axioms

Article

Distinguished Property in Tensor Products and Weak* Dual Spaces

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Abstract: A local convex space E is said to be distinguished if its strong dual E'_β has the topology $\beta(E', (E'_\beta)')$, i.e., if E'_β is barrelled. The distinguished property of the local convex space $C_p(X)$ of real-valued functions on a Tychonoff space X, equipped with the pointwise topology on X, has recently aroused great interest among analysts and C_p-theorists, obtaining very interesting properties and nice characterizations. For instance, it has recently been obtained that a space $C_p(X)$ is distinguished if and only if any function $f \in \mathbb{R}^X$ belongs to the pointwise closure of a pointwise bounded set in $C(X)$. The extensively studied distinguished properties in the injective tensor products $C_p(X) \otimes_\varepsilon E$ and in $C_p(X, E)$ contrasts with the few distinguished properties of injective tensor products related to the dual space $L_p(X)$ of $C_p(X)$ endowed with the weak* topology, as well as to the weak* dual of $C_p(X, E)$. To partially fill this gap, some distinguished properties in the injective tensor product space $L_p(X) \otimes_\varepsilon E$ are presented and a characterization of the distinguished property of the weak* dual of $C_p(X, E)$ for wide classes of spaces X and E is provided.

Keywords: distinguished space; injective and projective tensor product; vector-valued continuous function; Fréchet space; nuclear space

MSC: 46M05; 54C35; 46A03; 46A32

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1. Introduction

In this paper, X is an infinite Tychonoff space and $C(X)$ is the linear space of all real-valued continuous functions over X. $C_p(X)$ and $C_k(X)$ denote the space $C(X)$ equipped with the pointwise and compact-open topology, respectively. $L_p(X)$ represents the weak* dual of $C_p(X)$, i.e., the topological dual $L(X)$ of $C_p(X)$ endowed with the weak topology $\sigma(L(X), C(X))$ of the dual pair $\langle L(X), C(X)\rangle$, i.e., $L_p(X)$ has the topology of pointwise convergence on $C(X)$.

Moreover, all local convex spaces are assumed to be real and Hausdorff and the symbol '$\simeq$' indicates some canonical algebraic isomorphism or linear homeomorphism. The strong dual E'_β of a local convex space E is the topological dual E' of E equipped with the strong topology $\beta(E', E)$, which is the topology of uniform convergence on the bounded subsets of E. $\langle E, E'\rangle$ is a dual pair. For a subset A of E the *polar* A^0 of A with respect to a dual pair $\langle E, F\rangle$ is

$$A^0 = \{x \in F : |\langle a, x\rangle| \leq 1, \forall a \in A\}.$$

A local convex space E is barrelled if for each pointwise bounded subset M of E' there exists a neighborhood of the origin U in E such that M is uniformly bounded on U. Hence E is barrelled if and only if its topology is the topology $\beta(E, E')$, i.e., $[E'(\text{weak}^*)]'_\beta = E$.

Roughly speaking, E is barrelled if it verifies the local convex version of the Banach–Steinhaus uniform boundedness theorem.

The local convex space E is called *distinguished* if E'_β is barrelled. In [1–7] the distinguished property of the space $C_p(X)$ has been extensively studied. Furthermore, [8] [Proposition 6.4] is connected with distinguished $C_p(X)$ spaces. It is observed in [3] [Theorem 10] that $C_p(X)$ is distinguished if and only if $C_p(X)$ is a large subspace of $\mathbb{R}^X$, i.e., if each bounded set in $\mathbb{R}^X$ is contained in the closure in $\mathbb{R}^X$ of a bounded set of $C_p(X)$, or, equivalently, if the strong bidual of $C_p(X)$ is $\mathbb{R}^X$ [5]. In [7], [Theorem 2.1] it is shown that $C_p(X)$ is distinguished if and only if X is a Δ-space in the sense of Knight [9], and several applications of this fact are given. Equivalently, $C_p(X)$ is distinguished if for each countable partition $\{X_k : k \in \mathbb{N}\}$ of X into nonempty pairwise disjoint sets, there are open sets $\{U_k : k \in \mathbb{N}\}$ with $X_k \subseteq U_k$, for each $k \in \mathbb{N}$, such that each point $x \in X$ belongs to U_n for only finitely many $n \in \mathbb{N}$, [5] [Theorem 5].

If E and F are local convex spaces, $E \otimes_\varepsilon F$ and $E \otimes_\pi F$ represent the *injective* and *projective tensor product* of E and F, respectively. A basis of neighborhoods of the origin in $E'_\beta \otimes_\varepsilon F'_\beta$ is determined by the sets $\varepsilon(A,B) := \left(A^{00} \otimes B^{00}\right)^0$, where A is a bounded set in E, B is a bounded set in F, $A^0 \subseteq E'$, $A^{00} \subseteq E''$, $B^0 \subseteq F'$, $B^{00} \subseteq F''$ and $\left(A^{00} \otimes B^{00}\right)^0 \subseteq E' \otimes F'$. Analogously, a basis of neighborhoods of the origin in the tensor product space $E'_\beta \otimes_\pi F'_\beta$ is formed by the sets $\pi(A,B) := \mathbf{acx}\left(A^0 \otimes B^0\right)$, where A is a bounded set in E, B is a bounded set in F and $\mathbf{acx}\left(A^0 \otimes B^0\right)$ denotes the absolutely convex cover of the tensor product $A^0 \otimes B^0$. Recall that if E carries the weak topology, then $(E \otimes_\varepsilon F)' \simeq (E \otimes_\pi F)' \simeq E' \otimes F'$, [10] [41.3 (9) and 45.1 (2)]. A local convex space E is called *nuclear* if $E \otimes_\varepsilon F = E \otimes_\pi F$ for every local convex space F, [11] [21.2].

The distinguished property of $C_p(X)$ under the formation of some tensor products is examined in [2]. Among other results it is showed in [2] [Corollary 6] that for a local convex space E the injective tensor product $C_p(X) \otimes_\varepsilon E$ is distinguished if both $C_p(X)$ is distinguished and $\mathbb{R}^{(X)} \otimes_\varepsilon E'_\beta$ is barrelled, where $\mathbb{R}^{(X)}$ the local convex direct sum of $|X|$ real lines.

If E is a local convex space $C_p(X,E)$ and $C_k(X,E)$ will denote the linear space of all E-valued continuous functions defined on X equipped with the pointwise topology and compact-open topology, respectively. It is also proved in [2] [Corolary 21] that, for any Tychonoff space X and any normed space E, the vector-valued function space $C_p(X,E)$ is distinguished if and only if $C_p(X) \otimes_\varepsilon E$ is distinguished. In particular, if X is a countable Tychonoff space and E a normed space, then $C_p(X,E)$ is distinguished. Indeed, if X is countable, on the one hand $C_p(X)$ is distinguished by [5] [Corollary 6] and on the other hand $\mathbb{R}^{(X)}$ is both barrelled and nuclear (the latter because [11] [21.2.3 Corollary]), so that $\mathbb{R}^{(X)} \otimes_\varepsilon E'_\beta = \mathbb{R}^{(X)} \otimes_\pi E'_\beta$ is barrelled by [12] [Theorem 1.6.6]. Thus, $C_p(X) \otimes_\varepsilon E$ is distinguished by the already mentioned [2] [Corolary 6] and, since E is normed, $C_p(X,E)$ is distinguished too by [2] [Corollary 21]. A corresponding result for the compact-open topology, due to Díaz and Domański [13] [Corolary 2.5], states that the space $C_k(K,E)$ of continuous functions defined on a compact Hausdorff space K and with values in a reflexive Fréchet space E is also distinguished, being its strong dual naturally isomorphic to $L_1(\mu)\hat{\otimes}_\pi E'_\beta$.

According to [1] [Theorem 3.9], the strong dual $L_\beta(X)$ of $C_p(X)$ is always distinguished. The distinguished property of the weak* dual $L_p(X)$ of $C_p(X)$ is investigated in [5], where the following theorem is proved.

Theorem 1 ([5] [Theorem 27])**.** *If X is a μ-space, then the weak* dual $L_p(X)$ of $C_p(X)$ is distinguished.*

Recall that a Tychonoff space X is called a μ-space if each functionally bounded set is relatively compact.

The extensively studied distinguished properties in the injective tensor products $C_p(X) \otimes_\varepsilon E$ and in $C_p(X,E)$ contrasts with the few distinguished properties related with the injective tensor products $L_p(X) \otimes_\varepsilon E$ and with the weak* dual of $C_p(X, E)$. Theorem 1 and the fact that $L_p(X)$ spaces are studied so extensively as $C_p(X)$ spaces motivated us to fill partially this gap in this paper obtaining distinguished properties of injective tensor products $L_p(X) \otimes_\varepsilon E$ and providing a characterization of the distinguished property of the weak* dual of $C_p(X, E)$ for wide classes of spaces X and E. To reach these goals we require [2] [Theorem 5] and [2] [Proposition 19], which we include here for convenience.

Theorem 2 ([2] [Theorem 5]). *Let E and F be local convex spaces, where E carries the weak topology. If τ_ε and τ_π denote the injective and projective topologies of $E'_\beta \otimes F'_\beta$, the following properties hold*

1. *If $E'_\beta \otimes_\varepsilon F'_\beta$ is barrelled, then $\tau_\varepsilon = \beta(E' \otimes F', E \otimes F)$ and $E \otimes_\varepsilon F$ is distinguished.*
2. *If $E'_\beta \otimes_\pi F'_\beta$ is barrelled then $\tau_\varepsilon \leq \beta(E' \otimes F', E \otimes F) \leq \tau_\pi$.*

Theorem 3 ([2] [Proposition 19]). *For any local convex space E, the dual of the space $C_p(X, E)$ is algebraically isomorphic to $L(X) \otimes E'$, i.e., $C_p(X, E)' \simeq L(X) \otimes E'$.*

It should be noted that if $\Sigma_{i=1}^n f_i \otimes u_i$ is a representation of $\varphi \in C(X) \otimes E$ then Theorem 3 is due to the fact that the canonical map $T : C_p(X) \otimes_\varepsilon E \to C_p(X, E)$ given by

$$(T\varphi)(x) = \Sigma_{i=1}^n f_i(x) u_i,$$

is a linear homeomorphism from $C_p(X) \otimes_\varepsilon E$ into a dense linear subspace of $C_p(X, E)$. Furthermore, $\left(C_p(X) \otimes_\varepsilon E\right)' \simeq L(X) \otimes E'$, because $C_p(X)$ carries the weak topology, so one has $C_p(X, E)' \simeq L(X) \otimes E'$, as stated.

2. Distinguished Tensor Products of $L_p(X)$ Spaces

This section deals mainly with the injective tensor product of $L_p(X)$ and a nuclear metrizable space E. It should be noted that the class of nuclear metrizable spaces is large. Recall that the space s of all rapidly decreasing sequences, as well as the test space of distributions $\mathcal{D}(\Omega)$, where Ω is an open set in $\mathbb{R}^n$, with their usual local convex inductive topologies, are examples of nuclear Fréchet spaces [11] [Section 21.6]. The strong dual of $\mathcal{D}(\Omega)$ is the space of distributions on Ω and it is denoted by $\mathcal{D}'(\Omega)$.

Theorem 4. *Assume that X is a μ-space and let E be a nuclear metrizable local convex space. If every countable union of compact subsets of X is relatively compact, then $L_p(X) \otimes_\varepsilon E$ is distinguished.*

Proof. The space X is a μ-space if and only if $C_k(X)$ is barrelled, by the Nachbin-Shirota theorem [14] [Proposition 2.15]. On the other hand, as every countable union of compact subsets of X is assumed to be relatively compact, the space $C_k(X)$ is also a (DF)-space [15] [Theorem 12]. In addition, the strong dual E'_β of a metrizable local convex space E it is a complete (DF)-space by [16] [see 29.3 -in "By 2(1)"-]. Moreover, nuclearity of E implies that E'_β is nuclear too by [11] [21.5.3 Theorem]. As E'_β is a nuclear (DF)-space, one has that E'_β is a quasi-barrelled space [11] [21.5.4 Corollary]. Finally, the completeness of the quasi-barrelled space E'_β implies that E'_β is barrelled [16] [27.1.(1)], so E is distinguished.

The projective tensor product $C_k(X) \otimes_\pi E'_\beta$ is barrelled by [11] [15.6.8 Proposition]. Thus, taking into consideration E'_β nuclearity, it can be obtained that $C_k(X) \otimes_\varepsilon E'_\beta$ is also barrelled. On the other hand, since X is a μ-space it follows from [5] [Theorem 27] that $C_k(X)$ coincides with the strong dual of $L_p(X)$, i.e., $\left(L_p(X)\right)'_\beta = C_k(X)$, hence

$$\left(L_p(X)\right)'_\beta \otimes_\varepsilon E'_\beta$$

is barrelled. Finally, as $L_p(X)$ carries the weak topology, the first statement of Theorem 2, ensures that the space $L_p(X) \otimes_\varepsilon E$ is distinguished. □

Example 1. *In particular, for each compact topological space X and for each nuclear metrizable local convex space E it follows that $L_p(X) \otimes_\varepsilon E$ is distinguished.*

Hence, if X is the Cantor space K or the interval $[0,1]$, and if E is one of the local convex spaces $\mathcal{D}(\Omega)$ or s, then the injective tensor products $L_p(K) \otimes_\varepsilon \mathcal{D}(\Omega)$, $L_p(K) \otimes_\varepsilon s$, $L_p([0,1]) \otimes_\varepsilon \mathcal{D}(\Omega)$ and $L_p([0,1]) \otimes_\varepsilon s$ are distinguished.

Corollary 1. *If X is a compact space and Y is a countable Tychonoff space, then the space $L_p(X) \otimes_\varepsilon C_p(Y)$ is distinguished.*

Proof. Clearly, $C_p(Y)$ is metrizable (hence distinguished [1] [Theorem 3.3]) and nuclear (by [11] [21.2.3 Corollary]), so the statement follows from the previous theorem. □

If we apply this Corollary with X equal to the Stone-Čech compactification $\beta\mathbb{N}$ of the topological space $\mathbb{N}$ formed by the natural numbers endowed with the discrete topology and Y equal to the space $\mathbb{Q}$ of rational numbers endowed with the usual metrizable topology then we get that $L_p(\beta\mathbb{N}) \otimes_\varepsilon C_p(\mathbb{Q})$ is a distinguished space.

If the factor E of $L_p(X) \otimes_\varepsilon E$ is a normed space, the following theorem holds true.

Theorem 5. *If X is a μ-space with finite compact sets (equivalently, if every functionally bounded subset of X is finite) and E is a normed space, then $L_p(X) \otimes_\varepsilon E$ is distinguished.*

Proof. If X is a μ-space with finite compact sets, the space $C_k(X) = C_p(X)$ is barrelled and nuclear. As E'_β is a Banach space, [12] [Corollary 1.6.6] assures that $C_k(X) \otimes_\pi E'_\beta$ is a barrelled space, and $C_k(X)$ nuclearity yields that $C_k(X) \otimes_\varepsilon E'_\beta$ is also a barrelled space. Bearing in mind that $\left(L_p(X)\right)'_\beta = C_k(X)$, as a consequence of the fact that X is a μ-space (*cf.* [5] [Theorem 27]), Theorem 2 ensures that $L_p(X) \otimes_\varepsilon E$ is distinguished. □

A P-space in the sense of Gillman–Henriksen is a topological space in which every countable intersection of open sets is open.

Corollary 2. *If X is a P-space and E is a normed space, then $L_p(X) \otimes_\varepsilon E$ is distinguished.*

Proof. Every P-space is a μ-space with finite compact sets (*cf.* [17] [Problem 4K]). □

Example 2. *If $L(\mathfrak{m})$ denotes the Lindelöfication of the discrete space of cardinal $\mathfrak{m} \geq \aleph_1$, the space $L_p(L(\mathfrak{m})) \otimes_\varepsilon C_k([0,1])$ is distinguished.* In this case $L(\mathfrak{m})$ is a Lindelöf P-space.

Theorem 6. *If X is a μ-space with finite compact sets and E is normed space, then $L_p(X) \otimes_\varepsilon E'(\text{weak}^*)$ is distinguished.*

Proof. By [12] [Theorem 1.6.6] the projective tensor product $C_k(X) \otimes_\pi E$ is a barrelled space, hence $C_k(X)$ nuclearity yields that $C_k(X) \otimes_\varepsilon E$ is barrelled. So, the conclusion follows from the first statement of Theorem 2. □

Example 3. *The space $L_p(L(\mathfrak{m})) \otimes_\varepsilon \ell_p(\text{weak}^*)$ is distinguished for $1 \leq p < \infty$.*

A topological space is said to be *hemicompact* if it has a sequence of compact subsets such that every compact subset of the space lies inside some compact set in the sequence.

Theorem 7. *If X is a hemicompact space and E is a nuclear metrizable barrelled space (for instance a nuclear Fréchet space), then $L_p(X) \otimes_\varepsilon E'(\text{weak}^*)$ is distinguished.*

Proof. Clearly X is a Lindelöf space, hence it is a μ-space, and then both $C_k(X)$ and E are metrizable and barrelled spaces. Then [12] [Corollary 1.6.4] ensures that $C_k(X) \otimes_\pi E$ is also a (metrizable) barrelled space. This property and the E nuclearity imply that $C_k(X) \otimes_\varepsilon E$ is a barrelled space. Consequently, using that $\left(L_p(X)\right)'_\beta = C_k(X)$ and $E'(\text{weak}^*)'_\beta = E$, we get

$$L_p(X)'_\beta \otimes_\varepsilon E'(\text{weak}^*)'_\beta = C_k(X) \otimes_\varepsilon E.$$

So, Theorem 2 applies to guarantee that $L_p(X) \otimes_\varepsilon E'(\text{weak}^*)$ is distinguished. □

By Theorem 7 the injective tensor product $L_p(\mathbb{R}) \otimes_\varepsilon \mathcal{D}'(\Omega)(\text{weak}^*)$ is distinguished since $\mathbb{R}$ is hemicompact and $\mathcal{D}'(\Omega)(\text{weak}^*)$ is a nuclear Fréchet space. Theorem 7 is also applied in the next Example 4.

Example 4. *If $\mathbb{N}$ is equipped with the discrete topology, $p \in \beta\mathbb{N} \setminus \mathbb{N}$ and $Z = \mathbb{N} \cup \{p\}$ has the topology induced by $\beta\mathbb{N}$, then $L_p(Z) \otimes_\varepsilon L_p(Z)$ is distinguished.*

Proof. The subspace $Z = \mathbb{N} \cup \{p\}$ of $\beta\mathbb{N}$ is countable and has finite compact sets, so that it is hemicompact. Since Z is countable, $C_p(Z)$ is metrizable and, on the other hand, as a subspace of the nuclear space $\mathbb{R}^X$, the space $C_p(Z)$ is nuclear. In addition, since Z is a μ-space with finite compact sets, the space $C_p(Z)$ is barrelled [18]. So, according to the previous theorem, $L_p(Z) \otimes_\varepsilon L_p(Z)$ is distinguished. □

3. Distinguished Property of the Weak* Dual of $C_p(X) \otimes_\varepsilon E$

The preceding theorems are going to be applied to examine the distinguished property of the weak* dual of the injective tensor product $C_p(X) \otimes_\varepsilon E$. To get this property we need the following lemma.

Lemma 1. *The injective topology of the tensor product $L_p(X) \otimes E'(\text{weak}^*)$ coincides with the weak topology $\sigma(L(X) \otimes E', C(X) \otimes E)$.*

Proof. Since $L_p(X)$ carries the weak topology

$$\left(L_p(X) \otimes_\varepsilon E'(\text{weak}^*)\right)' \simeq C(X) \otimes E.$$

Hence, the injective topology τ_ε of $L_p(X) \otimes E'(\text{weak}^*)$ is stronger than the weak topology $\sigma(L(X) \otimes E', C(X) \otimes E)$. We prove that both topologies are the same. Indeed, if U is a closed absolutely convex neighborhood of the origin in $L_p(X)$ and V is a closed absolutely convex neighborhood of the origin in $E'(\text{weak}^*)$, there are finite sets Φ in $C(X)$ and Δ in E such that $\Phi^0 = U$ and $\Delta^0 = V$. Setting $\Lambda = \Phi \otimes \Delta$, then Λ is a finite set in $C(X) \otimes E$ such that

$$\varepsilon_{U,V}(\psi) = \sup_{f \in U^0, u \in V^0} \left| \sum_{i=1}^n f(x_i) \langle v_i, u \rangle \right| = \sup_{f \in U^0, u \in V^0} \left| \left\langle \sum_{i=1}^n \delta_{x_i} \otimes v_i, f \otimes u \right\rangle \right| \leq \sup_{F \in \Lambda} |\langle \psi, F \rangle|$$

for any $\psi = \sum_{i=1}^n \delta_{x_i} \otimes v_i \in L(X) \otimes_\varepsilon E'$, since

$$U^0 \otimes V^0 = \Phi^{00} \otimes \Delta^{00} = \mathbf{acx}(\Phi) \otimes \mathbf{acx}(\Delta) \subseteq \mathbf{acx}(\Lambda).$$

Therefore $\tau_\varepsilon = \sigma(L(X) \otimes E', C(X) \otimes E)$. □

Corollary 3. *If X is a hemicompact space and E is a nuclear Fréchet space, the weak* dual of $C_p(X) \otimes_\varepsilon E$ is distinguished.*

Proof. According to Lemma 1 the weak* dual of $C_p(X) \otimes_\varepsilon E$ is linearly homeomorphic to $L_p(X) \otimes_\varepsilon E'(\text{weak}^*)$, so Theorem 7 applies. □

The space Z considered in Example 4 is hemicompact, hence from Corollary 3 we have that the weak* duals of $C_p(\mathbb{R}) \otimes_\varepsilon C_p(Z)$ and $C_p(Z) \otimes_\varepsilon \mathcal{D}(\Omega)$ are distinguished.

Corollary 4. *If X is a μ-space with finite compact sets and E is a normed space, the weak* dual of $C_p(X) \otimes_\varepsilon E$ is distinguished.*

Proof. The proof is analogous to the proof of Corollary 3, with the difference of using Theorem 6 instead of Theorem 7. □

Example 5. *The weak* dual of $C_p(L(\mathfrak{m})) \otimes_\varepsilon C_k([0,1])$ is distinguished.*

4. A Characterization of the Distinguished Weak* Dual of $C_p(X, E)$

Let E be a local convex space. We will denote by $L_p(X, E')$ the weak* dual of $C_p(X, E)$. Since by Theorem 3 the dual space $C_p(X, E)'$ is algebraically isomorphic to $L(X) \otimes E'$, one has

$$L_p(X, E') \simeq \left(L(X) \otimes E', \sigma(L(X) \otimes E', C(X, E))\right).$$

A completely regular topological space X is a $k_{\mathbb{R}}$-space if every real function f defined on X whose restriction to every compact subset K of X is continuous, is continuous on X.

Theorem 8. *Let X be a hemicompact $k_{\mathbb{R}}$-space and let E be a nuclear Fréchet space. The space $L_p(X, E')$ is distinguished if and only if the strong dual of $L_p(X, E')$ coincides with $C_k(X, E)$.*

Proof. We will denote by $C_\beta(X, E)$ the linear space $C(X, E)$ equipped with the strong topology $\beta(C(X, E), L(X) \otimes E')$, i.e., the strong dual of $L_p(X, E')$. Since X is a $k_{\mathbb{R}}$-space and E is complete, [11] [16.6.3 Corollary] ensures that

$$C_k(X, E) \simeq C_k(X) \widehat{\otimes}_\varepsilon E. \tag{1}$$

So, as both $C_k(X)$ and E are metrizable, $C_k(X, E)$ is a Fréchet space. Consequently, if $C_\beta(X, E) = C_k(X, E)$ then $C_\beta(X, E)$ is barrelled and $L_p(X, E')$ is distinguished.

Assume, conversely, that $L_p(X, E')$ is distinguished. From $C_k(X, E) \simeq C_k(X) \widehat{\otimes}_\varepsilon E$ it follows that $C_k(X, E)' = (C_k(X) \otimes_\varepsilon E)'$. Since $L(X) \otimes E'$ is algebraically isomorphic to a subspace of $(C_k(X) \otimes_\varepsilon E)'$, it follows that the compact-open topology of $C(X, E)$ is stronger than $\beta(C(X, E), L(X) \otimes E')$. Hence, the identity map $J : C_k(X, E) \to C_\beta(X, E)$ is continuous.

Since X is a hemicompact, $C_k(X)$ is metrizable. As a consequence of E nuclearity, $C_k(X) \otimes_\varepsilon E = C_k(X) \otimes_\pi E$ is a metrizable space. Hence, by (1) $C_k(X, E)$ is a Fréchet space. If $L_p(X, E')$ is distinguished, then $C_\beta(X, E)$ is barrelled. So J is a linear homeomorphism by the closed graph theorem. Thus, $C_\beta(X, E) = C_k(X, E)$. □

5. Conclusions and Two Open Problems

This paper has been motivated by the contrast between the extensively distinguished properties obtained recently in the injective tensor products $C_p(X) \otimes_\varepsilon E$ and in the spaces $C_p(X, E)$ with the few distinguished properties of injective tensor products related to the dual space $L_p(X)$ of $C_p(X)$ endowed with the weak* topology, as well as to the weak* dual of $C_p(X, E)$. In Section 2, distinguished properties in the injective tensor product space $L_p(X) \otimes_\varepsilon E$ are provided and in Sections 3 and 4, the distinguished property of the weak* dual of $C_p(X) \otimes_\varepsilon E$ and a characterization of the distinguished property of the weak* dual of $C_p(X, E)$ for wide classes of spaces X and E are provided.

We do not know the answer for the following two problems when the Tychonoff space X is uncountable. It is easy to prove that the answer of these two problems is positive if X is countable.

Problem 1. *Is it true that if X is an uncountable P-space and E is a Fréchet space, then $L_p(X) \otimes_\varepsilon E'(\text{weak}^*)$ is distinguished?*

Problem 2. *Is it true that if X is an uncountable P-space and E is a Fréchet space, then the weak* dual of $C_p(X) \otimes_\varepsilon E$ is distinguished?*

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References

1. Ferrando, J.C.; Kąkol, J. Metrizable bounded sets in $C(X)$ spaces and distinguished $C_p(X)$ spaces. *J. Convex. Anal.* **2019**, *26*, 1337–1346. Available online: https://www.heldermann.de/JCA/JCA26/JCA264/jca26070.htm (accessed on 7 July 2021).
2. Ferrando, J.C.; Kąkol, J. Distinguished metrizable spaces E and injective tensor products $C_p(X) \otimes_\varepsilon E$. **2021**, submitted
3. Ferrando, J.C.; Kąkol, J.; Leiderman, A.; Saxon, S.A. Distinguished $C_p(X)$ spaces. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **2021**, *115*, 1–18. [CrossRef]
4. Ferrando, J.C.; Kąkol, J.; Saxon, S.A. Examples of Nondistinguished Function Spaces $C_p(X)$. *J. Convex. Anal.* **2019**, *26*, 1347–1348. Available online: https://www.heldermann.de/JCA/JCA26/JCA264/jca26071.htm (accessed on 7 July 2021).
5. Ferrando, J.C.; Saxon, S.A. If not distinguished, is $C_p(X)$ even close? *Proc. Am. Math. Soc.* **2021**, *149*, 2583–2596. [CrossRef]
6. Ferrando, J.C.; Saxon, S.A. Distinguished $C_p(X)$ spaces and the strongest locally convex topology. **2021**, submitted
7. Kąkol, J.; Leiderman, A. A characterization of X for which spaces $C_p(X)$ are distinguished and its applications. *Proc. Am. Math. Soc. Ser. B* **2021**, *8*, 86–99. [CrossRef]
8. Banakh, T.; Kąkol, J.; Schürz, J.P. ω^ω-base and infinite-dimensional compact sets in locally convex spaces. *Rev. Mat. Complut.* **2021**. [CrossRef]
9. Knight, R.W. Δ-Sets. *Trans. Amer. Math. Soc. Ser. B* **1993** , *339*, 45–60. Available online: https://www.ams.org/journals/tran/1993-339-01/home.html (accessed on 7 July 2021). [CrossRef]
10. Köthe, G. *Topological Vector Spaces II*; Grundlehren der Mathematischen Wissenschaften 237; Springer: New York, NY, USA; Berlin/Heidelberg, Germany, 1979. Available online: https://link.springer.com/book/10.1007/978-1-4684-9409-9 (accessed on 7 July 2021).
11. Jarchow, H. *Locally Convex Spaces*; Mathematical Textbooks; B. G. Teubner: Stuttgart, Germany, 1981. [CrossRef]
12. Ferrando, J.C.; López-Pellicer, M.; Sánchez Ruiz, L.M. *Metrizable Barrelled Spaces*; Pitman Research Notes in Mathematics Series 332; Longman: Harlow, UK, 1995.
13. Díaz, J.C.; Domański, P. On the injective tensor product of distinguished Fréchet spaces. *Math. Nachr.* **1999**, *198*, 41–50. [CrossRef]
14. Kąkol, J.; Kubiś, W.; López-Pellicer, M. *Descriptive Topology in Selected Topics of Functional Analysis*; Developments in Mathematics 24; Springer: New York, NY, USA, 2011. [CrossRef]
15. Warner, S. The topology of compact convergence on continuous function spaces. *Duke Math. J.* **1958**, *25*, 265–282. [CrossRef]
16. Köthe, G. *Topological Vector Spaces I*; Die Grundlehren der Mathematischen Wissenschaften Band 159; Springer: New York, NY, USA, 1969. Available online: https://link.springer.com/book/10.1007%2F978-3-642-64988-2 (accessed on 7 July 2021).
17. Gillman, L.; Jerison, M. *Rings of Continuous Functions*, 1960 ed.; Graduate Texts in Mathematics No. 43; Springer: New York, NY, USA; Berlin/Heidelberg, Germany, 1976. Available online: https://link.springer.com/book/10.1007/978-1-4615-7819-2 (accessed on 7 July 2021).
18. Buchwalter, H.; Schmets, J. Sur quelques propiértés de l'espace $C_s(T)$. *J. Math. Pures Appl. IX Sér.* **1973**, *52*, 337–352.