weaker ones. The concept of reiterative recurrence was specially interesting since it implied the existence of invariant measures, at least in our dual/reflexive setting. Even though this is not true in general (as it happens for some operators acting on non-reflexive Banach spaces), the following is a natural question:

Open Problem 5. *Let $T \in \mathcal{L}(X)$ be an operator acting on an F-space X. Suppose that there exists some vector $x_0 \in \mathrm{RRec}(T) \setminus \{0\}$:*

(a) *Under which extra conditions (on the operator T or on the space X) does T admits a non-trivial invariant measure containing the vector x_0 in its support?*

(b) *What happens when X is a (possibly reflexive) Fréchet space?*

To show the existence of unimodular eigenvectors we needed, in Chapter 2, a particularly restrictive setting (either an underlying Hilbert space or else a power-bounded operator acting on a reflexive Banach space). Following Questions 1.8 and 8.1 of Chapter 2:

Open Problem 6. *Let $T \in \mathcal{L}(X)$ be an operator acting on a complex F-space X:*

(a) *If T is uniformly recurrent, is $\mathrm{span}(\mathcal{E}(T))$ a dense set in X?*

(b) *What happens when X is a complex (reflexive) Fréchet space?*

(c) *What if X is a complex (reflexive) Banach space?*

(d) *What about the case where T is an adjoint operator on a separable dual Banach space X?*

(e) *What can we say about this question if T is a power-bounded operator?*

It seems to us that a more general "*unimodular eigenvectors' constructing machine*", not strictly restricted to the invariant measures or power-bounded assumptions, should be developed in order to provide a positive answer to **Problem 6**. What we know for the moment are the next two facts proved in Chapter 2, Section 8:

– *Let $T \in \mathcal{L}(H)$ an operator acting on a complex separable Hilbert space H. Given a T-invariant w-compact subset K of H for which $0 \notin K$, we have that $\overline{\mathrm{span}(\mathcal{E}(T))} \cap K \neq \varnothing$.*

– *Let $T \in \mathcal{L}(X)$ be an adjoint operator acting on a complex separable dual Banach space X. Let $n \in \mathbb{N}$ and $\lambda \in \mathbb{T}$. Given a $[\lambda T]^n$-invariant w^*-compact and convex subset K of H for which $0 \notin K$, we have that $\mathcal{E}(T) \cap \mathrm{span}(\mathrm{Orb}(x, T)) \neq \varnothing$ for some $x \in K$.*

The questions above where originated by Question 1.6 of Chapter 1, and in fact, a positive answer to **Problem 6** would automatically imply a negative one to the next problem also stated in Chapter 2, Question 8.2:

Open Problem 7. *Let $T \in \mathcal{L}(X)$ be an operator acting on a (real or complex) F-space X:*

(a) *Can T be $\mathcal{IP}^*$-recurrent but not Δ^*-recurrent?*

(b) *Can T be uniformly recurrent but not $\mathcal{IP}^*$-recurrent?*

As we comment in Section 8 of Chapter 2, the notions of uniform and $\mathcal{IP}^*$-recurrence are completely distinguished for compact dynamical systems (see the construction from [38], its properties in [27] and then use [41, Theorems 1.15 and 9.12]), so the question here is if the linearity assumption avoids that distinction.

2.3 On Chapter 3

One of the main results obtained in article [51] (i.e. Chapter 3) is the negative answer to the $T \oplus T$*-recurrence problem*. The counterexample constructed is valid for **discrete** linear dynamical systems, but we could also consider semigroups of operators: recall that an operator family $(T(t))_{t\geq 0} \subset \mathcal{L}(X)$ on a Banach space X is said to be a C_0*-semigroup* if the following conditions hold

($C_0$1) $T(0) = I$;

($C_0$2) $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$;

($C_0$3) the mapping $t \mapsto T(t)x \in X$, $t \geq 0$, is continuous for every $x \in X$.

Following [26] we have that: a vector x in a Banach space X is called *recurrent* for a C_0-semigroup $(T(t))_{t\geq 0} \subset \mathcal{L}(X)$ if there exist an increasing sequence $(t_n)_{n\in\mathbb{N}}$ of positive real numbers such that $T(t_n)x \to x$ as $n \to \infty$. We denote by $\mathrm{Rec}(T(t))$ the set of recurrent vectors for the C_0-semigroup $(T(t))_{t\geq 0}$. A kind of Costakis-Manoussos-Parissis theorem holds for recurrent C_0-semigroups (see [26, Theorem 2.1]) and the following was asked in [26]:

Open Problem 8. *Let $(T(t))_{t\geq 0}$ be a recurrent C_0-semigroup. Is the direct sum $(T(t)\oplus T(t))_{t\geq 0}$ a recurrent C_0-semigroup?*

As far as we know **Problem 8** is still open if we replace "*recurrent*" by "*hypercyclic*", but solving the recurrence version could be a first step to achieve the hypercyclicity one.

Returning to our **discrete** dynamical systems, we also include here those problems related to the existence of dense $\mathcal{C}_T$-orbits: recall that, for an operator $T \in \mathcal{L}(X)$, we have denoted by

$$\mathcal{C}_T := \{S \in \mathscr{C}(X) : S \circ T = T \circ S\}.$$

the (non-linear) *commutant* of the operator T, and that given any vector $x \in X$ we define the $\mathcal{C}_T$*-orbit* of x as $\mathcal{C}_T(x) := \{Sx : S \in \mathcal{C}_T\}$. The following is asked in Chapter 3:

Open Problem 9. *Let $\mathcal{F} \subset \mathscr{P}(\mathbb{N}_0)$ be any Furstenberg family and suppose that $T \in \mathcal{L}(X)$ is an $\mathcal{F}$-recurrent operator acting on a (real or complex) F-space X:*

(a) *If T does not admit any dense $\mathcal{C}_T$-orbit, can $T_{(N)}$ be $\mathcal{F}$-recurrent for all $N \in \mathbb{N}$?*

(b) *Can T admit a dense $\mathcal{C}_T$-orbit and fulfill that $\mathcal{F}\mathrm{Rec}(T) \cap \{x \in X : \overline{\mathcal{C}_T(x)} = X\} = \varnothing$?*

These seem to be tough questions since the commutator of an operator is usually difficult to describe. Related to **Problem 9**, and as we comment in Chapter 3, the general dense lineability of the set of $\mathcal{F}$-recurrent vectors is for the moment an open question:

Open Problem 10. *Let $\mathcal{F}$ be a Furstenberg family, assume that it is not a filter and suppose that $T \in \mathcal{L}(X)$ is an $\mathcal{F}$-recurrent operator acting on a (real or complex) F-space X:*

Is the set $\mathcal{F}\mathrm{Rec}(T)$ necessarily dense lineable?

2.4 On Chapter 4

In article [69] (i.e. Chapter 4) we have just looked at operators acting on Banach spaces. We have already mentioned along this work that there also exists a *hypercyclic spaceability theory* for operator acting on *Fréchet spaces* (see for instance [55, Chapter 10, Section 10.5]).

The following is then asked in Chapter 4:

Open Problem 11. *Let X be a Fréchet space:*

(a) *Under which conditions does a quasi-rigid operator $T \in \mathcal{L}(X)$ have a recurrent subspace?*

(b) *Suppose that $T \in \mathcal{L}(X)$ is a weakly-mixing operator. Is it true that T has a hypercyclic subspace if and only if it has a recurrent subspace?*

This last question seems to be non-trivial since for Fréchet spaces we loose the techniques, involving the essential spectrum, used in Chapter 4.

3 Recent advances in the previous problems

Although the final version of this memoir was presented in June 2023, the body of this document was written between February and the end of March 2023. From April to the end of June 2023 the author did a research stay in Mons where significant progress was made on the posed open problems. In particular:

– **Problem 2** has been solved in the negative by finding a counterexample.

– **Problem 5** is now partially solved: we can construct invariant measures for operators acting on reflexive Fréchet spaces, under some extra boundedness assumption.

– **Problem 6** is now partially solved: a positive answer has been obtained for power-bounded operators acting on arbitrary Fréchet spaces.

– **Problem 8** has been solved in the negative by modifying the counterexample exhibited here in Theorem 3.2 of Chapter 3.

– **Problem 10** is now partially solved: a negative answer has been obtained for usual recurrence by modifying the counterexample exhibited here in Theorem 3.2 of Chapter 3.

– **Problem 11** is now partially solved: similar sufficient conditions to those used in the Banach case are valid for quasi-rigid operators acting on Fréchet spaces.

These new ideas and results will appear in forthcoming works.

We end here this ***Conclusions*** chapter together with the entire thesis.

Appendix

Combinatorial Number Theory

Combinatorial Number Theory plays a fundamental role when we generalize concepts such as recurrence or hypercyclicity in terms of Furstenberg families. In this Appendix we gather some definitions, easy well-known facts and examples that help to understand and clarify the various ideas taken for granted along the different articles forming this memoir. Our aim is to provide a solid knowledge base to be able to work with $\mathcal{F}$-hypercyclicity and $\mathcal{F}$-recurrence.

The first section is devoted to introduce some necessary concepts of size, for infinite sets of natural numbers (see Definition 1.1), related to the topological and algebraic properties of the Stone-Čech compactification $\beta\mathbb{N}_0$. We then include in Proposition 1.3 some relations between them, which have been already used in various chapters of this work. We also introduce the notions of asymptotic and Banach density (see Definitions 2.1 and 2.6), studying their basic properties (see Propositions 2.4 and 2.8). In Section 3 we elaborate on the concept of Furstenberg family, and we give detailed examples of upper families (see Example 3.2). We finally check, in Section 4, how important are the $\mathcal{IP}$, Δ, $\mathcal{IP}^*$ and Δ^* families in dynamics.

Let us establish the following notation (used throughout the work): $\mathbb{N}$ will be the set of strictly positive integers while $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Moreover, $\mathbb{Z}$ will be the set of all integers and given any $n \in \mathbb{N}$ we will denote by $\mathbb{Z}/n\mathbb{Z}$ the quotient space of modulus n. Given $n, m \in \mathbb{Z}$ with $n \leq m$ and $A \subset \mathbb{N}_0$ we will write $[n, m] := \{n, n+1, ..., m\}$, $nA = n \cdot A := \{n \cdot x : x \in A\}$, $A + n := \{x + n : x \in A\}$, $A - n := \{x - n : x \in A\}$ and

$$A + [n, m] := \bigcup_{j=n}^{m} (A + j).$$

In addition, if we consider two sets of numbers $A, B \subset \mathbb{N}_0$, then we will denote their *sum set* by $A + B := \{x + y : x \in A, y \in B\}$, their *difference set* by $A - B := \{x - y : x \in A, y \in B\}$, and also their *symmetric difference* by $A \triangle B := (A \cup B) \setminus (A \cap B)$, where $\cup$ and $\cap$ are the usual union and intersection symbols. We will denote by $\#A$ the *cardinal* of the set A.

1 The natural numbers

As stated in [58, Part I], the internal operation defined on a discrete topological semigroup $(S, \cdot)$ has a natural extension to the Stone-Čech compactification

$$\beta S := \{p \subset \mathscr{P}(S) : p \text{ is an ultrafilter on } S\}.$$

By this extension $(\beta S, \cdot)$ becomes a compact topological semigroup containing S as its centre in the form of principal ultrafilters (see [58, Part II]). The study of the algebraic and topological properties of $(\beta S, \cdot)$ has applications in several fields such as Ramsey Theory, Combinatorial Number Theory [58, Part III], Ergodic Theory and Topological Dynamics [58, Part IV].

In particular, if we denote by $\mathbb{N}$ the set of strictly positive integers and by $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ we can consider the discrete topological semigroup $(\mathbb{N}_0, +)$, where the operation $+ : \mathbb{N}_0 \times \mathbb{N}_0 \longrightarrow \mathbb{N}_0$ is the usual sum that brings $(n, m) \mapsto n + m$ for each $n, m \in \mathbb{N}_0$. Then

$$\beta\mathbb{N}_0 := \{p \subset \mathscr{P}(\mathbb{N}_0) : p \text{ is an ultrafilter on } \mathbb{N}_0\},$$

with the extended operation $+ : \beta\mathbb{N}_0 \times \beta\mathbb{N}_0 \longrightarrow \beta\mathbb{N}_0$, is a compact topological semigroup that provides deep relations between Combinatorial Number Theory and Topological Dynamics.

In fact, many notions originated in Topological Dynamics such as *syndetic sets*, *piecewise syndetic sets* or *IP-sets* are important to describe the algebraic and topological structure of $(\beta\mathbb{N}_0, +)$. For example, a point $p \in \beta\mathbb{N}_0$ is in the closure of the smallest ideal of $\beta\mathbb{N}_0$ if and only if every element $A \in p$ satisfies that $A \subset \mathbb{N}_0$ is piecewise syndetic [58, Theorem 4.40]. It is also true that for a set $B \subset \mathbb{N}_0$ there is a sequence $(x_n)_{n\in\mathbb{N}} \in \mathbb{N}_0^{\mathbb{N}}$ fulfilling that

$$\left\{\sum_{n\in F} x_n : F \text{ finite subset of } \mathbb{N}\right\} \subset B,$$

if and only if there exists some (non-zero) idempotent $p \in \beta\mathbb{N}_0$ (i.e. $p + p = p$) such that $B \in p$, see [58, Theorem 5.12]. Due to the extension of this theory we will reduce ourselves to include the needed concepts of size for infinite subsets of natural numbers together with their basic properties. For a deeper and more complete development of the theory see [58].

Definition 1.1. We say that a set $A \subset \mathbb{N}_0$ is:

(a) *thick* if for each $m \in \mathbb{N}$ there exists $x \in A$ such that $[x, x + m] \subset A$.

(b) *syndetic* if there exists $m \in \mathbb{N}$ such that $[x, x + m] \cap A \neq \varnothing$ for all $x \in \mathbb{N}_0$.

(c) *thickly syndetic* if for each $m \in \mathbb{N}$ there is a syndetic set $A_m \subset \mathbb{N}_0$ such that $A_m + [0, m] \subset A$.

(d) *piecewise syndetic* if there exist two sets $B, C \subset \mathbb{N}_0$ fulfilling that $A = B \cap C$ and where the set B is thick and the set C is syndetic.

(e) an *AP-set* if for each $l \in \mathbb{N}$ there exist numbers $x, m \in \mathbb{N}$ such that

$$\{x + km : 0 \leq k \leq l\} \subset A.$$

(f) an *IP-set* if there exists a sequence $(x_n)_{n\in\mathbb{N}} \in \mathbb{N}_0^{\mathbb{N}}$ such that

$$\left\{\sum_{n\in F} x_n : F \text{ finite subset of } \mathbb{N}\right\} \subset A.$$

(g) a Δ-*set* if there exists an infinite set $B \subset \mathbb{N}_0$ such that $(B - B) \cap \mathbb{N} \subset A$.

Remark 1.2. The sets fulfilling the previous definitions are included in the collection of sets with infinite cardinality

$$\mathcal{I} := \{A \subset \mathbb{N}_0 : A \text{ is infinite}\}.$$

Moreover:

(a) Informally speaking, the thick sets are those that contain "intervals" of natural numbers of arbitrarily long "length". We will denote the family of thick sets by

$$\mathcal{T} := \{A \subset \mathbb{N}_0 : A \text{ is thick}\}.$$

(b) Informally speaking, the syndetic sets are those that have bounded "holes" or "gaps", that is, those $A \subset \mathbb{N}_0$ such that if $(n_k)_{k\in\mathbb{N}}$ is the strictly increasing sequence of natural numbers forming A, then

$$\sup_{k\in\mathbb{N}} (n_{k+1} - n_k) < \infty.$$

We will denote the family of syndetic sets by

$$\mathcal{S} := \{A \subset \mathbb{N}_0 : A \text{ is syndetic}\}.$$

(c) The thickly syndetic sets contain "intervals" arbitrarily "long" but at "bounded distance". We will denote the family of thickly syndetic sets by

$$\mathcal{TS} := \{A \subset \mathbb{N}_0 : A \text{ is thickly syndetic}\}.$$

(d) Piecewise syndetic sets are those containing "intervals" of natural numbers of arbitrarily long "length" but intersected with a syndetic set. Indeed, if $A = B \cap C$ for a thick set B and a syndetic set C, then we can find $m \in \mathbb{N}$ such that $[x, x+m] \cap C \neq \varnothing$ for all $x \in \mathbb{N}_0$. Hence, for each $n > m$ we can find $j \in B$ satisfying that $[j, j+n] \subset B$, so

$$[x, x+m] \cap A = [x, x+m] \cap C \neq \varnothing \quad \text{for each } x \in [j, j+(n-m)].$$

We will denote the family of piecewise syndetic sets by

$$\mathcal{PS} := \{A \subset \mathbb{N}_0 : A \text{ is piecewise syndetic}\}.$$

(e) The AP-sets contain "arbitrarily long arithmetic progressions" since, for each $l, x, m \in \mathbb{N}$ the set $\{x+km : 0 \leq k \leq l\}$ is the *arithmetic progression* of length $l+1$, common difference $m \in \mathbb{N}$ and initial term $x \in \mathbb{N}_0$. We will denote the family of AP-sets by

$$\mathcal{AP} := \{A \subset \mathbb{N}_0 : A \text{ is an AP-set}\}.$$

(f) The IP-sets contain the "finite sums" of a sequence of natural numbers. We will denote the family of IP-sets by

$$\mathcal{IP} := \{A \subset \mathbb{N}_0 : A \text{ is an IP-set}\}.$$

(g) The Δ-sets contain the difference set of an infinite set. We will denote the family of Δ-sets by

$$\Delta := \{A \subset \mathbb{N}_0 : A \text{ is a } \Delta\text{-set}\}.$$

The following well-known facts will be used in Section 3 of this Appendix:

Proposition 1.3. *Consider two sets $A, B \subset \mathbb{N}_0$. The following statements hold:*

(a) *A is syndetic if and only if $\mathbb{N}_0 \setminus A$ is not thick.*

(b) *A is piecewise syndetic if and only if $\mathbb{N}_0 \setminus A$ is not thickly syndetic.*

(c) *If A and B be are thickly syndetic, then $A \cap B$ is thickly syndetic.*

(d) *If A is thickly syndetic, then A is thick and syndetic.*

(e) *If A is thick or syndetic, then A is piecewise syndetic.*

(f) *If A is piecewise syndetic, then A is an AP-set.*

(g) *If A is thick, then A is an IP-set.*

(h) *If A is an IP-set, then A is a Δ-set.*

Proof. (a) The set A is syndetic if and only if there exists $m \in \mathbb{N}$ such that $[x, x+m] \cap A \neq \varnothing$ for all $x \in \mathbb{N}_0$, and therefore, if and only if there exists $m \in \mathbb{N}$ with $[x, x+m] \not\subset \mathbb{N}_0 \setminus A$ for all $x \in \mathbb{N}_0$, that is, if and only if $\mathbb{N}_0 \setminus A$ is not thick.

(b) A is piecewise syndetic if and only if there exists $m \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ with $n > m$ we can find $j \in \mathbb{N}$ satisfying that if $x \in [j, j + (n-m)]$ then $[x, x+m] \not\subset \mathbb{N}_0 \setminus A$ (see Remark 1.2), that is, if and only if there does not exist a syndetic set B_m satisfying $B_m + [0, m] \subset \mathbb{N}_0 \setminus A$, and therefore if and only if $\mathbb{N}_0 \setminus A$ is not thickly syndetic.

(c) Given $m \in \mathbb{N}$ consider a syndetic set $A_m \subset \mathbb{N}_0$ with $A_m + [0, m] \subset A$ and fix $k \in \mathbb{N}$ such that $A_m \cap [x, x+k] \neq \varnothing$ for each $x \in \mathbb{N}_0$. Considering now a syndetic set $B_{m+k} \subset \mathbb{N}_0$ with $B_{m+k} + [0, m+k] \subset B$ we get that $C_m := A_m \cap (B_{m+k} + [0, k])$ is syndetic and satisfies

$$C_m + [0, m] \subset A \cap B.$$

(d) Trivial from the definitions.

(e) Trivial since $A = A \cap \mathbb{N}_0$ and $\mathbb{N}_0$ is both thick and syndetic.

(f) This is a consequence of the (non-trivial) Szemerédi theorem (see [85]).

(g) Suppose that for some $k \in \mathbb{N}$ we have constructed a finite sequence $(x_n)_{n=1}^{k} \subset \mathbb{N}_0$ satisfying that

$$\left\{ \sum_{n \in F} x_n : F \subset \{1, ..., k\} \right\} \subset A.$$

Since A is thick we can choose an element $x_{k+1} \in A$ satisfying that

$$\left[x_{k+1}, x_{k+1} + \sum_{n=1}^{k} x_n \right] \subset A \qquad \text{and hence} \qquad \left\{ \sum_{n \in F} x_n : F \subset \{1, ..., k+1\} \right\} \subset A.$$

Recursively we obtain a sequence $(x_n)_{n \in \mathbb{N}}$ guaranteeing that A is an IP-set.

(h) Let $(x_n)_{n\in\mathbb{N}} \in \mathbb{N}_0^{\mathbb{N}}$ be a sequence satisfying that

$$\left\{\sum_{n\in F} x_n : F \text{ finite subset of } \mathbb{N}\right\} \subset A,$$

and consider the infinite set $B := \left\{\sum_{n=1}^{k} x_n : k \in \mathbb{N}\right\} \subset A$. Then it is clear that

$$(B - B) \cap \mathbb{N} \subset \left\{\sum_{n\in F} x_n : F \text{ finite subset of } \mathbb{N}\right\} \subset A,$$

so A is a Δ-set. □

In Section 3 of this Appendix we will go into detail about Furstenberg families, and in particular, we will easily check that the collections of sets $\mathcal{I}$, $\mathcal{T}$, $\mathcal{S}$, $\mathcal{TS}$, $\mathcal{PS}$, $\mathcal{AP}$, $\mathcal{IP}$ and Δ fulfill the definition of Furstenberg family. Moreover, Proposition 1.3 will be used to establish different (inclusion and dual) relations between these families.

2 Densities

In the different chapters/articles forming this memoir we have used several concepts of size for sets of natural numbers. In the previous section these notions were taken from the algebraic and topological properties of the semigroup $(\beta\mathbb{N}_0, +)$. Another way to "measure" the size of a set of natural numbers is by using the so-called *densities*.

The concept of density plays a fundamental role in the development of Probabilistic, Additive and Combinatorial Number Theory, and in certain areas of Analysis and Ergodic Theory. Indeed, the densities are a really effective tool when one wants to measure and study the relationship between the "structure" and the "width" of a set of natural or integer numbers.

The two most common density concepts in Linear Dynamics are the so-called *asymptotic* and *Banach* densities. These notions are the ones used in this work, and we have included here their definitions, some equivalences and their elementary properties.

Let us start by the asymptotic ones, which we will simply call *densities*:

Definition 2.1. Let $A \subset \mathbb{N}_0$. We define the (asymptotic) *lower density* of the set A as

$$\underline{\text{dens}}(A) := \liminf_{N\to\infty} \frac{\#(A \cap [0, N])}{N + 1}.$$

We define the (asymptotic) *upper density* of the set A as

$$\overline{\text{dens}}(A) := \limsup_{N\to\infty} \frac{\#(A \cap [0, N])}{N + 1}.$$

If the equality $\underline{\text{dens}}(A) = \overline{\text{dens}}(A)$ holds we will say that $\text{dens}(A) := \underline{\text{dens}}(A) = \overline{\text{dens}}(A)$ is the (asymptotic) *density* of the set A.

Remark 2.2. For each set $A \subset \mathbb{N}_0$ we have that

$$0 \leq \underline{\text{dens}}(A) \leq \overline{\text{dens}}(A) \leq 1.$$

Moreover, if the equality $\underline{\text{dens}}(A) = \overline{\text{dens}}(A)$ holds, then the following limit exists

$$\lim_{N\to\infty} \frac{\#(A \cap [0,N])}{N+1},$$

and this is exactly equal to $\text{dens}(A)$. On the other hand, the next equalities are always true:

$$\begin{aligned}
\underline{\text{dens}}(A) &= \liminf_{N\to\infty} \frac{\#(A \cap [0,N])}{N+1} = \liminf_{N\to\infty} \left(1 - \frac{\#((\mathbb{N}_0 \setminus A) \cap [0,N])}{N+1}\right) \\
&= 1 - \limsup_{N\to\infty} \frac{\#((\mathbb{N}_0 \setminus A) \cap [0,N])}{N+1} = 1 - \overline{\text{dens}}(\mathbb{N}_0 \setminus A).
\end{aligned}$$

If a set $A \subset \mathbb{N}_0$ is finite then its densities are equal to 0, the complementary set $\mathbb{N}_0 \setminus A$ is cofinite and its densities are equal to 1. For infinite sets the following holds:

Theorem 2.3 ([78, **Theorem 11.1**]). *Let $A \subset \mathbb{N}_0$ be an infinite set and suppose that $(n_k)_{k\in\mathbb{N}}$ is the strictly increasing sequence of numbers forming the set A. Then we have that:*

$$\underline{\text{dens}}(A) = \liminf_{k\to\infty} \frac{k}{n_k} \qquad \text{and} \qquad \overline{\text{dens}}(A) = \limsup_{k\to\infty} \frac{k}{n_k}.$$

If both limits coincide, then $\text{dens}(A) = \lim_{k\to\infty} \frac{k}{n_k}$.

Proof. Given $k \in \mathbb{N}$ with $0 < n_k \leq N < n_{k+1}$, then

$$\frac{k}{n_{k+1}} \leq \frac{\#(A \cap [0,N])}{N+1} < \frac{k}{n_k}.$$

The result follows from taking inferior and superior limits in the previous inequalities. □

These equivalences allow us to quickly prove some of the following well-known properties:

Proposition 2.4. *Let $A, B \subset \mathbb{N}_0$ and $p, q \in \mathbb{N}_0$. The following statements hold:*

(a) *If $A \subset B$, then $\overline{\text{dens}}(A) \leq \overline{\text{dens}}(B)$ and $\underline{\text{dens}}(A) \leq \underline{\text{dens}}(B)$.*

(b) $\overline{\text{dens}}(A \cup B) \leq \overline{\text{dens}}(A) + \overline{\text{dens}}(B)$. *If $\text{dens}(A \cap B) = 0$, $\underline{\text{dens}}(A) + \underline{\text{dens}}(B) \leq \underline{\text{dens}}(A \cup B)$.*

(c) *If $\text{dens}(A \triangle B) = 0$, then $\overline{\text{dens}}(A) = \overline{\text{dens}}(B)$ and $\underline{\text{dens}}(A) = \underline{\text{dens}}(B)$.*

(d) *If $p \geq 1$, then $\overline{\text{dens}}(p \cdot A) = \frac{1}{p} \cdot \overline{\text{dens}}(A)$ and $\underline{\text{dens}}(p \cdot A) = \frac{1}{p} \cdot \underline{\text{dens}}(A)$.*

(e) $\overline{\text{dens}}((A-q) \cap \mathbb{N}_0) = \overline{\text{dens}}(A) = \overline{\text{dens}}(A+q)$ *and* $\underline{\text{dens}}((A-q) \cap \mathbb{N}_0) = \underline{\text{dens}}(A) = \underline{\text{dens}}(A+q)$.

(f) *If $\overline{\text{dens}}(A \cup B) > 0$, then we have that $\overline{\text{dens}}(A) > 0$ or $\overline{\text{dens}}(B) > 0$, or both.*

(g) *If $\text{dens}((A+p) \cap (A+q)) = 0$, then $\overline{\text{dens}}((A+p) \cup (A+q)) = \overline{\text{dens}}(A+p) + \overline{\text{dens}}(A+q)$.*

Proof. (a) Trivial from the definitions.

(b) We just show the upper density case (the lower one being completely analogous):

$$\begin{aligned}\overline{\mathrm{dens}}(A\cup B) &= \limsup_{N\to\infty}\frac{\#((A\cup B)\cap[0,N])}{N+1}\\ &\le \limsup_{N\to\infty}\left(\frac{\#(A\cap[0,N])}{N+1}+\frac{\#(B\cap[0,N])}{N+1}\right)\\ &\le \limsup_{N\to\infty}\left(\frac{\#(A\cap[0,N])}{N+1}\right)+\limsup_{M\to\infty}\left(\frac{\#(B\cap[0,M])}{M+1}\right)\\ &= \overline{\mathrm{dens}}(A)+\overline{\mathrm{dens}}(B).\end{aligned}$$

(c) We just show the upper density case (the lower one being completely analogous): note that the hypothesis $\mathrm{dens}(A\triangle B)=0$ implies that $\overline{\mathrm{dens}}(A\cup B)=\overline{\mathrm{dens}}(A\cap B)$. Indeed

$$\begin{aligned}\overline{\mathrm{dens}}(A\cup B) &= \overline{\mathrm{dens}}((A\cap B)\cup(A\triangle B))\\ &= \limsup_{N\to\infty}\left(\frac{\#((A\cap B)\cap[0,N])}{N+1}+\frac{\#((A\triangle B)\cap[0,N])}{N+1}\right)\\ &= \overline{\mathrm{dens}}(A\cap B).\end{aligned}$$

Since $A\cap B\subset A,B\subset A\cup B$, using statement (a) we obtain $\overline{\mathrm{dens}}(A)=\overline{\mathrm{dens}}(B)$.

(d) If A is a finite set the result is clear. Suppose that A is infinite and let $(n_k)_{k\in\mathbb{N}}$ be the increasing sequence of integers forming the set A, then $p\cdot A=\{p\cdot n_k : k\in\mathbb{N}\}$ and by Theorem 2.3 we get that

$$\overline{\mathrm{dens}}(A)=\limsup_{k\to\infty}\frac{k}{p\cdot n_k}=\frac{1}{p}\cdot\overline{\mathrm{dens}}(A),\quad \underline{\mathrm{dens}}(p\cdot A)=\liminf_{k\to\infty}\frac{k}{p\cdot n_k}=\frac{1}{p}\cdot\underline{\mathrm{dens}}(A).$$

(e) If A is finite the result si clear. Suppose that $(n_k)_{k\in\mathbb{N}}$ is the infinite increasing sequence of integers forming the set A. Then $A+q=\{n_k+q : k\in\mathbb{N}\}$ and $(A-q)\cap\mathbb{N}_0=\{n_k-q : k\ge k_0\}$ where $k_0=\min\{k\in\mathbb{N} : n_k\ge q\}$. By Theorem 2.3 and writing "$\frac{1}{\infty}=0$" we get that

$$\begin{aligned}\overline{\mathrm{dens}}(A+q) &= \limsup_{k\to\infty}\frac{k}{n_k+q}=\left(\liminf_{k\to\infty}\frac{n_k+q}{k}\right)^{-1}\\ &= \left(\liminf_{k\to\infty}\frac{n_k}{k}\right)^{-1}=\limsup_{k\to\infty}\frac{k}{n_k}=\overline{\mathrm{dens}}(A),\end{aligned}$$

and also

$$\begin{aligned}\overline{\mathrm{dens}}((A-q)\cap\mathbb{N}_0) &= \limsup_{k\to\infty}\frac{(k-k_0)+1}{n_k-q}=\limsup_{k\to\infty}\frac{k}{n_k-q}\\ &= \left(\liminf_{k\to\infty}\frac{n_k-q}{k}\right)^{-1}=\left(\liminf_{k\to\infty}\frac{n_k}{k}\right)^{-1}=\limsup_{k\to\infty}\frac{k}{n_k}=\overline{\mathrm{dens}}(A).\end{aligned}$$

The lower density case is completely analogous.

(f) This a consequence of statement (b) since whenever $\overline{\text{dens}}(A) = 0 = \overline{\text{dens}}(B)$ we have that $0 \leq \overline{\text{dens}}(A \cup B) \leq \overline{\text{dens}}(A) + \overline{\text{dens}}(B) = 0$.

(g) Without lost of generality assume that $q < p$. Using statement (b) we have the inequality $\overline{\text{dens}}((A+p) \cup (A+q)) \leq \overline{\text{dens}}(A+p) + \overline{\text{dens}}(A+q)$. Moreover, from statement (e) we know that $\overline{\text{dens}}(A+p) = \overline{\text{dens}}(A) = \overline{\text{dens}}(A+q)$ and then there exists an increasing sequence of natural numbers $(N_k)_{k\in\mathbb{N}}$ such that

$$\overline{\text{dens}}(A+p) = \lim_{k\to\infty} \frac{\#((A+p)\cap[0,N_k])}{N_k+1}.$$

For each $k \in \mathbb{N}$ we have that $\#((A+p)\cap[0,N_k]) \leq \#((A+q)\cap[0,N_k])$ and with the same sequence we get that

$$\begin{aligned}\overline{\text{dens}}(A+q) &\geq \limsup_{k\to\infty} \frac{\#((A+q)\cap[0,N_k])}{N_k+1} \geq \liminf_{k\to\infty} \frac{\#((A+q)\cap[0,N_k])}{N_k+1} \\ &\geq \liminf_{k\to\infty} \frac{\#((A+p)\cap[0,N_k])}{N_k+1} = \overline{\text{dens}}(A+p) = \overline{\text{dens}}(A+q).\end{aligned}$$

Since the inferior and superior limits coincide we can exchange them by a limit. Finally:

$$\overline{\text{dens}}((A+p)\cup(A+q)) \geq \limsup_{k\to\infty} \frac{\#(((A+p)\cup(A+q))\cap[0,N_k])}{N_k+1},$$

which is equal to

$$\limsup_{k\to\infty}\left(\frac{\#((A+p)\cap[0,N_k])}{N_k+1} + \frac{\#((A+q)\cap[0,N_k])}{N_k+1} - \frac{\#((A+p)\cap(A+q)\cap[0,N_k])}{N_k+1}\right),$$

and taking limits on each term this is equal to $\overline{\text{dens}}(p+A) + \overline{\text{dens}}(q+A)$. □

Remark 2.5. Statement (g) of the previous proposition implies that the upper density is at least additive for translations of a fixed set. Moreover: given $A \subset \mathbb{N}_0$ and a finite sequence $(p_n)_{n=1}^{k} \subset \mathbb{N}_0$ satisfying the equality $\text{dens}((A+p_n)\cap(A+p_l)) = 0$ for every $1 \leq n < l \leq k$, then we can repeat the previous proof but using the inclusion-exclusion principle to obtain that

$$\overline{\text{dens}}\left(\bigcup_{n=1}^{k} A+p_n\right) = \sum_{n=1}^{k} \overline{\text{dens}}(A+p_n) = k \cdot \overline{\text{dens}}(A).$$

Let us now introduce the *Banach densities*:

Definition 2.6. Let $A \subset \mathbb{N}_0$. We define the *lower Banach density* of the set A as

$$\underline{\text{Bd}}(A) := \liminf_{N\to\infty}\left(\inf_{m\geq 0} \frac{\#(A\cap[m,m+N])}{N+1}\right).$$

We define the *upper Banach density* of the set A as

$$\overline{\text{Bd}}(A) := \limsup_{N\to\infty}\left(\sup_{m\geq 0} \frac{\#(A\cap[m,m+N])}{N+1}\right).$$

If the equality $\underline{\text{Bd}}(A) = \overline{\text{Bd}}(A)$ holds we will say that the value $\text{Bd}(A) := \underline{\text{Bd}}(A) = \overline{\text{Bd}}(A)$ is the *Banach density* of the set A.

These notions have a very similar behaviour to that of the (asymptotic) densities: first, for each set $A \subset \mathbb{N}_0$ the following inequalities are satisfied

$$0 \leq \underline{\mathrm{Bd}}(A) \leq \overline{\mathrm{Bd}}(A) \leq 1,$$

and also, if A is a finite set its Banach densities are equal to 0. On the other hand, there is a clear relationship between these densities and previous ones since the inequalities

$$\inf_{m\geq 0} \frac{\#(A\cap[m,m+N])}{N+1} \leq \frac{\#(A\cap[0,N])}{N+1} \leq \sup_{m\geq 0} \frac{\#(A\cap[m,m+N])}{N+1},$$

imply that

$$0 \leq \underline{\mathrm{Bd}}(A) \leq \underline{\mathrm{dens}}(A) \leq \overline{\mathrm{dens}}(A) \leq \overline{\mathrm{Bd}}(A) \leq 1,$$

for all $A \subset \mathbb{N}_0$.

Remark 2.7. In the definitions of lower and upper Banach density we can change the infimum and supremum, to a minimum and a maximum respectively. Furthermore, it can be shown that the lower and upper limits are in this case limits, obtaining the alternative formulas:

$$\begin{aligned}\underline{\mathrm{Bd}}(A) &= \lim_{N\to\infty}\left(\inf_{m\geq 0} \frac{\#(A\cap[m,m+N])}{N+1}\right) = \lim_{N\to\infty}\left(\min_{m\geq 0} \frac{\#(A\cap[m,m+N])}{N+1}\right)\\ &= \lim_{N\to\infty}\left(\liminf_{m\to\infty} \frac{\#(A\cap[m,m+N])}{N+1}\right),\end{aligned}$$

and for the upper Banach density

$$\begin{aligned}\overline{\mathrm{Bd}}(A) &= \lim_{N\to\infty}\left(\sup_{m\geq 0} \frac{\#(A\cap[m,m+N])}{N+1}\right) = \lim_{N\to\infty}\left(\max_{m\geq 0} \frac{\#(A\cap[m,m+N])}{N+1}\right)\\ &= \lim_{N\to\infty}\left(\limsup_{m\to\infty} \frac{\#(A\cap[m,m+N])}{N+1}\right).\end{aligned}$$

These results are well-known and a proof can be found in [49], a paper entirely devoted to establishing these equalities in a simple way.

Using the formulas with limits, minimums and maximums of the Banach densities it is clear that the equality $\underline{\mathrm{Bd}}(A) = \overline{\mathrm{Bd}}(A)$ implies the existence of the limit

$$\lim_{N\to\infty} \frac{\#(A\cap[m_N,m_N+N])}{N+1},$$

for every sequence $(m_N)_{N\in\mathbb{N}_0} \subset \mathbb{N}_0$, and then the previous limit coincide with $\mathrm{Bd}(A) = \mathrm{dens}(A)$.

It is also straightforward to check the equality

$$\underline{\mathrm{Bd}}(A) + \overline{\mathrm{Bd}}(\mathbb{N}_0 \setminus A) = 1,$$

for every set $A \subset \mathbb{N}_0$. Moreover, the statements of Proposition 2.4 are still valid if we replace the asymptotic densities by their respective Banach version:

Proposition 2.8. *Let $A, B \subset \mathbb{N}_0$ and $p, q \in \mathbb{N}_0$. The following statements hold:*

(a) *If $A \subset B$, then $\overline{\mathrm{Bd}}(A) \leq \overline{\mathrm{Bd}}(B)$ and $\underline{\mathrm{Bd}}(A) \leq \underline{\mathrm{Bd}}(B)$.*

(b) *$\overline{\mathrm{Bd}}(A \cup B) \leq \overline{\mathrm{Bd}}(A) + \overline{\mathrm{Bd}}(B)$. If $\mathrm{Bd}(A \cap B) = 0$, then $\underline{\mathrm{Bd}}(A) + \underline{\mathrm{Bd}}(B) \leq \underline{\mathrm{Bd}}(A \cup B)$.*

(c) *If $\mathrm{Bd}(A \triangle B) = 0$, then $\overline{\mathrm{Bd}}(A) = \overline{\mathrm{Bd}}(B)$ and $\underline{\mathrm{Bd}}(A) = \underline{\mathrm{Bd}}(B)$.*

(d) *If $p \geq 1$, then $\overline{\mathrm{Bd}}(p \cdot A) = \frac{1}{p} \cdot \overline{\mathrm{Bd}}(A)$ and $\underline{\mathrm{Bd}}(p \cdot A) = \frac{1}{p} \cdot \underline{\mathrm{Bd}}(A)$.*

(e) *$\overline{\mathrm{Bd}}((A - q) \cap \mathbb{N}_0) = \overline{\mathrm{Bd}}(A) = \overline{\mathrm{Bd}}(A + q)$ and $\underline{\mathrm{Bd}}((A - q) \cap \mathbb{N}_0) = \underline{\mathrm{Bd}}(A) = \underline{\mathrm{Bd}}(A + q)$.*

(f) *If $\overline{\mathrm{Bd}}(A \cup B) > 0$, then we have that $\overline{\mathrm{Bd}}(A) > 0$ or $\overline{\mathrm{Bd}}(B) > 0$, or both.*

(g) *If $\mathrm{Bd}((A + p) \cap (A + q)) = 0$, then $\overline{\mathrm{Bd}}((A + p) \cup (A + q)) = \overline{\mathrm{Bd}}(A + p) + \overline{\mathrm{Bd}}(A + q)$.*

Proof. The proof is analogous to that of Proposition 2.4 but using, in each case, the appropriate expression (from those stated in Remark 2.7) for the lower and upper Banach densities. □

Remark 2.9. As in Remark 2.5 it should be noted that the upper Banach density is additive for translations of a fixed set, that is: given $A \subset \mathbb{N}_0$ and any finite sequence $(p_n)_{n=1}^k \subset \mathbb{N}_0$ satisfying $\mathrm{Bd}((A + p_n) \cap (A + p_l)) = 0$ for every $1 \leq n < l \leq k$, then

$$\overline{\mathrm{Bd}}\left(\bigcup_{n=1}^{k} A + p_n \right) = \sum_{n=1}^{k} \overline{\mathrm{Bd}}(A + p_n) = k \cdot \overline{\mathrm{Bd}}(A).$$

This fact will be used in Proposition 2.10 below.

Let us now include some well-known relations between the densities introduced in this section and the collections of sets of natural numbers already used in Section 1. These facts together with Propositions 1.3, 2.4 and 2.8 will be used in Section 3:

Proposition 2.10. *Let $A, B \subset \mathbb{N}_0$. The following statements hold:*

(a) *A is thick if and only if $\overline{\mathrm{Bd}}(A) = 1$. Equivalently, A is syndetic if and only if $\underline{\mathrm{Bd}}(A) > 0$.*

(b) *If $\underline{\mathrm{Bd}}(A) = 1$, then A is thickly syndetic.*

(c) *If A is piecewise syndetic, then $\overline{\mathrm{Bd}}(A) > 0$.*

(d) *If $\overline{\mathrm{Bd}}(A) > 0$, then A is an AP-set.*

(e) *If $\overline{\mathrm{Bd}}(A) > 0$, then $(A - A) \cap \mathbb{N}_0$ is syndetic (and even $(A - A) \cap B \neq \varnothing$ for every $B \in \Delta$).*

(f) *If $\underline{\mathrm{dens}}(A) + \overline{\mathrm{dens}}(B) > 1$ or $\underline{\mathrm{Bd}}(A) + \overline{\mathrm{Bd}}(B) > 1$, then $A \cap B \neq \varnothing$.*

Proof. (a) If A is thick for each $N \in \mathbb{N}$ there exists $m_N \in A$ such that $[m_N, m_N + N] \subset A$ so

$$\overline{\mathrm{Bd}}(A) = \lim_{N \to \infty} \left(\max_{n \geq 0} \frac{\#(A \cap [n, n + N])}{N + 1} \right) \geq \lim_{N \to \infty} \frac{\#(A \cap [m_N, m_N + N])}{N + 1} = 1.$$

If A is syndetic there exists $m_A \in \mathbb{N}$ such that $A \cap [n + 1, n + m_A] \neq \varnothing$ for all $n \in \mathbb{N}_0$ so

$$\begin{aligned}\underline{\mathrm{Bd}}(A) &= \lim_{N\to\infty}\left(\min_{n\geq 0}\frac{\#(A\cap[n,n+N])}{N+1}\right) = \lim_{N\to\infty}\left(\min_{n\geq 0}\frac{\#(A\cap[n,n+(m_A\cdot N)])}{m_A\cdot N+1}\right)\\ &\geq \lim_{N\to\infty}\frac{N}{m_A\cdot N+1} = \frac{1}{m_A} > 0.\end{aligned}$$

Finally, if $\overline{\mathrm{Bd}}(A)=1$ then $\underline{\mathrm{Bd}}(\mathbb{N}_0\setminus A)=0$ so $\mathbb{N}_0\setminus A$ is not syndetic and hence A is thick.

(b) If $\underline{\mathrm{Bd}}(A)=1$ then $\overline{\mathrm{Bd}}(\mathbb{N}_0\setminus A)=0$. Now it is enough to show (c), since then $\mathbb{N}_0\setminus A$ is not piecewise syndetic and by Proposition 1.3 the set A is thickly syndetic.

(c) Suppose that $A=B\cap C$ where B is thick and C is syndetic. Let $m\in\mathbb{N}$ such that $C\cap[n+1,n+m]\neq\varnothing$ for all $n\in\mathbb{N}_0$. For each $N\in\mathbb{N}$ there exists $x_N\in B$ fulfilling that $[x_N,x_N+(m\cdot N)]\subset B$. Hence

$$\#(A\cap[x_N,x_N+(m\cdot N)]) = \#(C\cap[x_N,x_N+(m\cdot N)]) \geq N,$$

so we obtain that

$$\begin{aligned}\overline{\mathrm{Bd}}(A) &= \lim_{N\to\infty}\left(\max_{n\geq 0}\frac{\#(A\cap[n,n+N])}{N+1}\right) = \lim_{N\to\infty}\left(\max_{n\geq 0}\frac{\#(A\cap[n,n+(m\cdot N)])}{m\cdot N+1}\right)\\ &\geq \lim_{N\to\infty}\frac{\#(A\cap[x_N,x_N+(m\cdot N)])}{m\cdot N+1} \geq \lim_{N\to\infty}\frac{N}{m\cdot N+1} = \frac{1}{m} > 0.\end{aligned}$$

(d) This is a (non-trivial) consequence of the Szemerédi theorem (see [85]).

(e) Suppose that there exists $B\in\Delta$, and hence an infinite set $C\subset\mathbb{N}_0$ with $(C-C)\cap\mathbb{N}\subset B$, such that $(A-A)\cap(C-C)\cap\mathbb{N}\subset(A-A)\cap B=\varnothing$. Let $(x_j)_{j\in\mathbb{N}}$ be the increasing sequence of integers forming the infinite set C and note that $(A+x_j)\cap(A+x_i)=\varnothing$ for all $j\neq i\in\mathbb{N}$. Finally, given $k\in\mathbb{N}$ such that $\overline{\mathrm{Bd}}(A)>1/k$, by Remark 2.9 we get the contradiction

$$\overline{\mathrm{Bd}}\left(\bigcup_{j=1}^{k} A+x_j\right) = \sum_{j=1}^{k}\overline{\mathrm{Bd}}(A+x_j) = k\cdot\overline{\mathrm{Bd}}(A) > 1.$$

Since a set is syndetic if and only if it intersects every thick set, and since $\mathcal{T}\subset\Delta$, the proof is finished (see [41, Theorem 3.18] for an alternative proof via symbolic dynamics).

(f) If $A\cap B=\varnothing$ we have that $B\subset\mathbb{N}_0\setminus A$ and we get the contradiction

$$1 < \underline{\mathrm{dens}}(A)+\overline{\mathrm{dens}}(B) \leq \underline{\mathrm{dens}}(A)+\overline{\mathrm{dens}}(\mathbb{N}_0\setminus A) = 1. \qquad \square$$

Remark 2.11. Following [17] we will denote the collection of sets with positive density as:

$$\underline{\mathcal{BD}} := \{A\subset\mathbb{N}_0 : \underline{\mathrm{Bd}}(A)>0\}, \qquad \overline{\mathcal{BD}} := \{A\subset\mathbb{N}_0 : \overline{\mathrm{Bd}}(A)>0\},$$
$$\underline{\mathcal{D}} := \{A\subset\mathbb{N}_0 : \underline{\mathrm{dens}}(A)>0\}, \qquad \overline{\mathcal{D}} := \{A\subset\mathbb{N}_0 : \overline{\mathrm{dens}}(A)>0\}.$$

Given $0<\delta\leq 1$ we will denote the collection of sets with density greater or equal to δ by:

$$\underline{\mathcal{BD}}_\delta := \{A\subset\mathbb{N}_0 : \underline{\mathrm{Bd}}(A)\geq\delta\}, \qquad \overline{\mathcal{BD}}_\delta := \{A\subset\mathbb{N}_0 : \overline{\mathrm{Bd}}(A)\geq\delta\},$$
$$\underline{\mathcal{D}}_\delta := \{A\subset\mathbb{N}_0 : \underline{\mathrm{dens}}(A)\geq\delta\}, \qquad \overline{\mathcal{D}}_\delta := \{A\subset\mathbb{N}_0 : \overline{\mathrm{dens}}(A)\geq\delta\}.$$

The following is used in [21] (see Chapter 1, Section 2, Proof of Theorem 2.1):

Lemma 2.12 (**Argument found in [16, Theorem 14]**). *Let $A, B \subset \mathbb{N}_0$ fulfilling that for each $n \in \mathbb{N}_0$ there exists $k_n \in \mathbb{N}_0$ such that $(A \cap [0,n]) + k_n \subset B$. Then $\overline{\mathrm{Bd}}(A) \leq \overline{\mathrm{Bd}}(B)$.*

Proof. Fixed any $N \in \mathbb{N}$ let n_N such that $\max_{n\geq 0} \#(A \cap [n, n+N]) = \#(A \cap [n_N, n_N + N])$. Let $n := n_N + N$. There exists $k_n \in \mathbb{N}_0$ such that $(A \cap [0,n]) + k_n \subset B$ so

$$(A \cap [n_N, n_N + N]) + k_n \subset B \quad \text{and then} \quad \max_{j\geq 0} \#(B \cap [j, j+N]) \geq \#(A \cap [n_N, n_N + N]).$$

We get that

$$\overline{\mathrm{Bd}}(B) = \lim_{N\to\infty} \left(\max_{j\geq 0} \frac{\#(B \cap [j, j+N])}{N+1} \right) \geq \lim_{N\to\infty} \left(\max_{j\geq 0} \frac{\#(A \cap [j, j+N])}{N+1} \right) = \overline{\mathrm{Bd}}(A). \qquad \square$$

3 Furstenberg families

The term "*Furstenberg family*" was first used by Akin in [1], where the different approaches regarding the study of the size of the return sets (used by Gottschalk and Hedlundand [48] and by Furstenberg [41]) were unified in the context of compact dynamical systems. In this memoir we have used Furstenberg families in a very similar way through the concepts of $\mathcal{F}$-recurrence and $\mathcal{F}$-hypercyclicity. However, if one looks at the definitions of Furstenberg family given in each chapter/article one may note some differences between them:

– In Chapter 1, Section 8, we said that $\mathcal{F} \subset \mathscr{P}(\mathbb{N}_0)$ was a Furstenberg family whenever it was *hereditarily upward*, i.e. if given $A \in \mathcal{F}$ the inclusion $A \subset B$ implies that B belongs to $\mathcal{F}$. This definition coincides with that given by Akin in [1], and has been also recently used in the context of $\mathcal{F}$-hypercyclicity; see [16], [23], [24] and [25].

– In Chapter 2, Section 1.2, we said that $\mathcal{F} \subset \mathscr{P}(\mathbb{N}_0)$ was a Furstenberg family if, besides being *hereditarily upward*, each set A from the collection $\mathcal{F}$ was *infinite*. This is a natural extra assumption that does not reduce the amount of Furstenberg families $\mathcal{F}$ for which we can study $\mathcal{F}$-recurrence, since the return sets of a recurrent vector are always infinite. Moreover, this definition has been also used in the $\mathcal{F}$-hypercyclicity context; see [17] and [50].

– In Chapter 3, Section 4, we said that $\mathcal{F} \subset \mathscr{P}(\mathbb{N}_0)$ was a Furstenberg family if, apart from the above properties (being *hereditarily upward* and containing *infinite sets*), each set A from $\mathcal{F}$ had the property that $A \cap [n, \infty[$ belonged to $\mathcal{F}$ for all $n \in \mathbb{N}$. The families with this last property have been called *finitely invariant* in the recent work [25]. One can easily check that all the families $\mathcal{F}$ considered in this memoir fulfill this property and, moreover, this is also a natural condition in our $\mathcal{F}$-recurrence context by the following reasoning:

  Let (X, T) be a dynamical system and suppose that there exists a point $x \in X \setminus \mathrm{Per}(T)$ such that: for each neighbourhood V of x, the set $N_T(x, V) \setminus \{0\}$ belongs to some collection of sets $\mathcal{F} \subset \mathscr{P}(\mathbb{N}_0)$. Since $x \notin \mathrm{Per}(T)$, given any neighbourhood U of x and any $n \in \mathbb{N}$ we can choose another neighbourhood $U_n \subset U$ of x such that $T^j x \notin U_n$ for all $1 \leq j < n$. Hence

$$N_T(x, U_n) \setminus \{0\} \subset N_T(x, U) \cap [n, \infty[,$$

  so that, if $\mathcal{F}$ is hereditarily upward, then also $N_T(x, U) \cap [n, \infty[$ must belong to $\mathcal{F}$.

This last *finitely invariant* condition allows us to work in a much easier way when considering filters and free filters in [51] (see Chapter 3, Section 4, Definition 4.3 and Remark 4.4). For the rest of the Appendix we will use this last definition of Furstenberg family given in Chapter 3.

The *dual family* of a Furstenberg family $\mathcal{F} \subset \mathscr{P}(\mathbb{N}_0)$ is defined as the collection of sets

$$\mathcal{F}^* := \{A \subset \mathbb{N}_0 \text{ infinite} : A \cap B \neq \varnothing \text{ for all } B \in \mathcal{F}\}.$$

Note that, if we denote by $\mathcal{F}^{**} := (\mathcal{F}^*)^*$, then we have that $\mathcal{F}^{**} = \mathcal{F}$. Indeed:

- given $A \in \mathcal{F}$ we have that $A \cap B \neq \varnothing$ for all $B \in \mathcal{F}^*$ so that $A \in \mathcal{F}^{**}$ and hence $\mathcal{F} \subset \mathcal{F}^{**}$;
- conversely, given $A \in \mathcal{F}^{**}$ it is clear that $\mathbb{N}_0 \setminus A \notin \mathcal{F}^*$ so there exists $B \in \mathcal{F}$ for which $B \cap (\mathbb{N}_0 \setminus A) = \varnothing$, then $B \subset A$ and therefore $A \in \mathcal{F}$.

This means that every family, as defined here, is a dual family. Moreover, for any two families $\mathcal{F}_1, \mathcal{F}_2 \subset \mathscr{P}(\mathbb{N}_0)$ we have that

$$\mathcal{F}_1 \subset \mathcal{F}_2 \quad \text{implies that} \quad \mathcal{F}_2^* \subset \mathcal{F}_1^*,$$

that is, the direction of the inclusion is exchanged when considering dual families.

Example 3.1. Let us exemplify the previous concepts by using the collections of sets introduced along Sections 1 and 2 of this Appendix:

(a) By Remark 1.2 we know that the collections of sets $\mathcal{I}$, $\mathcal{T}$, $\mathcal{S}$, $\mathcal{TS}$, $\mathcal{PS}$, $\mathcal{AP}$, $\mathcal{IP}$ and Δ are the Furstenberg families of infinite, thick, syndetic, thickly syndetic, piecewise syndetic, AP, IP and Δ sets respectively. By Proposition 1.3 we know that $\mathcal{T}^* = \mathcal{S}$, $\mathcal{TS}^* = \mathcal{PS}$,

$$\mathcal{TS} \subset \mathcal{T} \cap \mathcal{S} \subset \mathcal{PS} \subset \mathcal{AP},$$

and also that

$$\mathcal{AP}^* \subset \mathcal{TS} \subset \mathcal{T} \subset \mathcal{IP} \subset \Delta \subset \mathcal{I}.$$

Taking the dual inclusions and recalling that $\mathcal{I}^* = \{A \subset \mathbb{N}_0 : A \text{ is cofinite}\}$ we have that

$$\mathcal{I}^* \subset \Delta^* \subset \mathcal{IP}^* \subset \mathcal{S} \subset \mathcal{PS} \subset \mathcal{AP}.$$

(b) By Remark 2.11 and the monotony of the densities (see Propositions 2.4 and 2.8) we know that the collections of sets $\overline{\mathcal{D}}$, $\underline{\mathcal{D}}$, $\overline{\mathcal{BD}}$, $\underline{\mathcal{BD}}$ and for each $0 < \delta \leq 1$ the collections $\overline{\mathcal{D}}_\delta$, $\underline{\mathcal{D}}_\delta$, $\overline{\mathcal{BD}}_\delta$, $\underline{\mathcal{BD}}_\delta$ are also Furstenberg families. By Proposition 2.10 we know that

$$\overline{\mathcal{BD}}_1 = \mathcal{T} \qquad \text{and} \qquad \underline{\mathcal{BD}} = \mathcal{S},$$

$$\mathcal{PS} \subset \overline{\mathcal{BD}} \subset \mathcal{AP} \qquad \text{and} \qquad \mathcal{AP}^* \subset \underline{\mathcal{BD}}_1 \subset \mathcal{TS}.$$

By statement (f) of Proposition 2.10 we also have that

$$\underline{\mathcal{BD}}_1 = \overline{\mathcal{BD}}^*, \quad \underline{\mathcal{D}}_1 = \overline{\mathcal{D}}^*, \quad \overline{\mathcal{D}}_1 = \underline{\mathcal{D}}^*, \quad \overline{\mathcal{BD}}_1 = \underline{\mathcal{BD}}^*.$$

Another concept used in Chapters 1 and 3 is that of *u.f.i. upper* Furstenberg family. This was introduced in [20] where a kind of *Birkhoff transitivity theorem* for the *upper* families is shown. Let us recall the exact definition and some examples: a Furstenberg family $\mathcal{F}$ is said to be *upper* if it can be written as

$$\mathcal{F} = \bigcup_{\delta \in D} \mathcal{F}_\delta \quad \text{with} \quad \mathcal{F}_\delta = \bigcap_{m \in M} \mathcal{F}_{\delta,m}$$

for sets $\mathcal{F}_{\delta,m} \subset \mathscr{P}(\mathbb{N}_0)$ such that $\delta \in D$ and $m \in M$, where D is arbitrary, M is countable and:

(i) for any $\mathcal{F}_{\delta,m}$ and any $A \in \mathcal{F}_{\delta,m}$ there exists a finite set $F \subset \mathbb{N}_0$ such that

$$A \cap F \subset B \text{ implies } B \in \mathcal{F}_{\delta,m};$$

(ii) for any $A \in \mathcal{F}$ there exists some $\delta \in D$ such that $(A - n) \cap \mathbb{N}_0 \in \mathcal{F}_\delta$ for all $n \in \mathbb{N}_0$.

We say that an upper family $\mathcal{F}$ is *uniformly finitely invariant* (called *u.f.i.* for short), if for any $A \in \mathcal{F}$ there is some $\delta \in D$ such that $A \cap [n, \infty[\in \mathcal{F}_\delta$ for all $n \in \mathbb{N}$.

Example 3.2. The families $\mathcal{I}, \mathcal{AP}, \overline{\mathcal{BD}}$ and $\overline{\mathcal{D}}$ are easily checked to be *upper*:

$\mathcal{I}$: As it is done in [20] we can consider $\mathcal{F}_m \subset \mathscr{P}(\mathbb{N}_0)$ as the set

$$\mathcal{F}_m := \{A \subset \mathbb{N}_0 : \exists N \geq m \text{ with } N \in A\} \qquad \text{for each } m \in \mathbb{N}.$$

Clearly $\mathcal{I} = \bigcap_{m\in\mathbb{N}} \mathcal{F}_m$ and fixed any $A \in \mathcal{F}_m$ with $N \geq m$ and $N \in A$, we can take $F := \{N\}$ which implies (i). Condition (ii) is trivially fulfilled.

$\mathcal{AP}$: As it is done in [24] we can consider $\mathcal{F}_m \subset \mathscr{P}(\mathbb{N}_0)$ as the set

$$\mathcal{F}_m := \{A \subset \mathbb{N}_0 : \exists l \geq m \text{ and } x, n \in \mathbb{N} \text{ such that } \{x + kn : 0 \leq k \leq l\} \subset A\},$$

for each $m \in \mathbb{N}$. Clearly $\mathcal{AP} = \bigcap_{m\in\mathbb{N}} \mathcal{F}_m$ and fixed any $A \in \mathcal{F}_m$ we can consider an arithmetic progression of length greater than m included in A as the finite set F to fulfill property (i) of the definition. Condition (ii) is also trivially fulfilled.

$\overline{\mathcal{D}}$: As it is done in [20] we can consider $\mathcal{F}_{\delta,m} \subset \mathscr{P}(\mathbb{N}_0)$ as the set

$$\mathcal{F}_{\delta,m} := \left\{A \subset \mathbb{N}_0 : \exists N \geq m, n_N \in \mathbb{N}_0 \text{ with } \frac{\#(A \cap [n_N, n_N + N])}{N+1} > \delta\right\},$$

for each $0 < \delta < 1$ and $m \in \mathbb{N}$. Clearly $\overline{\mathcal{D}} = \bigcup_{0<\delta<1} \bigcap_{m\in\mathbb{N}} \mathcal{F}_{\delta,m}$ and for any $A \in \mathcal{F}_{\delta,m}$ with $N \geq m$ and $\#(A \cap [0, N]) > (N+1)\delta$ then $F := [0, N]$ implies (i). Moreover, condition (ii) comes from the left-invariant property of the densities (see Proposition 2.4).

$\overline{\mathcal{BD}}$: As it is done in [20] we can consider $\mathcal{F}_{\delta,m} \subset \mathscr{P}(\mathbb{N}_0)$ as the set

$$\mathcal{F}_{\delta,m} := \left\{A \subset \mathbb{N}_0 : \exists N \geq m, n_N \in \mathbb{N}_0 \text{ with } \frac{\#(A \cap [n_N, n_N + N])}{N+1} > \delta\right\},$$

for each $0 < \delta < 1$ and $m \in \mathbb{N}$. Conditions (i) and (ii) follow as in the previous case.

The *u.f.i.* property is also trivially fulfilled by these families.

4 The families $\mathcal{IP}$, Δ, $\mathcal{IP}^*$ and Δ^* in dynamics

The Furstenberg families $\mathcal{IP}$, Δ, and their dual families $\mathcal{IP}^*$ and Δ^*, have repeatedly appeared along our work. For instance, we showed in Section 4 of Chapter 3 that the respective concepts of $\mathcal{IP}$-recurrence and Δ-recurrence are equivalent to the weakest possible recurrence notion (i.e. that of usual recurrence), while the dual families $\mathcal{IP}^*$ and Δ^* characterize the strongest $\mathcal{F}$-recurrence notions (weaker than periodicity) among those studied here. Our aim in this section is to recall three basic (and really well-known) properties of these families that have been constantly used in Chapters 1, 2 and 3:

- the families $\mathcal{IP}^*$ and Δ^* are filters;
- the equality $\mathcal{IP}\mathrm{Rec}(T) = \Delta\mathrm{Rec}(T) = \mathrm{Rec}(T)$ holds for every dynamical system (X, T);
- the inclusions $\mathrm{Per}(T) \subset \Delta^*\mathrm{Rec}(T) \subset \mathcal{IP}^*\mathrm{Rec}(T)$ holds for every dynamical system (X, T).

Recall first that the family $\mathcal{IP}$ can be expressed as the union of all non-zero idempotents $0 \neq p \in \beta\mathbb{N}_0$ (see [13, Definition 1.2]), and that Δ is also the union of a collection of ultrafilters (see [14, Definition 1.6 and Lemma 1.9]). By the definition of ultrafilter it is then obvious that the families $\mathcal{IP}$ and Δ are *partition regular*, i.e. if we fix $\mathcal{F} = \mathcal{IP}$ or Δ, then for each $A \in \mathcal{F}$ and each partition $A = A_1 \cup A_2$ we necessarily have that either A_1 or A_2 belongs to $\mathcal{F}$.

The *partition regular* condition (also called *Ramsey property* in the literature) easily implies the *filter* condition of the dual families $\mathcal{IP}^*$ and Δ^* by the following general well-known fact (see for instance [17, Lema 2.1]):

Lemma 4.1. *Given a Furstenberg family $\mathcal{F} \subset \mathscr{P}(\mathbb{N})$ the following are equivalent:*

(i) *$\mathcal{F}$ is partition regular;*

(ii) *$\mathcal{F}^*$ is a filter.*

Proof. (i) $\Rightarrow$ (ii): Given $B, C \in \mathcal{F}^*$ we have to check that $B \cap C \in \mathcal{F}^*$: for each $A \in \mathcal{F}$ we have that $A \cap B \neq \varnothing$ and $A = (A \cap B) \cup (A \setminus B)$, so the partition regular condition of $\mathcal{F}$ implies that either $A \cap B \in \mathcal{F}$ or $A \setminus B \in \mathcal{F}$. Since $(A \setminus B) \cap B = \varnothing$ we necessarily have that $A \cap B \in \mathcal{F}$, and finally

$$A \cap (B \cap C) = (A \cap B) \cap C \neq \varnothing,$$

which implies that $B \cap C$ belongs again to $\mathcal{F}^*$.

(ii) $\Rightarrow$ (i): Given $A \in \mathcal{F}$ and $A_1, A_2 \subset \mathbb{N}_0$ with $A = A_1 \cup A_2$ we have to check that $A_i \in \mathcal{F} = \mathcal{F}^{**}$ for some $i \in \{1, 2\}$. If this was false there would be $B_1, B_2 \in \mathcal{F}^*$ such that $A_i \cap B_i = \varnothing$ for each $i \in \{1, 2\}$, and the filter condition would imply that $C = B_1 \cap B_2$ belongs to $\mathcal{F}^*$ arriving to the contradiction

$$\varnothing \neq A \cap C \subset (A_1 \cap B_1) \cup (A_2 \cap B_2) = \varnothing. \qquad \square$$

Since $\mathcal{F} = \mathcal{F}^{**}$ holds for every family $\mathcal{F}$, this lemma also shows that $\mathcal{F}$ is a filter if and only if $\mathcal{F}^*$ is partition regular. The fact that $\mathcal{IP}^*$ and Δ^* are filters has been repeatedly used along this memoir (see Chapter 1, Section 6; Chapter 2, Sections 4 and 8; Chapter 3, Section 6).

Let us now prove the equality

$$\mathcal{IP}\text{Rec}(T) = \Delta\text{Rec}(T) = \text{Rec}(T),$$

which has been used in Chapter 1 (see Section 6) and Chapter 3 (see Section 4.2):

Proposition 4.2 (Extension of [41, Theorem 2.17]). *Let $T : X \longrightarrow X$ be a continuous map on a topological space X. For every recurrent point $x \in \text{Rec}(T)$ (that is, $x \in \overline{\text{Orb}(Tx, T)}$) and every neighbourhood U of x the return set*

$$N_T(x, U) := \{n \in \mathbb{N}_0 : T^n x \in U\} \text{ belongs to } \mathcal{IP}.$$

As a result, the equality

$$\mathcal{IP}\text{Rec}(T) = \Delta\text{Rec}(T) = \text{Rec}(T)$$

holds for every dynamical system (X, T).

Proof. Set $U_1 := U$ and let $n_1 \in N_T(x, U_1)$. By continuity of T we can find a neighbourhood $U_2 \subset U$ of x such that

$$T^{n_1}(U_2) \subset U_1.$$

Pick any $n_2 \in N_T(x, U_2)$ with $n_2 > n_1$ and note that

$$T^n x \in U \quad \text{for each } n \in \{n_1, n_2, n_1 + n_2\}.$$

Assume that we have constructed $(U_j)_{j=1}^k$, neighbourhoods of x included in U, and $(n_j)_{j=1}^k$, increasing sequence of positive integers, fulfilling that $n_j \in N_T(x, U_j)$ for each $1 \leq j \leq k$ but also that

$$T^{n_j}(U_l) \subset U_j \quad \text{for each } 1 \leq j < l \leq k.$$

Using again continuity we find a neighbourhood $U_{k+1} \subset U$ such that

$$T^{n_j}(U_{k+1}) \subset U_j \quad \text{for each } 1 \leq j \leq k.$$

Picking $n_{k+1} \in N_T(x, U_{k+1})$ with $n_{k+1} > n_k$ we get that

$$T^n x \in U \quad \text{for each } n \in \left\{ \sum_{j \in F} n_j : F \subset \{1, 2, ..., k, k+1\} \right\}.$$

Recursively we get a sequence $(n_j)_{j \in \mathbb{N}}$ which clearly fulfills that

$$\left\{ \sum_{j \in F} n_j : F \text{ finite subset of } \mathbb{N} \right\} \subset N_T(x, U),$$

so $N_T(x, U)$ belongs to $\mathcal{IP}$. Finally, since $\mathcal{IP} \subset \Delta$ we get that

$$\mathcal{IP}\text{Rec}(T) \subset \Delta\text{Rec}(T) \subset \text{Rec}(T) \subset \mathcal{IP}\text{Rec}(T).$$

□

In Section 6 of Chapter 1 we included the definition of *product recurrent vector* (see page 44). In general, given a dynamical system (X, T) we say that $x \in X$ is a *product recurrent point* if, for any dynamical system (S, Y) and any recurrent point $y \in \mathrm{Rec}(S)$, the vector $(x, y) \in X \times Y$ is recurrent for the direct product dynamical system $T \times S : X \times Y \longrightarrow X \times Y$. In view of this definition, Proposition 4.2 shows that being $\mathcal{IP}^*$-recurrent is a **sufficient** condition for a point to be a product recurrent point: indeed, the return sets of an $\mathcal{IP}^*$-recurrent point always intersect the IP-sets (and hence the return sets of every recurrent point).

The **necessity** is also proved in the same result [41, Theorem 2.17]: using the construction given there, for each increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers fulfilling that

$$\sum_{k=1}^{j} n_k < n_{j+1} \quad \text{for all } j \in \mathbb{N},$$

then there exist an operator $T \in \mathcal{L}(X)$, a vector $x \in X$ and a neighbourhood U of x such that

$$N_T(x, U) = \left\{ \sum_{k\in F} n_k : F \text{ finite subset of } \mathbb{N} \right\}.$$

Note that each IP-set contains a set as $\{\sum_{k\in F} n_k : F \text{ finite subset of } \mathbb{N}\}$ for a sequence $(n_k)_{k\in\mathbb{N}}$ with the mentioned properties.

We finally focus on the "Δ^*-recurrent behaviour" of the periodic points:

Lemma 4.3. *For any $p \in \mathbb{N}$ the set $p \cdot \mathbb{N} = \{pn : n \in \mathbb{N}\}$ belongs to Δ^* (and hence to $\mathcal{IP}^*$). As a result, the inclusion*

$$\mathrm{Per}(T) \subset \Delta^*\mathrm{Rec}(T) \subset \mathcal{IP}^*\mathrm{Rec}(T)$$

holds for every dynamical system (X, T).

Proof. Let p be a positive integer, $B \subset \mathbb{N}_0$ be an arbitrary but fixed infinite set, and suppose that $(n_k)_{k\in\mathbb{N}}$ is the increasing sequence of integers forming the set B. It is then obvious that there exist a class $[j] \in \mathbb{Z}/p\mathbb{Z}$ and an infinite set $A \subset \mathbb{N}$ such that $[j] = [n_k] \in \mathbb{Z}/p\mathbb{Z}$ for all $k \in A$. Picking $k_1, k_2 \in A$ with $k_1 \neq k_2$ we get that

$$[n_{k_2} - n_{k_1}] = [0] \in \mathbb{Z}/p\mathbb{Z} \quad \text{and then} \quad n_{k_2} - n_{k_1} \in (B - B) \cap p\mathbb{N}.$$

Since $\Delta^* \subset \mathcal{IP}^*$ we have that $p\mathbb{N}$ also belongs to $\mathcal{IP}^*$. □

When (X, T) is a complex linear dynamical system then we also have, in between, the set of *unimodular eigenvectors*

$$\mathrm{Per}(T) \subset \mathrm{span}(\mathcal{E}(T)) \subset \Delta^*\mathrm{Rec}(T) \subset \mathcal{IP}^*\mathrm{Rec}(T).$$

This is shown in detail in [50] (see Chapter 2, Section 4, Proposition 4.1). Note also that:

(a) Proposition 4.2 + Lemma 4.3 $\Rightarrow$ $\mathrm{Rec}(T) = \mathrm{Rec}(T^p)$ for every $p \in \mathbb{N}$;

(b) Proposition 4.2 + Proposition 4.1 of Chapter 2 $\Rightarrow$ $\mathrm{Rec}(T) = \mathrm{Rec}(\lambda T)$ for every $\lambda \in \mathbb{T}$.

The precise argument has been included in Proposition 4.7 of Chapter 3.

We refer the reader to the textbook [41, Chapters 8 and 9] for more on the dynamical role of the families $\mathcal{IP}$, Δ, $\mathcal{IP}^*$ and Δ^*.

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Index of notation

Index