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Additional Information

ON CERTAIN PRODUCTS OF PERMUTABLE SUBGROUPS

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Abstract. In this paper the structure of finite groups $G = AB$ which are a weakly mutually sn-permutable product of the subgroups A and B, that is, such that A permutes with every subnormal subgroup of B containing $A \cap B$ and B permutes with every subnormal subgroup of A containing $A \cap B$, is studied. Some known results on mutually *sn*-permutable products are extended.

Dedicated to the memory of Alexander Grant Robinson Stewart

1. Introduction

All groups considered here will be finite.

Mutually permutable products, that is, products $G = AB$ such that A permutes with every subgroup of B and B permutes with every subgroup of A, have been extensively studied by many authors (see [1], [4], [5], [7], [10]). In recent years, some other permutability connections between the factors were also considered. In particular, the rich normal structure of a mutually permutable product of two nilpotent groups (see [4, Chapter 5]) motivates interest in the study of mutually *sn*-permutable products.

Definition 1.1. *We say that a group $G = AB$ is the mutually sn-permutable product of the subgroups A and B if A permutes with every subnormal subgroup of B and B permutes with every subnormal subgroup of A.*

Carocca [8] showed that a mutually sn-permutable product of two soluble groups is soluble as well. In [2], the authors analyse the structure of mutually sn-permutable products and proved the following extension of a classical result of Asaad and Shaalan [1].

Theorem 1.2 ([2, Theorem B]). *Let $G = AB$ be the mutually sn-permutable product of the subgroups A and B, where A is supersoluble and B is nilpotent. If B permutes with each Sylow subgroup of A, then the group G is supersoluble.*

Following [12], we say that a subgroup H of a group G is $\mathbb{P}$-subnormal in G whenever either $H = G$ or there exists a chain of subgroups $H = H_0 \leq H_1 \leq \cdots \leq H_{n-1} \leq H_n = G$, such that $\mid H_i : H_{i-1} \mid$ is a prime for every $i = 1, \ldots, n$. It turns out that supersoluble groups are exactly those groups in which every subgroup if $\mathbb{P}$-subnormal. Having in mind this result and the influence of the embedding of Sylow subgroups on the structure of a group, the following extension of the class of supersoluble groups introduced in [12] seems to be natural.

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Definition 1.3. *A group G is called widely supersoluble, w-supersoluble for short, if every Sylow subgroup of G is $\mathbb{P}$-subnormal in G.*

The class of all finite w-supersoluble groups, denoted by $w\mathcal{U}$, is a saturated formation of soluble groups containing $\mathcal{U}$, the class of all supersoluble groups, which is locally defined by a formation function f, such that for every prime p, $f(p)$ is composed of all soluble groups G whose Sylow subgroups are abelian of exponent dividing $p-1$ ([12, Theorems 2.3 and 2.7]). Not every group in $w\mathcal{U}$ is supersoluble ([12, Example 1]). However, every group in $w\mathcal{U}$ has an ordered Sylow tower of supersoluble type ([12, Proposition 2.8]).

In [3] mutually sn-permutable products in which the factors are w-supersoluble are analysed. The following extension of Theorem 1.2 holds.

Theorem 1.4 ([3, Theorem 4])**.** *Let $G = AB$ be the mutually sn-permutable product of the subgroups A and B, where A is w-supersoluble and B is nilpotent. If B permutes with each Sylow subgroup of A, then the group G is w-supersoluble.*

Assume that $G = AB$ is the mutually sn-permutable product of the subgroups A and B. Then, by [4, Proposition 4.1.16 and Corollary 4.1.17], $A \cap B$ is subnormal in G and permutes with every subnormal subgroup of A and B. Assume now that $G = AB$ and $A \cap B$ satisfies the above condition. Then G is the mutually sn-permutable product of A and B if and only if A permutes with every subnormal subgroup V of B such that $A \cap B \leqslant V$, and B permutes with every subnormal subgroup U of A such that $A \cap B \leqslant U$. This motivates the following definition.

Definition 1.5. *Let A and B be two subgroups of a group G such that $G = AB$. We say that G is the weakly mutually sn-permutable product of A and B if A permutes with every subnormal subgroup V of B such that $A \cap B \leqslant V$, and B permutes with every subnormal subgroup U of A such that $A \cap B \leqslant U$.*

Obviously, mutually sn-permutable products are weakly mutually sn-permutable, but the converse is not true in general as the following example shows.

Example 1.6. *Let $G = \Sigma_4$ be the symmetric group of degree 4. Consider a maximal subgroup A of G which is isomorphic to Σ_3 and $B = A_4$, the alternating group of degree 4. Then $G = AB$ is the weakly mutually sn-permutable product of the subgroups A and B. However, G is not a mutually sn-permutable product of A and B because A does not permute with a subnormal subgroup of order 2 of B.*

Our first main result shows that Theorem 1.4 holds for weakly mutually sn-permutable products.

Theorem A. *Let $G = AB$ be the weakly mutually sn-permutable product of the subgroups A and B, where A is w-supersoluble and B is nilpotent. If B permutes with each Sylow subgroup of A, then the group G is w-supersoluble.*

The following corollary follows from the proof of Theorem A and generalises Theorem 1.2.

Corollary B. *Let $G = AB$ be the weakly mutually sn-permutable product of the subgroups A and B, where A is supersoluble and B is nilpotent. If B permutes with each Sylow subgroup of A, then the group G is supersoluble.*

The second part of the paper is concerned with weakly mutually sn-permutable products with nilpotent derived subgroup. Our starting point is the following extension of a classical result of Asaad and Shaalan [1].

Theorem 1.7 ([2, Theorem C])**.** *Let $G = AB$ be the mutually sn-permutable product of the supersoluble subgroups A and B. If the derived subgroup G' of G is nilpotent, then G is supersoluble.*

A natural question is whether this result is true for weakly mutually sn-permutable products under the same conditions. The following example answers this question negatively:

Example 1.8. *Let A be a cyclic group of order* 6*. It is known that A has an inrreducible and faithful module V over the field of* 5 *elements of dimension* 2 *(*[9, Theorem A.9.8]*). Let $G = [V]A$ be the corresponding semidirect product. Let $B = VC$, where C is the Sylow 2-subgroup of A. Then $G = AB$. Since B is normal in G, $A \cap B = C$ and B is the unique subnormal subgroup of B containing C, it follows that G is the weakly mutually sn-permutable product of A and B. It is clear that A and B are supersoluble and G' is nilpotent. However, G is not supersoluble.*

Note that in the above example B permutes with every Sylow subgroup of A. If A also permutes with every Sylow subgroup of B, we get supersolubility.

Theorem C. *Let $G = AB$ be the weakly mutually sn-permutable product of the supersoluble subgroups A and B. If B permutes with each Sylow subgroup of A, A permutes with every Sylow subgroup of B, and the derived subgroup G' of G is nilpotent, then G is supersoluble.*

By [11, Theorem 2.6], a group G is w-supersoluble if and only if every metanilpotent subgroup of G is supersoluble. In particular, if G' nilpotent, every w-supersoluble subgroup of G is supersoluble. Therefore we have:

Corollary D. *Let $G = AB$ be the weakly mutually sn-permutable product of the w-supersoluble subgroups A and B. If B permutes with each Sylow subgroup of A, A permutes with every Sylow subgroup of B, and the derived subgroup G' of G is nilpotent, then G is w-supersoluble.*

2. Preliminary Results

In this section we will prove some results needed in the proofs of our main results. We begin by showing that factor groups of weakly mutually sn-permutable products are also weakly mutually sn-permutable products.

Lemma 2.1. *Let $G = AB$ be the weakly mutually sn-permutable product of A and B and let N be a normal subgroup of G. Then $G/N = (AN/N)(BN/N)$ is the weakly mutually sn-permutable product of AN/N and BN/N.*

Proof. We have that $G/N = (AN/N)(BN/N)$. Suppose that H/N is a subnormal subgroup of AN/N such that $AN/N \cap BN/N \leqslant H/N$. Then $U = H \cap A$ is a subnormal subgroup of A such that $H = UN$ and $A \cap B \leq U$. Since U permutes with B and $H = UN$, it follows that H permutes with BN. Analogously, it can be showed that AN/N permutes with every subnormal subgroup of BN/N containing $AN/N \cap BN/N$ and therefore G/N is the weakly mutually sn-permutable product of AN/N and BN/N. □

Lemma 2.2. *Let $G = AB$ be the weakly mutually sn-permutable product of A and B.*

(a) *If H is a subnormal subgroup of A such that $A \cap B \leqslant H$, then HB is a weakly mutually sn-permutable product of H and B.*

(b) *If $A \cap B = 1$, then every subnormal subgroup of A permutes with every subnormal subgroup of B.*

Proof. Since every subnormal subgroup of H is a subnormal subgroup of A, we have that B permutes with every subnormal subgroup L of H such that $A \cap B \leqslant L$. Let M be a subnormal subgroup of B such that $A \cap B \leqslant M$. Then $HM = H(A \cap B)M = (A \cap HB)M = AM \cap HB = MA \cap BH = M(A \cap BH) = M(A \cap B)H = MH$. Hence A permutes with M and HB is the weakly mutually sn-permutable product of H and B.

Assume that $A \cap B = 1$. Let H be a subnormal subgroup of A and let K be a subnormal subgroup of B . By Statement (a), the product HB is weakly mutually sn-permutable and $H \cap B = 1$. Therefore H permutes with K, and Statement (b) holds. □

Observe that Lemma 2.2 implies that if $G = AB$ is the weakly mutually sn-permutable product of A and B, H is a subnormal subgroup of A such that $A \cap B \leqslant H$, and K is a subnormal subgroup of B such that $A \cap B \leqslant K$, then HK is a weakly mutually sn-permutable product of H and K.

Our next lemma analyses the behaviour of minimal normal subgroups of weakly mutually sn-permutable products containing the intersection of the factors.

Lemma 2.3. *Let $G = AB$ be the weakly mutually sn-permutable product of A and B. If N is a minimal normal subgroup of G such that $A \cap B \leqslant N$, then either $A \cap N = B \cap N = 1$ or $N = (N \cap A)(N \cap B)$.*

Proof. Observe that $A \cap N$ is a normal subgroup of A such that $A \cap B \leqslant A \cap N$ and so $H = (A \cap N)B$ is a subgroup of G. Note that $N \cap H = N \cap (A \cap N)B = (A \cap N)(B \cap N)$. Since $N \cap H$ is a normal subgroup of H, we have that B normalizes $N \cap H = (A \cap N)(B \cap N)$.

Using the same argument as above, $K = A(B \cap N)$ is a subgroup of G such that $K \cap N = A(B \cap N) \cap N = (A \cap N)(B \cap N)$. Moreover A normalizes $K \cap N = (A \cap N)(B \cap N)$. Hence $(A \cap N)(B \cap N)$ is a normal subgroup of G. By the minimality of N, we have that $A \cap N = B \cap N = 1$ or $N = (N \cap A)(N \cap B)$ as required. □

Lemma 2.4. *Let $G = AB$ be the weakly mutually sn-permutable product of the subgroups A and B. Assume that B is nilpotent. If B permutes with each Sylow subgroup of A, then $A \cap B$ is a subnormal subgroup of G.*

Proof. Let A_1 be a Sylow subgroup of A. Then B permutes with A_1 and so BA_1 is a subgroup of G. Futhermore, $BA_1 \cap A = A_1(A \cap B)$. Therefore $A \cap B$ permutes with A_1. We have shown that $A \cap B$ permutes with every Sylow subgroup of A. Applying [4, Theorem 1.2.14(3)], $A \cap B$ is a subnormal subgroup of A. Since B is nilpotent, it follows that $A \cap B$ is also subnormal in B. By [4, Theorem 1.1.7], we have that $A \cap B$ is a subnormal subgroup of G. □

Lemma 2.5. *Let $G = AB$ be the weakly mutually sn-permutable product of the subgroups A and B, where A is soluble and B is nilpotent. If B permutes with each Sylow subgroup of A, then the group G is soluble.*

Proof. Suppose that the theorem is false and let G be a minimal counterexample. If N is a minimal normal subgroup of G, then $G/N = (AN/N)(BN/N)$ is the weakly mutually sn-permutable product of the subgroups AN/N and BN/N by Lemma 2.1. Since BN/N permutes with each Sylow subgroup of AN/N, we have that G/N is soluble by the minimality of G. If N_1 and N_2 are two minimal normal subgroups of G, then G/N_1 and G/N_2 are soluble and so $G \cong G/(N_1 \cap N_2)$ is soluble, a contradiction. Hence $N = \mathbf{Soc}(G)$ is a non-abelian minimal normal subgroup of G. In particular, $\mathbf{F}(G) = 1$.

By Lemma 2.4, $A \cap B \leqslant \mathbf{F}(G)$. Therefore $A \cap B = 1$ and then every subnormal subgroup of A permutes with every subnormal subgroup of B by Lemma 2.2. The result then follows applying [8, Theorem 6]. □

Lemma 2.6. [2, Lemma 3] *Let G be a primitive group and let N be its unique minimal normal subgroup. Assume that G/N is supersoluble. If N is a p-group, where p is the largest prime dividing $|G|$, then $N = \mathbf{F}(G) = \mathbf{O}_p(G)$ is a Sylow p-subgroup of G.*

3. Main Results

We are ready to prove our main results.

Proof of Theorem A. Suppose the theorem is not true and let G be a minimal counterexample. Then A and B are proper subgroups of G. We proceed in a number of steps.

(a) G is a primitive soluble group with a unique minimal normal subgroup N and $N = \boldsymbol{C}_G(N) = \boldsymbol{F}(G) = \boldsymbol{O}_p(G)$ for a prime p.

Note that A is soluble. Therefore, by Lemma 2.5, G is soluble. Let N be a minimal normal subgroup of G. By Lemma 2.1, $G/N = (AN/N)(BN/N)$ is the weakly mutually sn-permutable product of AN/N and BN/N, and it is clear that BN/N permutes with every Sylow subgroup of AN/N. Moreover AN/N is w-supersoluble and BN/N is nilpotent. By the minimality of G, it follows that G/N is w-supersoluble. Note that the class of all w-supersoluble groups is a saturated formation of soluble groups by [12, Theorems 2.3 and 2.7]. This implies that G is a primitive soluble group and so G has a unique minimal normal subgroup N with $N = \mathbf{C}_G(N) = \mathbf{F}(G) = \mathbf{O}_p(G)$ for some prime p, as required.

(b) BN is w-supersoluble, $1 \neq A \cap B \leqslant N$ and $N = (N \cap A)(N \cap B)$.

If $A \cap B = 1$, then G is w-supersoluble by Lemma 2.2 and Theorem 1.4. This contradiction yields $A \cap B \neq 1$. Applying Lemma 2.4, it follows that $A \cap B$ is a nilpotent subnormal subgroup of G. Therefore $A \cap B \leqslant \mathbf{F}(G) = N$ and so $N = (N \cap A)(N \cap B)$ by Lemma 2.3. Hence $NB = (N \cap A)(N \cap B)B = (N \cap A)B$ is the weakly mutually sn-permutable product of $N \cap A$ and B. Also note that B permutes with every Sylow subgroup of $N \cap A$. If $NB < G$, then NB is w-supersoluble by the choice of G. Assume that $G = NB$. Let $1 \neq N_1 \leqslant A \cap B \leqslant N$. Note that N_1 is normal in N since N is abelian. Hence $N = N_1^G = N_1^{NB} = N_1^B \leqslant B$ and $G = B$, a contradiction. Therefore NB is a w-supersoluble proper subgroup of G.

(c) N is the Sylow p-subgroup of G and p is the largest prime dividing $|G|$.

Let q be the largest prime dividing $|G|$ and suppose that $q \neq p$. Suppose first that q divides $|BN|$. Since BN has a Sylow tower of supersoluble type, we have that BN has a unique Sylow q-subgroup, $(BN)_q$ say. This means that $(BN)_q$ centralises N. Thus $(BN)_q = 1$, since $\mathbf{C}_G(N) = N$, a contradiction. Therefore we may assume that q divides

$|A|$ but does not divide $|BN|$. Since A has a Sylow tower of supersoluble type, we have that A has a unique Sylow q-subgroup, A_q say. This means that A_q is normalised by $N \cap A$. Then $A_q(N \cap B) = A_q(A \cap B)(N \cap B)$ is the weakly mutually permutable product of $A_q(A \cap B)$ and $N \cap B$ by Lemma 2.2. Also $N \cap B$ permutes with each Sylow subgroup of $A_q(A \cap B)$. Suppose that $A_q(N \cap B) < G$. Then $A_q(N \cap B)$ is w-supersoluble by the choice of G. It follows that $A_q(N \cap B)$ has a unique Sylow q-subgroup since it has a Sylow tower of supersoluble type. In other words, A_q is normalised by $N \cap B$. Hence A_q is normalised by $(N \cap A)(N \cap B) = N$. This means that A_q centralises N, a contradiction. We may assume that $A_q(N \cap B) = G$. Then $N \cap B = B$ and so B is an elementary abelian p-group. Moreover $A = A_q(A \cap B)$. Then $A \cap B$ is a normal Sylow p-subgroup of A. Hence $A \cap B$ is normal in G because B is abelian. By the minimality of N, we have that $N = A \cap B$, that is, $G = A_q(N \cap B) = A_q(A \cap B) = A$, a contradiction. Therefore p is the largest prime dividing $|G|$.

Since G is a primitive soluble group, it follows that $G = NM$, where M is a maximal subgroup of G and $N \cap M = 1$. Then $M \cong G/N$ is w-supersoluble. By [9, Theorem A.15.6], $\mathbf{O}_p(M) = 1$. Note that M is a p'-group because it has a Sylow tower of supersoluble type. Therefore N is the unique Sylow p-subgroup of G.

(d) N is a subgroup of A and N is not contained in B.

Suppose that N is contained in B. Then a Hall p'-subgroup $B_{p'}$ of B must centralise $N = \mathbf{C}_G(N)$. Hence $B_{p'} = 1$ and B is a p-group. Then $G = AN$. Let $1 \neq N_1 \leqslant A \cap B$. Then $N \leq N_1^G = N_1^{AN} = N_1^A \leqslant A$ and so $G = A$, a contradiction. Therefore N is not contained in B. Hence B has a non-trivial Hall p'-subgroup, $B_{p'}$ say, which is normal in B. Consequently, $AB_{p'} = A(A \cap B)B_{p'}$ is a subgroup of G. Then $1 \neq B_{p'}^G \leqslant AB_{p'}$ and so $N \leqslant AB_{p'}$. Hence $N \leqslant A$, as required.

(e) Final Contradiction

Let $A_{p'}$ be a Hall p'-subgroup of A. If $A_{p'} = 1$, then $G = BN$ is w-supersoluble by Step (b), a contradiction. Hence $A_{p'} \neq 1$. Since B permutes with every Sylow subgroup of A, it follows that $A_{p'}B$ is a subgroup of G. By Step (d), N is not contained in B. Hence $A_{p'}B$ is a proper subgroup of G. Since $NA_{p'}B = G$, it follows that $N \cap A_{p'}B = N \cap B$ is normal in G. The minimality of N implies that $N = N \cap B$ or $N \cap B = 1$. By Step (d), $N \neq N \cap B$. Therefore $N \cap B = 1$, and then $A \cap B \leqslant N \cap B = 1$, contradicting Step (b). □

Proof of Theorem C. Assume the result is not true and let G be a minimal counterexample. It is clear that A and B are proper subgroups of G and $G' \neq 1$. Since the hypotheses of the theorem hold in every epimorphic image of G, it follows that G is a primitive soluble group. Hence G has a unique minimal normal subgroup N, and $N = F(G) = C_G(N)$. Moreover $G' = N$ because G' is nilpotent. We may assume that $A' \neq 1$ and $B' \neq 1$, otherwise the result follows from Corollary B. If $A \cap B = 1$, we have that G is the mutually sn-permutable product of A and B. By Theorem 1.7, G is supersoluble, a contradiction. Thus we may assume $A \cap B \neq 1$. Since A permutes with every Sylow subgroup of B and B permutes with every Sylow subgroup of A, we have that $A \cap B$ permutes with every Sylow subgroup of A and every Sylow subgroup of B. Hence $A \cap B$ is subnormal in A and B. Let N_1 be a minimal normal subgroup of A such that $N_1 \leq A'$. Then N_1 is of prime order since A is supersoluble. Since $N_1(A \cap B)$ is subnormal in A, it follows that $BN_1(A \cap B) = BN_1$ is a subgroup of G. Therefore

$1 \neq N_1^G = N_1^B \leq BN_1$ and so $N \leq BN_1$. In particular, $N = N_1(N \cap B)$, and either $N_1 \leq N \cap B$ or $N_1 \cap (N \cap B) = 1$. Write $T = BN$. If $N_1 \leq N \cap B$ we have that $BN = B$ is a supersoluble normal subgroup of G. Assume $N_1 \cap (N \cap B) = 1$. Then $N \cap B$ is a maximal subgroup of N, and so T is the weakly mutually sn-permutable product of B and N, and T satisfies the hypotheses of the theorem. Suppose that $G = BN$. Then $N \cap B = 1$ and $B' \leq N \cap B = 1$. Hence B is abelian. By Corollary B, G is supersoluble. If T is a proper subgroup of G, we have that $T = BN$ is a supersoluble normal subgroup of G. Consequently, either B is normal in G or BN is a supersoluble normal subgroup of G. We can argue in a similar way with A to conclude that either A is normal in G or AN is a normal supersoluble subgroup of G. In any case, we have that G is a product of two normal supersoluble subgroups. Applying Theorem 1.7, we conclude that G is supersoluble. This final contradiction proves the theorem. □

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