UNIVERSITAT POLITÈCNICA DE VALÈNCIA

Dept. of Applied Mathematics

Convergence of iterates of operators on Banach spaces

Master's Thesis

Master's Degree in Mathematical Research

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ACADEMIC YEAR: 2022/2023

M.U. en Investigación Matemática

Convergence of iterates of operators on Banach spaces

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Academic year 2022/2023

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Ad Augusta e Piero, per avermi insegnato l'arte del cercare il bello.

Introduction

This works sits in the framework of the course *Topología descriptiva. Aplicaciones*, which is part of the syllabus of the Master's programme. Our aim is to present some of the main results of the course in a complete, systematic and self-contained way. To be more precise, our main goal is to give a proof of the Yosida Theorem and some consequences. To do that we present some of the basic facts of Functional Analysis needed to state and understand the result. The work is divided in five chapters.

In the first chapter we present some of the basics on normed spaces, paying particular attention to continuity and linear mappings. We consider the space $L(X, Y)$ of continuous linear operators between two normed spaces X and Y, and the dual of a normed space X, defined as the space of scalar valued continuous linear functionals.

The dual space is principle endowed with the topology defined by the natural norm induced by the norm of X. The second chapter is devoted to the definition of two other topologies that play an important role within the theory: the *weak* and *weak** topologies. In order to do that we introduce the needed basic definitions and facts on seminorms and locally convex topologies.

In the third chapter we deal with the space $L(X)$ of continuous linear operators from X (a normed space) to itself. Again, it is endowed with the topology given by the natural operator norm. In this chapter two other topologies are defined and studied: the *strong* and *weak* operator topologies. We prove the Banach-Steinhaus theorem.

The Yosida Theorem deals with the behavior of the iterates of an operator T (that is, the iterative composition of T with itself), and the convergence in some sense in some of the topologies introduced in the previous chapter. This sits in a wide area known as *Ergodic operator theory*, and is developed in the fourth chapter. To handle this issue we introduce the Cesaro means of an operator, and introduce the notions of *power boundedness, mean ergocidity and uniform mean ergodicity.*

The final chapter of the work we look closer into uniform mean ergodic operators, in particular its connection with compact and quasi compact operators.

Chapter 1

Basics on normed spaces

1.1 Normed spaces and linear operators

Definition 1.1. *A norm* on a real or complex vector space X is a mapping

$$\begin{aligned} ||\cdot||: X &\to \mathbb{R}^+ \\ x &\mapsto ||x|| \end{aligned}$$

which satisfies the following conditions:

1. $||x|| \geq 0$, for all $x \in X$;
2. $||x|| = 0$ if and only if $x = 0$;
3. $||\lambda x|| = |\lambda| \cdot ||x||$, for all $\lambda \in \mathbb{C}$;
4. $||x + y|| \leq ||x|| + ||y||$, for all x, y in X.

Definition 1.2. A *normed space* is a pair $(X, ||\cdot||)$ where X is a real or complex vectorial space and $||\cdot||$ a norm.

Definition 1.3. A sequence of elements in a normed space X is a mapping from the set $\mathbb{N}$ of the natural numbers in X:

$$f : \mathbb{N} \longrightarrow X$$

The element x_n of the sequence is the image:

$$x_n = f(n)$$

of the number n trough the function f.

A point l in X is called *limit of the sequence* $\{x_n\}_n$ if and only if, for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that, if $n \geq n_0$ then $||x_n - l|| < \varepsilon$. In this case we will write

$$\lim_{n\to\infty} x_n = l$$

and we will say that the sequence is *convergent*.

Example 1.4. The sequence

$$\left(\frac{n+1}{n}\right)_{n\in\mathbb{N}}$$

is a sequence with values in $\mathbb{R}$ and its limit is $l = 1$.

Example 1.5. Let $p \in [1, \infty[$. The space $l_p := \{(x_n)_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}} : ||(x_n)||_p < \infty\}$ where

$$||(x_n)||_p := \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}$$

is a normed space.
For $p = \infty$ the space l_∞ is defined as

$$l_\infty := \{(x_n)_{n\in\mathbb{N}} : \sup_{n\,\in\,\mathbb{N}} |x_n| < \infty\}$$

and it is endowed with the norm

$$||(x_n)||_\infty := \sup_{n\,\in\,\mathbb{N}} |x_n|.$$

Also the space $c_0 := \{(x_n)_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}} : \lim_{n\to\infty} x_n = 0\}$ endowed with the norm $||(x_n)||_\infty$ is a normed space.

Definition 1.6. Let $\varepsilon > 0$ and $x \in X$. We define the set

$$B_\varepsilon(x) := \{y \in X : ||x - y|| < \varepsilon\}$$

and we will call it the ball of centre x and radius ε.

We will always use the notation B to indicate the ball of radius 1 and centre 0. Let us see various interesting properties of the balls. First of all we can see that

$$B_r(-x) = -B_r(x), \quad x \in X, r > 0$$

because if $y \in B_r(-x)$ then $||y - (-x)|| < r$. Thanks to the norm properties we have $||(-1)(y + x)|| < r$ that implies $|| - y - x|| < r$ and so $-y \in B_r(x)$.
Another interesting property of the balls is the following

$$B_r(x) + B_r(-x) = B_{2r}(0), \quad x \in X, r > 0.$$

Definition 1.7. A set $C \subseteq X$ is said to be *bounded* if there exists $\varepsilon > 0$ such that $C \subseteq B_\varepsilon(0)$.

Definition 1.8. $A\subseteq$ X is an *open set* if for every point x in A there exists a positive real number ε (depending on x) such that $B_\varepsilon(x) \subseteq A$.

Definition 1.9. $U \subseteq$ X is a *neighborhood* of x in X if there exists an open set A such that $x \in A \subseteq U$.
We call

$$\mathcal{N}(x) := \{U \subseteq X : U \quad neighborhood \quad of \quad x\}$$

the set of the all neighborhoods of x.

Definition 1.10. A subset $B(x)$ of $\mathcal{N}(x)$ with x in X it is called *basis of neighborhoods* if for all U in $\mathcal{N}(x)$ there exists V in $B(x)$ such that $V \subseteq U$.

Definition 1.11. Let $(X, ||\cdot||)$ be a normed space. A sequence $(x_n)_n \subseteq X$ is said to be Cauchy if the following condition is satisfied: for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $||x_n - x_m|| < \varepsilon$ for each $m, n \geq n_0$.

Theorem 1.12. *Let X be a normed space. Then every convergent sequence is a Cauchy sequence.*

Proof. The sequence $(x_n)_n$ is convergent so there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$ we have $||x_n - l|| < \varepsilon/2$. Let us now consider $n, m \in \mathbb{N}$ such that $n, m \geq n_0$. We clearly have, thanks to the triangle inequality,

$$||x_n - x_m|| \leq ||x_n - l|| + ||l - x_m|| < \varepsilon.$$

□

Definition 1.13. A normed space X is called *complete* (or *Banach*) if every Cauchy sequence is convergent.

Proposition 1.14. *Let $(X, ||\cdot||_X)$ be a Banach space and Y a linear subspace of X. Then $(Y, ||\cdot||_X)$ is a Banach space if and only if Y is a closed subset of X.*

The converse of the *Theorem* 1.12 does not hold in general. We see now a space in which there are Cauchy sequences that do not converge in the space (that is, we give an example of a normed space that is not Banach).

Example 1.15. The space c_{00} which consists of sequences $(a_n)_{n\in\mathbb{N}}$ of real number that are *eventually zero*, that is to say that there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, $a_n = 0$, equipped with the $l_1 - norm$ is not a *Banach space.*

If we consider the sequence $(v_n)_{n\in\mathbb{N}} \subseteq c_{00}$ as follow

$$v_n = \left(1, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n}, 0, 0, 0, \ldots\right)$$

it is clear that in l_1 we have that $v_n \longrightarrow v = \left(1, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n}, \frac{1}{2^{n+1}}, \ldots\right)$ as $n \longrightarrow \infty$.
It is clear that $v \in l_1$ as $\sum_{i=0}^{\infty} \frac{1}{2^i}$ converges. On the other hand we have that

$$||v - v_n||_1 = ||\left(0, 0, \ldots, 0, \frac{1}{2^n}, \frac{1}{2^{n+1}}, \ldots\right)||_1 = \frac{2}{2^{n+1}} \longrightarrow 0$$

as $n \to \infty$. However $v \notin c_{00}$ and so we can conclude that c_{00} does not contain all its accumulation points. Thanks to the *Proposition* 1.14 we can conclude that $(c_{00}, ||\cdot||_1)$ is not a Banach space.

1.2 Continuity of the operators

We now remind the characterization of continuity for functions in normed spaces. Let $f : X \to Y$ a mapping between two normed spaces X and Y.

Definition 1.16. The mapping $f : X \to Y$ is said *continuous* at the point x_0 in X if for all $\varepsilon > 0$ exists $\delta > 0$ such that for all x in X with $||x - x_0||_X < \delta$ we have that $||f(x) - f(x_0)||_Y < \varepsilon$.

The mapping f is said to be continuous on X when is continuous at every point of X. The continuity can also be expressed through the analysis of the behavior of sequences:

Proposition 1.17. *The mapping f is continuous on the point x_0 if and only if for all sequences $(x_n)_{n\in\mathbb{N}}$ converging to x_0 in X we have that the sequence of the images $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x_0)$ in Y.*

Proof. Let us suppose that f is continuous on x_0 and that $(x_n)_{n\in\mathbb{N}}$ converges to x_0. For each $\varepsilon > 0$ there exists $\delta > 0$ such that, if $||x - x_0|| < \delta$ then $||f(x) - f(x_0)|| < \varepsilon$. As the sequence is convergent to x_0 we know that there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, $||x_n - x_0|| < \varepsilon$. For this reason , if we now take $n \geq n_0$ we will have, thanks to the continuity, $||f(x_n) - f(x_0)|| < \varepsilon$. This shows that $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x_0)$ in Y. Vice versa, if f is not continuous on x_0 there exists $\varepsilon_* > 0$ such that, for all $\delta > 0$ we can find $x \in X$ with $||x - x_0|| < \delta$ but $||f(x) - f(x_0)|| > \varepsilon_*$. For all $n \in \mathbb{N}$ and choosing $\delta = 1/n$ we can now find a point $x_n \in X$ such that $||x_n - x_0|| < 1/n$ and $||f(x_n) - f(x_0)|| \geq \varepsilon_*$. This shows us that the sequence $(x_n)_{n\in\mathbb{N}}$ converges to x_0 but the sequence of the images $(f(x_n))_{n\in\mathbb{N}}$ does not converge to $f(x_0)$. □

We are now going to introduce another type of continuity which is going to be called *uniform continuity*.

Definition 1.18. The mapping $f : X \to Y$ is said *uniformly continuous* on X if for all $\varepsilon > 0$ exists $\delta > 0$ such that, for all $x, y \in X$ that satisfy $||x - y||_X < \delta$, we have $||f(x) - f(y)||_Y < \varepsilon$.

This new type of continuity is called *uniform* because the choice of δ is *uniform*, in the sense that it depends just on the choice of ε (and not on the point). Let us now introduce another definition which is going to be closely related with the concept of continuity.

Definition 1.19. A mapping $f : X \to Y$ is said to be *bounded* if $f(B)$ is bounded in Y for every $B \subseteq X$ bounded.

If the mapping is linear, the continuity can be characterized in several ways. Let us recall when a mapping $T : X \longrightarrow Y$ is said to be linear.

Definition 1.20. Let $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ be two normed spaces on $\mathbb{K}$ (where $\mathbb{K}$=$\mathbb{R}$ or $\mathbb{K}$=$\mathbb{C}$). An application $T : X \longrightarrow Y$ is said to be *linear operator* from X to Y when

- $T(x + y) = T(x) + T(y)$, for all x,y in X
- $T(\lambda x) = \lambda T(x)$ for all x in X and $\lambda \in \mathbb{K}$.

We are now going to prove an important result that allows us to understand how continuity and being bounded are equivalent for linear operators. It is clear that to be continuous an operator has to be bounded, but the following theorem shows that this condition is even sufficient.

Theorem 1.21. *Let $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ be normed spaces and $T : X \longrightarrow Y$ a linear operator. The following statements are equivalent:*

1. *the operator T is* bounded*;*
2. *there exists a constant $C \geq 0$ such that $||T(x)||_Y \leq C||x||_X$, for all $x \in X$ and so sup$\{||T(x)_Y|| : ||x||_X \leq 1\} < \infty$;*
3. *for all $x, y \in X$ we have that $||T(x) - T(y)||_Y \leq C||x - y||_X$, for a $C > 0$;*
4. *the operator T is uniformly continuous;*
5. *the operator T is continuous on X;*
6. *the operator T is continuous at a point;*
7. *the operator T is continuous at the origin.*

Proof. (1) $\implies$ (2) The unit sphere $U := \{x \in X : ||x||_X = 1\}$ is a bounded set of X and so its image T(U) will be contained on a ball $B(0, C)$ of W, for a $C > 0$, that is to say that when $||x||_X = 1$ we have that $||T(x)||_Y < C$. Given a vector $x \in X \setminus \{0\}$, we consider $y := \dfrac{1}{||x||_X} x \in X$. As $x = ||x||_X y$, from the linearity we obtain $T(x) = ||x||_X T(y)$ and we have

$$||T(x)||_Y = ||x||_X ||T(y)||_Y < C||x||_X$$

(2) $\implies$ (3) Given a pair of vectors $x, y \in X$ from the linearity we have that $T(x) - T(y) = T(x - y)$ and so

$$||T(x) - T(y)||_Y = ||T(x - y)||_Y \leq C||x - y||_X.$$

(3) $\Longrightarrow$ (4) For all $\varepsilon > 0$ if we take $\delta = \varepsilon/C$ then we have that, for all $x, y \in X$ such that $||x - y||_X < \delta$,

$$||T(x) - T(y)||_Y \leq C||x - y||_X < C\delta = \varepsilon.$$

(4) $\Longrightarrow$ (5) This is trivial.
(5) $\Longrightarrow$ (6)This is trivial.
(6) $\Longrightarrow$ (7) Let us suppose that T is continuous at the point $x \in X$. Let $(y_n)_{n\in\mathbb{N}}$ a sequence converging to 0. The sequence $x + y_n$ converges to x and so, by the continuity of T in x, we have that $T(x + y_n)$ converges to $T(x)$. Now, by the linearity, we have that $T(y_n) = T(x+v_n) - T(x)$ and it converges to $T(x) - T(x) = 0 = T(0)$. Thanks to Proposition 1.17 we have that T is continuous at the origin.
(7) $\Longrightarrow$ (1) The continuity at the origin implies that there exists $\delta > 0$ such that, if $||x||_X < \delta$ we then have $||T(x)||_Y < 1$. Let now A be a bounded subset of X, which means that there exists $R > 0$ such that $A \subseteq B(0, R)$. If we take a point $x \in A$, let $u := \frac{\delta}{R}x$; we now have that $||u||_X = \frac{\delta}{R}||x||_X < \delta$ and so $||T(u)||_Y < 1$. But by the linearity of T we have $||T(u)||_Y = \frac{\delta}{R}||T(x)||_Y$ and so $||T(x)||_Y < \frac{R}{\delta}$. We now deduce that $T(A) \subseteq B(0, R/\delta)$, that is to say that $T(A)$ is a bounded subset of Y. □

From now whenever $T : X \longrightarrow Y$ is linear, we will use indistinctly "continuous" and "bounded".
From what we have seen in *Theorem* 1.21 we can assure that verifying the continuity of a linear operator $T : X \longrightarrow Y$ is equivalent to verify the estimation

$$||T(x)||_Y \leq C \cdot ||x||_X$$

for a $C \geq 0$ and for all $x \in X$, that is to say that we have to estimate

$$\sup_{x \in X, x \neq 0} \frac{||T(x)||_Y}{||x||_X} < \infty.$$

Definition 1.22. Let X, Y be two normed spaces. The *Operator Norm* of a linear and continuous operator $T : X \longrightarrow Y$ is defined as

$$||T||_{X\to Y} = \sup_{x \in X, x \neq 0} \frac{||T(x)||_Y}{||x||_X}.$$

It is clear from this *Definition* that

$$\frac{||T(x)||_Y}{||x||_X} \leq ||T||_{X\to Y}, \quad for \quad all \quad x \in X.$$

1.3 The normed space L(X,Y)

If we now denote by $L(X,Y)$ the space of all bounded linear operators $T : X \longrightarrow Y$ we want to see that $(L(X,Y), ||\cdot||_{X\to Y})$ is a normed space but before we need to check that is a vector space. We can define the sum of two operators $S,T : X \longrightarrow Y$ as

$$(S+T)x := S(x) + T(x), \quad for \quad all \quad x \in X.$$

The product between a linear operator $T : X \longrightarrow Y$ and a scalar $\lambda \in \mathbb{K}$ can be defined as

$$(\lambda T)x := \lambda(Tx) \quad for \quad all \quad x \in X.$$

From the linearity of S and T follows that the operators $S+T$ and λT are still linear operators. The null operator $0 \in L(X,Y)$ is the operator that associate $0 \in Y$ to every $x \in X$.

Proposition 1.23. *If $S,T : X \longrightarrow Y$ are continuous and linear operators between normed spaces and $\lambda \in \mathbb{K}$ then also $S+T$ and λT are continuous and we have*

$$||S+T||_{X\to Y} \leq ||S||_{X\to Y} + ||T||_{X\to Y}, \qquad ||\lambda T||_{X\to Y} = |\lambda| \cdot ||T||_{X\to Y}.$$

Proof. Let us consider $x \in X$. We have, being $(Y, ||\cdot||_Y)$ a normed space,

$$\begin{aligned} ||(S+T)x||_Y \leq ||Sx||_Y + ||Tx||_Y \leq ||S||_{X\to Y} \cdot ||x||_X + ||T||_{X\to Y} \cdot ||x||_X = \\ = (||S||_{X\to Y} + ||T||_{X\to Y}) \cdot ||x||_X \end{aligned}$$

and so we get that $||S+T||_{X\to Y} \leq ||S||_{X\to Y} + ||T||_{X\to Y}$. Now, being $||\cdot||_Y$ a norm, we have that

$$||\lambda T||_{X\to Y} = \sup_{x \in X \setminus \{0\}} \frac{||\lambda T(x)||_Y}{||x||_X} = \sup_{x \in X \setminus \{0\}} |\lambda| \frac{||T(x)||_Y}{||x||_X} = |\lambda| \cdot ||T||_{X\to Y}.$$

□

Observation 1.24. Note that if $||T||_{X\to Y} = 0$ then $||Tx||_Y = 0$ for every $x \neq 0$ and so T has to be the null operator.

This Proposition allows us to claim that $S+T$ and λT are continuous because of the *Theorem* 1.21 and so we get that $L(X,Y)$ is a vector space. Furthermore, as a consequence of *Observation* 1.24, we have that $(L(X,Y), ||\cdot||_{X\to Y})$ is a normed space.

Theorem 1.25. *Let X, Y be normed spaces. If Y is a Banach space then $L(X,Y)$ is also Banach.*

Proof. Let $(T_n)_{n\in\mathbb{N}}$ a Cauchy sequence in $L(X,Y)$. We have that, for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $||T_n - T_m|| < \varepsilon$ for all $n, m \geq n_0$. We then have that the sequence $(T_n)_{n\in\mathbb{N}}$ is bounded in $L(X,Y)$ because it is a Cauchy sequence, there exists $C > 0$ such that

$$\sup_{n\in\mathbb{N}} ||T_n|| \leq C.$$

For all $x \in X$ we have

$$||T_n(x) - T_m(x)||_Y \leq ||T_n - T_m|| \cdot ||x||_X \leq \varepsilon ||x||_X, \tag{1.1}$$

if $n, m \geq n_0$. This shows that the sequence $(T_n(x))_{n\in\mathbb{N}}$ is Cauchy in Y and so converges to a point of the space because Y is a complete space. We can now define the mapping $T : X \to Y$ in this way

$$T(x) := \lim_{n\to\infty} T_n(x).$$

As T_n is linear and thanks to the linearity of the limit operation it is clear that the mapping $T : X \to Y$ is linear. T is continuous because

$$||T(x)||_Y = \lim_{n\to\infty} ||T_n(x)||_Y \leq \sup_{n\in\mathbb{N}} ||T_n(x)||_Y \leq$$
$$\leq \sup_{n\in\mathbb{N}} ||T_n|| \cdot ||x||_X \leq C||x||_X$$

and so $T \in L(X,Y)$. From the equation (1.1) taking the limit we have

$$||T_n - T|| < \varepsilon,$$

for all $n \geq n_0$. This shows that the sequence $(T_n)_{n\in\mathbb{N}}$ is convergent on $L(X,Y)$ to the operator T. □

1.4 The dual of a normed space

The *dual of a normed space* is defined as $X^* = L(X,\mathbb{K})$. The Theorem 1.25 shows that the *dual space* X^* of a normed space X is always a Banach space because $\mathbb{K}$ is Banach.

We can see that, as an immediate consequence of Theorem 1.21, we have the following

Proposition 1.26. *Let $x^* : X \to \mathbb{K}$ be linear. The following statements are equivalent:*

1. *$x^* \in X^*$;*
2. *There exists $C > 0$ such that $|x^*(x)| \leq C||x||_X$, for all $x \in X$;*
3. *$\sup\{|x^*(x)| : ||x||_X \leq 1\} < \infty$.*

The expression (3) defines a norm on X^* and furthermore

$$|x^*(x)| \leq ||x^*||_{X\to\mathbb{K}} \cdot ||x||_X \tag{1.2}$$

for all $x^* \in X^*$ and for all $x \in X$.

Chapter 2

Topologies on the dual space

2.1 Definition of topologies

Let us now remind how is possible to define a topology over a set X starting from a family of its subsets.

Theorem 2.1. *Let* $X \neq \emptyset$ *and* $\mathcal{B} \subseteq \mathcal{P}(X)$ *such that*

1. $X = \bigcup_{B \in \mathcal{B}} B$;
2. *For all* $B_1, B_2 \in \mathcal{B}$ *and for all* $x \in B_1 \cap B_2$ *exists* $B_3 \in \mathcal{B}$ *such that* $x \in B_3 \subseteq B_1 \cap B_2$.

so there exists over X *a unique topology* τ *such that* $\mathcal{B}$ *is a basis for* (X, τ). *Furthermore this topology* τ *is defined as*

$$\tau := \{\bigcup_{i \in I} B_i : B_i \in \mathcal{B}, \quad any \quad I\}.$$

We are now going to introduce the following comparison between topologies over the same set.

Definition 2.2. Let τ, σ be two topologies over the same set $X \neq \emptyset$. The topology τ is said to be *coarser* than σ if every τ open set is σ open (and we will write $\sigma \preceq \tau$).

Let now be $X \neq \emptyset$ and let $(Y_i, \tau_i)_{i \in I}$ be a family of fixed topological spaces and $(\psi_i)_{i \in I}$ a family of mappings $\psi_i : X \to Y_i$. We now apply the *Theorem* 2.1 in order to build a topology σ over X that makes all the functions $\psi_i : (X, \sigma) \to (Y_i, \tau_i)$ continuous. This topology σ has to contain the pre-images of the open set, that is to say that, for all $A_i \in \tau_i$, $\psi_i^{-1}(A_i) \in \sigma$. We even have that the finite intersections of $\psi_i^{-1}(A_i)$ have to be in σ. For this reason we define

$$\mathcal{B} := \{\bigcap_{\text{finite}} \psi_i^{-1}(A_i) : A_i \in \tau_i\}.$$

This family of sets verifies the properties (1) and (2) of the *Theorem* 2.1 and so there exists a topology on X that has $\mathcal{B}$ as a basis. The open sets of this topology are

$$\bigcup_{\text{any}} \bigcap_{\text{finite}} \psi_i^{-1}(A_i)$$

where A_i is a τ_i-open set.

2.2 Locally convex spaces

We have already seen, in the past chapter, that X^* is a normed space endowed by the norm $||x^*|| = \sup\{|x^*(x)| : ||x||_X \leq 1\}$ and is even a Banach space. On the dual X^* we can define different topologies but we now need first some concepts.

Definition 2.3. *A semi-norm* on a real or complex vector space X is a mapping

$$p: X \to \mathbb{R}^+$$
$$x \mapsto p(x)$$

which satisfies the following conditions:

1. $p(x) \geq 0$, for all $x \in X$;
2. $p(\lambda x) = |\lambda| \cdot p(x)$, for all $\lambda \in \mathbb{C}$;
3. $p(x + y) \leq p(x) + p(y)$, for all x, y in X.

Note that the only difference with a norm is that if we have a $x \in X$ such that $p(x) = 0$ we can not conclude that $x = 0$.

Definition 2.4. A family of semi-norm $(p_i)_{i\in I}$ is said to be *directed* if, for all pair of $i, j \in I$ there exists $k \in I$ such that $p_i \leq p_k$ and $p_j \leq p_k$.

Definition 2.5. A *locally convex space* is a vector space endowed with the topology defined by a directed family of *semi-norms.*

Suppose that $(p_i)_{i\in I}$ is a *directed family of semi-norms* on a vector space X. We consider the family

$$\mathcal{B} := \{\bigcap_{\text{finite}} p_i^{-1}(A_i) : A_i \quad open \quad of \quad \mathbb{R}\}$$

which satisfies the conditions of *Theorem* 2.1. Then there exists a topology over X that has $\mathcal{B}$ as a basis and its open sets are

$$\bigcup_{\text{any}} \bigcap_{\text{finite}} p_i^{-1}(A_i)$$

where A_i is an open set of $\mathbb{R}$, for all $i \in I$.

We also have that, for all x in X, a neighborhoods of x is

$$V_{i\varepsilon}(x) := \{y \in X : p_i(x-y) < \varepsilon\}, \quad i \in I, \varepsilon > 0.$$

Following the same idea as in *Definition* 1.11, we introduce the idea of convergence of sequence in a *locally convex space.*

Definition 2.6. Let X be a locally convex space whose topology is defined by a family of semi-norms $(p_i)_{i\in I}$. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ in X converges to $x \in X$ if, for every p_i and every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $p_i(x_n - x) < \varepsilon$, for every $n \geq n_0$.

Definition 2.7. Let X be a locally convex space and $(x_n)_{n=1}^{\infty}$ a sequence on X. We define the *Cesaro means* of $(x_n)_{n\in\mathbb{N}}$ as

$$x_{[n]} = \frac{1}{n}(x_1 + \ldots + x_n)$$

and we will say that $(x_n)_{n=1}^{\infty}$ is *Cesaro convergent to* x if

$$\lim_{n\to\infty} x_{[n]} = x.$$

Proposition 2.8. *Let $(x_n)_{n=1}^{\infty}$ be a sequence on X. If $\lim_{n\to\infty} x_n = x$ then we have that $\lim_{n\to\infty} x_{[n]} = x$.*

Proof. Let $(p_i)_{i\in I}$ be a set of *semi-norms* which generates the topology of X. As $\lim_{n\to\infty} x_n = x$, therefore we have that for all $\varepsilon > 0$, for all $i \in I$, there exists $n_0 \in \mathbb{N}$ such that

$$p_i(x_n - x) < \varepsilon$$

for all $n \geq n_0$. We now observe that being $(x_n)_{n=1}^{\infty}$ convergent implies that $(x_n)_{n=1}^{\infty}$ is bounded, so for all $i \in I$ exists $M_i > 0$ such that

$$p_i(x_n) < M_i \qquad \forall n \in \mathbb{N}.$$

We now study

$$\begin{aligned} p_i(x_{[n]} - x) = p_i(\frac{(x_1 - x) + \ldots + (x_n - x)}{n}) \leq \\ \leq \frac{1}{n}p_i(x_1 - x) + \ldots + \frac{1}{n}p_i(x_{n_0-1} - x) + \frac{1}{n}p_i(x_{n_0} - x) + \ldots + \frac{1}{n}p_i(x_n - x) \leq \\ \leq \frac{1}{n}[p_i(x_1 - x) + \ldots + p_i(x_{n_0-1} - x)] + \frac{1}{n}\varepsilon + \ldots + \frac{1}{n}\varepsilon = \\ = \frac{1}{n}[p_i(x_1 - x) + \ldots + p_i(x_{n_0-1} - x)] + \frac{n - n_0 + 1}{n}\varepsilon \end{aligned}$$

and the last term goes to 0 as $n \longrightarrow \infty$. □

The converse of *Proposition* 2.8 is not true in general, that is to say that there are some *non-convergent* sequences that are *Cesaro convergent* as we see in the following examples.

Example 2.9. The first sequence is

$$x_n = \begin{cases} m & \text{if } n = 2^m \\ 0 & \text{else} \end{cases}$$

which is not bounded and then does not converge. However

$$|x_{[2^m]}| = \frac{1 + \ldots + m}{\sum_{j=1}^{m} 2^j}$$

goes to 0 as $m \longrightarrow \infty$.

Example 2.10. The second example of a sequence which is not convergent but is Cesaro convergent is

$$x_n = \begin{cases} \sqrt{n} & \text{if } n = 2k, \quad k \in \mathbb{Z} \\ -\sqrt{n} & \text{if } n = 2k - 1, \quad k \in \mathbb{Z} \end{cases}$$

which has no limit but $x_{[n]}$ goes to 0 as $n \longrightarrow \infty$. To see this we note that if n is even, then $x_{[n]} = 0$, and if n is odd we have

$$x_{[n]} = \frac{1}{n}\left(-\sqrt{n} + \sqrt{n} + \ldots - \sqrt{n}\right) = -\frac{\sqrt{n}}{n} = -\frac{1}{\sqrt{n}}$$

which goes to 0 as $n \longrightarrow \infty$.

We introduce now a new concept, that extends the idea of sequence, to a more general setting. We need some definitions.

Definition 2.11. A set I is said to be *partially ordered* if there exists a binary relationship " $\leq$ " such that

1. $x \leq x$;
2. if $x \leq y$ and $y \leq z$ so $x \leq z$;
3. $x \leq y$ and $y \leq x$ implies that $x = y$.

and we will denote this set as $(I, \leq)$.

The order is said to be **total** if, for all pairs $i, j \in I$ we have

$$i \leq j \quad or \quad j \leq i$$

and so $(I, \leq)$ will be called *totally ordered.*

Example 2.12. It is easy to check that the following sets are all *totally ordered*:

- $(\mathbb{N}, \leq)$;
- $(\mathbb{R}^2, \leq)$;
- $(P(X), \subseteq)$ where $A \leq B$ if $A \subseteq B$.

Definition 2.13. The partially ordered set $(I, \leq)$ is said to be a *directed set* if, for all $i, j \in I$, there exists $k \in I$ such that

$$i \leq k \quad and \quad j \leq k.$$

It is clear that if $(I, \leq)$ is *totally ordered* it is also a *directed set.*

Definition 2.14. Let X be a *locally convex space* and $(I, \leq)$ a directed set. A **net** on X with indexes on I is an application

$$\begin{aligned} s \colon I &\longrightarrow X \\ i &\longmapsto x_i \end{aligned}$$

and we will denote the net as $s := (x_i)_{i \in I}$.

Definition 2.15. Let X be a *locally convex space.* A net $(x_j)_{j \in I}$ in X is said *convergent* to x_0 if, for all $\varepsilon > 0$ and for all semi-norms p_i, there exists $j_0 \in I$ such that

$$p_i(x_j - x_0) < \varepsilon$$

for all $j \geq j_0$.

We can also reformulate this definition using the neighborhood of x_0; it is to say that $(x_i)_{i \in I}$ is said to be *convergent* to x_0 if for all neighborhood U of x_0 there exists $j_0 \in I$ such that $j_n \in U$ for all $j \geq j_0$.

2.3 Weak Topology

Let E be a vector space. We want to study how the algebraic structure can define a topology on E. To do that we are going to consider the finest topology that makes continuous all linear maps. Let us recall that the algebraic dual of E consists of all scalar-values linear maps defined on E. $\gamma : E \longrightarrow \mathbb{K}$.

Definition 2.16. Let F be a subset of algebraic dual of E. We say that F separates points if, for all $x, y \in E$, there exists a mapping $\gamma \in F$ such that

$$\gamma(x) \neq \gamma(y).$$

Now, if F is a subset of E^* that separates points we can define, for all $\gamma \in F$,

$$p_\gamma(x) := |\gamma(x)|$$

for all $x \in E$.

Proposition 2.17. *Let $F \subseteq E^*$ and $\gamma \in F$. The mapping*

$$\begin{aligned} p_\gamma \colon E &\longrightarrow \mathbb{R} \\ x &\longmapsto p_\gamma(x) := |\gamma(x)| \end{aligned}$$

is a semi-norm.

Proof. For all $x \in E$ we have that $|\gamma(x)| \geq 0$ because $|\cdot|$ is a norm on $\mathbb{R}$. Let us now consider $p_\gamma(\lambda x) = |\gamma(\lambda x)|$ for all $\lambda \in \mathbb{R}$. As $\gamma \in F$ is linear and being $|\cdot|$ a norm on $\mathbb{R}$ we have that $|\gamma(\lambda x)| = |\lambda \cdot \gamma(x)| = \lambda|\gamma(x)| = \lambda p_\gamma(x)$. To show the triangle inequality we now have to consider that, for all $x, y \in F$,

$$p_\gamma(x+y) = |\gamma(x+y)| = |\gamma(x)+\gamma(y)| \leq |\gamma(x)| + |\gamma(y)| = p_\gamma(x) + p_\gamma(y)$$

being γ linear and $|\cdot|$ a norm.

□

Let us now suppose that $p_\gamma(x) = 0$. This condition clearly does not imply that $x = 0$ because if $p_\gamma(x) = 0$ we can have $x = 0$ or $\gamma = 0$.

Proposition 2.18. *The family of semi-norm $\{p_\gamma : \gamma \in F\}$ is a* directed family.

Proof. We just have to show that, for all pairs $\gamma, \phi \in F$, there exists $\psi \in F$ such that $p_\gamma \leq p_\psi$ and $p_\phi \leq p_\psi$.
Suppose that there exist $\gamma, \phi \in F$ such

$$p_\gamma > p_\psi \quad or \quad p_\phi > p_\psi$$

that for all $\psi \in F$. If this were the case we would have that $p_\gamma > p_\gamma$ or $p_\phi > p_\phi$ and this is clearly not possible. □

We can now claim, thanks to *Proposition* 2.17 and *Proposition* 2.18, that the family of semi-norms $\{p_\gamma : \gamma \in F\}$ defines a locally convex topology on E.
We consider the finest topology that makes continuous all the mappings $\gamma : E \longrightarrow \mathbb{R}$ in F, and call this topology **Weak Topology** (denoted by $\sigma(E, F)$). A basis of open sets for this topology is the family of finite intersections

$$\mathcal{B} := \{\bigcap_{\text{finite}} \gamma_i^{-1}(V_i) : V_i \quad open \quad in \quad \mathbb{K}, \gamma_i \in F\}$$

We are now interested in studying the neighborhoods and the basis of neighborhood of this new topology.

Given $\varepsilon > 0$, $n \in \mathbb{N}$ and $\gamma_1, ..., \gamma_n \in F$ we define the set

$$U(x_0; \varepsilon, n, \gamma_1, ..., \gamma_n) := \{x \in E : |\gamma_i(x) - \gamma_i(x_0)| < \varepsilon, \quad k = 1, ..., n\}.$$

which is a neighborhood of x_0 for the weak topology $\sigma(E, F)$. Furthermore, the family

$$\{U(x_0; \varepsilon, n, \gamma_1, ..., \gamma_n) : \varepsilon > 0, n \in N, \gamma_1, ..., \gamma_n \in F\}$$

is a basis of neighborhoods of x_0.

Proposition 2.19. *A net $(x_i)_{i \in I}$ in E converges to x_0 in the* weak topology $\sigma(E, F)$ *if and only if $\gamma(x_n)$ converges to $\gamma(x_0)$ on $\mathbb{R}$ for all γ in F.*

Proof. Let us the prove the first implication.
We now have that, for all neighborhood U of x_0 of the weak topology, which means that U has the form of

$$\{x \in X : |\gamma(x) - \gamma(x_0)| < \varepsilon\}$$

for fixed $\gamma \in F$, $\varepsilon > 0$, there exists $i_0 \in I$ such that $x_i \in U$, for all $i \geq i_0$. As $x_i \in U$ we have that

$$|\gamma(x_i) - \gamma(x_0)| < \varepsilon$$

for all $\gamma \in F$.
For the converse implication we consider a neighborhood V of x_0 in the weak topology of X, so

$$\{x \in X : |\gamma_i(x) - \gamma_i(x_0)| < \varepsilon, \quad k = 1, ..., m\} \subseteq V$$

with $\varepsilon > 0$, $m \in \mathbb{N}$ and $\gamma_1, ..., \gamma_m \in F$.
We now know that $\{\gamma_i(x_n)\}$ converges to $\gamma_i(x_0)$ for all $i \in \{1, ..., m\}$ so there exists $n_i \in \mathbb{N}$ such that, if $n \geq n_i$, we have

$$|\gamma_i(x_n) - \gamma_i(x_0)| < \varepsilon.$$

If we now consider $N = max\{n_i : i = 1, ..., m\}$ we have that, for all $n \geq N$

$$|\gamma_i(x_n) - \gamma_i(x_0)| < \varepsilon, \quad \forall i = 1, ..., m$$

and this shows us that $x_n \in V$, for $n \geq N$. □

If we now consider $E = X$ a normed space we have that $F = X^*$, the dual of X, consisting of all continuous linear maps $x^* : X \longrightarrow \mathbb{K}$, is a set of linear mappings that separates points. Then the topology $\sigma(X, X^*)$ will simply be called the **Weak Topology** of X, denoted ω.

Proposition 2.20. *A net $(x_i)_{i \in I}$ converges to x_0 in the* weak topology ω *if and only if $x^*(x_i)$ converges to $x^*(x_0)$ on $\mathbb{K}$ for all x^* in X^*.*

This result is a clear consequence of *Proposition* 2.19.

2.4 Weak* Topology

If X is a normed space, then its dual X^* is a vector space, where we may consider the topology defined by the norm or the *weak topology* $\sigma(X^*, X^{**})$ that we defined in the previous section. Our aim now is to define a third topology. To do this we need some preliminary results.

Proposition 2.21. *Let $x \in X$, then the following mapping*

$$\delta_x : X^* \to \mathbb{R} \tag{2.1}$$
$$x^* \mapsto \delta_x(x^*) = x^*(x) \tag{2.2}$$

called ***evaluation on x*** *is linear and continuous.*

Proof. The mapping is clearly linear because, for all $x^*, y^* \in X^*$

$$\delta_x(x^* + y^*) = (x^* + y^*)(x) = x^*(x) + y^*(x) = \delta_x(x^*) + \delta_x(y^*).$$

To show the continuity we just need to show that its norm is finite. Let us remind that

$$||\delta_x||_{X^{**}} = \sup_{x^* \in X^* \setminus \{0\}} \frac{|x^*(x)|}{||x^*||_{X^*}}.$$

We have that, for all $x^* \in X^*$, $|x^*(x)| \leq ||x^*||_{X^*} \cdot ||x||_X$ and so

$$||x||_X \geq \sup_{x^* \in X^* \setminus \{0\}} \frac{|x^*(x)|}{||x^*||_{X^*}}.$$

We have obtained that $||\delta_x||_{X^{**}} \leq ||x||_X < \infty$ and so this shows that δ_x is continuous.

□

Theorem 2.22. *Given a not null vector x of a normed space E there exists (at least) a mapping $\psi_\star \in X^*$ such that $||\psi_\star||_{X^*} = 1$ and $\psi_\star(x) = ||x||_X$.*

This is a consequence of the Hahn-Banach Theorem, and the proof can be found at [7, Corollary 3.3] (see also [8]). We have stated the Theorem 2.22 because is going to be necessary for the proof of the following

Theorem 2.23. *Let x be a vector of a normed space X and δ_x the mapping defined in* (2.2)*. We have that*

$$||\delta_x||_{X^{**}} = ||x||_X.$$

Proof. We have already seen that $||\delta_x||_{X^{**}} \leq ||x||_X$.

Now, if $x \neq 0$, thanks to Theorem 2.22, there exists $\psi_\star \in X^*$ such that $||\psi_\star||_{X^*} = 1$ and $\psi_\star(x) = ||x||_X$. It implies that

$$||x||_X = \frac{|\psi_\star(x)|}{||\psi_\star||_{X^*}} \leq \sup_{x^* \in X^* \setminus \{0\}} \frac{|x^*(x)|}{||x^*||_{X^*}}$$

□

This result shows us that X can be seen as a subset of X^{**} and furthermore

Proposition 2.24. *The subset* $X \subseteq X^{**}$ *separates points.*

Proof. We have to show that, for all $x^*, y^* \in X^*$, $x^* \neq y^*$, there exists $x \in X$ such that

$$x^*(x) = \gamma_x(x^*) \neq \gamma_x(y^*) = y^*(x).$$

Suppose that there exists $x^*, y^* \in X^*$ such that, for all $x \in X$,

$$x^*(x) = y^*(x).$$

If this were the case we would have $x^* = y^*$ and this is a contradiction. □

The Weak* Topology $\sigma(X^*, X)$. Let X be a normed space and X^* its dual. We want to construct over X^* the *coarser* topology that makes all the mappings $(\delta_x)_{x\in X}$ continuous. We construct this topology applying what we have done in the *Section 2.1* using $Y_i = \mathbb{R}$ and $(\psi_i)_{i\in I} = (\delta_x)_{x\in X}$. We will call this topology **Weak* Topology** and it will be indicated by $\sigma(X^*, X)$ or ω^*.
A basis of open sets for this topology is the family of finite intersections

$$\mathcal{B} := \{\bigcap_{\text{finite}} \delta_x^{-1}(A_i) : A_i \quad open \quad of \quad \mathbb{R}, x \in X\}.$$

In order to define the neighborhoods of the *weak* topology* we have to consider $f_0 \in X^*$ $\varepsilon > 0$, $n \in \mathbb{N}$ and $x_1, ..., x_n \in X$. Now the set

$$U(f_0; \varepsilon, n, x_1, ..., x_n) := \{f \in X^* : |x_i(f) - x_i(f)| < \varepsilon, \quad i = 1, ..., n\}.$$

is a neighborhood of f_0 for the weak* topology of X^*. Furthermore, the family

$$\{U(f_0; \varepsilon, n, x_1, ..., x_n) : \varepsilon > 0, n \in N, x_1, ..., x_n \in X^*\}$$

is a basis of neighborhoods of f_0.
It is interesting now to see a property for the convergence of sequences in this new topology because allows us to see a similarity with the convergence of nets in the weak topology. For this reason, we are now interested on the proof of the following

Proposition 2.25. *A sequence* $\{f_n\}$ *in* X^* *converges to* $f_0 \in X^*$ *in the* weak* topology $\sigma(X^*, X)$ *if and only if* $\{f_n(x)\}$ *converges to* $f_0(x)$ *in* $\mathbb{K}$ *for all* $x \in X$.

Proof. Let us the prove the first implication.
Let x be in X and $\varepsilon > 0$. We know that

$$\{f \in X^* : |x(f) - x(f_0)| < \varepsilon\}$$

is a neighborhood of f_0 in the topology $\sigma(X^*, X)$. From the hypothesis we have that $\{f_n\}$ converges to f_0, so that there exists $m \in \mathbb{N}$ such that, for all $n \geq m$ we have

$$|x(f_n) - x(f_0)| < \varepsilon.$$

For the converse implication we now consider V a neighborhood of f_0 on the weak* topology, so that

$$\{f \in X^* : |x_i(f) - x_i(f_0)| < \varepsilon, \quad i = 1, ..., m\} \subseteq V$$

with $\varepsilon > 0$, $m \in \mathbb{N}$ and $x_1, ..., x_m \in X$.
We know that $\{f_(x_i)\}$ converges to $f_0(x_i)$ for each $i \in \{1, ..., m\}$ so there exists $n_i \in \mathbb{N}$ such that, if $n \geq n_i$, we have

$$|x_i(f_n) - x_i(f_0)| < \varepsilon, \quad i = 1, ..., m.$$

If we now consider $N = max\{n_i : i = 1, ..., m\}$ we have that, for all $n \geq N$

$$|x_i(f_n) - x_i(f_0)| < \varepsilon, \quad i = 1, ..., m$$

and this shows us that $f_n \in V$, for $n \geq N$. □

Chapter 3

Topologies on L(X)

3.1 Norm, SOT and WOT Topologies

In this chapter we are going to define different topologies on the space $L(X)$, where X is a normed space.

Definition 3.1. Let X be a normed space. The *Natural locally convex topology* on the space $L(X)$ is the topology defined by the norm, that is $(L(X), \|\cdot\|_X)$ where

$$||T|| = \sup_{x \in B_X} ||T(x)||_X.$$

It is clear that $(L(X), \|\cdot\|_X)$ is a Banach space if X is Banach, thanks to what we have studied in the *Chapter 1.* We now want to define a family of semi-norms which is going to define a new type of topology on $L(X)$.

Proposition 3.2. *Let X be a normed space and $F \subseteq X$ a finite set. The mapping*

$$\begin{aligned} P_F \colon L(X) &\longrightarrow \mathbb{R} \\ T &\longmapsto P_F(T) := sup\{||T(x)||_X : x \in F\} \end{aligned}$$

is a semi-norm.

Proof. Being $|| \cdot ||_X$ a norm on X it is clear that $P_F(T) \geq 0$ for all $T \in L(X)$. Now let us consider $\lambda \in \mathbb{R}$ and we get that

$$sup\{||\lambda T(x)||_X : x \in F\} = |\lambda| \cdot sup\{||T(x)||_X : x \in F\}$$

being $|| \cdot ||_X$ a norm. On the other hand we have, for all $T, S \in L(X)$,

$$sup\{||T(x) + S(x)||_X : x \in F\} \leq sup\{||T(x)||_X : x \in F\} + sup\{||S(x)||_X : x \in F\}$$

and so we get the *triangle inequality.* Finally $T = 0$ implies that $P_F(T) = 0$. ◻

Observation 3.3. Let us notice that if we have $P_F(T) = 0$ we can not conclude that $T = 0$ because we can have $F = \{0\}$ and in that case $T(0) = 0$, for any $T \neq 0$. For this reason we can not conclude that the mapping P_F is a norm.

Definition 3.4. Let X be a normed space and $F \subseteq X$ a finite set. The *Strong Operator Topology, (SOT)* is the topology defined by the family of semi-norms described on the Proposition 3.2. With this we have that a net $(T_i)_{i\in I} \subseteq L(X)$ converges to $T \in L(X)$ in the SOT topology and we will write

$$(T_i)_{i\in I} \overset{\text{SOT}}{\longrightarrow} T,$$

if and only if $\lim_{i\in I} ||T_i(x) - T(x)||_X = 0$ for all $x \in X$.

A neighborhood of a fixed $T_0 \in L(X)$ in this topology is given by the set

$$U(T_0, x, \varepsilon) := \{T \in L(X) : ||T(x) - T_0(x)||_X < \varepsilon\}$$

where x is any vector of X and ε any positive real number.
In order to define the last type of topology on $L(X)$ we now consider the following

Proposition 3.5. *Let X be a normed space, $F \subseteq X$ a finite set and let $J \subseteq X^*$ to be finite. The mapping*

$$\begin{aligned} P_{F,J} \colon L(X) &\to \mathbb{R} \\ T &\mapsto P_{F,J}(T) := sup\{|y^*(T(x))| : x \in F, y* \in J\} \end{aligned}$$

is a semi-norm.

Proof. Being $|\cdot|$ a norm on $\mathbb{K}$ we get that $P_{F,J}(T) \geq 0$, for every $T \in L(X)$. Thanks to the linearity of any $y^* \in J$ and being $|\cdot|$ a norm we get

$$sup\{|y^*(\lambda T(x))| : x \in F, y* \in J\} = |\lambda| \cdot sup\{|y^*(T(x))| : x \in F, y* \in J\}$$

and also

$$\begin{aligned} P_{F,J}(T+S) &= sup\{|y^*(Tx) + y^*(Sx)| : x \in F, y* \in J\} \leq \\ &\leq sup\{|y^*(Tx)| + |y^*(Sx)| : x \in F, y* \in J\} \leq \\ &\leq sup\{|y^*(Tx)| : x \in F, y* \in J\} + sup\{|y^*(Sx)| : x \in F, y* \in J\} = \\ &= P_{F,J}(T) + P_{F,J}(S) \end{aligned}$$

for any $\lambda \in \mathbb{K}$, and any pair $S, T \in L(X)$.
Then, it is clear that $T = 0$ implies $|y^*(0)|$ for any $y^* \in J$. □

Observation 3.6. Let us notice that $P_{F,J}(T) = 0$ does not imply that T is the null operator because we can have that $sup\{|y^*(T(x))| : x \in F, y* \in J\} = 0$ can be given by $y^* = 0$ or $x = 0$. For this reason we can not conclude that $P_{F,J}$ is a norm.

Definition 3.7. Let X be a normed space, $F \subseteq X$ a finite set and let $J \subseteq X^*$ be finite. The *Weak Operator Topology, (WOT)* is the topology defined by the seminorms described in Proposition 3.5. With this we have that a net $(T_i)_{i\in I} \subseteq L(X)$ converges to $T_0 \in L(X)$ in the WOT topology and we will write

$$(T_i)_{i\in I} \xrightarrow{\text{WOT}} T_0,$$

if and only if $\lim_{i\in I} T_i(x) \overset{\text{w}}{=} T_0(x)$ for all $x \in X$, that is that $\lim_{i\in I} |y * (T_i x) - y * (T_0 x)| = 0$, for all $x \in X$.

A neighborhood of a fixed $T_0 \in L(X)$ in this topology is given by the set

$$N(T_0, F, J, \varepsilon) := \{T \in L(X) : |y^*((T - T_0)(x))| < \varepsilon, x \in F, y^* \in J\}$$

where F is a subset of X.

Observation 3.8. Let X be a normed space. If we have a sequence $\{T_i\}_{i\in\mathbb{I}} \subseteq L(X)$ and $T_0 \in L(X)$ such that $\lim_{i\in\mathbb{I}} ||T_i - T_0||_{L(X)} = 0$ then we clearly have that

$$\lim_{i\in\mathbb{I}} ||T_i(x) - T_0(x)||_X = 0 \tag{3.1}$$

for any $x \in X$. This shows that the convergence in norm of a net implies the SOT convergence of the net to the same operator. Now, from (3.1) and thanks to the continuity of every $y^* \in X^*$, we get

$$\lim_{i\in\mathbb{I}} |y^*(T_i x - T_0 x)| = 0$$

that is the convergence in the WOT-topology.

Let us see now some examples that show that the converse implications of in *Observation* 3.8 do not hold in general

Example 3.9. Let us consider the space c_0 endowed with the norm $||(x_n)||_\infty$. We consider the map

$$B\colon c_0 \longrightarrow c_0$$
$$(x_1, x_2, ...) \mapsto (x_2, x_3, ...)$$

as an element of $L(c_0)$. We denote B^n for the composition of B with itself $n-1$ times (see *Section 4.1* for more details) then

$$B^n\colon c_0 \longrightarrow c_0$$
$$(x_1, x_2, ...) \mapsto (x_{n+1}, x_{n+2}, ...).$$

We want to see that B^n is SOT-convergent to 0 but does not converge to 0 in the Norm-Topology. Let us take $x = (x_i)_{i\in\mathbb{N}} \in c_0$, that is to say that, for all $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$, $|x_k| < \varepsilon$. Now if we take $n \geq k_0$ we have that $B^n(x_i)_i = (x_{n+i})_i$ and so

$$||B^n(x_i)_i||_\infty = ||(x_{n+i})_i||_\infty < \varepsilon.$$

It shows that B^n is SOT-Convergent to 0. Now if we check the norm of the operator B^n we have that

$$||B^n|| = \sup_{x \in B(0,1)} ||B^n(x)||_\infty = ||(x_{n+1}, x_{n+2}, ...)||_\infty \leq 1.$$

If we take the sequence $e_{n+1} = (0, ..., 0, 1, 0, ...)$ with all zeros except 1 on the $(n+1) - th$ entrance we get that $||B^n(e_{n+1})||_\infty = 1$ and so $||B^n|| = 1$ and it shows that B^n does not converge to 0 in the Norm-Topology.

Example 3.10. Let us consider the same space c_0 and the map

$$F\colon c_0 \longrightarrow c_0$$
$$(x_1, x_2, ...) \mapsto (0, x_1, x_2, ...).$$

As in the previous example, we write F^n for the composition of F with itself $n-1$ times.

$$F^n\colon c_0 \longrightarrow c_0$$
$$(x_1, x_2, ...) \mapsto (0, ..., 0, x_1, x_2, ...),$$

and then $||F^n(x)||_\infty = ||x||_\infty$, for all $x \in c_0$, and for any $n \in \mathbb{N}$, and for this reason we can claim that F^n is not SOT-convergent to 0. Let us now study the WOT-convergence. We have to see that, for any $y^* \in c_0^* = l_1$, $\lim_{n\to\infty} \langle F^n, y^* \rangle = 0$. Being y^* an element of l_1 we have that, for any $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{k=k_0+1}^{\infty} |y_k| < \frac{\varepsilon}{||x||_\infty}.$$

If we now consider $n \geq k_0$ then

$$\langle F^n(x), y^* \rangle = \langle (0, ..., 0, x_1, x_2, \ldots), (y_1, y_2, \ldots) \rangle = \sum_{n=1}^{\infty} x_i \cdot y_{n+i}$$

and so

$$|\langle F^n(x), y^* \rangle| \leq \sum_{n=1}^{\infty} |x_i| \cdot |y_{n+i}| \leq ||x||_\infty \sum_{n=1}^{\infty} |y_{n+1}| < \varepsilon.$$

This clearly shows that F^n is *WOT-Convergent* to 0.

3.2 Banach Steinhaus theorem and consequences

We now state the *Baire's Theorem* without the proof, which is given in [7, Theorem 3.10] (see also [5] and [1]), because it is essential on the formulation of the *Banach-Steinhaus Theorem.*

Theorem 3.11. ***Baire's Theorem***
Let (X, d) be a complete metric space. If $(V_n)_n$ is a sequence of open and dense subsets of X then

$$\bigcap_{n \in \mathbb{N}} V_n$$

is dense.

Corollary 3.12. *Let (X, d) be a complete metric space and $X = \bigcup_{n=1}^{\infty} F_n$ where F_n is closed for all $n \in \mathbb{N}$. Then there exists $n_0 \in \mathbb{N}$ such that $\overset{\circ}{F}_{n_0} \neq \emptyset$.*

Proof. Let us suppose that

$$X = \bigcup_{n=1}^{\infty} F_n$$

where F_n are closed for all $n \in \mathbb{N}$ and $\overset{\circ}{F}_n = \emptyset$ for all $n \in \mathbb{N}$. Now if $X \setminus F_n$ were not dense for all $n \in \mathbb{N}$ then there would exists $V \subseteq X$ such that $V \cap F_n^C = \emptyset$. It implies that $V \subseteq F_n$ that is clearly not possible since $\overset{\circ}{F}_n = \emptyset$ for all $n \in \mathbb{N}$. So we have that $X \setminus F_n$ is dense for all $n \in \mathbb{N}$ and also an open. Then the *Baire's Theorem* implies that $\bigcap_{n=1}^{\infty}(X \setminus F_n)$ is dense but

$$\bigcap_{n=1}^{\infty}(X \setminus F_n) = X \setminus (\bigcup_{n=1}^{\infty} F_n) = \emptyset.$$

This is a contradiction and proves the Theorem. □

Thanks to the *Baire's Theorem* and its Corollary we are now ready to prove the following

Theorem 3.13. ***(Banach-Steinhaus Theorem)***
Let X be a Banach space *and $(T_i)_{i \in I} \subseteq L(X)$. If $M_x := \sup_{i \in I} ||(T_i(x))|| < \infty$, for all $x \in X$, then*

$$M := \sup_{i \in I} ||T_i|| < \infty \tag{3.2}$$

Proof. Let us define the sets $F_n := \{x \in X : ||T_i(x)|| \leq n, i \in I\}$ for all $n \in \mathbb{N}$. This sets are clearly closed because

$$F_n := \bigcap_{i \in I} T_i^{-1}(\overline{B(0, n)})$$

and furthermore $X = \bigcup_{n=1}^{\infty} F_n$. Thanks to *Corollary* 3.12 we have that there exists $n_0 \in \mathbb{N}$ such that $\overset{\circ}{F}_{n_0} \neq \emptyset$. Take now $x \in X$ and $r_0 > 0$ such that

$$B(x_0, r_0) \subseteq F_{n_0}$$

and observe that, by the definition of the set, if $x \in F_{n_0}$ then $-x \in F_{n_0}$ and so also $-B(x_0, r_0) = B(-x_0, r_0) \subseteq F_{n_0}$. Now we have

$$B(0, 2r_0) = B(x_0, r_0) + B(-x_0, r_0) \subseteq F_{n_0} + F_{n_0} \subseteq F_{2n_0}$$

and this implies that, for all $i \in I$,

$$\sup_{x \in B(0, 2r_0)} ||T_i(x)|| \leq 2n_0.$$

Let us now take $x \in B(0, 1)$ and fix $i \in I$, then

$$||T_i(x)|| = ||T_i(\frac{2r_0}{2r_0} x)|| = \frac{1}{2r_0} ||T_i(2r_0 \cdot x)|| \leq \frac{1}{2r_0} 2n_0 = \frac{n_0}{r_0},$$

which shows that $\sup_{i \in I} ||T_i|| < \infty$. □

Corollary 3.14. *A subset $B \subseteq X$ is bounded if and only if it is weakly bounded.*

Proof. Let $B = \{x_i : i \in I\}$. Since $X \hookrightarrow X^{**} = L(X^*, \mathbb{K})$ isometrically, the result follows directly from Banach-Steinhauss theorem. □

Proposition 3.15. *Let X, Y be a Banach spaces. Then $L(X, Y)$ have the same bounbded sets in the WOT, SOT and norm topology*

Proof. Since WOT is coarser that SOT and SOT is coarser than the norm topology, we only need to show that each WOT bounded set in $L(X, Y)$ is norm bounded. Let $B = \{T_i : i \in I\}$ be WOT bounded. Then, for each $x \in X$, the set $\{T_i(x) : i \in I\}$ is weakly bounded in Y, hence it is norm bounded in Y by Corollary 3.14. We conclude that B is norm bounded in $L(X, Y)$ by the Banach-Steinhaus theorem. □

Chapter 4

Ergodic Operators

4.1 Basics on Ergodic Operators

Our aim in this chapter is to study the behavior of an operator when composed with itself. To be more precise, if X is a *Banach space* and $T \in L(X)$, then we write

$$\begin{aligned} T^0 &= Id \\ T^1 &= T \\ T^2 &= T \circ T \\ &\vdots \\ T^n &= T^{n-1} \circ T \end{aligned}$$

and call this the **iterate of T**. We want to study the behavior of $(T^n)_{n\in\mathbb{N}}$ as n grows.

Definition 4.1. Let X be a *Banach space* and $T \in L(X)$. We define its **Cesaro mean** as

$$T_{[n]} := \frac{1}{n}(T + T^2 + ... + T^n).$$

Observation 4.2. Let us see that, if the *Cesaro means* $(T_{[n]})_n$ converges to an operator $P(x)$, in some topology τ, we have that

$$\tau - \lim_{n\to\infty} T_{[n]}(x) = P(x), \quad for \quad all \quad x \in X$$

and so, as a consequence,

$$\tau - \lim_{n\to\infty} \frac{T^n(x)}{n} = \tau - \lim_{n\to\infty} \left(T_{[n]}(x) - \frac{n-1}{n} T_{[n-1]}(x) \right) = P(x) - P(x) = 0$$

for all $x \in X$. It clearly shows that $\tau - \lim_{n\to\infty} \frac{T^n(x)}{n} = 0$ for any topology τ that we are considering.

Definition 4.3. We will say that

1. T es *Weakly Mean Ergodic* if $T_{[n]}$ is *convergent* to some $P \in L(X)$ on *WOT*, and we will write $T_{[n]} \xrightarrow{\text{WOT}} P$. That is to say, thanks to what we have seen in Observation 4.2,
$$w - \lim_{n\to\infty} \frac{T^n(x)}{n} = 0, \quad for \quad all \quad x \in X.$$
2. T is *Mean Ergodic* if $T_{[n]}$ is *convergent* to some $P \in L(X)$ on *SOT*, and we will write $T_{[n]} \xrightarrow{\text{SOT}} P$. Thanks to Observation 4.2 we have that
$$\lim_{n\to\infty} \frac{||T^n(x)||}{n} = 0, \quad for \quad all \quad x \in X.$$
3. T is *Uniformly Mean Ergodic* if exists $P \in L(X)$ such that $\lim_n ||T_{[n]} - P|| = 0$. It implies that
$$\lim_{n\to\infty} \frac{||T^n||}{n} = 0$$
4. T is *Power Bounded* if $(||T^n||)_n$ is bounded.

It is now clear that *Uniformly Mean Ergodic* implies *Mean Ergodic* which implies *Weakly Mean Ergodic*. We can see the article [4] for other properties. We now want to observe that being $(\|T^n\|)_{n\in\mathbb{N}}$ bounded is equivalent, thanks to the *Banach-Steinhause Theorem*, to $(T^n(x))_{n\in\mathbb{N}}$ to be bounded on X, for all $x \in X$.

Definition 4.4. An operator $P \in L(X)$ is said to be a *projection* if
$$P^2 = P.$$

Proposition 4.5. *The mappings P of the previous Definition are all projections.*

Proof. Let us consider
$$x = P(y) = \lim_n T_{[n]}(y)$$
and so, thanks to the continuity of T we have
$$\begin{aligned} T(P(y)) &= T(\lim_n T_{[n]}(y)) = \lim_n T(T_{[n]}(y)) = \\ &= \lim_n \frac{1}{n}(T^2(y) + \ldots + T^{n+1}(y)) = \\ &= \lim_n [\frac{1}{n}(T(y) + \ldots + T^n(y)) + \frac{1}{n}T^{n+1}(y) - \frac{1}{n}T(y)] = \\ &= P(y). \end{aligned}$$

This shows us that, in all cases, we have

$$TP = PT = P.$$

Inductively we can show in the same way that, for all $n \in \mathbb{N}$,

$$T^n P = PT^n = P,$$

and we deduce

$$\begin{aligned} PT_{[n]} &= P(\frac{1}{n}(T + T^2 + ... + T^n)) = \\ &= \frac{1}{n}(PT + PT^2 + ... + PT^n) = \frac{1}{n}(nP) = P. \end{aligned}$$

Now, thanks to the continuity of P, we have that

$$P = \lim_n PT_{[n]} = P \lim_n T_{[n]} = P^2.$$

□

4.2 Ergodic properties of an operator versus those of its powers

Proposition 4.6. *Let $T \in L(X)$. Then the sequence of iterates $(T^n)_n$ is convergent in norm SOT or WOT. Then $(T^{n_0 n})_n$ is convergent in the corresponding topology $n_0 \in \mathbb{N}$.*

Proof. Just observe that $(T^{n_0 n})_n$ is a subsequence of $(T^n)_n$, and then it is convergent in the corresponding topology. □

Remark 4.7. The converse in the above proposition is not true. Just observe that $T = -I \in L(X)$ satisfies $T^2 = I$ and $(T(x))$ is not convergent for any $x \neq 0$.

Proposition 4.8. *Let $T \in L(X)$. Then the sequence of iterates $(T^n)_n$ is convergent to 0 in norm SOT or WOT if and only if $(T^{n_0 n})_n$ is convergent in the corresponding topology $n_0 \in \mathbb{N}$.*

Proof. By Proposition 4.6, we only need to show that $(T^n)_n$ is convergent to 0 when $(T^{n_0 n})_n$ is. But this convergence implies that $(T^{n_0 n + j})_n = (T^{n_0 n} \circ T^j)_n$ is convergent to 0 for each $j \in \{0, \dots, n_0 - 1\}$. Then $(T^n)_n$ can be decomposed in n_0 convergent subsequences. □

Proposition 4.9. *Let X be a Banach space and let $T \in L(X)$ be an operator. If there is $n_0 \in \mathbb{N}$ such that T^{n_0} is (uniformly) mean ergodic then T is (uniformly) mean ergodic.*

Proof. We give the proof for mean ergodicity. The uniformly mean ergodic case is completely analogous. Let $P \in L(X)$ be the projection such that $\lim T^{n_0}_{[n]} = P \lim(T^{n_0})_{[n]} = P$ in the SOT. Let $k \in \mathbb{N}$, for every $j = 0, 1, \ldots, n_0 - 1$ we have

$$\lim_k \frac{T^{n_0+j} + T^{2n_0+j} + \cdots + T^{kn_0+j}}{k} = \lim_k T^{n_0}_{[k]} T^j = PT^j \tag{4.1}$$

in the SOT.

Fix now $n \in \mathbb{N}$ that, without loss of generality, we may assume $n \geq n_0$. There exist $k \in \mathbb{N}$ a $j_0 \in \{0, \ldots, n_0 - 1\}$ such that $n = kn_0 + j_0$. Then, we have

$$T_{[n]} = \frac{T + T^2 + \cdots + T^{n_0-1}}{n} + \sum_{j=0}^{n_0-1} \frac{T^{n_0+j} + T^{2n_0+j} + \cdots + T^{kn_0+j}}{n} - \sum_{i=j_0+1}^{n_0-1} \frac{T^{kn_0+i}}{n}. \tag{4.2}$$

Let observe that $k = k(n)$ and note that, for each n we have $k = \frac{n-j}{n_0}$ and, then,

$$\lim_n \frac{k}{n} = \frac{1}{n_0} \tag{4.3}$$

Let us study the limit of each one of the three terms in (4.2). The first term in (4.2) is unifomly convergent to 0. For the third term let us note that the mean ergodicity of T implies that $\lim \frac{T^n}{n} = 0$ in the SOT. Hence, for each $i \in \{j_0 + 1, \ldots, n_0 - 1\}$ we get

$$\lim_n \frac{T^{kn_0+i}}{n} = \lim_n \frac{kn_0 + i}{n} \frac{T^{kn_0+i}}{kn_0 + i} = 0,$$

in the SOT. Thus, for the third term in (4.2) we get

$$\lim_n \sum_{i=j_0+1}^{n_0-1} \frac{T^{n_0+i} + T^{2n_0+i} + \cdots + T^{kn_0+i}}{n} = 0.$$

Finally, for the second term, simply observe that, for $j \in \{0, \ldots, n_0 - 1\}$, we apply (4.3) and (4.1) to get

$$\lim_n \frac{T^{n_0+j} + T^{2n_0+j} + \cdots + T^{kn_0+j}}{n} = \lim_n \frac{k}{n} \frac{T^{n_0+j} + T^{2n_0+j} + \cdots + T^{kn_0+j}}{k} = \frac{1}{n_0} PT^j.$$

From this we get the convergence of the second term in (4.2). This shows that $(T_{[n]})_n$ i SOT-convergent and, then, T is mean ergodic. □

Remark 4.10. From the proof of Proposition 4.9 it follows that, if $T \in L(X)$ and $\lim_n (T^{n_0})_{[n]} = P$, uniformly or in SOT, then $\lim T_{[n]} = \frac{1}{n_0}(P + P \circ T + \cdots + P \circ T^{n_0-1})$ in the corresponding topology.

Remark 4.11. An operator $T \in L(X)$ is said to be *periodic* if $T^k = I$ for some k. Note that then T^{nk} is again I for every $n \in \mathbb{N}$. If T is periodic and n_0 is the first natural number satisfying $T^{n_0} = I$, then clearly $\lim T_{[n]} = \frac{1}{n_0}(I + T + \cdots + T^{n_0-1})$ in norm and T is uniformly mean ergodic.

4.3 The Yosida's Theorem

We are now ready to study the **Yosida's Theorem** that gives us a sufficient condition for T to be *Mean Ergodic*. Before we need to state the following theorem.

Theorem 4.12. ***(Eberlein)*** *Let X be a Banach space and A a subset of X. Then the following statements are equivalent*

- *each sequence of elements of A has a subsequence that is weakly convergent in X;*
- *each sequence of elements of A has a weak accumulation point in X;*
- *The weak closure of A is weakly compact.*

The proof of this Theorem can be found at [9, Chapter V] (see also [3]). This result but it is going to be essential in order to prove the

Theorem 4.13. ***(Yosida's Theorem)***
Let X be a Banach space. Then there is a bounded linear operator P on X such that $(T_{[n]}x)_n$ is SOT-convergent to $P(x)$ for each $x \in X$ if and only if the following two conditions hold for each $x \in X$:

1. $\|\cdot\| - \lim \frac{T^n(x)}{n} = 0$
2. $\{T_{[n]}(x) \colon n \in \mathbb{N}\}$ *is relatively weakly compact.*

Proof. The necessity of condition (1) comes from

$$\frac{T^n(x)}{n} = T_{[n]}(x) - \frac{n-1}{n} T_{[n-1]}(x), \quad \text{for any } x \in X.$$

Since $\|\cdot\|$ is stronger than the weak topology, Condition (2) is obviously also necessary.

We now check the sufficiency of (1) and (2). We first observe that $T_{[n]}(I-T) = \frac{1}{n}(T - T^{n+1})$, hence $T_{[n]}(I-T)(x)$ is $\|\cdot\|$-convergent to 0 for every $x \in X$. Since, by the Banach Steinhaus theorem, $(\|T_{[n]}\|)_n$ is a bounded sequence, we get

$$\|\cdot\| - \lim_n T_{[n]}(v) = 0 \text{ for every } v \in \overline{(I-T)(X)}. \tag{4.4}$$

Let now $x \in X$. As $\{T_{[n]}(x) \colon n \in \mathbb{N}\}$ is relatively weakly compact, we get from Eberlein's theorem an increasing sequence (n_k) of natural numbers and $y_x \in X$ such that $(T_{[n_k]}(x))$ is weakly convergent to y_x. Then, by the preceding paragraph, and using that $TT_{[n_k]} = T_{[n_k]}T$,

$$0 = \sigma(X, X^*) - \lim_k T_{[n_k]}((I-T)(x)) = y_x - T(y_x). \tag{4.5}$$

On the other hand

$$x - y_x = \sigma(X, X^*) - \lim_k \frac{1}{n_k} \left(\sum_{n=1}^{n_k} (x - T^n(x)) \right)$$
$$= \sigma(X, X^*) - \lim_k \frac{1}{n_k} \Big(\sum_{n=1}^{n_k} (I - T) \left((I + T + \cdots + T^{n-1})(x) \right) \Big).$$

From this and $(I - T)(X)$ being a vector subspace we deduce, as a consequence of Hahn-Banach theorem, that $x - y_x$ is in the norm closure $\overline{(I - T)(X)}$. Applying (4.4) once more we see that $\| \cdot \| - \lim_n T_{[n]}(x - y_x) = 0$. From this and (4.5) we conclude that, putting $Px = y_x$,

$$\lim_n T_{[n]}x = Px, \qquad \text{for every } x \in X.$$

Since $\|y_x\| \leq \sup_k \|T_{[n_k]}\| \|x\|$, it is clear that $P \in L(X)$. □

We are now interested in studying the consequences of this theorem that allow us to see other sufficient conditions for T to be *mean ergodic*. The first important corollary of the *Yosida's Theorem* is the following

Proposition 4.14. *Let X be a Banach space and $T \in L(X)$. If $(T^n(x))_n$ is* weakly convergent *for all $x \in X$ then T is* Mean Ergodic.

Proof. From the hypothesis we have that $T^n(x) \longrightarrow P(x)$ *weakly* for all $x \in X$. This condition implies that the sequence $(T^n(x))_n$ is weakly bounded and so, thanks to the Banach-Steinhause Theorem, we get that $(||T^n||)_n$ is bounded. This condition says us that T is *power bounded* and even that

$$\lim_{n \to \infty} \frac{T^n(x)}{n} = 0.$$

So we got the condition *1)* of the *Yosida-Kakutani's Theorem.* Now if $T^n \overset{\text{WOT}}{\longrightarrow} P$, thanks to the *Proposition* 2.8, we get

$$T_{[n]} \overset{\text{WOT}}{\longrightarrow} P$$

and so it says that $\{T_{[n]}(x) : n \in \mathbb{N}\}$ is relatively weakly compact. Now the *Yosida-Kakutani's Theorem* gives that T is *Mean Ergodic.* □

Let us now study an example that shows that if $T^n(x)$ is weakly convergent we can not affirm that $T^n(x)$ is *SOT-convergent.*

Example 4.15. Let $p > 1$ and F the map defined in the example 3.10. We know that the dual space of c_0 is l_1 and so, for all $y \in l_1$, we have that

$$\langle y, F^n(x) \rangle \to 0.$$

This condition affirms that that $F^n(x)$ is weakly convergent and so, by *Proposition* 4.14, we get that $F_{[n]} \overset{\text{SOT}}{\to} 0$. On the other hand we have that $||F^n(x)||_\infty = ||x||_\infty$ that does not converges to 0 as $n \to \infty$.

Chapter 5

Uniform Convergence

Let T be a continuous linear operator on a Banach space X. We are now interested in proving that the space X is the direct sum of $F := \{x \in X : Tx = x\}$ and $N = (I - T)(X)$ (we will write this as $X = F \oplus N$).
Before to do this we need the statement of the following classical result in *Functional Analysis.*

Theorem 5.1. ***(Open Mapping Theorem)*** *If X and Y are Banach spaces and $T : X \longrightarrow Y$ is a surjective continuous linear operator, then T is an open map (that is, if U is an open set of X, then $T(U)$ is open in Y).*

The proof of this fact is given in [7, Theorem 3.16]. Having this theorem we are now ready to prove the following

Theorem 5.2. *Let $T \in L(X)$ where X is a Banach space. If $\lim_{n\to\infty} \frac{||T^n||}{n} = 0$ then T is* Uniformly Mean Ergodic *if and only if $N = (I - T)(X)$ is closed.*

Proof. All through the proof we write $Y = \overline{N}$. Let T be *Uniformly Mean Ergodic.* We have that Y is invariant under T and the mapping $S := T_{|Y}$ satisfies $||S_{[n]}|| \longrightarrow 0$. Now, if $n \in \mathbb{N}$ is such that $||S_{[n]}|| < 1$, then $I - S_{[n]}$ is invertible. As we have the following identity

$$(I - S)\left(\frac{n-1}{n}I + \frac{n-2}{n}S + \dots + \frac{1}{n}S^{n-1}\right) = I - S_{[n]}$$

we get that $I - S$ is invertible. So we now have that $Y = (I - S)Y = (I - T)Y \subseteq (I - T)X = N$ and so, being $Y = \overline{N}$, we get $Y = N$ and so N is closed.
Conversely if we assume that N is closed (i.e. $Y = N$), the *Open Mapping Theorem* affirms that $(I-T)U$ is an open set of Y for any open $U \subseteq X$. This shows that there exists $K > 0$ such that, for all $y \in Y$, there exists $x \in X$ such that $(I - T)x = y$ and $||x|| \leq K||y||$. Now we have that

$$||T_{[n]}(y)|| = ||T_{[n]}(I - T)z|| \leq \frac{1}{n}||I - T^{n+1}|| \cdot ||z|| \leq \frac{K}{n}||I - T^{n+1}|| \cdot ||y||$$

and it shows that $S := T_{|Y}$ is *Uniformly Mean Ergodic* because

$$||T_{[n]}(y)|| \leq \frac{K}{n}||I - T^{n+1}|| \cdot ||y|| \longrightarrow 0$$

as $n \longrightarrow \infty$, for all $y \in Y$. We now have, as above, that $I - S$ is invertible on Y and $(I - T)X = Y = (I - S)Y = (I - T)Y$ and so there exists, for any $z \in X$, a $y \in Y$ such that

$$(I - T)x = (I - T)y \tag{5.1}$$

and, thanks to the invertibility of $(I - S)$ we may assume that there exists $C < \infty$ succh that $||y|| \leq C \cdot ||(I - T)x||$. Let us now write $x = (x - y) + y$. We have, thanks to (5.1), that $(x - y)$ is fixed under T and so also under T^k, for all $k > 0$. This shows us that

$$||T_{[n]}(x) - (x - y)|| = ||T_{[n]}(y)|| \leq \frac{K}{n}||I - T^{n+1}|| \cdot ||y|| \leq \frac{K}{n}||I - T^{n+1}|| \cdot ||I - T|| \cdot ||x||.$$

This last identity shows that $T_{[n]}$ converges uniformly to the projection P given by $Px = x - y$ is uniform. □

5.1 Quasi-compact operators

We now introduce a new class of operators for which uniform convergence can be obtained.

Definition 5.3. Let X be a Banach space and $T \in L(X)$. The operator T is called *(weakly) compact* if $T(B)$ is (weakly) compact, where B is the unit sphere of X.
T is called *(weakly) quasi-compact* if there exists an integer m and a (weakly) compact operator Q such that

$$||T^m - Q|| \leq 1.$$

Proposition 5.4. *T is (weakly) quasi compact if and only if there exists a sequence Q_n of (weakly) compact operators with*

$$||T^n - Q_n|| \to 0.$$

Proof. Being T quasi-compact we can put $M := T^m - Q$. For some $n \geq 0$ there exists two integers r, s such that $n = ms + r$, with $0 \leq r < m$.
We now put $Q_n = T^n - T^r M^s$ in order to obtain

$$Q_n = T^{ms+r} - T^r M^s = T^r[T^{ms} - M^s] = T^r[T^{ms} - (T^m - Q)^s].$$

In the expansion of $(T^m - Q)^s$ the only term that does not contain Q as a factor is T^{ms}. Therefore Q_n is (weakly) compact. As $||M|| < 1$ we have that $||T^n - Q_n|| = ||T^r M^s|| \leq ||M^s|| \cdot sup\{||T^r|| : 0 \leq r < m\} \to 0$.
The converse implication is trivial because, being $||T^n - Q_n|| \to 0$, there exists for sure an integer $k > 0$ such that $||T^k - Q_k|| < 1$ where Q_k is (weakly) compact. Therefore T is (weakly) quasi-compact.

□

5.2 Yosida-Kakutani's Theorem

We are now ready to give the proof of the following

Theorem 5.5. ***(Yosida-Kakutani's Theorem)***
Let T be a power bounded quasi compact operator. Then T is uniformly mean ergodic.

Proof. We only give the proof when T is *power compact*, i.e. there is $n_0 \in \mathbb{N}$ such that T^{n_0} is a compact operator. Due to Proposition 4.9, we can even assume $n_0 = 1$, i.e. that T itself is compact and, then, $T(B)$ is relatively compact in X. Since T is power bounded, there is $M \geq 1$ such that $\|T^n\| \leq M$ for every $n \in \mathbb{N}$. Then $\|T^n x\| \leq M\|x\|$ for every $x \in X$ and therefore $T^n(B) \subseteq MB$ for every $n \in \mathbb{N}$. Hence, for $n > 1$, we have

$$T^n(B) = T(T^{-n-1}(B)) \subseteq T(MB) = MT(B),$$

and this gives that $\{T^n(x) : x \in B, n \in \mathbb{N}\}$ is relatively compact in X. Since the absolutely convex hull of a compact set of a Banach space is again compact, we conclude that $\{T_{[n]}(x) : x \in B, n \in \mathbb{N}\}$ is relatively compact in X. Now the result follows immediately from Yosida Theorem. □

Theorem 5.5 here is a simplified version of the original Yosida-Kakutani theorem given in [10] (see also Theorem 2.8 in [6]). We now state the following theorem without the proof, in order to see an example of an operator T that is *uniformly mean ergodic* but T^n is not *uniformly convergent*, although is *SOT-convergent.*

Theorem 5.6. *Let T be a* power bounded *linear operator. The operator T is* uniformly mean ergodic *if and only if $(I-T)X$ is closed. The Banach space X can be decomposed as*

$$X = (I-T)X \oplus ker(I-T).$$

The proof of this fact can be found at [2]. We are now ready to see the next

Example 5.7. Let $a = (a_n)$ be a sequence such that $\lim_{n\to\infty} a_n = -1$ and $|a_n| < 1$ for all $n \in \mathbb{N}$. We now consider the operator

$$M_a : l_1(\mathbb{N}) \to l_1(\mathbb{N})$$
$$(b_n) \mapsto (a_n b_n).$$

It is clear that $|M_a| = 1$ and $1 \notin \sigma(M_a) = \{a_n : n \in \mathbb{N}\} \bigcup \{-1\}$, so we have that M_a is uniformly mean ergodic by the *Theorem 14.* Moreover M_a is even *SOT-convergent* to 0. However $|M_a^n| = 1$ for all $n \in \mathbb{N}$. We consider now $M_a^2 : l_1(\mathbb{N}) \to l_1(\mathbb{N})$. Actually, $M_a^2 = M_{a^2}$, when $a^2(n) = a_n^2$ for all $n \in \mathbb{N}$. Now $\sigma(M_{a^2}) = \{a_n^2 : n \in \mathbb{N}\} \cup \{1\}$. Hence 1 is an accumulation point of $\sigma(M_{a^2})$. This implies that M_{a^2} is not uniformly mean ergodic by Theorem 5.6.

A power bounded operator $T \in L(l_\infty)$ is mean ergodic if and only if T is uniformly mean ergodic. $M_a : l_\infty \to l_\infty$ is an example of operator mean ergodic such that M_a^2 is not mean ergodic by Theorem 5.6.

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