Improved results on stability analysis of time-varying delay systems via delay partitioning method and Finsler's Lemma

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Abstract

This paper proposes novel conditions based on linear matrix inequalities (LMI) for stability analysis of arbitrarily-fast time-varying delays systems. The time-varying delay interval is divided into smaller pieces in order to obtain an equivalent switched model with multiple time-varying delays of smaller interval, which differently from other existing approaches, the maximum switching frequency is not required for stability analysis. Thus, by the use of augmented Lyapunov-Krasovskii functionals and the Finsler's lemma, together with some relationships among state variables intentionally defined, the inherent conservatism can be progressively reduced by refining more and more the delay partition. The superiority of the proposed method is illustrated through two benchmark examples.

Keywords: Stability analysis, Time-varying delay, Delay partitioning, Switched system, Lyapunov-Krasovskii method, Finsler's Lemma

1. Introduction

Time-varying delays frequently appear in many engineering applications, such as network control systems [**?**], multiagent systems [**?**] and vehicle suspension systems [**?**]. It is well known that the presence of delays generally leads to poor performance or even instability if they are large enough, although some exceptions have been reported in [**? ?**] where delays may improve the system performance. Motivated by the influence of time delays in the system behavior, the research of more efficient methods for stability analysis of time delay systems has been addressed in last decades, where the survey [?] gives an overview of the most recent contributions in this topic. To cope with their infinite dimensional nature, the use of Lyapunov-Krasovskii Functional (LKF) methods, among others, has been extensively applied in this aim. The advantage of LKF is that the stability analysis can be guaranteed by checking a finite number of Linear Matrix Inequalities (LMI). Nevertheless, some degree of conservatism is unavoidable, depending on the choice of the LKF and the bounding techniques used to linearize the stability conditions.

Therefore, all efforts are aimed at reducing the inherent conservatism of LKF-based approaches, where two main approaches can be distinguished: (i) the research of more advanced structures of LKF: augmented LKF [? ?], delay-partitioning method [? ? ?], triple integral [? ?] and multiple integral based LKF [? ?] among others, and (ii) the research of bounding methods aimed at reducing the gap of the integral inequalities of the form $\int_{t-h}^{t} x^T(s)Zx(s)$ by extending classical Jensen's inequality to Wirtinger's inequality [? ?] and other techniques, such as auxiliary-matrix inequality [?], free-matrix-based inequality [? ? ? ? ?], and Bessel-Legendre inequality [? ?]. For time-varying delay systems, different methods have been provided to handle the reciprocal convexity (see e.g., reciprocally convex combination lemma [?]), where further refinements of this method have been proposed in [? ? ? ?] by including more slack variables using different strategies in order to obtain tighter boundings. Other recent works introduced additional degree of freedom for stability analysis by means of delay partitoning with delay-mode LKF in [?], overlapped switching LKF in [?], and some relaxation techniques to deal with quadratic functions [? ?] and cubic functions [?] of time-varying delays, whose negativity is proved by exploiting some geometrical properties [?]. The wide number of recent contributions in this field reveals that there still exists room for improvement in the sense of conservatism reduction for stability analysis of time-varying delay systems, which motivates this work.

The objective of this paper is to obtain new LMI based conditions for time-varying delay systems allowing arbitrarily fast or discontinuous time-varying delays, as usual in communication over networks [?]. The proposed method first divides the time-varying delay interval into smaller pieces so as the time-varying delay system is modeled as a switched system with multiple *auxiliary* delay functions of smaller delay interval. Hence, an augmented LKF is proposed to obtain LMI conditions for stability analysis, where differently from [?], the LKF does not depend on the switching signal between the defined auxiliary delays. In addition, some relationships among different variables of the augmented state vector intentionally defined are treated by Finsler's Lemma [?], leading to further conservatism reduction.

The remainder of the paper is organized as follows: Section ?? describes the problem statement and gives some preliminary results. Section ?? describes the proposed delay partition method and the defined equivalence between state variables. Section ?? presents the stability analysis criteria, Section ?? shows and discusses the achieved improvements through two benchmark examples, and finally some conclusions are gathered in Section ??.

2. Problem statement and preliminaries

Consider the following delayed system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t - d(t)), \\ x(t) &= \phi(t), \quad t \in [-\bar{d}, 0] \end{aligned} \tag{1}$$

where $x(t) \in \mathcal{R}^n$ is the state variable, the initial condition $\phi(t) : \mathcal{R} \to \mathcal{R}^n$ is a continuous function defined in $[-\bar{d}, 0]$, and $0 \leq d(t) \leq \bar{d}$ is a time-varying delay function with no constraint on the time-derivative $\dot{d}(t)$, that is to say, arbitrarily fast time-varying delays are allowed.

The objective of the paper is to establish stability criteria for system (??) based on the delay partition method described later in Section ?? so as the conservatism can be reduced as long as the number of delay intervals is higher. To this end, the following preliminary results are given:

Lemma 1 *[? , Lemma 1] Let n be a positive integer and a, b two real values. Given a symmetric matrix $Z \in \mathcal{R}^n > 0$, for any continuous function ω in $[a, b] \to \mathcal{R}^n$, the following inequality holds:*

$$(b-a)\int_a^b \dot{\omega}^T(s) Z \dot{\omega}(s) ds \geq \mathcal{W}^T \begin{bmatrix} Z & 0 & 0 \\ 0 & 3Z & 0 \\ 0 & 0 & 5Z \end{bmatrix} \mathcal{W} \tag{2}$$

where

$$\mathcal{W} = \begin{bmatrix} \omega(b) - \omega(a) \\ \omega(b) + \omega(a) - \frac{2}{b-a}\int_a^b \omega(s) ds \\ \omega(b) - \omega(a) + \frac{6}{b-a}\int_a^b \omega(s) ds - \frac{12}{(b-a)^2}\int_a^b \int_\theta^b \omega(s) ds d\theta \end{bmatrix} \tag{3}$$

Lemma 2 (Extended Reciprocally Convex Inequality) *[? , Lemma 2] Let n be a positive integer, and $Z \in \mathcal{R}^n > 0$ be a symmetric definite matrix. If there exist symmetric matrices X_1, X_2, X_3, X_4 and full matrices Y_1, Y_2, Y_3, Y_4 of appropriate dimensions such that*

$$\begin{bmatrix} Z - \alpha X_1 - \alpha^2 X_3 & -\alpha Y_1 - (1-\alpha)Y_2 - \alpha^2 Y_3 - (1-\alpha)^2 Y_4 \\ (*) & Z - (1-\alpha)X_2 - (1-\alpha)^2 X_4 \end{bmatrix} \geq 0 \tag{4}$$

for $\alpha = 0, 1$, then the following inequality holds $\forall \alpha \in (0,1)$:

$$\begin{bmatrix} \frac{1}{\alpha}Z & 0 \\ (*) & \frac{1}{1-\alpha}Z \end{bmatrix} \geq \begin{bmatrix} Z + S_1 & S_2 \\ (*) & Z + S_3 \end{bmatrix} \tag{5}$$

where

$$\begin{aligned} &S_1 = (1-\alpha)X_1 + \alpha(1-\alpha)X_3, \qquad S_3 = \alpha X_2 + \alpha(1-\alpha)X_4, \\ &S_2 = \alpha Y_1 + (1-\alpha)Y_2 + \alpha^2 Y_3 + (1-\alpha)^2 Y_4 \end{aligned} \tag{6}$$

Lemma 3 *[? , Lemma 4] Let $f(\alpha) = g_0 + g_1\alpha + g_2\alpha^2$, where $\alpha \in [\underline{\alpha}, \overline{\alpha}]$ and $g_0, g_1, g_2 \in \mathcal{R}$. Suppose that the following conditions are satisfied for a certain integer $N_p > 1$, (i) $f(\underline{\alpha}) < 0$, (ii) $f(\overline{\alpha}) < 0$, and (iii) $\frac{\delta_\alpha}{2^{N_p+1}}\dot{f}\left(\frac{j-1}{2^{N_p}}\delta_\alpha + \underline{\alpha}\right) + f\left(\frac{j-1}{2^{N_p}}\delta_\alpha + \underline{\alpha}\right)$, $\forall j = 1, ..., 2^{N_p}$ with $\delta_\alpha = \overline{\alpha} - \underline{\alpha}$. Then $f(\alpha) < 0$, $\forall \alpha \in [\underline{\alpha}, \overline{\alpha}]$.*

Lemma 4 (Finsler's Lemma) *[? , Lemma 2] Let n be a positive integer, $\Xi \in \mathcal{R}^n$ be a symmetric definite matrix, and $\mathcal{T} \in \mathcal{R}^{m\times n}$ such that $m < n$. The following statements are equivalent:*

- *(i) $\xi^T \Xi \xi < 0, \quad \forall \xi \in \mathcal{R}^n \ / \ \mathcal{T}\xi = 0, \quad \xi \neq 0$,*
- *(ii) $\exists M \in \mathcal{R}^{n\times m}$ such that $\Xi + (M\mathcal{T})^T + (M\mathcal{T}) < 0$,*
- *(iii) $\mathcal{T}^{\perp T}\Xi\mathcal{T}^{\perp} < 0$*

where $\mathcal{T}^{\perp}$ is a right orthogonal complement of $\mathcal{T}$.

3. Description of the proposed delay partition method

Before proceeding with the stability analysis, the idea is to divide the time-varying delay interval $\mathcal{I} = [0, \bar{d}]$ into N smaller pieces $\mathcal{I}_q$ such that $\mathcal{I} = \mathcal{U}_{q=1}^N \mathcal{I}_q$, where

$$\mathcal{I}_q = \begin{cases} [\delta_{q-1}, \quad \delta_q[, \quad q = 1, ..., N-1, \\ [\delta_{q-1}, \quad \delta_q] \quad q = N. \end{cases} \tag{7}$$

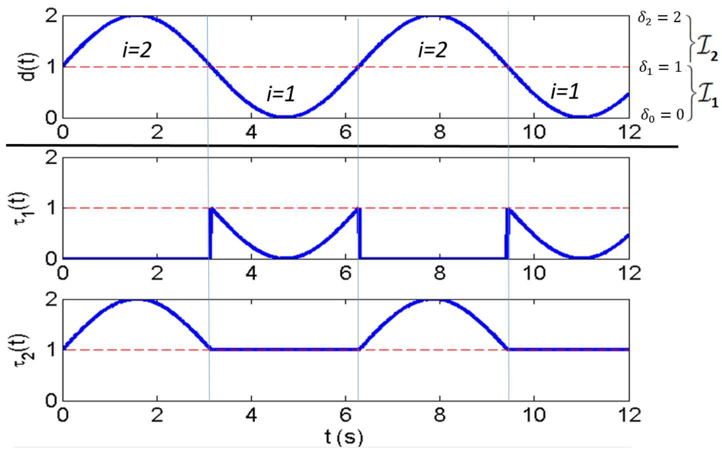

Figure 1: Example of the proposed delay partition of a time-varying delay with $d(t) = 1 + sin(t)$ and $N = 2$ subintervals $\mathcal{I}_1$,$\mathcal{I}_2$ with $\delta_0 = 0, \delta_1 = 1, \delta_2 = 2$.

and

$$\delta_0 = 0, \qquad \delta_q := q\delta, \quad q = 1, ..., N, \quad \delta = \frac{\bar{d}}{N} \tag{8}$$

Now, let us introduce the following auxiliary time-varying delay functions $\tau_q(t) \equiv \tau_q(d(t)),\ q = 1, ..., N$ defined as:

$$\tau_q(t) := \begin{cases} d(t) & \text{if } d(t) \in \mathcal{I}_q, \\ \delta_{q-1} & \text{otherwise} \end{cases} \tag{9}$$

which can be reformulated as

$$\tau_q(t) = \delta_{q-1} + \delta\alpha_q(t) \tag{10}$$

where $0 \leq \alpha_q(t) \leq 1$ can be viewed as the normalized time-varying delay corresponding to each $\tau_q(t)$, defined as:

$$\alpha_q(t) = \frac{\tau_q(t) - \delta_{q-1}}{\delta_q - \delta_{q-1}} = \frac{\tau_q(t) - \delta_{q-1}}{\delta} \tag{11}$$

Note that $\alpha_q(t)$ can also be expressed as the convex sum:

$$\alpha_q(t) = \sum_{i=1}^{N} \lambda_i(t) \left((1 - \alpha_q(t)) \times 0 + \alpha_q(t) \times \hat{\gamma}_{q,i}\right) = \sum_{i=1}^{N} \lambda_i(t)\alpha_q(t)\hat{\gamma}_{q,i} \tag{12}$$

where

$$\hat{\gamma}_{q,i} = \begin{cases} 1 & \text{if } q = i \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

and $\lambda_i(t),\ i = 1, ..., N$ are scalar functions defined as:

$$\lambda_i(t) = \begin{cases} 1 & \text{if } d(t) \in \mathcal{I}_i \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

satisfying the convex properties: $0 \leq \lambda_i(t) \leq 1$ and $\sum_{i=1}^{N} \lambda_i(t) = 1, \forall i = 1, ..., N$. Note that system (**??**) can equivalently be modeled as the following switched system with multiple time-varying delays with reduced delay interval $\delta = \bar{d}/N$:

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{N} \lambda_i(t) A_d x(t - \tau_i(t)) \tag{15}$$

understanding that i denotes the number of the delay subinterval $\mathcal{I}_q,\ q = 1, ..., N$ where $d(t)$ is contained at instant t.

To better illustrate the above definitions, an example is provided in Fig. 1 for a time-varying delay $d(t) = 1 + sin(t)$ (depicted in the top part of the figure). Choosing $N = 2$, the two delay subintervals $\mathcal{I}_1 = [\delta_0, \delta_1[$ and $\mathcal{I}_2 = [\delta_1, \delta_2[$ are obtained with $\delta_0 = 0$, $\delta_1 = 1$ and $\delta_2 = 2$. The second and third pictures (from top to bottom) in Fig. 1 represent the time-evolution of each time-varying delay function $\tau_q(t),\ q = 1, 2$. It can be seen that $d(t) \in \mathcal{I}_2,\ \forall t \in [0, \pi[$ $(i = 2)$ and $d(t) \in \mathcal{I}_1,\ \forall t \in [\pi, 2\pi[$ $(i = 1)$. Therefore, $\lambda_1(t) = 0$, $\lambda_2(t) = 1$, $\tau_1(t) \equiv \delta_0 = 0$, $\tau_2(t) = d(t)$ $\forall t \in [0, \pi[$, and $\lambda_1(t) = 1$, $\lambda_2(t) = 0$, $\tau_1(t) = d(t)$, $\tau_2(t) \equiv \delta_1 = 1$, $\forall t \in [\pi, 2\pi[$.

Remark 1 *A similar approach was recently proposed in [**?**] by modeling the varying delay $d(t)$ with the composition of multiple time-varying functions (denoted as delay modes) defined in different delay intervals. However, differently from [**?**], each delay mode $\tau_q(t)$ is intentionally set to the lower bound of the delay subinterval $\mathcal{I}_q$, say, $\tau_q(t) = \delta_{q-1}$ when $d(t) \notin \mathcal{I}_q$. This definition is helpful to reduce conservatism in the stability analysis, as later discussed in Remark* **??**.

Remark 2 *The polytopic model adopted in (**??**) describes the switching between delay modes through the functions $\lambda_i(t)$. This differs from [**?**], where a delay-mode based LKF is proposed with different conservatism depending on the maximum switching frequency. In this paper, the stability is proven here by convexity using continuous LKF (later depicted in (**??**), (**??**)) and the polytopic description of system (**??**). Therefore, any arbitrarily fast switching frequency between delay modes is allowed for the proposed stability analysis.*

Remark 3 *From the definition of $\lambda_i(t)$ in (**??**), it can be deduced that $\lambda_i(t)\lambda_j(t) = 0,\ \forall i \neq j$ and $\lambda_i^2(t) = \lambda_i(t)$. Therefore, the equivalence:*

$$\left(\sum_{i=1}^{N} \lambda_i(t)\mathcal{X}_i\right)\left(\sum_{i=1}^{N} \lambda_i(t)\mathcal{Y}_i\right) = \sum_{i=1}^{N} \lambda_i(t)\mathcal{X}_i\mathcal{Y}_i \tag{16}$$

is fulfilled for any arbitrary matrices $\mathcal{X}_i, \mathcal{Y}_i$ of proper dimensions. This property will play a key role to simplify the stability analysis by handling cross product terms with linear dependence on $\lambda_i(t)$ without increasing the number of LMI constraints. More details about this advantage are given in Remark **??**.

Remark 4 *Let i the delay interval where $d(t)$ is contained at instant t, say, $d(t) \in \mathcal{I}_i$, and $\lambda_i(t) = 1$. Then, by setting $\tau_q(t) = \delta_{q-1}, \forall q \neq i$, the following equivalences can be deduced:*

$$\begin{aligned} &\sum_{i=1}^{N} \lambda_i(t)\left(x(t-\tau_{q^*}(t)) - x(t-\delta_{q^*-1})\right) = 0, \\ &q^* = 1, ..., N, \quad q^* \neq i \end{aligned} \tag{17}$$

*The equivalences (**??**) will be useful to further reduce conservatism in stability analysis by exploiting the above relationships by virtue of Lemma **??**.*

The equivalences (**??**) can be illustrated following the example of Fig. 1: for instance, $x(t-\tau_1(t)) - x(t) = 0$ is true since $\tau_1(t) = 0\ \forall t \in [0, \pi[$. Analogously, $x(t-\tau_2(t)) - x(t-1) = 0$ is true since $\tau_1(t) = 1\ \forall t \in [\pi, 2\pi[$.

4. Stability analysis

First, let us introduce the helpful notation for next results:

$$e_k = \begin{bmatrix} 0_{1\times k-1} & 1 & 0_{1\times 8N-k} \end{bmatrix} \otimes I_n, \quad 1 \leq k \leq 8N \tag{18}$$

understanding that $0_{1\times 0}$ represents the empty set. Given a set of positive integers denoted by $\chi = [k_1, ..., k_m]$, let $e_\chi = \begin{bmatrix} e_{k_1}^T & ... & e_{k_m}^T \end{bmatrix}^T$. Then, consider the matrices $e_{\chi_j}, j = 1, ..., 9$ and $e_{\chi_{10,i}}$, $e_{\chi_{11,i}}$ with:

$$\begin{aligned}
&\chi_1 = [k \ / \ 1 \leq k \leq N], && \chi_2 = [k \ / \ 2 \leq k \leq N+1], \\
&\chi_3 = [k \ / \ 2N+2 \leq k \leq 3N+1], && \chi_4 = [k \ / \ 3N+2 \leq k \leq 4N+1], \\
&\chi_5 = [k \ / \ 4N+2 \leq k \leq 5N+1], && \chi_6 = [k \ / \ 5N+2 \leq k \leq 6N+1], \\
&\chi_7 = [k \ / \ 6N+2 \leq k \leq 7N+1], && \chi_8 = [k \ / \ 7N+2 \leq k \leq 8N-1], \\
&\chi_9 = [k \ / \ 7N+2 \leq k \leq 8N], && \chi_{10,i} = [k \ / \ 1 \leq k \leq N, k \neq i] \\
&\chi_{11,i} = [k \ / \ N+2 \leq k \leq 2N+1, k-N-1 \neq i], && i = 1, ..., N
\end{aligned} \tag{19}$$

Also, let us introduce the augmented vectors:

$$\begin{aligned}
\xi_1(t) &= \big[\underbrace{x^T(t),\ x^T(t-\delta_1), \cdots,\ x^T(t-\delta_{N-1})}_{N},\ \underbrace{u_1(t), \cdots, u_N(t)}_{N}, \underbrace{\tilde{u}_1(t), \cdots, \tilde{u}_N(t)}_{N}\big]^T, \\
\xi_2(t) &= \big[\underbrace{x^T(t),\ x^T(t-\delta_1), \cdots,\ x^T(t-\delta_{N-1})}_{N}, \underbrace{\dot{x}^T(t),\ \dot{x}^T(t-\delta_1), \cdots, \dot{x}^T(t-\delta_{N-2})}_{N-1}\big]^T, \\
u_q(t) &= \int_{t-\delta_q}^{t-\delta_{q-1}} x(s)ds, \qquad \tilde{u}_q = \int_{t-\delta_q}^{t-\delta_{q-1}} \int_s^{t-\delta_{q-1}} x(\theta)d\theta ds, \qquad q = 1, ..., N.
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
\bar{\xi}(t) = \big[&\underbrace{x^T(t),\ x^T(t-\delta_1),\ \cdots,\ x^T(t-\delta_N)}_{N+1},\ \underbrace{x^T(t-\tau_1(t)),\ \cdots,\ x^T(t-\tau_N(t))}_{N}, \\
&\underbrace{v_1^T(t),\ \cdots,\ v_N^T(t)}_{N},\ \underbrace{w_1^T(t),\ \cdots, w_N^T(t)}_{N},\ \underbrace{\tilde{v}_1^T(t),\ \cdots,\ \tilde{v}_N^T(t)}_{N},\ \underbrace{\tilde{w}_1^T(t),\ \cdots, \tilde{w}_N^T(t)}_{N} \\
&\underbrace{u_1^T(t), \cdots, u_N^T(t)}_{N},\ \underbrace{\dot{x}^T(t-\delta_1),\ \cdots,\ \dot{x}^T(t-\delta_{N-1})}_{N-1}\big]^T,
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
v_q(t) &= \frac{1}{\tau_q(t)-\delta_{q-1}} \int_{t-\tau_q(t)}^{t-\delta_{q-1}} x(s)ds, \quad w_q(t) = \frac{1}{\delta_q - \tau_q(t)} \int_{t-\delta_q}^{t-\tau_q(t)} x(s)ds, \\
\tilde{v}_q(t) &= \frac{1}{(\tau_q(t)-\delta_{q-1})^2} \int_{t-\tau_q(t)}^{t-\delta_{q-1}} \int_s^{t-\delta_{q-1}} x(s)d\theta ds, \\
\tilde{w}_q(t) &= \frac{1}{(\delta_q - \tau_q(t))^2} \int_{t-\delta_q}^{t-\tau_q(t)} \int_s^{t-\tau_q(t)} x(\theta)d\theta ds, \qquad q = 1, ..., N
\end{aligned} \tag{22}$$

The notation $He(.)$ denotes $He(.) = (.) + (.)^T$ for any input matrix, and the symbol $(*)$ in a matrix denotes the term induced by symmetry.

The following theorem proves the stability of (**??**) for any arbitrarily-fast varying delay $0 \leq d(t) \leq \bar{d}$, given a certain number of delay subintervals $N > 1$:

Theorem 1 *Given a maximum delay $\bar{d}$ and some partition $N > 1$, system (**??**) is stable for a certain integer $N_p > 1$ if there exist symmetric matrices $P \in \mathcal{R}^{3nN}, Q \in \mathcal{R}^{2nN-n}$, $Z_i \in \mathcal{R}^n > 0$, symmetric matrices $X_{1,i}, X_{2,i}, X_{3,i}, X_{4,i} \in \mathcal{R}^{3n}$, $i = 1, ..., N$, and full matrices $Y_{1,i}$, $Y_{2,i}$, $Y_{3,i}, Y_{4,i} \in \mathcal{R}^{3n}, M_{1,i}, M_{2,i} \in \mathcal{R}^{(8nN)\times(nN)}$ such that the following LMIs hold $\forall [i, j] = [1, ..., N] \times [1, ..., 2^{N_p}]$:*

$$\Pi_{8,i}^{\perp T} \left(\mathcal{F}_{1,i}\right) \Pi_{8,i}^{\perp} < 0, \qquad \Pi_{8,i}^{\perp T} \left(\mathcal{F}_{1,i} + \mathcal{F}_{2,i} + \mathcal{F}_{3,i}\right) \Pi_{8,i}^{\perp} < 0, \tag{23}$$
$$\Pi_{8,i}^{\perp T} \left(\mathcal{F}_{1,i} + \left(\frac{2j-1}{2^{N_p+1}}\right) \mathcal{F}_{2,i} + \left(\frac{j^2-j}{2^{2N_p}}\right) \mathcal{F}_{3,i}\right) \Pi_{8,i}^{\perp} < 0$$

and

$$\mathcal{G}_{1,i} \geq 0, \qquad \mathcal{G}_{1,i} + \mathcal{G}_{2,i} + \mathcal{G}_{3,i} \geq 0, \tag{24}$$
$$\mathcal{G}_{1,i} + \left(\frac{2j-1}{2^{N_p+1}}\right) \mathcal{G}_{2,i} + \left(\frac{j^2-j}{2^{2N_p}}\right) \mathcal{G}_{3,i} \geq 0$$

where $\Pi_{8,i}^{\perp}$ denote a right orthogonal complement of $\Pi_{8,i} = e_{\chi_{10,i}} - e_{\chi_{11,i}}$ (say, *any matrix* $\Pi_{8,i}^{\perp}$ *satisfying* $\Pi_{8,i}\Pi_{8,i}^{\perp}=0$*), and*

$$
\begin{aligned}
&\mathcal{F}_{1,i}=He\left(\Pi_{11}^{T}P\Pi_{2,i}+M_{1,i}\Pi_{71}\right)+\Pi_{3,i}^{T}Q\Pi_{3,i}-\Pi_{4}^{T}Q\Pi_{4}+\Pi_{5,i}^{T}\bar{Z}\Pi_{5,i}-\Pi_{61},\\
&\mathcal{F}_{2,i}=He\left(\Pi_{12,i}^{T}P\Pi_{2,i}+M_{1,i}\Pi_{72,i}+M_{2,i}\Pi_{71}\right)-\Pi_{62,i}, \qquad (25)\\
&\mathcal{F}_{3,i}=He\left(\Pi_{13,i}^{T}P\Pi_{2,i}+M_{2,i}\Pi_{72,i}\right)-\Pi_{63,i},\\
&\mathcal{G}_{1,i}=\begin{bmatrix}\mathcal{Z}_i & -Y_{2,i}-Y_{4,i}\\ (*) & \mathcal{Z}_i-X_{2,i}-X_{4,i}\end{bmatrix},\quad \mathcal{G}_{2,i}=\begin{bmatrix}-X_{1,i} & -Y_{1,i}+Y_{2,i}+2Y_{4,i}\\ (*) & X_{2,i}+2X_{4,i}\end{bmatrix},\\
&\mathcal{G}_{3,i}=\begin{bmatrix}-X_{3,i} & -Y_{3,i}-Y_{4,i}\\ (*) & -X_{4,i}\end{bmatrix},\\
&\Pi_{11}=\begin{bmatrix}e_{\chi_1}\\ e_{\chi_7}\\ \delta^2 e_{\chi_6}\end{bmatrix},\ \Pi_{12,i}=\begin{bmatrix}0\\0\\ \delta^2\hat{\Gamma}_i\left(e_{\chi_3}-2e_{\chi_6}\right)\end{bmatrix},\Pi_{13,i}=\begin{bmatrix}0\\0\\ \delta^2\hat{\Gamma}_i\left(e_{\chi_5}+e_{\chi_6}-e_{\chi_3}\right)\end{bmatrix},\\
&\Pi_{2,i}=\begin{bmatrix}\hat{\Omega}_i\\ e_{\chi_9}\\ e_{\chi_1}-e_{\chi_2}\\ \delta e_{\chi_1}-e_{\chi_7}\end{bmatrix},\quad \Pi_{3,i}=\begin{bmatrix}e_{\chi_1}\\ \hat{\Omega}_i\\ e_{\chi_8}\end{bmatrix},\quad \Pi_4=\begin{bmatrix}e_{\chi_2}\\ e_{\chi_9}\end{bmatrix},\quad \Pi_{5,i}=\begin{bmatrix}\hat{\Omega}_i\\ e_{\chi_9}\end{bmatrix},\\
&\Pi_{61}=\sum_{q=1}^{N}\mathcal{W}_q\Phi_{1,q}\mathcal{W}_q^T,\qquad \Pi_{62,i}=\mathcal{W}_i\Phi_{2,i}\mathcal{W}_i^T,\qquad \Pi_{63,i}=\mathcal{W}_i\Phi_{3,i}\mathcal{W}_i^T,\\
&\Pi_{71,i}=\delta e_{\chi_4}-e_{\chi_7},\qquad \Pi_{72,i}=\delta\hat{\Gamma}_i\left(e_{\chi_3}-e_{\chi_4}\right),
\end{aligned}
$$

$$
\begin{aligned}
&\Phi_{1,i}=\begin{bmatrix}\mathcal{Z}_i+X_{1,i} & Y_{2,i}+Y_{4,i}\\ (*) & \mathcal{Z}_i\end{bmatrix},\quad \Phi_{2,i}=\begin{bmatrix}X_{3,i}-X_{1,i} & Y_{1,i}-Y_{2,i}-2Y_{4,i}\\ (*) & X_{2,i}+X_{4,i}\end{bmatrix},\\
&\Phi_{3,i}=\begin{bmatrix}-X_{3,i} & Y_{3,i}+Y_{4,i}\\ (*) & -X_{4,i}\end{bmatrix}, \qquad (26)
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{Z}_i=diag\left(Z_i,3Z_i,5Z_i\right),\quad \bar{Z}=\delta^2\left(diag_{q=1}^{N}Z_q\right),\quad \mathcal{W}_i=\begin{bmatrix}\mathcal{W}_{1,i}^T & \mathcal{W}_{2,i}^T\end{bmatrix},\\
&\mathcal{W}_{1,i}=\begin{bmatrix}e_i-e_{N+1+i}\\ e_i+e_{N+1+i}-2e_{2N+1+i}\\ e_i-e_{N+1+i}+6e_{2N+1+i}-12e_{4N+1+i}\end{bmatrix}, \qquad (27)\\
&\mathcal{W}_{2,i}=\begin{bmatrix}e_{N+1+i}-e_{1+i}\\ e_{N+1+i}+e_{1+i}-2e_{3N+1+i}\\ e_{N+1+i}-e_{1+i}+6e_{3N+1+i}-12e_{5N+1+i}\end{bmatrix},\\
&\hat{\Omega}_i=Ae_1+A_d e_{N+1+i},\qquad \hat{\Gamma}_i=diag\left(\begin{bmatrix}0_{1\times i-1} & 1 & 0_{1\times N-i}\end{bmatrix}\right)\otimes I_n
\end{aligned}
$$

Proof: Consider the Lyapunov functional:

$$V(t) = V_1(t) + V_2(t) + V_3(t) \tag{28}$$

where

$$\begin{aligned} V_1(t) &= \xi_1^T(t) P \xi_1(t), \\ V_2(t) &= \int_{t-\delta}^{t} \xi_2^T(s) Q \xi_2(s) ds \\ V_3(t) &= \delta \sum_{q=1}^{N} \int_{-\delta}^{0} \int_{t+s}^{t} \dot{x}^T(\theta - \delta_{q-1}) Z_q \dot{x}(\theta - \delta_{q-1}) d\theta ds \end{aligned} \tag{29}$$

Time-derivative of $V_1(t), V_2(t)$ and $V_3(t)$ render:

$$\begin{aligned} \dot{V}_1(t) &= \dot{\xi}_1^T(t) P \xi_1(t) + \xi_1^T(t) P \dot{\xi}_1(t), \\ \dot{V}_2(t) &= \xi_2^T(t) Q \xi_2(t) - \xi_2^T(t-\delta) Q \xi_2(t-\delta), \\ \dot{V}_3(t) &= \sum_{q=1}^{N} \dot{x}^T(t-\delta_{q-1}) \left(\delta^2 Z_q\right) \dot{x}(t-\delta_{q-1}) - \delta \sum_{q=1}^{N} \int_{t-\delta_q}^{t-\delta_{q-1}} \dot{x}^T(s) Z_q \dot{x}(s) ds \end{aligned} \tag{30}$$

The integral terms in the rightmost part of $\dot{V}_3(t)$ given in (**??**) can be decomposed as:

$$\begin{aligned} &\delta \int_{t-\delta_q}^{t-\delta_{q-1}} \dot{x}^T(s) Z_q \dot{x}(s) ds = \\ &\frac{1}{\alpha_q(t)} (\tau_q(t) - \delta_{q-1}) \int_{t-\tau_q(t)}^{t-\delta_{q-1}} \dot{x}^T(s) Z_q \dot{x}(s) ds \\ &+ \frac{1}{1-\alpha_q(t)} (\delta_q - \tau_q(t)) \int_{t-\delta_q}^{t-\tau_q(t)} \dot{x}^T(s) Z_q \dot{x}(s) ds \end{aligned} \tag{31}$$

where $0 \leq \alpha_q(t) \leq 1$ is the normalized time-varying delay defined in (**??**). Applying the extended Wirtinger's inequality (Lemma **??**), we obtain:

$$\begin{aligned} &-(\tau_q(t) - \delta_{q-1}) \int_{t-\tau_q(t)}^{t-\delta_{q-1}} \dot{x}^T(s) Z_q \dot{x}(s) ds \leq -\bar{\xi}^T(t) \mathcal{W}_{1,q}^T \mathcal{Z}_q \mathcal{W}_{1,q} \bar{\xi}(t), \\ &-(\delta_q - \tau_q(t)) \int_{t-\delta_q}^{t-\tau_q(t)} \dot{x}^T(s) Z_q \dot{x}(s) \leq -\bar{\xi}^T(t) \mathcal{W}_{2,q}^T \mathcal{Z}_q \mathcal{W}_{2,q} \bar{\xi}(t). \end{aligned} \tag{32}$$

Hence, from (??)-(??), one has that

$$\delta \int_{t-\delta_q}^{t-\delta_{q-1}} \dot{x}^T(s) Z_q \dot{x}(s) ds \leq \bar{\xi}^T(t) \left(\frac{1}{\alpha_q(t)} \mathcal{W}_{1,q}^T \mathcal{Z}_q \mathcal{W}_{1,q} + \frac{1}{1-\alpha_q(t)} \mathcal{W}_{2,q}^T \mathcal{Z}_q \mathcal{W}_{2,q} \right) \bar{\xi}(t), \tag{33}$$

where $\alpha_q(t)$ is above defined in (??). By applying the extended reciprocally convex inequality (see Lemma ??), the following inequality is obtained:

$$\frac{1}{\alpha_q(t)} \mathcal{W}_{1,q}^T \mathcal{Z}_q \mathcal{W}_{1,q} + \frac{1}{1-\alpha_q(t)} \mathcal{W}_{2,q}^T \mathcal{Z}_q \mathcal{W}_{2,q} \leq \mathcal{W}_q \Phi_q(t) \mathcal{W}_q^T, \tag{34}$$

where $\Phi_q(t)$ is defined as:

$$\Phi_q(t) = \begin{bmatrix} \mathcal{Z}_q + S_{1,q}(t) & S_{2,q}(t) \\ (*) & \mathcal{Z}_q + S_{3,q}(t) \end{bmatrix} \tag{35}$$

with

$$\begin{aligned} S_{1,q}(t) &= (1-\alpha_q(t))\, X_{1,q} + \alpha_q(t)\,(1-\alpha_q(t))\, X_{3,q}, \\ S_{2,q}(t) &= \alpha_q(t) Y_{1,q} + (1-\alpha_q(t))\, Y_{2,q} + \alpha_q^2(t) Y_{3,q} + (1-\alpha_q(t))^2\, Y_{4,q} \\ S_{3,q}(t) &= \alpha_q(t) X_{2,q} + \alpha_q(t)\,(1-\alpha_q(t))\, X_{4,q} \end{aligned} \tag{36}$$

being $X_{1,q}, X_{2,q}, X_{3,q}, X_{4,q}$ symmetric matrices and $Y_{1,q}, Y_{2,q}, Y_{3,q}, Y_{4,q}$ full matrices satisfying:

$$\begin{aligned} &\begin{bmatrix} \mathcal{Z}_q & 0 \\ (*) & \mathcal{Z}_q \end{bmatrix} - \alpha_q(t) \begin{bmatrix} X_{1,q} & Y_{1,q} \\ (*) & 0 \end{bmatrix} - (1-\alpha_q(t)) \begin{bmatrix} 0 & Y_{2,q} \\ (*) & X_{2,q} \end{bmatrix} \\ &- \alpha_q^2(t) \begin{bmatrix} X_{3,q} & Y_{3,q} \\ (*) & 0 \end{bmatrix} - (1-\alpha_q(t))^2 \begin{bmatrix} 0 & Y_{4,q} \\ (*) & X_{4,q} \end{bmatrix} \geq 0, \end{aligned} \tag{37}$$

or equivalently expressed with the definition of $\mathcal{G}_{1,q}$, $\mathcal{G}_{2,q}$, $\mathcal{G}_{3,q}$ in (??) as

$$\mathcal{G}_{1,q} + \mathcal{G}_{2,q}\alpha_q(t) + \mathcal{G}_{3,q}\alpha_q^2(t) \geq 0 \tag{38}$$

From the above definitions and $u_q(t)$, $\tilde{u}_q(t)$ in (??), the following equivalences can be deduced:

$$u_q(t) = (\tau_q(t) - \delta_{q-1})\, v_q(t) + (\delta_q - \tau_{q-1}(t))\, w_q(t) \tag{39}$$

and

$$
\begin{aligned}
\tilde{u}_q(t) &= \left(\tau_q(t)-\delta_{q-1}\right)^2 \tilde{v}_q(t) + \left(\delta_q - \tau_{q-1}(t)\right)^2 \tilde{w}_q(t) \\
&+ \left(\tau_q(t)-\delta_{q-1}\right)\left(\delta_q - \tau_{q-1}(t)\right) v_q(t)
\end{aligned} \tag{40}
$$

Considering the definitions of $\xi_1(t)$ and $\xi_2(t)$ in (??), the above equivalence (??), and the system model (??), the terms given below can be expressed as a function of the augmented state $\bar{\xi}(t)$ given in (??):

$$
\begin{aligned}
&\xi_1(t) = \Pi_1(t)\bar{\xi}(t), && \dot{\xi}_1(t) = \Pi_2(t)\bar{\xi}(t), \\
&\xi_2(t) = \Pi_3\bar{\xi}(t), && \xi_2\left(t-\delta\right) = \Pi_4\bar{\xi}(t), \\
&\dot{x}(t) = \Omega(t)\bar{\xi}(t), && \dot{x}\left(t-\delta_{q-1}\right) = e_{7N+1+q}\ \bar{\xi}(t), \quad q = 2, ..., N,
\end{aligned} \tag{41}
$$

where

$$
\begin{aligned}
&\Pi_1(t) = \begin{bmatrix} e_{\chi_1} \\ e_{\chi_7} \\ \mathcal{D}_0(t)e_{\chi_5} + \mathcal{D}_1(t)e_{\chi_6} + \mathcal{D}_2(t)e_{\chi_3} \end{bmatrix}, \\
&\Pi_2(t) = \begin{bmatrix} \Omega(t) \\ e_{\chi_9} \\ e_{\chi_1} - e_{\chi_2} \\ \delta e_{\chi_1} - e_{\chi_7} \end{bmatrix},\ \Pi_3(t) = \begin{bmatrix} e_{\chi_1} \\ \Omega(t) \\ e_{\chi_8} \end{bmatrix},\ \Pi_5(t) = \begin{bmatrix} \Omega(t) \\ e_{\chi_9} \end{bmatrix}, \\
&\mathcal{D}_0(t) = diag_{q=1}^{N}\left(\tau_q(t)-\delta_{q-1}\right)^2 \otimes I_n \ = \ \delta^2\Lambda^2(t)e_{\chi_5}, \\
&\mathcal{D}_1(t) = diag_{q=1}^{N}\left(\delta_q-\tau_q(t)\right)^2 \otimes I_n \ = \ \delta^2\left(I-\Lambda(t)\right)^2 e_{\chi_6}, \\
&\mathcal{D}_2(t) = diag_{q=1}^{N}\left(\tau_q(t)-\delta_{q-1}\right)\left(\delta_q-\tau_q(t)\right) \otimes I_n \ = \ \delta^2\Lambda(t)\left(I-\Lambda(t)\right) e_{\chi_3}
\end{aligned} \tag{42}
$$

and

$$
\Lambda(t) = diag_{q=1}^{N}\left(\alpha_q(t)\right), \qquad \Omega(t) = \sum_{i=1}^{N} \lambda_i(t)\hat{\Omega}_i \tag{43}
$$

From (??), (??), (??) and (??), the time-derivatives $\dot{V}_1(t)$ and $\dot{V}_2(t)$ can be expressed as:

$$
\begin{aligned}
\dot{V}_1(t) &= \bar{\xi}^T(t) He\left(\Pi_1^T(t)P\Pi_2(t)\right)\bar{\xi}(t), \\
\dot{V}_2(t) &= \bar{\xi}^T(t)\left(\Pi_3^T(t)Q\Pi_3(t) - \Pi_4^T Q\Pi_4\right)\bar{\xi}(t),
\end{aligned} \tag{44}
$$

and $\dot{V}_3(t)$ can be bounded by applying (**??**)-(**??**) as:

$$\dot{V}_3(t) \leq \bar{\xi}^T(t)\left(\Pi_5^T(t)\bar{Z}\Pi_5(t) - \Pi_6(t)\right)\bar{\xi}(t) \tag{45}$$

where $\Pi_6(t) = \sum_{q=1}^{N} \mathcal{W}_q \Phi_q(t) \mathcal{W}_q^T$ with $\Phi_q(t)$ is defined in (**??**). Then, the stability of system (**??**) can be ensured by proving

$$\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t) \leq \bar{\xi}^T(t)\bar{\Pi}(t)\bar{\xi}(t) < 0, \qquad \forall t \geq 0 \tag{46}$$

where

$$\begin{aligned}\bar{\Pi}(t) = He\left(\Pi_1^T(t)P\Pi_2(t)\right) + \Pi_3^T(t)Q\Pi_3(t) \\ - \Pi_4^T Q\Pi_4 + \Pi_5^T(t)\bar{Z}\Pi_5(t) - \Pi_6(t)\end{aligned} \tag{47}$$

provided that the following constraint obtained from (**??**) holds:

$$\Pi_7(t)\bar{\xi}(t) = 0 \tag{48}$$

where

$$\begin{aligned}\Pi_7(t) &= diag_{q=1}^{N}\left(\tau_q(t) - \delta_{q-1}\right)e_{\chi 3} + diag_{q=1}^{N}\left(\delta_q - \tau_q(t)\right)e_{\chi 4} - e_{\chi 7} \\ &= \delta\Lambda(t)e_{\chi 3} + \delta(I - \Lambda(t))e_{\chi 4} - e_{\chi 7}\end{aligned} \tag{49}$$

Moreover, as pointed out in Remark **??**, if we take into consideration the new relationships defined in (**??**), then we have the following extra conditions:

$$\sum_{i=1}^{N} \lambda_i(t)\left(e_{q^*} - e_{N+1+q^*}\right)\bar{\xi}(t) = 0, \quad \forall q^* = 1, ..., N,\ q^* \neq i. \tag{50}$$

which can be gathered into a single constraint that depends on the delay interval $\mathcal{I}_i$ where $d(t)$ is contained at instant t:

$$\Pi_8(t)\bar{\xi}(t) = 0 \tag{51}$$

where

$$\Pi_8(t) = \sum_{i=1}^{N} \lambda_i(t)\Pi_{8,i}, \qquad \Pi_{8,i} = e_{\chi 10,i} - e_{\chi 11,i}$$

Applying Lemma **??** (Finsler's lemma), it can be ensured that (**??**) subjected to (**??**) and (**??**) holds if there exist matrices of suitable dimensions $M_1(t)$ and $M_2(t)$ such that the inequality below is true:

$$\bar{\Pi}(t) + He\left(M_1(t)\Pi_7(t)\right) + He\left(M_2(t)\Pi_8(t)\right) < 0 \tag{52}$$

It should be noticed that condition (**??**) implies an infinite number of matrix inequalities to be checked due to its dependence on time. To circumvent this issue, in what follows a polytopic description for all terms depending on time will be found. First, taking into consideration the normalized delays $\alpha_q(t)$ given in the form (**??**) and the fact that $\alpha_q(t) = 0, \forall q \neq i$ and $\alpha_i(t) = 1$, it can be seen that $\Lambda(t)$ in (**??**) can be expressed as:

$$\Lambda(t) = \sum_{i=1}^{N} \lambda_i(t) \left((1 - \alpha_i(t)) \times 0_{nN} + \alpha_i(t) \times \hat{\Gamma}_i \right) = \sum_{i=1}^{N} \lambda_i(t) \alpha_i(t) \hat{\Gamma}_i \tag{53}$$

From the property of functions $\lambda_i(t)$ given in (**??**), one has that

$$\begin{aligned} \Lambda^2(t) &= \sum_{i=1}^{N} \lambda_i(t) \alpha_i^2(t) \hat{\Gamma}_i^2, \\ \Lambda(t)\left(I - \Lambda(t)\right) &= \sum_{i=1}^{N} \lambda_i(t) \left(\alpha_i(t) \hat{\Gamma}_i - \alpha_i^2(t) \hat{\Gamma}_i^2 \right) \end{aligned} \tag{54}$$

Noting from (**??**) that $\hat{\gamma}_{q,i}^2 = \hat{\gamma}_{q,i} = 0,\ \forall q \neq i,\ \hat{\gamma}_{i,i}^2 = \hat{\gamma}_{i,i} = 1$ and taking into account (**??**), it is easy to see that $\hat{\Gamma}_i^2 = \hat{\Gamma}_i$, and hence the matrix $\Pi_1(t)$ can be rewritten as $\Pi_1(t) = \begin{bmatrix} e_{\chi_1}^T & e_{\chi_7}^T & \delta^2 g^T(t) \end{bmatrix}^T$, where

$$\begin{aligned} &g(t) \equiv \Lambda^2(t) e_{\chi_5} + \left(I - \Lambda(t)\right)^2 e_{\chi_6} + \Lambda(t)\left(I - \Lambda(t)\right) e_{\chi_3} \\ &\sum_{i=1}^{N} \lambda_i(t) \left(e_{\chi_6} + \alpha_i(t) \hat{\Gamma}_i \left(e_{\chi_3} - 2e_{\chi_6}\right) + \alpha_i^2(t) \hat{\Gamma}_i \left(e_{\chi_5} + e_{\chi_6} - e_{\chi_3}\right) \right). \end{aligned} \tag{55}$$

Therefore, the time-varying terms $\Pi_1(t)$, $\Pi_2(t)$, $\Pi_3(t)$, $\Pi_5(t)$ and $\Pi_7(t)$ in

(**??**) and (**??**) can be reformulated as:

$$\Pi_1(t) = \sum_{i=1}^{N} \lambda_i(t) \left(\Pi_{11} + \alpha_i(t)\Pi_{12,i} + \alpha_i^2(t)\Pi_{13,i}\right), \tag{56}$$

$$\Pi_2(t) = \sum_{i=1}^{N} \lambda_i(t)\Pi_{12,i}, \quad \Pi_3(t) = \sum_{i=1}^{N} \lambda_i(t)\Pi_{3,i}, \quad \Pi_5(t) = \sum_{i=1}^{N} \lambda_i(t)\Pi_{5,i}$$

$$\Pi_7(t) = \sum_{i=1}^{N} \lambda_i(t) \left(\delta e_{\chi_4} - e_{\chi_7} + \alpha_i(t)\delta\hat{\Gamma}_i \left(e_{\chi_3} - e_{\chi_4}\right)\right)$$

$$= \sum_{i=1}^{N} \lambda_i(t) \left(\Pi_{71} + \alpha_i(t)\Pi_{72,i}\right)$$

Analogously, the time-varying terms $S_{1,q}(t), S_{2,q}(t), S_{3,q}(t)$ in (**??**) can be reformulated as:

$$S_{1,q}(t) = \sum_{i=1}^{N} \lambda_i(t) \left(X_{1,q} + \alpha_q(t)\hat{\gamma}_{q,i}(X_{3,q} - X_{1,q}) - \alpha_q^2(t)\hat{\gamma}_{q,i}X_{3,q}\right), \tag{57}$$

$$= \sum_{i=1}^{N} \lambda_i(t) \left(X_{1,q} + \alpha_i(t)(X_{3,i} - X_{1,i}) - \alpha_i^2(t)X_{3,i}\right),$$

$$S_{2,q}(t) = \sum_{i=1}^{N} \lambda_i(t) \left(Y_{2,q} + Y_{4,q} + \alpha_q(t)\hat{\gamma}_{q,i}(Y_{1,q} - Y_{2,q} - 2Y_{4,q}) + \alpha_q^2(t)\hat{\gamma}_{q,i}(Y_{3,q} + Y_{4,q})\right),$$

$$= \sum_{i=1}^{N} \lambda_i(t) \left(Y_{2,q} + Y_{4,q} + \alpha_i(t)(Y_{1,i} - Y_{2,i} - 2Y_{4,i}) + \alpha_i^2(t)(Y_{3,i} + Y_{4,i})\right),$$

$$S_{3,q}(t) = \sum_{i=1}^{N} \lambda_i(t) \left(\alpha_q(t)\hat{\gamma}_{q,i}(X_{2,q} + X_{4,q}) - \alpha_q^2(t)\hat{\gamma}_{q,i}X_{4,q}\right)$$

$$= \sum_{i=1}^{N} \lambda_i(t) \left(\alpha_i(t)(X_{2,i} + X_{4,i}) - \alpha_i^2(t)X_{4,i}\right),$$

and therefore from (**??**) and (**??**) one has that

$$\Pi_6(t) = \sum_{q=1}^{N} \mathcal{W}_q\Phi_q(t)\mathcal{W}_q^T = \sum_{i=1}^{N} \lambda_i(t) \left(\Pi_{61} + \alpha_i(t)\Pi_{62,i} + \alpha_i^2(t)\Pi_{63,i}\right) \tag{58}$$

Hence, noting again the property of $\lambda_i(t)$ given in (**??**) and defining

$$M_1(t) = \sum_{i=1}^{N} \lambda_i(t) \left(M_{1,i} + \alpha_i(t) M_{2,i}\right), \qquad M_2(t) = \sum_{i=1}^{N} \lambda_i(t) \tilde{M}_i, \tag{59}$$

the time-dependent condition (**??**) can be written as the convex sum of the functions $f_i(\alpha_i(t)) = \mathcal{F}_{1,i} + \alpha_i(t)\mathcal{F}_{2,i} + \alpha_i^2(t)\mathcal{F}_{3,i},\ i = 1, ..., N$, that is to say:

$$\sum_{i=1}^{N} \lambda_i(t) \left(f_i(\alpha_i(t)) + He\left(\tilde{M}_i \Pi_{8,i}\right)\right) < 0 \tag{60}$$

where $\mathcal{F}_{1,i}, \mathcal{F}_{2,i}, \mathcal{F}_{3,i}$ are defined in (**??**). By convexity, the inequality (**??**) is true if the following conditions are true $\forall i = 1, ..., N$

$$f_i(\alpha_i(t)) + He\left(\tilde{M}_i \Pi_{8,i}\right) < 0 \tag{61}$$

Applying again the Finsler's Lemma, the slack variables $\tilde{M}_i$ can be removed from above inequalities (**??**), leading to the equivalent conditions:

$$\Pi_{8,i}^{\perp T} \left(f_i(\alpha_i(t))\right) \Pi_{8,i}^{\perp} < 0, \quad i = 1, ..., N \tag{62}$$

Finally, noting that $0 \le \alpha_i(t) \le 1$ and the quadratic dependence on $\alpha_i(t)$ in $f_i(\alpha_i(t))$, applying Lemma **??** with $\underline{\alpha} = 0$, $\overline{\alpha} = 1$, the above conditions (**??**) and (**??**) hold if LMIs (**??**) and LMIs (**??**) are respectively true, concluding the proof. ■

Remark 5 *Note that the Lyapunov functional $V(t)$ (**??**) is independent of $\lambda_i(t)$ (**??**) and the discontinuous time-varying delays functions defined in (**??**). From this fact and the proof of Theorem **??**, it can be deduced that $V(t)$ is continuous and decreasing at the switching time instants between delay modes.*

The following corollary is obtained from Theorem **??** without taking into account the equivalences (**??**)

Corollary 1 *Given a maximum delay $\bar{d}$ and some partition $N > 1$, system (**??**) is stable for a certain integer $N_p > 1$ if there exist symmetric matrices $P \in \mathcal{R}^{3nN}, Q \in \mathcal{R}^{2nN-n}$, $Z_i \in \mathcal{R}^n > 0$, symmetric matrices $X_{1,i}, X_{2,i}, X_{3,i}, X_{4,i} \in \mathcal{R}^{3n},\ i = 1, ..., N$, and full matrices $Y_{1,i}$, $Y_{2,i}$, $Y_{3,i}, Y_{4,i} \in$*

$\mathcal{R}^{3n}, M_{1,i}, M_{2,i} \in \mathcal{R}^{(8nN)\times(nN)}$ *such that LMIs* (**??**) *and* (**??**) *(depicted below) hold* $\forall[i,j] = [1,...,N] \times [1,...,2^{N_p}]$*:*

$$\mathcal{F}_{1,i} < 0, \qquad \mathcal{F}_{1,i} + \mathcal{F}_{2,i} + \mathcal{F}_{3,i} < 0, \qquad (63)$$
$$\mathcal{F}_{1,i} + \left(\frac{2j-1}{2^{N_p+1}}\right)\mathcal{F}_{2,i} + \left(\frac{j^2-j}{2^{2N_p}}\right)\mathcal{F}_{3,i} < 0$$

where $\mathcal{F}_{1,i}, \mathcal{F}_{2,i}, \mathcal{F}_{3,i}$ *are defined in* (**??**).

Proof: The proof is similar to Theorem **??**, but without including the constraints (**??**) derived from (**??**). Hence, the condition (**??**) renders:

$$\bar{\Pi}(t) + He\left(M_1(t)\Pi_7(t)\right) < 0 \qquad (64)$$

which can equivalently be formulated as

$$\sum_{i=1}^{N} \lambda_i(t) f_i(\alpha_i(t)) < 0 \qquad (65)$$

Finally, the above condition can be proved checking the LMIs (**??**) by applying Lemma **??**. ■

Remark 6 *The property* (**??**)*, together with the polytopic model* (**??**) *and the definition of* $M_1(t), M_2(t)$ *in* (**??**)*, leads to a reduction of the number of LMIs in Theorem 1 and Corollary 1 since the product terms* $\Pi_1(t)P\Pi_2(t)$*,* $\Pi_3^T(t)Q\Pi_3(t)$*,* $\Pi_5^T(t)\bar{Z}\Pi_5(t)$*,* $\Lambda^2(t)$*,* $M_1(t)\Pi_7(t)$*,* $M_2(t)\Pi_8(t)$ *in* (**??**)*,* (**??**)*,* (**??**) *contain* N *terms, instead of* N^2*.*

Remark 7 *In view of the elimination of the slack variables* $\tilde{M}_i$ *via Finsler's Lemma (see* (**??**) *and* (**??**)*), the number of decision variables (NoV) are the same Theorem 1 and Corollary 1:* $NoV = (22.5N^2 + 25.5N + 0.5)\,n^2 + (6N - 0.5)\,n$*. This fact reveals that the defined relationships* (**??**) *are useful to reduce conservatism by applying Finsler's Lemma without increasing the complexity of the stability analysis method given in Theorem 1, as illustrated in next section.*

5. Numerical examples

In this section, two benchmark examples are provided to compare the maximum delay bound estimation obtained with the proposed method respect to other ones reported in the literature where an upper bound for the time-derivative of the varying delay function is not available.

5.1. Example 1

Method	$\bar{d}$	NoV
[? , Theorem 1]	1.64	143
[? , Theorem iv]	1.862	93
[? , Theorem 1]	1.975	-
[? , Theorem 1]	2.395	365
[? , Proposition 6-(ii)]	2.53	485
[? , Theorem 2]	2.542	424
Theorem 1 (N=2, $N_p = 3$)	2.82	589
Theorem 1 (N=3, $N_p = 3$)	3.40	1153
Theorem 1 (N=4, $N_p = 3$)	3.77	1897
Corollary 1 (N=2, $N_p = 3$)	2.79	589
Corollary 1 (N=3, $N_p = 3$)	3.34	1153
Corollary 1 (N=4, $N_p = 3$)	3.69	1897

Table 1: Maximum delay bound $\bar{d}$ for different number of delay subintervals N (Example 1)

Consider system (**??**) with system matrices

$$A = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & 0.1 \\ 0.1 & 0.2 \end{bmatrix} \tag{66}$$

The maximum delay bound $\bar{d}$ obtained by means of Theorem **??** and Corollary **??** for different numbers of delay subintervals N is compared in Table 1 with previous results [**? ? ? ?**]. It can be seen that both stability criterions allow to obtain a more accurate estimation of the maximum delay $\bar{d}$ as long as N increases, outperforming the best estimation of $\bar{d}$ reported in previous results for $N \geq 2$.

Moreover, the benefits of the relationships (**??**) considered in Theorem **??** by the Finlser's Lemma can be appreciated comparing the results given by Corollary **??** and Theorem **??**: for a given number of delay intervals N, Theorem **??** gives less conservative estimations of $\bar{d}$ in comparison to Corollary 1 with the same number of decision variables, showing the effectiveness of these equivalences, as above discussed in Remark **??**. For a fair comparison between the results obtained with different number of delay intervals N, the same value for $N_p = 3$ has been chosen. In this example, $N_p > 3$ does not

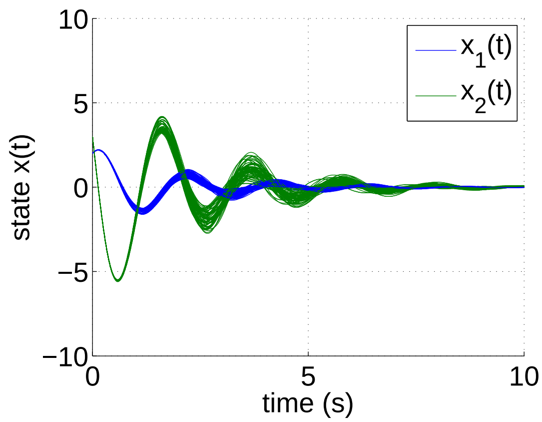

Figure 2: Simulation of the state responses obtained for 100 simulations with a time-varying delay randomly generated satisfying $0 \leq d(t) \leq 3.77$ (Example 1)

give appreciable conservatism reduction in the estimation of the maximum delay $\bar{d}$.

Fig. **??** represents the state evolution of system in Example 1 with initial condition $\phi(t) = [2 \quad 3]^T, t \in [\bar{d}, 0]$. In order to illustrate the effectiveness of the proposed method, a large number of simulations with different time-varying delay functions randomly generated satisfying $0 \leq d(t) \leq \bar{d}$ have been performed with $\bar{d} = 3.77$ (the larger delay bound, obtained with Theorem 1 and $N = 3$, $N_p = 3$ depicted in Table 1). It can be appreciated that the system is stable, as could be expected from the results in Table 1.

5.2. Example 2

Consider system (**??**) with system matrices

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \tag{67}$$

The achieved improvements are illustrated in Table 2, where tighter estimations of the maximum delay bound $\bar{d}$ are obtained by Theorem **??** with respect to previous results reported in [**? ? ?**]. An exception is found in [**?**], where less conservative estimations of $\bar{d}$ are given. This fact can be explained from the relaxation given by the maximum switching frequency between delay modes (namely f_i) in [**?**], whereas the proposed method proves the stability, no matter how fast delay modes switch (see Remark

??). Analogously, the parameter N_p used in Lemma ?? has been chosen to be $N_p = 2$ in all cases for a fair comparison between the results obtained with different number of delay intervals N. In this example, $N_p > 2$ does not give appreciable conservatism reduction in the estimation of the maximum delay $\bar{d}$.

Method	$\bar{d}$	Number of variables
[?]	2.24	$34.5n^2 + 3.5n$
[?]	2.243	$28.5n^2 + 5.5n$
[?]	2.26	$154.5n^2 + 4.5n$
[?] $\left(N = 2, \quad f_i < \frac{0.01}{ln(1.5)}\right)$	2.33	$19n^2 + 5n$
[?] $\left(N = 3, \quad f_i < \frac{0.01}{ln(1.5)}\right)$	2.42	$28.5n^2 + 7.5n$
Theorem ?? (N=2, $N_p = 2$)	2.34	$141.5n^2 + 11.5n$
Theorem ?? (N=3, $N_p = 2$)	2.35	$279.5n^2 + 17.5n$

Table 2: Maximum delay bound $\bar{d}$ for different number of partitions N (Example 2)

6. Conclusions and perspectives

This paper has proposed a novel LMI-based condition for stability analysis of arbitrarily fast time-varying delay systems based on a delay partitioning of the time-varying delay interval. A switched system model with multiple delays of smaller interval has been obtained for this purpose. The Lyapunov-Krasovskii method has been combined with the Finsler's lemma by including some defined relationships between state variables without involving extra decision variables. As a result, conservatism can be reduced in comparison to previous methods reported in the literature. Finally, two benchmark examples have been provided to show the effectiveness of the proposed approach.

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