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SURVEY

Weighted Banach spaces of analytic functions with sup-norms and operators between them: a survey

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Abstract

In this survey we report about recent work on weighted Banach spaces of analytic functions on the unit disc and on the whole complex plane defined with sup-norms and operators between them. Results about the solid hull and core of these spaces and distance formulas are reviewed. Differentiation and integration operators, Cesàro and Volterra operators, weighted composition and superposition operators and Toeplitz operators on these spaces are analyzed. Boundedness, compactness, the spectrum, hypercyclicity and (uniform) mean ergodicity of these operators are considered.

Keywords Weighted Banach spaces of holomorphic functions · Associated weights · Composition operators · Integral operators · Mean ergodic operators · Hypercyclic operators

Mathematics Subject Classification Primary 46E15 · 47B38; Secondary 46E10 · 47A10 · 47A16 · 47A35 · 47B07 · 47B33 · 47B35

1 Introduction

Given a strictly positive continuous weight v on an open connected subset of the complex plane, an analytic function f belongs to the weighted Banach space H_v^∞ if $v|f|$ is bounded in the open set. The norm is defined as the supremum of $v|f|$. The author published in 2003 his survey article [40] in which he collected many results obtained a few years before about weighted Banach spaces H_v^∞ of analytic functions of type H^∞ and inductive limits of them, interpolation and sampling and composition, multiplication and differentiation operators between them. Since that time much progress has been obtained by many authors, and it is my feeling that it might be time to review, at least part, of all this work. In this sense, this article could be considered as a continuation of [40]. The selection of material reflects the research interests of the author. There are certainly many important related results which are not included here, and I apologize to those authors whose work is not mentioned. Despite the necessary selection, the list of references is enormous. I am also sorry about that, but it

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seems unavoidable. The author hopes that this survey might be informative for the reader, and that it might show how many interesting results have been obtained recently and, at the same time, that some questions remain open in this area of research.

We briefly describe the content of the article. Precise statements can be seen at the right place in the paper. No proofs are given in the survey. The interested reader should look at the original articles. Precise references are given. After the notation and terminology in the next section, we recall the notion of associated weight in Sect. 3. Associated weights were introduced and studied in [36] and they have played a significant role in characterizing properties of weighted Banach spaces H_v^∞. Two important results due to Abakumov and Doubtsov [3] characterizing those radial weights which are equivalent to their associated weight are stated in Theorems 2 and 3. In Sect. 4 we report on our joint work with Vukotic [69] about conditions on a non-negative function $v : G \to [0, \infty)$, not necessarily bounded or strictly positive, defined on an open connected domain G in the complex plane, to ensure that the semi-normed space $H_v^\infty(G)$ is in fact normed and complete. Section 5 is dedicated to review some results of Lusky, Taskinen and the author about the solid hull and core of H_v^∞. A distance formula from an element $f \in H_v^\infty$ to the closed subspace H_v^0, which is the closure of the polynomials, was obtained by Perfekt [141, 142]. A direct, elementary proof was presented in [57] and it is explained in Sect. 6.

Necessary definitions about bounded and linear operators, spectrum, hypercyclicity, power boundedness and (uniform) mean ergodicity are briefly recalled in Sect. 7. They are important in the second part of the article. The operators of differentiation and integration are treated in Sect. 8. We first state characterizations of boundedness and compactness due to Harutyunyan and Lusky [110] and to Abanin and Tien [6]. Then we include a few results about the spectrum, mean ergodicity and hypercyclicity of these two operators when they act on weighted Banach spaces of entire functions, see [30]. The proper closed invariant subspaces of the integration operator on H_v^0 were determined by Abanin and Tien [7]. Their statement is presented in Theorem 39. In Sect. 9 we collect some results of Albanese, Ricker and the author [9] about the Cesàro operator on weighted Banach spaces for standard weights. A few results about the Volterra integral operator are collected in Sect. 10. Several questions about weighted composition operators on H_v^∞ spaces are discussed in Sect. 11: boundedness, (weak) compactness, invertibility, the spectrum, (uniform) mean ergodicity and hypercyclicity. We briefly report on the deep work of many authors. Section 12 presents a few results by Boyd and Rueda [76, 77] and by Vukotic and the author [68] concerning superposition operators on H_v^∞ of analytic functions on the disc. Lusky, Taskinen and the author have investigated bounded Toeplitz operators acting on H_v^∞ in the case of the disc in the papers [60, 61]. Some of these results are stated in Sect. 13. The final Sect. 14 includes a few results about the Hilbert matrix operator, the Libera operator and the Hausdorff operators.

2 Notation and preliminaries

Let G be an open subset of the complex plane $\mathbb{C}$. A *general weight* $v : G \to \mathbb{R}$ is a continuous and strictly positive function on G. We define the following *weighted Banach spaces of analytic functions* on G

$$H_v^\infty(G) := \{f \in H(G);\ \|f\|_v := \sup_{z \in G} v(z)|f(z)| < \infty\},$$

$$H_v^0(G) := \{f \in H(G);\ v|f| \text{ vanishes at } \infty \text{ on } G\}.$$

Recall that a function g vanishes at infinity on G if for every $\varepsilon > 0$ there is a compact subset K of G such that $|g(z)| < \varepsilon$ if $z \notin K$. If G is an open subset of $\mathbb{C}$, we denote by $H(G)$ the Fréchet space of all analytic functions on G endowed with the topology τ_{co} of uniform convergence on the compact subsets of G.

We are mainly interested in the case of radial weights defined on the unit disc or the whole complex plane. To be more precise, we fix some notation and terminology in this case. We set $R = 1$ (for the case of analytic functions on the unit disc) and $R = +\infty$ (for the case of entire functions). A *weight* v is a continuous function $v : [0, R[\to]0, \infty[$, which is non-increasing on $[0, R[$ and satisfies $\lim_{r \to R} r^n v(r) = 0$ for each $n \in \mathbb{N}$. Observe that in the case of $R = \infty$, the condition $\lim_{r \to \infty} r^n v(r) = 0$, for each $n \in \mathbb{N}$, is equivalent to the fact that $H_v^\infty(\mathbb{C})$ contains the polynomials, and also to the fact that $H_v^0(\mathbb{C})$ contains the polynomials. We extend v to $\mathbb{D}$ if $R = 1$ and to $\mathbb{C}$ if $R = +\infty$ by $v(z) := v(|z|)$. For an analytic function $f \in H(\{z \in \mathbb{C}; |z| < R\})$ and $r < R$, we denote $M(f, r) := \max\{|f(z)| \,;\ |z| = r\}$. Using the notation O and o of Landau, $f \in H_v^\infty(G)$ if and only if $M(f, r) = O(1/v(r)), r \to R$. For these radial weights, it is known that the closure of the polynomials in $H_v^\infty(G)$ coincides with the Banach space $H_v^0(G)$, which is the set of those analytic functions on $\{z \in \mathbb{C}; |z| < R\}$ such that $M(f, r) = o(1/v(r)), r \to R$, see e.g. [35]. It was proved in [37] that, under the present conditions on the radial weight v on the disc or the complex plane, the space $H_v^\infty(G)$ is canonically isometric to the bidual of $H_v^0(G)$.

It will be clear from the context in the rest of the article when we refer to analytic functions on the disc or entire functions. Hence, we might write simply H_v^∞ and H_v^0. Anyway, if it is necessary to distinguish at some point, we will use the notations $H_v^\infty(\mathbb{D})$ and $H_v^\infty(\mathbb{C})$, respectively.

We recall some examples of weights:

For $R = 1$,

(i) $v(r) = (1 - r)^\gamma$ with $\gamma > 0$, which are the standard weights on the disc,
(ii) $v(r) = \exp(-b(1 - r)^{-a})$, $a, b > 0$, which are called exponential weights, and
(iii) $v(z) = (\log \frac{e}{1-r})^{-\alpha}$, $\alpha > 0$, which are called logarithmic weights.

For $R = +\infty$,

(i) $v(r) = \exp(-r^p)$ with $p > 0$,
(ii) $v(r) = \exp(-\exp r)$, and
(iii) $v(r) = \exp\big(-(\log^+ r)^\alpha\big)$, where $\alpha > 1$ and $\log^+ r = \max(\log r, 0)$.

Banach spaces of the type mentioned above appear naturally in the study of growth conditions of analytic functions and have been considered in many papers. We refer to [35–37, 151]. Composition operators on weighted Banach spaces of this type when $G = \mathbb{D}$ have been studied in [49, 52, 88, 137, 155]. Pointwise multiplication operators were considered in [50], and sampling and interpolation in these spaces in [92]. Lusky presented in [125–127] a complete isomorphic classification of the spaces H_v^∞ and H_v^0. We reported about these and related results in our survey [40]. Much progress has been done recently about these spaces and operators between them. We present some of the new interesting developments in the next pages.

The Banach spaces $H_v^\infty(\mathbb{D})$ and $H_v^0(\mathbb{D})$ for the standard weight $v(r) = (1-r)^\gamma,\ \gamma > 0$, are the growth spaces of Korenblum type $A^{-\gamma}$ and $A_0^{-\gamma}$, which are also denoted H_γ^∞ and H_γ^0. They play an important role in connection with the interpolation and sampling of analytic functions, [111, Ch. 4 and 5]. Observe that, since $(1 - r)^\gamma \le (1 - r^2)^\gamma \le 2^\gamma(1 - r)^\gamma$, for $0 < r < 1$, these weights define the same spaces (with an equivalent norm).

The space $H_v^\infty(\mathbb{C})$ is denoted as the weighted Fock space $\mathcal{F}_\infty^\phi$ of order infinity (i.e. with sup-norms) with $v(z) = \exp(-\phi(|z|))$, and $\phi : [0, \infty[\to]0, \infty[$ is a twice continuously differentiable increasing function. See for example [84, 166].

For each $0 < p < \infty$, the *Bloch space of order* p is

$$\mathcal{B}_p = \left\{ f \in H(\mathbb{D}) : \sup_{z\in\mathbb{D}} (1 - |z|^2)^p |f'(z)| < \infty \right\},$$

and the *little Bloch space of order* p is

$$\mathcal{B}_p^0 = \left\{ f \in H(\mathbb{D}) : \lim_{|z|\to 1} (1 - |z|^2)^p |f'(z)| = 0 \right\}.$$

It is a well-known fact that $\mathcal{B}_p$ is a Banach space when it is endowed with the norm

$$\|f\|_p = |f(0)| + \sup_{z\in\mathbb{D}} (1 - |z|^2)^p |f'(z)|,$$

and that $\mathcal{B}_p^0$ is a closed subspace of $\mathcal{B}_p$ for each $p > 0$. For $p = 1$, the spaces $\mathcal{B}_1$ and $\mathcal{B}_1^0$ are called the *classical Bloch space* and the *little Bloch space*, and they are denoted by $\mathcal{B}$ and $\mathcal{B}_0$ [165]. All these spaces are continuously embedded in $H(\mathbb{D})$. The Bloch spaces are closely related to weighted Banach spaces of analytic functions. In fact, $f \in \mathcal{B}_p$ if and only if $f' \in H_{v_p}^\infty(\mathbb{D})$ for $v_p(z) = (1 - |z|^2)^p$ and $p > 0$. We have the isometric identification of $\mathcal{B}_p$ with the ℓ_1-sum $H_{v_p}^\infty \oplus_{\ell_1} \mathbb{C}$, via $g \to (g', g(0))$. With this identification, several operators on Bloch type spaces can be treated as operators on $H_v^\infty(\mathbb{D})$ spaces; in particular, composition operators on Bloch spaces can be considered as weighted composition operators. See for example [56, 88]. We will not state in this survey all the possible consequences for Bloch type spaces, and refer to the original papers. Moreover, we will not report here about important, related work on weighted Hardy and Bergman spaces.

Our notation for complex analysis, functional analysis and operator theory is standard. We refer the reader to [96, 112, 131, 146]. If E is a Hausdorff locally convex space space, for example a Banach space, its topological dual is denoted by E'. The weak topology on E is denoted by $\sigma(E, E')$ and the weak* topology on E' by $\sigma(E', E)$. The linear span of a subset A of E is denoted by $\mathrm{span}(A)$. In what follows, we set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

3 The associated weight and the characterization of essential weights.

Let G be an open connected subset of $\mathbb{C}$ and let $v : G \to \mathbb{R}$ be a general weight on G. *We assume in what follows that the norm $||\delta_z||$ of the Dirac measure, $\delta_z(f) := f(z), z \in G$ as an element of $H_v^\infty(G)'$ is strictly positive.*

The *associated weight* $\tilde{v}$ of v is defined by

$$\tilde{v}(z) := 1/\|\delta_z\|_{H_v^\infty(G)'}.$$

By our assumption above, $\tilde{v}(z)$ is finite for every $z \in G$. Moreover $v \le \tilde{v}$ on G, $1/\tilde{v}$ is continuous and subharmonic, and the Banach spaces $H_v^\infty(G)$ and $H_{\tilde{v}}^\infty(G)$ coincide isometrically. A weight v is called *essential* if there is $C \ge 1$ such that $v \le \tilde{v} \le Cv$ on G. Associated weights were thoroughly studied in [36, 52].

Example 1 (1) If $G = \mathbb{D}$ or $G = \mathbb{C}$ and v is radial, then $\tilde{v}$ is also radial.

(2) If $G = \mathbb{D}$ or $G = \mathbb{C}$, v is radial and $\lim_{r\to R} r^n v(r) = 0$ for each $n \in \mathbb{N}$, then $\lim_{r\to R} r^n \tilde{v}(r) = 0$.
(3) The following weights are essential: (i) $v(r) = (1-r)^\alpha, 0 \le r < 1$, with $\alpha > 0$, (ii) $v(r) = \exp(-b(1-r)^{-a}), a, b > 0$, and (iii) $v(z) = (\log \frac{e}{1-r})^{-\alpha},\ \alpha > 0$ for the unit disc; and (iv) $v(r) = \exp(-r^n)$ with $n \in \mathbb{N}$, for $\mathbb{C}$. See [36, Example 1.7].
(4) ([49]) A radial and non-increasing weight v on $\mathbb{D}$ is essential if and only if it is equivalent to a log-convex radial weight w on $\mathbb{D}$. We recall that the radial weight w is *log-convex* on $G = \mathbb{D}$ or $G = \mathbb{C}$ if the function $t \to -\log w(e^t)$ is convex. The associated weight to a radial weight is log-convex.
(5) Every normal weight on $\mathbb{D}$ in the sense of Shields and Williams is essential. We refer the reader to [92, 151].

A thorough investigation of essential radial weights on the disc $\mathbb{D}$ or the complex plane was undertaken by Abakumov and Doubtsov [1–3]. We recall their main results. A few definitions are necessary. We consider in the rest of this section only radial, non-increasing, continuous, strictly positive weights $v : G \to \mathbb{R}$ for $G = \mathbb{D}$ or $G = \mathbb{C}$.

Given two functions $f, g : X \to (0, +\infty)$, we write $f \approx g$ if there are positive constants $A, B > 0$ such that $Af(x) \le g(x) \le Bf(x)$ for each $x \in X$.

A weight v is called *approximable by a finite sum of moduli* if there exist $f_1, \ldots, f_s \in H(G)$ such that $1/v(z) \approx |f_1(z)| + \cdots + |f_s(z)|$, $z \in G$, it is said to be *approximable by the maximum of an analytic function modulus* if there is $f \in H(G)$ such that $1/v(r) \approx M(f, r)$ for each $r \in (0, R)$, it is *approximable by power series with positive coefficients* if there exist $a_k \ge 0$, $k = 0, 1, 2, \ldots$ such that $\sum_{k=0}^{\infty} a_k r^k \approx 1/v(r)$ for each $r \in (0, R)$.

The norm of a monomial $z^k, k = 0, 1, 2, \ldots$ in the space $H_v^\infty(G)$ is given by $\|z^k\|_v = \sup_{0\le r<R} r^k v(r)$. We set $P_v(r) := \sup_{k\in\mathbb{N}_0} r^k/\|z^k\|_v,\ 0 \le r < R$. Clearly $P_v(r) \le 1/v(r), 0 \le r < R$. We say that v is *approximable from below by monomials* if there is $C > 0$ such that $1/v(r) \le C P_v(r)$ for each $0 \le r < R$. It was proved in [49] that every essential weight on the unit disc is approximable from below by monomials. Abakumov and Doubtsov used a result of Erdös and Kövári to show in [3, Lemma 1] that the associate weight $\tilde{v}$ of a radial, non-increasing, continuous, strictly positive weight v on $\mathbb{C}$ such that $\lim_{r\to\infty} r^n \tilde{v}(r) = 0$ for each $n \in \mathbb{N}$ satisfies $P_v(r) \le 1/\tilde{v}(r) \le 6P_v(r), 0 \le r < \infty$.

Theorem 2 (Abakumov, Doubtsov [3]) *The following conditions are equivalent for a radial, non-increasing, continuous, strictly positive weight $v : \mathbb{D} \to \mathbb{R}$ on the unit disc such that* $\lim_{r\to 1-} v(r) = 0$*:*

(i) *v is essential.*
(ii) *v is equivalent to a log-convex radial weight w on $\mathbb{D}$.*
(iii) *v is approximable by a finite sum of moduli of (two) analytic functions.*
(iv) *v is approximable by the maximum of an analytic function modulus.*
(v) *v is approximable by power series with positive coefficients.*
(vi) *v is approximable from below by monomials.*

It follows from [36, Example 3.3] that there exist log-convex, radial, non-increasing, continuous, strictly positive weights $v : \mathbb{C} \to \mathbb{R}$ on the complex plane with $\lim_{r\to\infty} r^n v(r) = 0$ for each $n \in \mathbb{N}$, which are not approximable by a finite sum of moduli of entire functions. Accordingly condition (ii) in Theorem 2 does not imply (iii) for weights on the complex plane. We have the following result.

Theorem 3 (Abakumov, Doubtsov [3]) *The following conditions are equivalent for a radial, non-increasing, continuous, strictly positive weight $v : \mathbb{C} \to \mathbb{R}$ on the complex plane such that* $\lim_{r\to\infty} r^n v(r) = 0$ *for each $n \in \mathbb{N}$.*

(i) *v is essential.*
(ii) *v is approximable by a finite sum of moduli of analytic functions.*
(iii) *v is approximable by the maximum of an analytic function modulus.*
(iv) *v is approximable by power series with positive coefficients.*
(v) *v is approximable from below by monomials.*

Each of these conditions imply that v is equivalent to a log-convex radial weight w on $\mathbb{C}$.

The following example is taken from [3, Example 1].

Proposition 4 *Let $\alpha > 1$ and let v_α be a weight on the complex plane such that $v_\alpha(r) := \exp(-(\log r)^\alpha), e \leq r < \infty$. The weight v_α is essential if $\alpha \geq 2$ and not essential if $1 < \alpha < 2$.*

4 The seminormed space $H_v^\infty(G)$ for non-negative weight v, not necessarily bounded or strictly positive

Vukotic and the author investigated in [69] some necessary and some sufficient conditions on a non-negative function $v : G \to [0, \infty[$, not necessarily bounded or strictly positive, defined on an *open connected domain* G in the complex plane, to ensure that the semi-normed space $H_v^\infty(G)$ is in fact normed and complete. The completeness of weighted Bergman spaces was studied by Arcozzi and Björn [19]. They obtained complete characterizations when the weight $v(z) = \chi_E(z), z \in G$, is the characteristic function χ_E of a subset E of G in [19, Theorem 2.1]. We review here some of the results in [69].

For a non-negative function $v : G \to [0, \infty[$, the space $H_v^\infty(G)$ associated with v is also defined by

$$H_v^\infty(G) := \left\{ f \in H(G)\ ;\ \|f\|_v = \sup_{z \in G} v(z)|f(z)| < +\infty \right\},$$

and it is endowed with the natural *seminorm* $\|f\|_v := \sup_{z\in G} v(z)|f(z)|$.

We use the following notation in this section:

$$E_v := \{z \in G\ ;\ v(z) > 0\}.$$

Proposition 5 *Let $v : G \to [0, \infty[$ be a weight on a planar domain G. Then the space $H_v^\infty(G)$ is normed if and only if E_v is not a discrete set (that is, it has a limit point in G). Moreover, if $H_v^\infty(G)$ is normed, then the inclusion map $J : H_v^\infty(G) \to (H(G), \tau_{co})$ has closed graph.*

Proposition 6 *Assume that the space $H_v^\infty(G)$ is normed. The following conditions are equivalent:*

(i) *The space $H_v^\infty(G)$ is a Banach space.*
(ii) *The inclusion map $J : H_v^\infty(G) \to (H(G), \tau_{co})$ is continuous.* (*Equivalently, every sequence in $H_v^\infty(G)$ bounded in the norm is a normal family.*)
(iii) *The closed unit ball B_v^∞ of $H_v^\infty(G)$ is bounded in $(H(G), \tau_{co})$.*
(iv) *For each $z \in G$, the point evaluation functional $\delta_z(f) := f(z), z \in G$, satisfies $\delta_z \in H_v^\infty(G)'$ and, moreover, $\sup_{z\in K} \|\delta_z\|_v' < \infty$ for each compact subset K of G.*

Theorem 7 *Let $v : G \to [0, \infty[$ be a bounded weight on a planar domain G such that the space $H_v^\infty(G)$ is normed. The space $H_v^\infty(G)$ is complete if and only if there is a bounded, continuous, strictly positive weight $\tilde{v}$ on G such that $H_v^\infty(G) = H_{\tilde{v}}^\infty(G)$.*

Proposition 8 *Let $v : G \to [0, \infty[$ be a weight on a planar domain G. If $H_v^\infty(G)$ is a Banach space containing non-zero functions, then the boundary ∂G is contained in the closure $\overline{E}_v$ of E_v in $\mathbb{C}$.*

Corollary 9 *Let $v : G \to [0, \infty[$ be a weight on a planar domain G (other than the plane itself) such that $H_v^\infty(G)$ is normed and it contains non-zero functions.*

(1) *If E_v is contained in a convex closed proper subset A of G, then $H_v^\infty(G)$ is not a Banach space.*
(2) *If the closure of E_v in $\mathbb{C}$ is a compact subset of G, then $H_v^\infty(G)$ is not a Banach space.*

As a consequence of Corollary 9 one easily deduces the following example: the weight $v(z) = \max\{0, \operatorname{Re} z\}$ is continuous on the unit disc $\mathbb{D}$, vanishes in the left-hand half of the disc, it is strictly positive in the remaining open right semi-disc, and $H_v^\infty(\mathbb{D})$ is not a Banach space.

Proposition 10 *Let $v : G \to [0, \infty[$ be a weight on a planar domain G. Suppose that for each $z \in G$ there is a bounded open set $U \subset G$ such that $z \in U$, $\partial U \subset E_v$, and v is bounded away from 0 on ∂U. Then $H_v^\infty(G)$ is a Banach space*

Corollary 11 *Let $v : G \to [0, \infty[$ be a continuous weight on a planar domain G such that $G \backslash E_v$ is discrete, i.e., the zeros of v are isolated. Then $H_v^\infty(G)$ is a Banach space.*

As a consequence of Corollary 11, if $F \in H(G)$ is a non-zero analytic function on a planar domain G and $v(z) := |F(z)|$, $z \in G$, then $H_v^\infty(G)$ is a Banach space.

Corollary 12 *Let $v : G \to [0, \infty[$ be a weight on a planar domain G such that $\overline{G \backslash E_v}$ is a compact subset of G. If $\inf_{z \in K} v(z) > 0$ for each compact subset $K \subset G \backslash (\overline{G \backslash E_v})$, then $H_v^\infty(G)$ is a Banach space.*

Corollary 13 *Let $G = \mathbb{D}$ (resp. $G = \mathbb{C}$). Let v be a bounded radial weight on G. The space $H_v^\infty(G)$ is a Banach space if and only if E_v is not compact in G or, equivalently, if and only if there is an increasing sequence $(r_k)_k$ in $]0, 1[$ tending to 1 (resp. $(r_k)_k$ in $]0, \infty[$ tending to ∞) such that $v(r_k) > 0$ for each $k \in \mathbb{N}$.*

In particular, if $v(z) := |F(|z|)|$, $z \in G$, for a non-zero function $F \in H(G)$, then $H_v^\infty(G)$ is a Banach space.

Let v be the weight on $\mathbb{D}$ defined by $v(z) := a_n > 0$ if $|z| = 1 - (1/n)$, and $v(z) = 0$ otherwise. Then $H_v^\infty(G)$ is a Banach space by Corollary 13. Observe that the sequence $(a_n)_n \subset]0, \infty[$ need not be bounded. Similar examples can be obtained by replacing $\mathbb{D}$ by $\mathbb{C}$ and $1 - (1/n)$ by n, $n \in \mathbb{N}$.

Proposition 14 *Let $F \in H(G)$ be a non-zero function on a planar domain G. Define $v(z) := 0$ if $F(z) = 0$ and $v(z) := 1/|F(z)|$ if $F(z) \neq 0$. Then $H_v^\infty(G)$ is a Banach space that coincides with the set of all $f \in H(G)$ such that there is $C = C(f) > 0$ with $|f(z)| \leq C|F(z)|$ for each $z \in G$.*

Example 15 (1) Let $q > 0$ and $v(z) = |\operatorname{Re} z|^q$, $z \in \mathbb{D}$. Then the normed space $H_v^\infty(\mathbb{D})$ is complete.
(2) Let v be a weight on $\mathbb{D}$ such that there is a strictly increasing sequence $(r_n)_n$ of positive numbers tending to 1 such that for each n there is $a_n > 0$ such that $v(r_n e^{i\theta}) \geq a_n$ almost everywhere in $[0, 2\pi]$. Then the normed space $H_v^\infty(\mathbb{D})$ is complete.
(3) Let w be a continuous weight on G such that $H_w^\infty(\mathbb{D})$ is a Banach space. Let v be a weight on G such that $\{z \in G \ ; \ v(z) = w(z)\}$ is dense in G (note that this implies that E_v is not discrete, so that $H_v^\infty(\mathbb{D})$ is normed). Then $H_v^\infty(\mathbb{D})$ is actually a Banach space.

5 Solid hull and solid core of the space H_v^∞

In this section we identify an analytic function $f(z) = \sum_{n=0}^\infty a_n z^n$ on $\mathbb{D}$ or $\mathbb{C}$ with the sequence of its Taylor coefficients $(a_n)_{n=0}^\infty$. Let A and B be vector spaces of complex sequences containing the space of all the sequences with finitely many non-zero coordinates. The space A is *solid* if $a = (a_n) \in A$ and $|b_n| \le |a_n|$ for each n implies $b = (b_n) \in A$. The *solid hull of* A is

$$S(A) := \{(c_n) \,:\, \exists (a_n) \in A \text{ such that } |c_n| \le |a_n| \, \forall n \in \mathbb{N}\}.$$

The *solid core of* A is

$$s(A) := \{(c_n) \,:\, (c_n a_n) \in A \, \forall (a_n) \in \ell_\infty\}.$$

It is easy to see that the Fréchet spaces $H(\mathbb{D})$ and $H(\mathbb{C})$ of analytic functions on the unit disc and on the whole complex plane are solid.

In [18], the solid hull $S_{\text{vect}}(A)$ of a space A of analytic functions on $\mathbb{D}$ is defined as the intersection of all solid *vector spaces* of analytic functions on $\mathbb{D}$ containing A. Clearly, $S(A) \subset S_{\text{vect}}(A)$ in general. In all the known examples of the solid hull of a space of analytic functions it turns out that it is already a vector space.

We refer the reader to [97, 165] for information about Hardy spaces. The solid hull of the Hardy spaces $S(H^p) = H^2, 2 \le p \le \infty$ is known. The proof for H^∞ depends on the following deep result of Kislyakov from 1981.

Theorem 16 (Kislyakov) *There is $C > 0$ such that for each $(b_j)_{j=n}^m, 0 < n < m$, there is a polynomial $P(z) = \sum_{j=n}^m c_j z^j$ such that $|b_j| \le |c_j|, j = n, \dots, m$ and $\|P\|_\infty \le C(\sum_{j=n}^m |b_j|^2)^{1/2}$.*

The solid hull $S(H^p)$ for $1 \le p < 2$ seems to be unknown. The solid core of the Hardy spaces $s(H^p) = H^2, 1 \le p \le 2$, is known. Moreover $s(H^\infty) = \ell_1$. In particular, the space H^∞ is not solid. The disc algebra $A(\mathbb{D})$ is also not solid. It is an open problem to describe the solid core $s(H^p)$ for $2 < p < \infty$.

Bennet, Stegenga and Timoney in their paper [34] determined the solid hull and the solid core of the weighted spaces $H_v^\infty(\mathbb{D})$ when the weight v is doubling. Recall that the weight v on $\mathbb{D}$ is *doubling* if there is $M > 0$ such that $v(1-r) \le Mv(1-(r/2))$ for each $0 < r < 1$. Exponential weights $v(r) = \exp(-a/(1-r)^b)$ with $a, b > 0$ are not doubling. The solid hull and core of spaces of analytic functions on the disc has been investigated by many authors. In addition to those mentioned above, Anderson, Dostanić, Blasco, Buckley, Jevtić, Pavlović, Ramanujan, Shields and Vukotić, among many others. We refer the reader to the book [117].

In the case of a standard weight $v_\alpha(z) = (1-|z|^2)^\alpha$, where $\alpha \ge 0$, we recall that $A^{-\alpha} = H_{v_\alpha}^\infty(\mathbb{D})$. The solid hull of $A^{-\alpha}$ is known:

$$S\big(A^{-\alpha}\big) = \left\{ (b_m)_{m=0}^\infty \,:\, \sup_{n \in \mathbb{N}_0} \left(\sum_{m=2^n}^{2^{n+1}-1} |b_m|^2 (m+1)^{-2\alpha} \right)^{1/2} < \infty \right\}.$$

This is Theorem 8.2.1 of [117]. Moreover, the solid core $s\big(A^{-\alpha}\big)$ can also be characterized, see Theorem 8.3.4 of [117]:

$$s\big(A^{-\alpha}\big) = \left\{ (b_m)_{m=0}^\infty \,:\, \sup_{n \in \mathbb{N}_0} \left(\sum_{m=2^n}^{2^{n+1}-1} |b_m| (m+1)^{-\alpha} \right) < \infty \right\}.$$

In our joint article [58] with Lusky and Taskinen we extended previous work in [66, 67] and determined the solid hull and solid core of weighted Banach spaces H_v^∞, both in the case of the analytic functions on the disc and on the whole complex plane, for a very general class of radial weights v. The case of Bergman spaces was treated later in [59]. We present some of these results here.

Recall that a sequence $(e_n)_{n=1}^\infty$ of elements of a separable Banach space X is a *Schauder basis*, if every element $f \in X$ can be presented as a convergent sum

$$f = \sum_{n=1}^{\infty} f_n e_n,$$

where the numbers $f_n \in \mathbb{K}$ are unique for $f \in X$.

We denote by $\Lambda = \{z^k : k = 0, 1, 2, \ldots\}$ the sequence of monomials.

Theorem 17 *If $S(H_v^\infty) = H_v^\infty$, then Λ is a Schauder basis of H_v^0.*

By a theorem of Lusky [126], Λ is never a basis for $H_v^0(\mathbb{D})$. This implies the following consequence.

Corollary 18 *In the case of analytic functions on the disc $\mathbb{D}$, one always has $S(H_v^\infty(\mathbb{D})) \neq H_v^\infty(\mathbb{D})$ and $s(H_v^\infty(\mathbb{D})) \neq H_v^\infty(\mathbb{D})$.*

Lusky [126] proved that the monomials $\Lambda = \{z^k : k = 0, 1, 2, \ldots\}$ are a basis of $H_v^0(\mathbb{C})$ for the weight $v(r) = \exp(-(\log(r))^2)$. In the case of weighted spaces of entire functions we have the following result to be found in [59].

Theorem 19 *Let v be a weight on the complex plane satisfying* ***condition (b)*** *(given later). The space H_v^∞ is solid if and only if Λ is a Schauder basis of H_v^0.*

Now we present the solid hull and solid core of weighted Banach spaces H_v^∞ for concrete weights on the disc and on the complex plane.

Theorem 20 *For $v(r) = \exp(-1/(1-r))$ the solid hull of $H_v^\infty(\mathbb{D})$ is*

$$\left\{ (b_m)_{m=0}^\infty \ : \ \sup_n \exp(-2n^2) \sum_{m=n^4+1}^{(n+1)^4} |b_m|^2 \left(1 - \frac{1}{n^2}\right)^{2m} < \infty \right\},$$

and the solid core is

$$\left\{ (b_m)_{m=0}^\infty \ : \ \sup_n \exp(-n^2) \sum_{m=n^4+1}^{(n+1)^4} |b_m| \left(1 - \frac{1}{n^2}\right)^{m} < \infty \right\},$$

Theorem 21 *Let v be the weight $v(r) = \exp(-ar^p)$ on $\mathbb{C}$, where $a > 0$ and $p > 0$ are constants. Then, the solid hull of $H_v^\infty(\mathbb{C})$ is*

$$\left\{ (b_m)_{m=0}^\infty \ : \ \sup_{n\in\mathbb{N}} \sum_{pn^2+1<m\le p(n+1)^2} |b_m|^2 e^{-2n^2} n^{4m/p} (ap)^{-m/p} < \infty \right\}.$$

and the solid core is

$$\left\{ (b_m)_{m=0}^\infty \ : \ \sup_{n\in\mathbb{N}} \sum_{pn^2+1<m\le p(n+1)^2} |b_m| e^{-n^2} n^{2m/p} (ap)^{-m/2p} < \infty \right\}.$$

These results are a consequence of a general theorem. To state it, we need a few definitions. Let $r_m \in]0, R[$ be a global maximum point of the function $r^m v(r)$ for any $m > 0$. The weight v is said to satisfy the *condition* (b) if there exist numbers $b > 2$, $K > b$ and $0 < m_1 < m_2 < \dots$ with $\lim_{n\to\infty} m_n = \infty$ such that

$$b \le \left(\frac{r_{m_n}}{r_{m_{n+1}}}\right)^{m_n} \frac{v(r_{m_n})}{v(r_{m_{n+1}})}, \left(\frac{r_{m_{n+1}}}{r_{m_n}}\right)^{m_{n+1}} \frac{v(r_{m_{n+1}})}{v(r_{m_n})} \le K.$$

Theorem 22 *If the weight v satisfies condition (b), we have*

$$S(H_v^\infty) = \left\{ (b_m)_{m=0}^\infty : \sup_n v(r_{m_n}) \left(\sum_{m_n < m \le m_{n+1}} |b_m|^2 r_{m_n}^{2m} \right)^{1/2} < \infty \right\},$$

and

$$s(H_v^\infty) = \left\{ (b_m)_{m=0}^\infty : \sup_n v(r_{m_n}) \left(\sum_{m_n < m \le m_{n+1}} |b_m| r_{m_n}^{m} \right) < \infty \right\}.$$

The proof is mainly based on results and techniques of Lusky [127] and methods due to Bennet, Stegenga and Timoney [34], in particular Theorem 16 of Kislyakov. We recall the main technical result of Lusky, since it is important in other contexts, for example in the study of Toeplitz operators.

Let $h(z) = \sum_{k=0}^\infty b_k z^k$, and put $M_\infty(h, r) = \sup_{|z|=r} |h(z)|$. For the numbers m_n as in condition (b) set

$$(R_n h)(z) = \sum_{k=0}^{m_{n-1}} b_k z^k + \sum_{m_{n-1} < k \le m_n} \frac{[m_n] - k}{[m_n] - [m_{n-1}]} b_k z^k.$$

Define the operators $V_n = R_n - R_{n-1}$ for $n \in \mathbb{N}$.

Theorem 23 (Lusky) *The norms*

$$\|h\|_v \quad \text{and} \quad \sup_n \sup_{r_{m_{n-1}} \le r \le r_{m_{n+1}}} M_\infty(V_n h, r) v(r)$$

are equivalent.

Moreover, the operators V_n are uniformly bounded on H_v^∞ and there are $c_1 > 0$ and $c_2 > 0$ with

$$c_1 \sup_n \|V_n h\|_v \le \|h\|_v \le c_2 \sup_n \|V_n h\|_v \quad \textit{for all } h \in H_v^\infty.$$

Here is the explicit calculation of the sequence m_n for concrete weights. They are needed to deduce Theorems 20 and 21 from Theorem 22.

(1) For the complex plane. Let $v(r) = \exp(-ar^p)$, $r \in [0, \infty[$, $a > 0$, $p > 0$ and let $b > 2$. The weight v satisfies condition (b) for the sequence $m_n := p(\log b)n^2$ with $K = b^5$.
(2) For the unit disc. Let $v(z) = \exp\left(-\frac{a}{(1-r)^b}\right)$. The weight v satisfies condition (b) for the sequence $m_n = b^{1+1/b} a^{-1/b} n^{2+2/b} - bn^2$. In particular, for $v(z) = \exp\left(-\frac{1}{(1-r)}\right)$, one can take $m_n = n^4 - n^2$. It can be shown that one can also take $m_n = n^4$ in this case.

The investigation of solid hull and cores of spaces of type H^∞ has been continued by Schindl in [147]. He observed that some of the weights which appear in the descriptions given above arise frequently in the theory of ultradifferentiable and ultraholomorphic function classes. This connection enabled him to see which growth behaviour must satisfy the sequences (m_n) which appear in the descriptions and also to study when these numbers can exist. New examples were obtained.

6 Distance formulas

Lusky, Taskinen and the author investigated in [57] the distance $d(f, H_v^0) = \inf_{g\in H_v^0} \|f - g\|_v$ of a function $f \in H_v^\infty$ to the closed subspace H_v^0. Perfekt in Example 4.4 of [141] had proved that $d(f, H_v^0) = \limsup_{r\to R} M(f,r)v(r)$ for each $f \in H_v^\infty$. This result is obtained by him as a consequence of an abstract result [141, Theorem 2.3] with an argument using duality and measures. Moreover, H_v^0 is a proximinal subspace of H_v^∞; that is, for each $f \in H_v^\infty$ the distance $d(f, H_v^0)$ is attained at a point $g \in H_v^0$. The proximinality in Theorem 24, i.e. the existence of the minimizer g, also appears in Perfekt [142] as a consequence of the fact that H_v^0 is an M-ideal of H_v^∞. Our approach gave an elementary, direct proof of the formula of the distance. Consequences about Bloch type spaces were obtained, but will not be stated here.

Theorem 24 *For every $f \in H_v^\infty$ there is $g \in H_v^0$ with*

$$d(f, H_v^0) = \|f - g\|_v = \limsup_{r\to R} M(f,r)v(r).$$

Our proof depends on a technical lemma, which could be of independent interest, in which we use the following notation. Given an analytic function f on $\mathbb{D}$ or $\mathbb{C}$, we denote by $\sigma_n f$ the n-th Cesàro mean of f; i.e. the arithmetic mean of the first n Taylor polynomials of f. In this case, one has $M(\sigma_n f, r) \le M(f,r)$ for each $0 < r < R$.

Lemma 25 *Let $f \in H_v^\infty$ and assume that there is $0 < \tau < 1$ with*

$$\tau\|f\|_v \le \limsup_{r\to R} M(f,r)v(r).$$

Then, for each $\varepsilon > 0$ and $m \in \mathbb{N}$ there is $n \in \mathbb{N}$, $n > m$, such that with $\rho = (1-\tau)/(1+\tau)$ we have

$$\left(\frac{1+\tau}{2(1+\varepsilon)}\right)\|f - \rho\sigma_n f\|_v \le \limsup_{r\to R} M(f,r)v(r) = \limsup_{r\to R} M(f - \rho\sigma_n f, r)v(r).$$

The following simple examples show that the distance $d(f, H_v^0)$ can be attained at many points of H_v^0 for a given function $f \in H_v^\infty$.

(1) Consider the weight $v(r) = e^{-r}$, $r \in [0, \infty[$, on the complex plane and the analytic function $f(z) = e^z$, $z \in \mathbb{C}$. Clearly $f \in H_v^\infty$ and $\|f\|_v = 1$. Set $P_n(z) = \sum_{k=0}^n \frac{z^k}{k!}$ for each $n \in \mathbb{N}$. We have, for each n, $P_n \in H_v^0$ and

$$\|f - P_n\|_v = \sup_{r>0} e^{-r} \sum_{k=n+1}^{\infty} \frac{r^k}{k!} = 1 = d(f, H_v^0).$$

(2) Now define the weight $v(r) = 1 - r$, $r \in [0, 1[$, on the unit disc. The function $f(z) = \frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$ belongs to H_v^{∞} and $\|f\|_v = 1$. Set $P_n(z) = \sum_{k=0}^{n} z^k$ for each $n \in \mathbb{N}$. We have, for each n, $P_n \in H_v^0$ and

$$M(f - P_n, r) = \sum_{k=n+1}^{\infty} r^k = \frac{r^{n+1}}{1-r}.$$

Therefore

$$\|f - P_n\|_v = \sup_{r \in [0,1[} (1-r) M(f - P_n, r) = 1 = d(f, H_v^0).$$

7 A few definitions concerning bounded linear operators

We recall a few definitions concerning operator theory, mean ergodic operators and linear dynamics which will be used in the rest of the article.

Let X, Y be Banach spaces. Let $\mathcal{L}(X, Y)$ denote the space of all continuous (or bounded) linear operators from X into Y. When $X = Y$, we denote by $\mathcal{L}(X)$ the space of all continuous linear operators from a Banach space X into itself. An operator $T \in \mathcal{L}(X, Y)$ is called *(weakly) compact* if the image $T(B_X)$ of the unit ball B_X of X is relatively (weakly) compact in Y. If $T \in \mathcal{L}(X, Y)$ is an operator on a Banach space X, the essential norm $\|T\|_{e,X,Y}$ of T is the distance to the space of compact operators from X into Y. If $X = Y$, we write $\|T\|_{e,X}$, and if the space is clear from the context, we simply write $\|T\|_e$. The operator $T \in \mathcal{L}(X, Y)$ is said to be *Fredholm* if $\ker T$ and $Y/\mathrm{Im} T$ are finite dimensional. An operator T is Fredholm if and only if there are $S \in \mathcal{L}(Y, X)$ and compact operators $K_1 \in \mathcal{L}(X)$ and $K_2 \in \mathcal{L}(Y)$ such that $ST = I + K_1$ and $TS = I + K_2$

For $T \in \mathcal{L}(X)$, the *resolvent set* $\rho(T)$ of T consists of all $\lambda \in \mathbb{C}$ such that $R(\lambda, T) := (\lambda I - T)^{-1}$ exists in $\mathcal{L}(X)$. The set $\sigma(T) := \mathbb{C} \setminus \rho(T)$ is called the *spectrum* of T. The *point spectrum* $\sigma_{pt}(T)$ of T consists of all $\lambda \in \mathbb{C}$ such that $(\lambda I - T)$ is not injective. If we need to stress the space X, then we also write $\sigma(T; X)$, $\sigma_{pt}(T; X)$ and $\rho(T; X)$. Given $\lambda, \mu \in \rho(T)$ the *resolvent identity* $R(\lambda, T) - R(\mu, T) = (\mu - \lambda) R(\lambda, T) R(\mu, T)$ holds.

The essential spectrum $\sigma_e(T, X)$ of an operator $T \in \mathcal{L}(X)$ on the Banach space X is the set of all $\lambda \in \mathbb{C}$ such that $\lambda I - T$ is not Fredholm. The essential spectral radius is

$$r_{e,X}(T) := \sup\{|\lambda| \; ; \; \lambda \in \sigma_e(T, X)\}.$$

It can be calculated as follows

$$r_{e,X}(T) = \lim_{n \to \infty} (\|T^n\|_{e,X})^{1/n}.$$

An operator $T \in \mathcal{L}(X)$ is *mean ergodic* if its sequence of Cesàro averages

$$T_{[n]} := \frac{1}{n} \sum_{m=1}^{n} T^m, \quad n \in \mathbb{N}, \tag{1}$$

converges to some operator $P \in \mathcal{L}(X)$ in the strong operator topology τ_s, i.e., $\lim_{n \to \infty} T_{[n]} x = Px$ for each $x \in X$, [96, Ch.VIII]. It follows from (1) that $\frac{T^n}{n} = T_{[n]} - \frac{n-1}{n} T_{[n-1]}$, for $n \geq 2$. Hence, τ_s-$\lim_{n \to \infty} \frac{T^n}{n} = 0$ whenever T is mean ergodic and, in particular, $\sup_n \frac{\|T^n\|}{n} < \infty$. According to [96, VIII Corollary 5.2, p.662], when T is mean ergodic one has the direct decomposition

$$X = \mathrm{Ker}(I - T) \oplus \overline{\mathrm{Im}(I - T)}. \qquad (2)$$

An operator $T \in \mathcal{L}(X)$ is called *uniformly mean ergodic* if there exists $P \in \mathcal{L}(X)$ such that $\lim_{n\to\infty} \|T_{[n]} - P\| = 0$. It is then immediate that necessarily $\lim_{n\to\infty} \frac{\|T^n\|}{n} = 0$. A result of Lin, [120, Theorem 2.1], states that $T \in \mathcal{L}(X)$ satisfying $\lim_{n\to\infty} \frac{\|T^n\|}{n} = 0$ is uniformly mean ergodic if and only if $\mathrm{Im}(I - T)$ is a *closed* subspace of X.

An operator $T \in \mathcal{L}(X)$ is called *power bounded* if $\sup_{n\in\mathbb{N}} \|T^n\| < \infty$. Since $T_{[n]}(I-T) = \frac{1}{n}(T - T^{n+1})$ for $n \in \mathbb{N}$, it follows that

$$\lim_{n\to\infty} T_{[n]}x = 0, \qquad x \in \mathrm{Im}(I - T), \qquad (3)$$

whenever T is power bounded.

An operator $T \in \mathcal{L}(X)$, with X a separable Banach space, is called *hypercyclic* if there exists $x \in X$ such that the orbit $\{T^n x : n \in \mathbb{N}_0\}$ is dense in X. If, for some $z \in X$, the projective orbit $\{\lambda T^n z : \lambda \in \mathbb{C},\ n \in \mathbb{N}_0\}$ is dense in X, then T is called *supercyclic*. Clearly, hypercyclicity implies supercyclicity. The operator T is called *chaotic* if it is hypercylic and has a dense set of periodic points.

More details for mean ergodic operators can be seen in [96, 120], and for linear dynamics in [26, 106].

8 The differentiation and integration operators

8.1 Continuity of differentiation and integration operators

The differentiation operator $D(f) := f'$ is continuous on the Fréchet space $H(G)$ of all analytic functions. A detailed study of continuity of the differentiation operator $D(f) := f'$ acting in the space H_v^∞ for a radial weight function on the disc or the complex plane was conducted by Harutyunyan and Lusky in [110]. They used methods developed by Lusky in [127], which were mentioned briefly in Sect. 5. As was observed by Abanin and Tien [6], the weights were assumed implicitly to be log-convex in [110]. This is a natural assumption as was explained in Sect. 3. In this section we first state some results of Abanin and Tien [6, 8]. They used a more direct approach than Harutyunyan and Lusky.

We assume that the radial weight $v : [0, R[\to]0, \infty[$ is continuous, non-increasing on $[0, R[$ and satisfies $\lim_{r\to R} r^n v(r) = 0$ for each $n \in \mathbb{N}$.

Every increasing log-convex weight on $(0, R)$ has a right derivative everywhere on its domain of definition. Accordingly, we state the results for differentiable weights v.

Theorem 26 (Abanin and Tien [6, 8]) *Let v be a radial, log-convex weight on $\mathbb{C}$.*

(a) *The following conditions are equivalent.*

  (i) *The operator $D : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ is continuous.*
  (ii) $-\log v(r) = O(r)$ *as* $r \to \infty$.
  (iii) $\limsup_{r\to\infty} v(r)(1/v)'(r) < \infty$.

(b) *The following conditions are equivalent.*

  (i) *The operator $D : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ is compact.*
  (ii) $-\log v(r) = o(r)$ *as* $r \to \infty$.
  (iii) $\lim_{r\to\infty} v(r)(1/v)'(r) = 0$.

The differentiation operator D is continuous and surjective on $H_v^\infty(\mathbb{C})$ for $v(r) = e^{-\alpha r}, \alpha > 0$, it is continuous but not surjective for $v(r) = \exp(-(\log r)^2)$ and it is not continuous for $v(r) = \exp(-e^r)$. See also [110, Theorems 4.1 and 4.2].

Theorem 27 [6, 8, 110] *The following conditions are equivalent for a radial, log-convex weight v on $\mathbb{D}$.*

(i) *The operator $D : H_v^\infty(\mathbb{D}) \to H_w^\infty(\mathbb{D})$, with $w(r) = (1-r)v(r)$ is continuous.*
(ii) $v(r)(1-r)^{-\alpha}$ *is increasing on* $[r_0, 1)$ *for some* $\alpha > 0$ *and some* $r_0 > 0$.
(iii) $\limsup_{r\to 1}\left(-\frac{(1-r)v'(r)}{v(r)}\right) < \infty$.
(iv) $\sup_n \frac{v(1-2^{-n})}{v(1-2^{-n+1})} < \infty$.
(v) $v(r^2) = O(v(r))$ *as* $r \to 1$.

If these equivalent conditions hold, then the continuous operator $D : H_v^\infty(\mathbb{D}) \to H_w^\infty(\mathbb{D})$ is not compact.

Theorem 27 can be considered as an extension of a classical result of Hardy and Littlewood [97, Theorem 5.5]: $f \in H(\mathbb{D})$ satisfies $\sup_{z\in\mathbb{D}}(1-|z|)^\gamma|f(z)| < \infty$ if and only if $\sup_{z\in\mathbb{D}}(1-|z|)^{\gamma+1}|f'(z)| < \infty$.

The *integration operator* $Jf(z) = \int_0^z f(\zeta)d\zeta$, $z \in G$, is also well-defined and continuous on the Fréchet space $H(G)$ for $G = \mathbb{D}$ and $G = \mathbb{C}$. The continuity of J on spaces of type H_v^∞ was also investigated in [6, 110]. We recall some results.

Theorem 28 *Let v be a radial, log-convex weight on $\mathbb{C}$.*

(a) *The following conditions are equivalent.*

  (i) *The operator $J : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ is continuous.*
  (ii) $\limsup_{r\to\infty} v(r)\int_0^r \frac{1}{v(t)}dt < \infty$.
  (iii) $\liminf_{r\to\infty} v(r)(1/v)'(r) > 0$.

(b) *The following conditions are equivalent.*

  (i) *The operator $J : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ is compact.*
  (ii) $\lim_{r\to\infty} v(r)\int_0^r \frac{1}{v(t)}dt = 0$.
  (iii) $\lim_{r\to\infty} v(r)(1/v)'(r) = \infty$.

As a consequence, the integration operator J is continuous on $H_v^\infty(\mathbb{C})$ for $v(r) = e^{-\alpha r}, \alpha > 0$.

Corollary 29 *Let v be a radial, log-convex weight on $\mathbb{C}$. The following conditions are equivalent.*

(i) *The operator $D : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ is continuous and surjective.*
(ii) $0 < \liminf_{r\to\infty} v(r)(1/v)'(r) \le \limsup_{r\to\infty} v(r)(1/v)'(r) < \infty$.
(iii) *There are $A, C \ge 1$ such that, for all $0 \le r < \infty$,*

$$\frac{1}{A}e^{-Cr} \le v(r) \le Ae^{-r/C}.$$

Theorem 30 *Let v be a radial, log-convex weight on $\mathbb{D}$ and $w(r) := v(r)/(1-r)$, $0 \le r < 1$. The following conditions are equivalent.*

(i) *The integration operator $J : H_w^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ is continuous.*

(ii) $v(r)(1-r)^\alpha$ *is increasing on* $[r_0, 1)$ *for some* $\alpha > 0$ *and some* $r_0 > 0$.
(iii) $\liminf_{r\to 1}\left(-\frac{(1-r)v'(r)}{v(r)}\right) > 0$.
(iv) *There is* $k \in \mathbb{N}$ *such that* $\limsup_{n\to\infty} \frac{v(1-2^{-n-k})}{v(1-2^{-n})} < 1$.
(v) *There is* $\gamma > 1$ *such that* $\limsup_{r\to 1} \frac{v(r^\gamma)}{v(r)} < 1$.

Corollary 31 *Let* v *be a radial, log-convex weight on* $\mathbb{C}$ *and* $w(r) := v(r)/(1-r)$, $0 \le r < 1$. *The differentiation operator* $D : H_v^\infty(\mathbb{D}) \to H_w^\infty(\mathbb{D})$ *is continuous and surjective if and only if*

$$0 < \liminf_{r\to 1}\left(-\frac{(1-r)v'(r)}{v(r)}\right) \le \limsup_{r\to 1}\left(-\frac{(1-r)v'(r)}{v(r)}\right) < \infty.$$

Remark 32 The integration and differentiation operators are continuous on H_v^∞ if and only if they are continuous on H_v^0. See e.g. [30, Lemma 2.1].

Examples are given in [6, 8] to show that the requirement that the weight is log-convex in the results in this section is really essential. Related results about continuity of differentiation and integration type operators in generalized Fock spaces can be seen in [132, 134].

8.2 Spectrum, mean ergodicity and linear dynamics

We survey now some results about the behaviour of the differentiation and the integration operators when they act on some weighted Banach spaces of entire functions of type $H_v^\infty(\mathbb{C})$ and $H_v^0(\mathbb{C})$. These results are taken from [30, 44]. The spectrum of the differentiation operator on weighted Banach spaces of entire functions had been studied by Atzmon and Brive [24].

In this subsection, when $v(r) = e^{-\alpha r}$ ($\alpha > 0$) we write $H_\alpha^\infty(\mathbb{C})$ and $H_\alpha^0(\mathbb{C})$ for the weighted Banach spaces and denote their norm by $\|\cdot\|_\alpha$. The operators D and J are bounded on these spaces. See Sect. 8.1. In what follows, if $T \in \mathcal{L}(H_v^\infty(\mathbb{C})))$ we write $\|T\|_v$ instead of $\|T\|_{L(H_v^\infty(\mathbb{C}))} = \|T\|_{L(H_v^0(\mathbb{C}))}$ and $\sigma_v(T)$ for the spectrum. The notation $\|T\|_\alpha, \sigma_\alpha(T)$ refers to the case $v(r) = e^{-\alpha r}$.

Theorem 33 *Assume that the differentiation operator* $D : H_v^0(\mathbb{C}) \to H_v^0(\mathbb{C})$ *is continuous. The following conditions are equivalent:*

(1) *D has a dense set of periodic points.*
(2) *D has a periodic point different from* 0.
(3) $\lim_{r\to\infty} v(r)e^r = 0$.

Theorem 34 *Assume that the differentiation operator* $D : H_v^0(\mathbb{C}) \to H_v^0(\mathbb{C})$ *is continuous. The following conditions are equivalent:*

(1) *D is hypercyclic on* $H_v^0(\mathbb{C})$.
(2) $\liminf_{n\to\infty} \frac{\|z^n\|_v}{n!} = 0$.

Proposition 35 *The operator J is never hypercyclic on* $H_v^0(\mathbb{C})$ *and it has no periodic points different from* 0 *in* $H_v^\infty(\mathbb{C})$.

Corollary 36 *Let v be a weight such that the differentiation operator* $D : H_v^0(\mathbb{C}) \to H_v^0(\mathbb{C})$ *is continuous. Then*

(1) *If there is* $A > 0$ *such that* $1/v(r) \le Ar^{-1/2}e^r$, $r > 0$, *then D is not hypercyclic on* $H_v^0(\mathbb{C})$. *In particular D is not hypercyclic for* $v(r) = \exp(-\log^2 r)$ *or for* $v(r) = \exp(-\alpha r)$, $0 < \alpha < 1$.

(2) *If there are* $B > 0, \alpha \geq 1, r_0 > 0$ *such that* $v(r) \leq B \exp(-\alpha r)$ *for each* $r \geq r_0$*, then* D *is hypercyclic on* $H_v^0(\mathbb{C})$*. In particular,* D *is hypercyclic for* $v(r) = \exp(-\alpha r), \alpha \geq 1$.
(3) *The differentiation operator on* $H_\alpha^0(\mathbb{C})$ *is not hypercyclic and has no periodic point different from* 0 *if* $\alpha < 1$*, it is hypercyclic and has a dense set of periodic points if* $\alpha > 1$ *and it is hypercyclic but has no periodic point different from* 0 *if* $\alpha = 1$.

Theorem 37 (1) *The differentiation operator* D *satisfies*

(i)

$$\|D^n\|_\alpha = n! \left(\frac{e\alpha}{n}\right)^n, \quad n \in \mathbb{N}.$$

(ii) *It is power bounded if and only if* $\alpha < 1$.
(iii) *The spectrum of* D *is the closed disc of radius* α.
(iv) *It is uniformly mean ergodic on* $H_\alpha^\infty(\mathbb{C})$ *and* $H_\alpha^0(\mathbb{C})$ *if* $\alpha < 1$*, not mean ergodic if* $\alpha > 1$*, and it is not mean ergodic on* $H_1^\infty(\mathbb{C})$ *and not uniformly mean ergodic on* $H_1^0(\mathbb{C})$.

(2) *The integration operator* J *satisfies*

(i) $\|J^n\|_\alpha = 1/\alpha^n, \quad n \in \mathbb{N}$.
(ii) *It is power bounded if and only if* $\alpha \geq 1$.
(iii) *It is never hypercyclic on* $H_\alpha^0(\mathbb{C})$.
(iv) *The spectrum of* J *is the closed disc of radius* $1/\alpha$.
(v) *If* $\alpha > 1$*, then* J *is uniformly mean ergodic on* $H_\alpha^\infty(\mathbb{C})$ *and* $H_\alpha^0(\mathbb{C})$*, and it is not mean ergodic on these spaces if* $\alpha < 1$.
(vi) *If* $\alpha = 1$*, then* J *is not mean ergodic on* $H_1^\infty(\mathbb{C})$*, and mean ergodic but not uniformly mean ergodic on* $H_1^0(\mathbb{C})$.

Extensions of these results for more general weighted Banach spaces of entire functions were obtained by Beltrán [27], Bonilla and the author [46] and Mengestie, Worku and the author in [64]. The case of differentiation and integration operators acting on Hörmander algebras was studied in [31]. Tien investigated in [157] the dynamical properties of translation operators on weighted Hilbert and Banach spaces of entire functions.

8.3 Invariant subspaces of the integration operator

Abanin and Tien in [7] described the proper closed invariant subspaces of the integration operator on various scales of weighted Banach spaces of analytic functions on the unit disc and the complex plane. We include some of their results for H_v^0.

Let E be a Banach space of analytic functions on the open unit disc $\mathbb{D}$ or the complex plane $\mathbb{C}$ which contains the polynomials and such that the inclusion map $E \subset H(G)$ is continuous. For each $N \in \mathbb{N}$, we set

$$A_N(E) := \{f \in E \,;\; f^{(j)}(0) = 0, 0 \leq j < N\}.$$

If the integration operator $J : E \to E$, $Jf(z) := \int_0^z f(\zeta)d\zeta,\ f \in E$, is continuous, then each $A_N(E)$ is a proper closed subspace of E which is invariant for J, that is, $J(A_N(E)) \subset A_N(E)$. The question is whether there are other closed proper invariant subspaces for J on E.

Lemma 38 *Let E be a Banach space of analytic functions on the open domain $G = \mathbb{D}$ or $G = \mathbb{C}$, such that the inclusion map $E \subset H(G)$ is continuous and the polynomials are contained and dense in E. For each $N \in \mathbb{N}$ we have*

$$A_N(E) = \overline{\text{span}(\{z^j \ ; \ j \geq N\})}.$$

Theorem 39 (Abanin, Tien [7]) *Let v be a radial, continuous, non-increasing, log-convex weight $v : [0, R[\to (0, \infty[$, which satisfies $\lim_{r \to R} r^n v(r) = 0$ for each $n \in \mathbb{N}$, with $R = 1$ for the disc and $R = +\infty$ for the complex plane. Assume that the integration operator $J : H_v^0 \to H_v^0$ is continuous. Moreover, in the case of the complex plane assume that*

$$\liminf_{r \to \infty} \frac{r v(r)(1/v)'(r)}{-\log v(r)} > 1.$$

Then every proper closed invariant subspace for J on H_v^0 is of the form

$$A_K(H_v^0) = \{f \in H_v^0 \ ; \ f^{(j)}(0) = 0, 0 \leq j < K\}$$

for some $K \in \mathbb{N}$.

The weight $v(z) := \exp(-|z|^\alpha)$, $\alpha > 1$, on the complex plane satisfies the assumption of Theorem 39. Galbis and the author utilized the results of Abanin and Tien to describe in [53] the proper closed invariant subspaces of the integration operator when it acts continuously on countable intersections and countable unions of weighted Banach spaces of analytic functions on the unit disc or the complex plane, in particular for Korenblum type spaces and for Hörmander algebras of entire functions.

9 The Cesàro operator of growth Banach spaces for standard weights

The classical Cesàro operator C is given by

$$f \mapsto \mathsf{C}(f)\colon z \mapsto \frac{1}{z}\int_0^z \frac{f(\zeta)}{1-\zeta} d\zeta, \quad z \in \mathbb{D}\backslash\{0\}, \quad \text{and} \quad \mathsf{C}(f)(0) = f(0), \tag{4}$$

for $f \in H(\mathbb{D})$. It is a Fréchet space isomorphism of $H(\mathbb{D})$ onto itself. In terms of the Taylor coefficients $\widehat{f}(n) := \frac{f^{(n)}(0)}{n!}$, for $n \in \mathbb{N}_0$, of functions $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n \in H(\mathbb{D})$ one has the description

$$\mathsf{C}(f)(z) = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} \widehat{f}(k) \right) z^n, \quad z \in \mathbb{D}.$$

It is known that there are many classical Banach spaces X of analytic functions on $\mathbb{D}$ such that the Cesàro operator C acts continuously from X into itself; for instance, the Hardy spaces $H^p(\mathbb{D})$, $1 \leq p < \infty$, the Bergman and the Dirichlet spaces, etc. See for example [14, 143] and the references therein. On the other hand, C fails to act in $H^\infty(\mathbb{D})$ since $\mathsf{C}(\mathbf{1})(z) = (1/z)\log(1/(1-z))$, for $z \in \mathbb{D}$.

We collect in this section several results about the behaviour of the Cesàro operator C when it acts on growth Banach spaces $A^{-\gamma}$ and $A_0^{-\gamma}$, for $\gamma > 0$. We denote the norm in these spaces by

$$\|f\|_{-\gamma} := \sup_{z \in \mathbb{D}} (1-|z|)^\gamma |f(z)|, \quad f \in A^{-\gamma}.$$

The spectrum of C on $A^{-\gamma}$ and $A_0^{-\gamma}$ was obtained by Aleman and Persson [14, 143]. We keep the notations $\mathsf{C}_{\gamma,0}\colon A_0^{-\gamma} \to A_0^{-\gamma}$ and $\mathsf{C}_\gamma : A^{-\gamma} \to A^{-\gamma}$ for the Cesàro operator when it acts in the corresponding spaces.

Theorem 40 *Let $\gamma > 0$. The Cesàro operator $\mathsf{C}_{\gamma,0}\colon A_0^{-\gamma} \to A_0^{-\gamma}$ is continuous and it has the following properties.*

(i) $\sigma_{pt}(\mathsf{C}_{\gamma,0}) = \{\frac{1}{m} : m \in \mathbb{N},\ m < \gamma\}$.
(ii) $\sigma(\mathsf{C}_{\gamma,0}) = \sigma_{pt}(\mathsf{C}_{\gamma,0}) \cup \{\lambda \in \mathbb{C}\colon |\lambda - \frac{1}{2\gamma}| \le \frac{1}{2\gamma}\}$.
(iii) *If* $|\lambda - \frac{1}{2\gamma}| < \frac{1}{2\gamma}$ *(equivalently* $\mathrm{Re}\left(\frac{1}{\lambda}\right) > \gamma$*), then* $\mathrm{Im}(\lambda I - \mathsf{C}_{\gamma,0})$ *is a closed subspace of* $A_0^{-\gamma}$ *and has codimension* 1.

Moreover, the Cesàro operator $\mathsf{C}_\gamma : A^{-\gamma} \to A^{-\gamma}$ is continuous and it satisfies

(iv) $\sigma_{pt}(\mathsf{C}_\gamma) = \{\frac{1}{m} : m \in \mathbb{N},\ m \le \gamma\}$*, and*
(v) $\sigma(\mathsf{C}_\gamma) = \sigma(\mathsf{C}_{\gamma,0})$.

Now we present several results obtained in collaboration with Albanese and Ricker [9].

Theorem 41 (i) *Let* $\gamma \ge 1$. *Then* $\|\mathsf{C}_\gamma^n\| = \|\mathsf{C}_{\gamma,0}^n\| = 1$ *for all* $n \in \mathbb{N}$.
(ii) *Let* $0 < \gamma < 1$. *Then* $\|\mathsf{C}_\gamma^n\| = \|\mathsf{C}_{\gamma,0}^n\| = 1/\gamma^n$ *for all* $n \in \mathbb{N}$.

Theorem 42 (i) *Let* $0 < \gamma < 1$. *Both of the operators* C_γ *and* $\mathsf{C}_{\gamma,0}$ *fail to be power bounded and are not mean ergodic. Moreover,*

$$\mathrm{Ker}(I - \mathsf{C}_\gamma) = \mathrm{Ker}(I - \mathsf{C}_{\gamma,0}) = \{0\},$$

and $\mathrm{Im}(I - \mathsf{C}_\gamma)$ *(resp.* $\mathrm{Im}(I - \mathsf{C}_{\gamma,0})$*) is a proper closed subspace of* $A^{-\gamma}$ *(resp. of* $A_0^{-\gamma}$*).*

(ii) *Both of the operators* C_1 *and* $\mathsf{C}_{1,0}$ *are power bounded but not mean ergodic. Moreover,* $\mathrm{Im}(I - \mathsf{C}_1)$ *(resp.* $\mathrm{Im}(I - \mathsf{C}_{1,0})$*) is not a closed subspace of* $A^{-\gamma}$ *(resp. of* $A_0^{-\gamma}$*).*

(iii) *Let* $\gamma > 1$. *Both of the operators* C_γ *and* $\mathsf{C}_{\gamma,0}$ *are power bounded and uniformly mean ergodic. Moreover,* $\mathrm{Im}(I - \mathsf{C}_\gamma)$ *(resp.* $\mathrm{Im}(I - \mathsf{C}_{\gamma,0})$*) is a proper closed subspace of* $A^{-\gamma}$ *(resp. of* $A_0^{-\gamma}$*). In addition,*

$$\mathrm{Im}(I - \mathsf{C}_\gamma) = \{h \in A^{-\gamma} : h(0) = 0\}. \tag{5}$$

Moreover, with $\varphi(z) := 1/(1-z)$*, for* $z \in \mathbb{D}$*, the linear projection operator* $P_\gamma : A^{-\gamma} \to A^{-\gamma}$ *given by*

$$P_\gamma(f) := f(0)\varphi, \qquad f \in A^{-\gamma},$$

is continuous and satisfies $\lim_{n\to\infty}(\mathsf{C}_\gamma)_{[n]} = P_\gamma$ *in the operator norm.*

Proposition 43 *The Cesàro operator* $\mathsf{C}_{\gamma,0}$ *is not supercyclic and hence, also not hypercyclic, in each space* $A_0^{-\gamma}$*, for* $\gamma > 0$.

This follows from the fact that C is not supercyclic on $H(\mathbb{D})$, since $A_0^{-\gamma}$ is dense in $H(\mathbb{D})$.

A vector space $X \subseteq H(\mathbb{D})$ is called a *Banach space of analytic functions on* $\mathbb{D}$ if it contains the polynomials and it is a Banach space relative to a norm for which the natural inclusion of X into $H(\mathbb{D})$ is continuous. Since evaluation at points of $\mathbb{D}$ are continuous linear functionals on $H(\mathbb{D})$, this is equivalent to each evaluation functional $\delta_z : f \mapsto f(z)$ at a point $z \in \mathbb{D}$ being an element of the dual Banach space X' of X.

The *optimal domain* of the Cesàro operator C when it acts on a Banach space of analytic functions X on $\mathbb{D}$ is defined by

$$[\mathsf{C}, X] := \{f \in H(\mathbb{D}) \ : \ \mathsf{C}(f) \in X\},$$

which is a Banach space for the norm

$$\|f\|_{[\mathsf{C},X]} := \|\mathsf{C}(f)\|_X, \qquad f \in [\mathsf{C}, X]. \tag{6}$$

If C acts on X, then $X \subseteq [\mathsf{C}, X]$ and the natural inclusion map is continuous. Moreover, $[\mathsf{C}, X]$ is the *largest* of all Banach spaces of analytic functions Y on $\mathbb{D}$ that C maps continuously into X.

Theorem 44 *Let $\gamma > 0$ and $\varphi(z) := 1/(1-z)$ for $z \in \mathbb{D}$.*
The optimal domain $[\mathsf{C}, A^{-\gamma}]$ of $\mathsf{C}_\gamma : A^{-\gamma} \to A^{-\gamma}$ is isometrically isomorphic to $A^{-\gamma}$ and is given by

$$[\mathsf{C}, A^{-\gamma}] = \{f \in H(\mathbb{D}) \ : \ f\varphi \in A^{-(\gamma+1)}\}. \tag{7}$$

Moreover, the norm $\|\cdot\|_{[\mathsf{C},A^{-\gamma}]}$ is equivalent to the norm $f \to \|f\varphi\|_{-(\gamma+1)}$ and the containment $A^{-\gamma} \subseteq [\mathsf{C}, A^{-\gamma}]$ is proper.

A similar result holds for the optimal domain $[\mathsf{C}, A_0^{-\gamma}]$ of $\mathsf{C}_{\gamma,0} : A_0^{-\gamma} \to A_0^{-\gamma}$.

The behaviour of the Cesàro operator on the Korenblum space $A^{-\infty}$ and related Fréchet and (LB)-spaces of analytic functions on the unit disc was investigated by Albanese, Ricker and the author in [10]. The spectrum is completely determined and some consequences concerning mean ergodicity are deduced.

10 Volterra operators

The Volterra integral operator is defined on the space of analytic functions of $G = \mathbb{D}$ or $G = \mathbb{C}$ in the following way. Given $g \in H(G)$, we set

$$V_g(f)(z) := \int_0^z f(\zeta)g'(\zeta)d\zeta \quad (z \in \mathbb{D}).$$

The analytic function g is called the symbol of V_g. Clearly $V_g : H(G) \to H(G)$ is continuous.

The Volterra operator for holomorphic functions on the unit disc was introduced by Pommerenke [144] and he proved that V_g is bounded on the Hardy space H^2, if and only if $g \in BMOA$. Aleman and Siskakis [15] extended this result for H^p, $1 \leq p < \infty$, and they considered later in [16] the case of weighted Bergman spaces. Volterra operators on weighted Banach spaces $H_v^\infty(\mathbb{D})$ of holomorphic functions on the disc of type have been investigated in [25], thus extending results in [113, 153]. Lin [122] obtained some further results in this direction.

Contreras, Peláez, Pommerenke and Rättyä [87] studied thoroughly the boundedness, compactness and weak compactness of the operators $V_g : X \to H^\infty$ from a Banach space of analytic functions X into H^∞. They obtained general results which they applied to particular choices of X. Smith, Stoljarov and Volberg [152] presented a necessary and sufficient condition for the operator of integration to be bounded on the space of bounded analytic functions on a simply connected domain, thus solving a conjecture in [17] about the boundedness of V_g with values on H^∞ when the function g is univalent. A counterexample to the general case of the conjecture about the boundedness of V_g is also included in [152]. Motivated by the results of this article, Abakumov and Doubtsov [5] characterized the boundedness of $V_g : H_v^\infty \to H_w^\infty$ for general weights v, w and a univalent symbol g. In the article [100] the authors characterize the continuity and (weak) compactness of $V_g : H_v^\infty(\mathbb{D}) \to H^\infty$ when $v(z) = (1-|z|^2)^\alpha$ is a standard weight and g is univalent.

Aleman, Constantin, Peláez and Persson [11, 13, 14] investigated the spectra of Volterra and Cesàro operators on several spaces of holomorphic functions on the disc. The spectra of Volterra operators acting on growth spaces has been investigated by Malman [128]. Constantin started in [83] the study of the Volterra operator on spaces of entire functions. She characterized the continuity of V_g on the classical Fock spaces and investigated its spectrum. Constantin and Peláez [84] characterize the entire functions $g \in H(\mathbb{C})$ such that V_g is bounded or compact on a large class of Fock spaces induced by smooth radial weights. See also [4, 38, 39, 65]. The investigation of the spectrum of the Volterra operator for weighted spaces of entire functions was continued in [43, 84]. Volterra operators on Korenblum type Fréchet and (LB)-spaces and on Hörmander algebras are studied in [42, 43].

The first result is a consequence of more general Theorems [25, Theorems 1 and 2].

Proposition 45 *Let $g \in H(\mathbb{D})$ be an analytic function and let $\alpha > 0$ and $\beta > 0$.*

(i) *The operator $V_g : A^{-\alpha} \to A^{-\beta}$ is continuous if and only if $V_g : A_0^{-\alpha} \to A_0^{-\beta}$ is continuous and if and only if*

$$\sup_{z\in\mathbb{D}} (1-|z|)^{\beta-\alpha+1}|g'(z)| < \infty.$$

(ii) *If $\alpha < \beta$, then $V_g : A^{-\alpha} \to A^{-\beta}$ is continuous if and only if $V_g : A_0^{-\alpha} \to A_0^{-\beta}$ is continuous and if and only if $g \in A^{-(\beta-\alpha)}$.*

(iii) *The operator $V_g : A^{-\alpha} \to A^{-\beta}$ is compact if and only if it is weakly compact if and only if $V_g : A_0^{-\alpha} \to A_0^{-\beta}$ is compact and if and only if it is weakly compact. Moreover, these conditions are equivalent to*

$$\lim_{|z|\to 1} (1-|z|)^{\beta-\alpha+1}|g'(z)| = 0.$$

Theorem 46 (Abakumov and Doubtsov [5]) *Let $g \in H(\mathbb{D})$ be a univalent function and let v and w be weight functions.*

(i) *The operator $V_g : H_w^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ is bounded if and only if*

$$\sup_{0\le\theta\le 2\pi} \sup_{0\le r<1} v(r) \int_0^r \frac{|g'(te^{i\theta}|}{\tilde{w}(t)} dt < \infty.$$

(i) *The operator $V_g : H_w^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ is compact*

$$\lim_{\rho\to 1} \sup_{r\ge\rho} \sup_{0\le\theta\le 2\pi} v(r) \int_0^r \frac{|g'(te^{i\theta}|}{\tilde{w}(t)} dt = 0.$$

The proof of Theorem 46 uses methods of [152] and Theorem 2.

Lemma 47 *Let $X \subset H(\mathbb{D})$ be a Banach space that contains the constants and such that the inclusion $X \subset H(\mathbb{D})$ is continuous. Assume that $V_g : X \to X$ is continuous for some non-constant entire function g such that $g(0) = 0$. Then*

$$\{0\} \cup \overline{\{\lambda \in \mathbb{C}\setminus\{0\}\,;\; e^{\frac{g}{\lambda}} \notin X\}} \subset \sigma(V_g, X).$$

The corresponding result is valid for Banach spaces $X \subset H(\mathbb{C})$ containing the constants and such that the inclusion $X \subset H(\mathbb{C})$ is continuous.

Theorem 48 (Malman [128]) *Let $g \in H(\mathbb{D})$, $g(0) = 0$, and let $\alpha > 0$.*

(i) *If $g \in H^\infty$ or $g \in \mathcal{B}_0$, then $\sigma(V_g, A^{-\alpha}) = \sigma(V_g, A_0^{-\alpha}) = \{0\}$.*
(ii) *If $g(z) = c\log(1/(1-\overline{w}z))$, $z \in \mathbb{D}$ with $c, w \in \mathbb{C}$, $c \neq 0$, $|w| = 1$, then $V_g : A^{-\alpha} \to A^{-\alpha}$, $\alpha > 0$, is continuous and $\sigma(V_g, A^{-\alpha}) = \{\lambda \in \mathbb{C}\,;\ \mathrm{Re}(\frac{c}{\lambda}) \geq \alpha\}$.*

In the notation of Theorem 48, we understand $0 \in \{\lambda \in \mathbb{C}\,;\ \mathrm{Re}(\frac{c}{\lambda}) \geq \alpha\}$. With this in mind, this set coincides with the disc $\{\lambda \in \mathbb{C}\,;\ |\lambda - \frac{c}{2\alpha}| \leq \frac{|c|}{2\alpha}\}$.

Proposition 49 [65] *Assume that $v(r) = \exp(-\alpha r^p)$, $\alpha > 0$, $p > 0$.*

(i) *$V_g : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ is continuous if and only if g is a polynomial of degree less than or equal to the integer part of p.*
(ii) *$V_g : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ is compact if and only if g is a polynomial of degree strictly less than p.*

Theorem 50 [43] *Assume that $v(r) = \exp(-\alpha r^p)$, $\alpha > 0$, $p > 0$. Let g be a polynomial of degree $n \in \mathbb{N}$ less than or equal to the integer part of p with $g(0) = 0$.*

(i) *If the degree n of g satisfies $n < p$, then $\sigma(V_g, H_v^\infty(\mathbb{C})) = \{0\}$.*
(ii) *If $p = n$ and $g(z) = \beta z^n + k(z)$, k a polynomial of degree strictly less than n, then $\sigma(V_g, H_v^\infty(\mathbb{C})) = \{\lambda \in \mathbb{C}\,;\ |\lambda| \leq |\beta|/\alpha\}$.*

Moreover, we have $\sigma(V_g, H_v^\infty(\mathbb{C})) = \{0\} \cup \overline{\{\lambda \in \mathbb{C}\backslash\{0\}\,;\ e^{\frac{g}{\lambda}} \notin H_v^\infty(\mathbb{C})\}}$.

11 Weighted composition operators

Weighted composition operators on various spaces of analytic functions on the unit disc or the complex plane have been studied very thoroughly by a number of authors. For the unit disc, the books of Cowen, MacCluer [89] and Shapiro [150] are standard references. In this section we mainly concentrate on composition operators on the spaces $H_v^\infty(\mathbb{D})$ and $H_v^0(\mathbb{D})$ and we mention a few results about spaces of entire functions.

11.1 Continuity, (weak) compactness, isometries

We consider a non-constant self map $\varphi \in H(\mathbb{D})$ satisfying $\varphi(\mathbb{D}) \subset \mathbb{D}$ and a function $\psi \in H(\mathbb{D})$ which is not identically equal zero. They induce the *weighted composition operator*

$$W_{\varphi,\psi} f := \psi(f \circ \varphi).$$

This operator is continuous on $H(\mathbb{D})$ for the topology of uniform convergence on the compact subsets of $\mathbb{D}$.

If $\psi = 1$, then as usual we denote the *composition operator* $W_{\varphi,1}$ by C_φ

$$C_\varphi f := f \circ \varphi.$$

And if $\varphi(z) = z$, $z \in \mathbb{D}$, then $W_{\varphi,\psi}$ is the multiplication operator

$$M_\psi f := \psi f.$$

We first mention some results about continuity, compactness and the essential norm from [49, 52, 88, 116, 137]. All the weights we consider are radial, non-increasing and tending to zero at the boundary.

Theorem 51 *Let v and w be weights on $\mathbb{D}$.*

(a) *The following conditions are equivalent.*

  (1) $W_{\varphi,\psi}(H_v^\infty(\mathbb{D})) \subset H_w^\infty(\mathbb{D})$.
  (2) $W_{\varphi,\psi} : H_v^\infty(\mathbb{D}) \to H_w^\infty(\mathbb{D})$ *is continuous.*
  (3) $\sup_{z\in\mathbb{D}} \frac{|\psi(z)|w(z)}{\tilde{v}(\varphi(z))} < \infty$.
  (4) $\sup_{n\in\mathbb{N}_0} \frac{\|\psi\varphi^n\|_v}{\|z^n\|_v} < \infty$.

  In this case, $\psi \in H_w^\infty$ and $\|W_{\varphi,\psi}\| = \sup_{z\in\mathbb{D}} \frac{|\psi(z)|w(z)}{\tilde{v}(\varphi(z))}$. Moreover, this norm is comparable with the supremum which appears in (4).

(b) *The following conditions are equivalent.*

  (1) $W_{\varphi,\psi}(H_v^0(\mathbb{D})) \subset H_w^0(\mathbb{D})$.
  (2) $W_{\varphi,\psi} : H_v^0(\mathbb{D}) \to H_w^0(\mathbb{D})$ *is continuous.*
  (3) $\psi \in H_w^0$ *and* $\sup_{z\in\mathbb{D}} \frac{|\psi(z)|w(z)}{\tilde{v}(\varphi(z))} < \infty$.

  In this case, $\|W_{\varphi,\psi}\| = \sup_{z\in\mathbb{D}} \frac{|\psi(z)|w(z)}{\tilde{v}(\varphi(z))}$.

Theorem 52 *Let v and w be weights on $\mathbb{D}$. If the operator $W_{\varphi,\psi} : H_v^\infty(\mathbb{D}) \to H_w^\infty(\mathbb{D})$ is continuous, then its essential norm satisfies*

$$\|W_{\varphi,\psi}\|_e = \lim_{s\to 1} \sup_{|\varphi(z)|>s} \frac{|\psi(z)|w(z)}{\tilde{v}(\varphi(z))} = \limsup_{n\to\infty} \frac{\|\psi\varphi^n\|_v}{\|z^n\|_v}.$$

In particular, $W_{\varphi,\psi}$ is compact if and only if

$$\lim_{s\to 1} \sup_{|\varphi(z)|>s} \frac{|\psi(z)|w(z)}{\tilde{v}(\varphi(z))} = \lim_{n\to\infty} \frac{\|\psi\varphi^n\|_v}{\|z^n\|_v} = 0.$$

Theorem 53 *Let v and w be weights on $\mathbb{D}$. If the operator $W_{\varphi,\psi} : H_v^0 \to H_w^0$ is continuous, then its essential norm satisfies*

$$\|W_{\varphi,\psi}\|_e = \limsup_{|z|\to 1} \frac{|\psi(z)|w(z)}{\tilde{v}(\varphi(z))}.$$

In particular, $W_{\varphi,\psi}$ is compact if and only if

$$\limsup_{|z|\to 1} \frac{|\psi(z)|w(z)}{\tilde{v}(\varphi(z))} = 0.$$

Theorem 54 *Let v and w be weights on $\mathbb{D}$.*

(1) *Assume that the operator $W_{\varphi,\psi} : H_v^\infty(\mathbb{D}) \to H_w^\infty(\mathbb{D})$ is continuous. Then $W_{\varphi,\psi}$ is either compact or an isomorphism on a subspace isomorphic to ℓ_∞.*
(2) *Assume that the operator $W_{\varphi,\psi} : H_v^0(\mathbb{D}) \to H_w^0(\mathbb{D})$ is continuous. Then $W_{\varphi,\psi}$ is either compact or an isomorphism on a subspace isomorphic to c_0.*
  In particular, $W_{\varphi,\psi}$ is compact if and only if it is weakly compact in both cases.

Proposition 55 *Let φ be given. The following holds:*

(1) $C_\varphi : H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ *is continuous for all radial non-increasing weights* v *if and only if there is* $0 < s < 1$ *such that* $|\varphi(z)| \le |z|$ *for all* $|z| \ge s$.
(2) $C_\varphi : H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ *is compact for all radial non-increasing weights* v *if and only if* $\varphi(\mathbb{D}) \subset s\mathbb{D}$ *for some* $0 < s < 1$.

Proposition 56 *A radial non-increasing weight* v *satisfies that the operator* $C_\varphi : H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ *is continuous for every* φ *if and only if the weight* $\tilde{v}$ *satisfies the condition*

$$(L1) \quad \sup_n \frac{\tilde{v}(1-2^{-n})}{\tilde{v}(1-2^{-n-1})} < \infty.$$

If $\varphi(z) = (z+1)/2,\ z \in \mathbb{D}$, and $v(z) = \exp(-1/(1-|z|)),\ z \in \mathbb{D}$, the composition operator C_φ is not continuous on $H_v^\infty(\mathbb{D})$.

We refer to Lusky [125, 127] for the relevance of condition $(L1)$ in connection with the isomorphic classification of spaces H_v^∞. Some conditions of various types that are equivalent to $(L1)$ were stated in [6, Lemma 2.6]. Interesting extensions of Proposition 56 were obtained by Bourdon [72].

The characterization of nuclear weighted composition operators $W_{\varphi,\psi} : H_v^0(\mathbb{D}) \to H_w^0(\mathbb{D})$ under some conditions on the weights has been obtained in [55]. This result was inspired and motivated by the characterization of nuclear weighted composition operators on Bloch spaces due to Fares and Lefèvre [101].

Fredholm weighted composition operators on H_v^∞ and H_v^0 were investigated in [102, 105, 115].

Theorem 57 *Let* $W_{\varphi,\psi} : H_v^0(\mathbb{D}) \to H_w^0(\mathbb{D})$ *be continuous.*

(a) *The following conditions are equivalent.*
  (i) $W_{\varphi,\psi} : H_v^\infty(\mathbb{D}) \to H_w^\infty(\mathbb{D})$ *is Fredholm.*
  (ii) $W_{\varphi,\psi} : H_v^0(\mathbb{D}) \to H_w^0(\mathbb{D})$ *is Fredholm.*
  (iii) $\psi \in H^\infty$, *there is* $\varepsilon > 0$ *such that* $|\psi(z)| > \varepsilon$ *for all* $|z| \ge 1-\varepsilon$, *and* φ *is an automorphism.*

(b) *The following conditions are equivalent.*
  (i) $W_{\varphi,\psi} : H_v^\infty(\mathbb{D}) \to H_w^\infty(\mathbb{D})$ *is invertible.*
  (ii) $W_{\varphi,\psi} : H_v^0(\mathbb{D}) \to H_w^0(\mathbb{D})$ *is invertible.*
  (iii) ψ *and* $1/\psi$ *belong to* H^∞, *and* φ *is an automorphism.*

As a consequence a continuous composition operator $C_\varphi : H_v^0 \to H_w^0$ is Fredholm if and only if it is invertible and if and only if φ is an automorphism. These conditions are also equivalent to $C_\varphi : H_v^\infty(\mathbb{D}) \to H_w^\infty(\mathbb{D})$ being invertible or Fredholm. A general result about invertible weighted composition operators is due to Bourdon [73]. Extensions of these results have been obtained recently by Mas and Vukotić [130].

Martín and Vukotić [129] analyzed when composition operators on the Bloch space are (not necessarily surjective) isometries, and they show that every thin Blaschke product induces an isometric composition operator on the Bloch space. Motivated by these results, together with Lindström and Wolf, we characterize in [56] isometric weighted composition operators on $H_v^\infty(\mathbb{D})$. As a consequence a composition operator C_φ of $H_{v_p}^\infty(\mathbb{D})$ for a standard weight $v_p(z) = (1-|z|)^p$, $p > 0$, is an isometry if and only if φ is a rotation. Boyd and Rueda [78] present a detailed study of the question of under which conditions a given isometry between weighted spaces of holomorphic functions is surjective.

Boyd and Rueda have obtained interesting related results on isometries on weighted Banach spaces $H_v^\infty(U)$ of holomorphic functions defined on an open subset U of $\mathbb{C}^n$. We refer the reader to their papers [74, 75] and the references therein. They explain how the isometries of a weighted space of holomorphic functions are determined by a subgroup of the automorphisms of a subset of the domain, called the v-boundary of U. The relation between this group and the weight is investigated for bounded and unbounded domains. Examples are presented.

11.2 The spectrum

The following result was obtained in [164]. It extends theorems due to Aron, Lindström [23, 54].

Theorem 58 *Suppose φ is not an automorphism and has fixed point $a \in \mathbb{D}$. Then*

$$\sigma(W_{\psi,\varphi}, H_v^\infty(\mathbb{D})) = \{\lambda \in \mathbb{C} : |\lambda| \leq r_{e,H_v^\infty}(W_{\psi,\varphi})\} \cup \{\psi(a)\varphi'(a)^n\}_{n=0}^\infty.$$

A similar result holds for $H_v^0(\mathbb{D})$.

Further extensions and related results can be seen in [98, 99, 103, 104].

Let φ be an analytic self map on $\mathbb{D}$ which is not an automorphism and has a (necessarily unique) fixed point $a \in \mathbb{D}$. By Koenigs' Theorem [150, Chapter 6], if $\varphi'(a) = 0$, the equation $f \circ \varphi = \lambda f$ has a non-trivial solution if and only if $\lambda = 1$ and the constant functions are the only solutions. On the other hand, if $\varphi'(a) \neq 0$, then (i) the equation $f \circ \varphi = \lambda f$ has a non-trivial solution if and only if $\lambda = \varphi'(a)^n$, $n \in \mathbb{N}_0$, (ii) there is a unique function $\sigma \in H(\mathbb{D})$, called the Koenigs' eigenfunction of φ, such that $\sigma \circ \varphi = \varphi'(w)\sigma$, $\sigma'(w) = 1$, and (iii) a function $f \in H(\mathbb{D})$ satisfies $f \circ \varphi = \varphi'(a)^n f$ for some $n \in \mathbb{N}_0$ if and only if $f = c\sigma^n$ for some $c \in \mathbb{C}$.

As a consequence of Schwarz Lemma, an analytic self map φ on $\mathbb{D}$ which is not an automorphism and has a fixed point $a \in \mathbb{D}$ satisfies $\varphi'(a) \neq 0$ if and only if $0 < |\varphi'(a)| < 1$. Moreover, in this case, the Koenigs' eigenfunction of φ can be obtained as the limit of the sequence $(\sigma_n)_n$ with $\sigma_n := \varphi_n/\varphi'(a)^n$, $n \in \mathbb{N}$, which converges to σ uniformly on the compact subsets of $\mathbb{D}$. Here $\varphi_n = \varphi \circ \cdots \circ \varphi$ is the n-fold composition of φ.

The spectrum and essential spectrum of the composition operator $C_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$ for an analytic self map φ on $\mathbb{D}$ which is not an automorphism and has a fixed point $a \in \mathbb{D}$ with $0 \leq |\varphi'(a)| < 1$ have been determined recently in [20]: $\sigma(C_\varphi, H(\mathbb{D})) = \{0\} \cup \{\varphi'(a)^n \, ; \, n \in \mathbb{N}_0\}$, and its essential spectrum reduces to $\{0\}$. The proofs are based on explicit formulas for the spectral projections associated with the point spectrum found by Koenigs. As a consequence, information on the spectrum for bounded composition operators induced by a symbol as above on Banach spaces of analytic functions continuously embedded in $H(\mathbb{D})$ is obtained. The definitions of spectrum and essential spectrum for an operator on a locally convex space coincide with those given in Sect. 7. Recall that an operator between locally convex spaces is compact if it maps a neighbourhood of the domain into a relatively compact set in the image. The case of composition operators induced by rotations was analyzed in [41].

In [54, 98] it was investigated how the essential spectral radius of C_φ on both $H_v^\infty(\mathbb{D})$ and $H_v^0(\mathbb{D})$ determines whether the Koenigs eigenfunction σ of φ belongs to $H_v^\infty(\mathbb{D})$ and H_v^0 respectively. Let φ be an analytic self map on $\mathbb{D}$ which is not an automorphism such that $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$. Bourdon [71] proved that $\sigma \in H_{v_p}^0(\mathbb{D})$, for $v_p(z) = (1-|z|)^p$, $p > 0$, if and only if $|\varphi'(0)| > r_{e,H_{v_p}^0}$. Examples given in [54] show that Bourdon's characterization does not hold for more general radial weights. On the other hand, by [98, Theorem 3.1], $\sigma \in H^\infty$ if and only if $r_{e,H^\infty}(C_\varphi) = 0$.

Proposition 59 [98] *Let φ be an analytic self map on $\mathbb{D}$ which is not an automorphism such that $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$. Let v be a radial weight on $\mathbb{D}$. Then*

(i) *The Koenigs' eigenfunction σ belongs to $H_v^\infty(\mathbb{D})$ if and only if the sequence $\sigma_n := \varphi_n/\varphi'(0)^n$, $n \in \mathbb{N}$, is bounded in $H_v^\infty(\mathbb{D})$.*
(ii) *Assume that φ is univalent. The Koenigs' eigenfunction σ belongs to $H_v^0(\mathbb{D})$ if and only if $\lim_{n\to\infty} \|\sigma_n - \sigma\|_v = 0$.*

Hyvärinen, Lindström, Nieminen and Saukko [115], using ideas of Kamowitz and Gunatillake, calculated the spectrum of the invertible weighted composition operator $W_{\varphi,\psi}$ for an automorphic symbol φ on a wide class of analytic function spaces; in particular for spaces of type $H_v^\infty(\mathbb{D})$. The analysis of the spectral behaviour depends on the type of the symbol φ, that is, if it is an elliptic, parabolic or hyperbolic automorphism.

An *automorphism* $\varphi \in Aut(\mathbb{D})$ is an injective analytic function on $\mathbb{D}$ such that $\varphi(\mathbb{D}) = \mathbb{D}$. As a consequence of Schwarz Lemma, we get $\varphi \in Aut(\mathbb{D})$ if and only if there are $a \in \mathbb{D}$ and $\lambda \in \mathbb{C}$, $|\lambda| = 1$ such that $\varphi(z) = \lambda\frac{a-z}{1-\bar{a}z}$. For $z, a \in \mathbb{D}$, $\varphi_a(z) := \frac{a-z}{1-\bar{a}z}$ is the automorphism of $\mathbb{D}$ which interchanges 0 and a. An automorphism $\varphi \in Aut(\mathbb{D})$ is called

- *elliptic* if it has one fixed point in $\mathbb{D}$ and one outside $\overline{\mathbb{D}}$, which could be in the infinity point of the Riemann sphere.
  Example: $\varphi(z) = iz$. Fixed points: 0 and ∞.
- *hyperbolic* if it has two different fixed points in the boundary $\mathbb{D}$.
  Example: $\varphi(z) = \frac{z+0,5}{1+0,5z}$. Fixed points: 1 and -1.
- *parabolic* if it has one fixed point in the boundary of $\mathbb{D}$ with multiplicity 2.
  Example: $\varphi(z) = \frac{(1+i)z-i}{iz+1-i}$. Fixed point: 1.

Theorem 60 (Theorem of Denjoy–Wolff) *If $\varphi \in H(\mathbb{D})$ is a self map on $\mathbb{D}$ with no fixed point in $\mathbb{D}$ (for example if $\varphi \in Aut(\mathbb{D})$ is parabolic or hyperbolic), then there is a unique point w in the boundary of $\mathbb{D}$, called the Denjoy–Wolff point of φ, such that $(\varphi_n)_n$ converges to w uniformly on the compact subsets of $\mathbb{D}$.*

If $\varphi \in H(\mathbb{D})$ is a self map on $\mathbb{D}$ that fixes a point $p \in \mathbb{D}$, and is not a conformal automorphism, then $(\varphi_n)_n$ converges to p uniformly on the compact subsets of $\mathbb{D}$.

An elliptic automorphism has one fixed point in $\mathbb{D}$. A hyperbolic automorphism has two fixed points in the boundary $\partial\mathbb{D}$ of $\mathbb{D}$, one is attractive and the other one repulsive, and a parabolic automorphism has one fixed point in the boundary of $\mathbb{D}$ with multiplicity 2. See more details in [79], [89] and [150]. Recall that the disc algebra $A(\mathbb{D})$ is the space of continuous functions on the closed unit disc $\overline{\mathbb{D}}$ which are analytic on $\mathbb{D}$. It is a Banach space endowed with the supremum norm on $\mathbb{D}$.

Theorem 61 [115] *Let $v_p = (1 - |z|^2)^p$, $p > 0$ and let $W_{\varphi,\psi} : H_{v_p}^\infty(\mathbb{D}) \to H_{v_p}^\infty(\mathbb{D})$ be continuous. Assume that $\psi \in A(\mathbb{D})$ is bounded away from zero on $\mathbb{D}$. Then*

(i) *If φ is a parabolic automorphism with Denjoy–Wolff point $a \in \partial\mathbb{D}$, then*

$$\sigma(W_{\varphi,\psi}, H_{v_p}^\infty(\mathbb{D})) = \{\lambda \in \mathbb{C}\ ;\ |\lambda| = |\psi(a)|\}.$$

(ii) *If φ is a hyperbolic automorphism with attractive fixed point $a \in \partial\mathbb{D}$ and repulsive fixed point $b \in \partial\mathbb{D}$, such that $|\psi(b)/\varphi'(b)^p| \le |\psi(a)/\varphi'(a)^p|$, then*

$$\sigma(W_{\varphi,\psi}, H_{v_p}^\infty(\mathbb{D})) = \left\{\lambda \in \mathbb{C}\ ;\ \frac{|\psi(b)|}{\varphi'(b)^p} \le |\lambda| \le \frac{|\psi(a)|}{\varphi'(a)^p}\right\}.$$

(iii) *If φ is an automorphism such that there is a $j \in \mathbb{N}$ such that $\varphi_j(z) = z$ for each $z \in \mathbb{D}$, then, for the smaller such $n \in \mathbb{N}$, we have*

$$\sigma(W_{\varphi,\psi}, H^\infty_{v_p}(\mathbb{D})) = \overline{\left\{\lambda \in \mathbb{C}\ ;\ \lambda^n = \prod_{m=0}^{n-1} (\psi \circ \varphi_m)(z),\ z \in \mathbb{D}\right\}}.$$

(iv) *If φ is an automorphism such that φ_n does not coincide with the identity function $id(z) = z$ for each $n \in \mathbb{N}$, and $a \in \mathbb{D}$ is the unique fixed point of φ, then $\sigma(W_{\varphi,\psi}, H^\infty_{v_p}(\mathbb{D})) = \{\lambda \in \mathbb{C}\ ;\ |\lambda| = |\psi(a)|\}$.*

Further results can be seen in [103, 104]. Many questions remain open about the spectrum of weighted composition operators.

11.3 Hypercyclicity and mean ergodicity

Miralles and Wolf [135] investigated hypercyclic continuous composition operators $C_\varphi : H^0_v(\mathbb{D}) \to H^0_v(\mathbb{D})$ for an analytic self map φ on $\mathbb{D}$ and a radial weight on $\mathbb{D}$ satisfying the usual assumptions (non-increasing approaching 0 at the boundary).

Proposition 62 *If $C_\varphi : H^0_v(\mathbb{D}) \to H^0_v(\mathbb{D})$ is a continuous hypercyclic operator, then φ has no fixed point in $\mathbb{D}$ and it is injective.*

Theorem 63 *If φ is an automorphism which fixes no point in $\mathbb{D}$ and $C_\varphi : H^0_v(\mathbb{D}) \to H^0_v(\mathbb{D})$ is continuous, then C_φ is hypercyclic.*

We refer to the book of Shapiro [150] for linear fractional transformations of the unit disc.

Theorem 64 *Let φ be a linear fractional transformation of $\mathbb{D}$ such that $C_\varphi : H^0_v(\mathbb{D}) \to H^0_v(\mathbb{D})$ is continuous. If φ is a hyperbolic non-automorphism, then C_φ is hypercyclic.*

Proposition 65 *Let $v(z) = (1 - |z|)^{1/2}$ for every $z \in \mathbb{D}$. If φ be a linear fractional transformation of $\mathbb{D}$ which is a parabolic non-automorphism, then $C_\varphi : H^0_v(\mathbb{D}) \to H^0_v(\mathbb{D})$ is not hypercyclic.*

These results were extended to the case of weighted composition operators of the form $\lambda C_\varphi,\ \lambda \in \mathbb{C}$, by Liang and Zhou [121]. Colonna and Martínez-Avendaño [82] extend some of these results and present an informative brief summary of the literature on hypercyclic composition operators on Banach spaces of analytic functions.

Power bounded and (uniformly) mean ergodic composition operators on $H(G)$, G an open connected subset of $\mathbb{C}$, were investigated by Domanski and the author in [47]. This research was continued later for operators on spaces of real analytic functions in [48]. These papers triggered quite an amount of research. The study of mean ergodic weighted composition operators on $H(G)$ was done by Beltrán, Gómez-Collado, Jordá and Jornet in [33]. They continued their work in [32] about mean ergodicity of composition operators on the Banach spaces H^∞ and the disc algebra $A(\mathbb{D})$.

A systematic study of powers of (weighted) composition operators on Banach spaces of analytic functions on the unit disc has been done by Arendt, Chalendar, Kumar and Srivastava [21, 22], Jordá and Rodríguez [118] and Tien [158]. See the references in these papers for more related work in this direction. We state some results for composition operators C_φ on $H^\infty_v(\mathbb{D})$ and $H^0_v(\mathbb{D})$.

Proposition 66 *Let v be a log-convex weight on $\mathbb{D}$ satisfying condition $(L1)$. Let φ be an elliptic automorphism with fixed point $z(0) \in \mathbb{D}$. Then*

(i) *If φ is equivalent to a rational rotation, then C_φ is uniformly mean ergodic on $H_v^\infty(\mathbb{D})$ and $H_v^0(\mathbb{D})$. Moreover, there is $k \in \mathbb{N}$ such that $((C_\varphi)_{[n]})_n$ converges to $(1/k)(C_\varphi + \cdots + (C_\varphi)^k)$.*
(ii) *If φ is equivalent to a irrational rotation, then C_φ is not uniformly mean ergodic on $H_v^\infty(\mathbb{D})$ and $H_v^0(\mathbb{D})$.*
(iii) *If φ is equivalent to a irrational rotation, then C_φ is mean ergodic on $H_v^0(\mathbb{D})$, and $((C_\varphi)_{[n]} f)$ converges to $C_{z(0)} f = f(z(0))$ for each $f \in H_v^0(\mathbb{D})$.*

Theorem 67 *Let v be a log-convex weight on $\mathbb{D}$ satisfying condition $(L1)$. Let φ be a self map on $\mathbb{D}$ with Denjoy–Wolff point 0. Then the sequence $(C_{\varphi_n} f)_n$ converges to $f(0)$ in $H_v^0(\mathbb{D})$ for each $f \in H_v^0(\mathbb{D})$.*

Tien proves in [158, Theorem 4.8 (b)] that, under some mild assumption, the sequence $(C_{\varphi_n})_n$ converges to the operator $C_0(f) := f(0)$ in $\mathcal{L}(H_v^\infty(\mathbb{D}))$. Moreover, he gives an example of a weight v and self map φ with Denjoy–Wolff point 0 such that C_φ is power bounded, $(C_{\varphi_n})_n$ does not converge even in the weak operator topology on $H_v^\infty(\mathbb{D})$, but it converges in the strong operator topology on $H_v^0(\mathbb{D})$. This operator C_φ is mean ergodic and not uniformly mean ergodic on $H_v^0(\mathbb{D})$, and not mean ergodic on $H_v^\infty(\mathbb{D})$.

Theorem 68 *Let φ be a self map on $\mathbb{D}$ with Denjoy-Wolf point $z(0)$ in the boundary of the unit disc. Let v be a log-convex weight on $\mathbb{D}$ satisfying condition $(L1)$ such that $-\log(1-r) = O(1/v(r))$ as $r \to 1-$. Then C_φ is not power bounded and not mean ergodic on $H_v^\infty(\mathbb{D})$.*

11.4 Weighted composition operators on spaces of entire functions

Many properties of composition operators on spaces of entire functions have also been investigated. For instance, in the frame of Fock spaces, in 2003, Carswell, MacCluer and Schuster [81] characterized bounded and compact composition operators on the classical Fock spaces $\mathcal{F}^p$, $0 < p < \infty$. We refer to [166] for Fock spaces. They showed that only the class of affine mappings $\psi(z) = az + b$, $|a| \leq 1$ and $b = 0$ whenever $|a| = 1$ induce bounded composition operators. Compactness of C_ψ was described by the strict requirement $|a| < 1$. In 2008, Guo and Izuchi [107] studied various aspects of the composition operators on Fock type spaces.

In analogy to the notion of associated weights for weighted spaces of analytic functions with sup-norms, Mangino and the author introduced in [63] p-associated weights for spaces of entire p-integrable functions, $1 \leq p < \infty$. As an application, necessary conditions for the boundedness of composition operators acting between general Fock type spaces were proved.

Seyoum, Mengestie and the author [149] proved that every bounded composition operator C_φ defined by an analytic symbol φ on the complex plane when acting on generalized Fock spaces $\mathcal{F}_\varphi^p$, $1 \leq p \leq \infty$, and $p = 0$, is power bounded. Mean ergodic and uniformly mean ergodic bounded composition operators on these spaces are characterized in terms of the symbol. The behaviour for $p = 0$ and $p = \infty$ differs. The set of periodic points of these operators is also determined. This research is continued for weighted composition operators in [148].

Ueki [161, 162] investigated the boundedness, compactness and essential norm of weighted composition operators $W_{\varphi,\psi}$ on Hilbert Fock spaces of several variables in terms of a certain integral transform. Weighted composition operators on Fock spaces were investigated by Hai and Khoi [108], and between different Fock type spaces by Tien and Khoi [160]. The characterizations of the boundedness and compactness of these operators in the Fock space setting required that the symbol φ and the multiplier ψ satisfy certain uniform conditions. Carroll and Gilmore [80] present a more explicit characterization of bounded weighted composition operators, as well as compact weighted composition operators, on Fock spaces in terms of the order and type of the multiplier, and obtain a complete description of zero-free multipliers that admit bounded or compact operators. An explicit asymptotics for the iterates of the operator is also given. As an application it is shown that a weighted composition operator acting on Fock spaces cannot be supercyclic.

Tien [159] completely solves several problems, such as boundedness and compactness, topological structure, ergodic and dynamical properties, for composition operators on weighted Banach spaces of entire functions with sup-norms.

The dynamics of weighted composition operators on weighted Banach spaces of entire functions $H_v^\infty(\mathbb{C})$ and $H_v^0(\mathbb{C})$ has been investigated by Beltrán in [28]. The continuity and compactness are characterized. Moreover, in the case of affine symbols and exponential weights, it is analyzed when the operator is power bounded, (uniformly) mean ergodic and hypercyclic. This work was continued by Beltrán and Jordá [29] to investigate power boundedness, (uniform) mean ergodicity and hypercyclicity of certain weighted composition operators on spaces of entire functions of exponential and infraexponential type.

12 Superposition operators

The purpose of this section is to present a few results about superposition operators $f \to \varphi \circ f$ defined between weighted Banach spaces $H_v^\infty = H_v^\infty(\mathbb{D})$ of holomorphic functions on the disc by means of an entire function φ. If X and Y are linear spaces of holomorphic functions on the unit disc $\mathbb{D}$ of the complex plane and φ is an entire function, the superposition operator $S_\varphi : X \to Y$ with symbol φ is defined by $S_\varphi(f) := \varphi \circ f$. Since X and Y are assumed to be linear spaces, the operator S_φ is linear if and only if φ is a linear function that fixes the origin. The central question concerning superposition operators is to characterize those symbols φ such that the superposition operator maps X into Y. In case X and Y are Banach spaces, it is also important to determine when S_φ is *bounded*, in the sense that it maps bounded subsets of X into bounded subsets of Y, when S_φ is continuous or when it is *compact*, in the sense that it maps bounded sets into relatively compact sets.

We refer the reader to the introduction of [68] and to the survey [163] for references about superposition operators on different spaces of analytic functions on the disc. Superposition operators on weighted spaces of type $H_v^\infty(\mathbb{D})$ have been investigated in [68, 76, 77, 93, 145].

Lemma 69 (Boyd, Rueda) *Let u and v be weights. If the entire function φ satisfies that the superposition operator S_φ maps H_u^∞ into H_v^∞ and is bounded, then $S_\varphi : H_u^\infty \to H_v^\infty$ is continuous.*

Theorem 70 *Let u and v be weights, such that u is strictly decreasing.* (a) *If the entire function φ satisfies the following condition:*

$$\forall \varepsilon \in]0,1[\ \exists C > 0\ \exists R_0 > 0\ \forall R \geq R_0 :$$
$$v\left(u^{-1}\left(\frac{1}{\varepsilon R}\right)\right) \max_{|w|=R} |\varphi(w))| \leq C,$$

then the superposition operator S_φ maps $H_u^\infty(\mathbb{D})$ into $H_v^\infty(\mathbb{D})$ and is bounded.

(b) *If the entire function* φ *satisfies the following condition:*

$$\lim_{R\to\infty} v\left(u^{-1}\left(\frac{k}{R}\right)\right) M(\varphi, R) = 0$$

for each $k \in \mathbb{N}$*, then* $S_\varphi : H_u^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ *is compact.*

Proposition 71 *Let* $u(z) = (1-|z|)^\alpha$, $\alpha > 0$, $v(z) = (1-|z|)^\beta$, $\beta > 0$.

(1) *The following conditions are equivalent for an entire function* φ*:*

 (i) φ *is a polynomial of degree at most the integer part* $[\beta/\alpha]$ *of* β/α.
 (ii) *The superposition operator* S_φ *maps* $H_u^\infty(\mathbb{D})$ *into* $H_v^\infty(\mathbb{D})$.
 (iii) *The superposition operator* S_φ *maps* $H_u^\infty(\mathbb{D})$ *into* $H_v^\infty(\mathbb{D})$ *and it is bounded.*

(2) *The following conditions are equivalent for an entire function* φ*:*

 (i) φ *is a polynomial of degree* k *less than* β/α.
 (ii) *The superposition operator* S_φ *maps* $H_u^\infty(\mathbb{D})$ *into* $H_v^\infty(\mathbb{D})$ *and it is compact.*

Proposition 72 *Let* $u(z) = (1-|z|)^\alpha$, $z \in \mathbb{D}$ *and* $v(z) = \exp(-\frac{1}{(1-|z|)^\beta})$, $\alpha, \beta > 0$. *Let* φ *be an entire function. The following conditions are equivalent:*

(i) *The function* φ *is of order less than* β/α *or of order* β/α *and type zero.*
(ii) *For all* $0 < \varepsilon < 1$ *there are* $C \geq 1$, $R_0 > 0$ *such that* $|\varphi(z)| \leq C\exp(\varepsilon|z|^{\beta/\alpha})$ *for all* $z \in \mathbb{C}$ *with* $|z| \geq R_0$.
(iii) *The superposition operator* S_φ *maps* $H_u^\infty(\mathbb{D})$ *into* $H_v^\infty(\mathbb{D})$.
(iv) *The superposition operator* S_φ *maps* $H_u^\infty(\mathbb{D})$ *into* $H_v^\infty(\mathbb{D})$ *and it is bounded.*

Theorem 73 *Let* $u(r) = (\log\frac{e}{1-r})^{-\alpha}$ *and* $v(r) = (1-r)^\beta$ *with* $\alpha, \beta > 0$. *The following statements are equivalent for an entire function* φ*:*

(i) *The function* φ *is of order less than* $1/\alpha$ *or of order* $1/\alpha$ *and type zero.*
(ii) *For all* $0 < \varepsilon < 1$ *there are* $C > 0$, $R_0 > 0$ *such that* $|\varphi(z)| \leq C\exp(\varepsilon|z|^{1/\alpha})$ *for all* $z \in \mathbb{C}$ *with* $|z| \geq R_0$.
(iii) *The superposition operator* S_φ *maps* H_u^∞ *into* H_v^∞.
(iv) S_φ *is a bounded operator from* $H_u^\infty(\mathbb{D})$ *into* $H_v^\infty(\mathbb{D})$.
(v) S_φ *is a compact operator from* $H_u^\infty(\mathbb{D})$ *into* $H_v^\infty(\mathbb{D})$.

Proposition 74 *Let* $u(r) = (\log\frac{e}{1-r})^\alpha$, $\alpha > 0$, $v(r) = (\log\frac{e}{1-r})^\beta$, $\beta > 0$.

(1) *The superposition operator* S_φ *maps* $H_u^\infty(\mathbb{D})$ *into* $H_v^\infty(\mathbb{D})$ *and is bounded if and only if* φ *is a polynomial of degree at most* $[\beta/\alpha]$.
(2) *The superposition operator* S_φ *maps* $H_u^\infty(\mathbb{D})$ *into* $H_v^\infty(\mathbb{D})$ *and is compact if and only if* φ *is a polynomial of degree less than* β/α.

Proposition 75 *Let* $u(r) = \exp(-(1-|z|)^{-\alpha})$, $\alpha > 0$ *and let* φ *be an entire function.*

(1) *If there exist* $C > 0$ *and* $R_0 > 0$ *such that* $|\varphi(w)| \leq C\exp((\log|w|)^\gamma)$ *for* $|w| \geq R_0$, *then for each* $c > 1$ *the superposition operator* S_φ *maps* $H_u^\infty(\mathbb{D})$ *boundedly into the space* $H_{v_c}^\infty(\mathbb{D})$, *where* $v_c(r) = \exp(-\frac{c}{(1-|z|)^{\alpha\gamma}})$.
(2) *If the superposition operator* S_φ *maps* $H_u^\infty(\mathbb{D})$ *into* $H_v^\infty(\mathbb{D})$, $v(r) = \exp(-\frac{1}{(1-|z|)^\beta})$, $\beta > 0$, *then for every* $c > 1$ *there exist* $C > 0$ *and* $R_0 > 0$ *such that* $|\varphi(w)| \leq C\exp(c(\log|w|)^{\beta/\alpha})$ *for* $|w| \geq R_0$.

If $\gamma > 1$, there exist entire transcendental functions φ satisfying the assumptions of Proposition 75.

13 Toeplitz operators

Lusky, Taskinen and the author studied boundedness and compactness of Toeplitz operators on $H_v^\infty(\mathbb{D})$ in [60] and [61] for general weights on the unit disc, in particular for exponential weights $v(r) = \exp(-\alpha/(1-r)^\beta)$, $\alpha, \beta > 0$. Toeplitz operators on spaces of analytic functions have been investigated by many authors. We refer the reader to [165] and, for example, to the introduction of [60] or to the survey article [156]. We recall the necessary definitions to state some results.

Given a weight v on $\mathbb{D}$, we set

$$L_v^\infty = \big\{h : \mathbb{D} \to \mathbb{C} : h \text{ measurable}\ ,\ \|h\|_v := \operatorname*{ess\,sup}_{z\in\mathbb{D}} |h(z)|v(|z|) < \infty\big\}.$$

Let μ be the Lebesgue area measure on $\mathbb{D}$ endowed with v as density, i.e. $d\mu(re^{i\varphi}) = v(r)rdrd\varphi$ and denote the weighted L^p- and Bergman spaces by

$$L_v^p = \left\{g : \mathbb{D} \to \mathbb{C} : g \text{ measurable},\ \|g\|_{p,v}^p := \int_{\mathbb{D}} |g|^p d\mu < \infty\right\}$$

and

$$A_v^p = \{h \in L_v^p : h \text{ holomorphic}\},$$

where $1 \le p < \infty$. In the unweighted case v is omitted in the notation.

In the Hilbert spaces L_v^2 and A_v^2, with inner product $\langle f, g\rangle = \int_{\mathbb{D}} f\overline{g}\, d\mu$, the functions $e_k(z) = \Gamma_{2k}^{-1/2} z^k,\ k \in \mathbb{N}_0$, with

$$\Gamma_k = 2\pi \int_0^1 r^{k+1} v(r)dr \quad \text{for } k \in \mathbb{N}_0,$$

form an orthonormal basis of A_v^2.

Since the convergence in the space A_v^2 implies pointwise convergence, we find the reproducing kernel, i.e. a family of functions $K_z \in A_v^2$, $z \in \mathbb{D}$, such that

$$g(z) = \langle g, K_z\rangle = \int_{\mathbb{D}} g(w)\overline{K_z(w)}\, d\mu(w)$$

for all $g \in A_v^2$. The integral operator defined by the right-hand side can be extended to L_v^2, and it actually defines the orthogonal projection from L_v^2 onto A_v^2, i.e. the Bergman projection P_v; see [85, 94]. Using the orthonormal basis we can write, for all $z \in \mathbb{D}$,

$$P_v g(z) = \sum_{k=0}^\infty \langle g, e_k\rangle e_k(z) = \int_{\mathbb{D}} \sum_{k=0}^\infty \frac{z^k \overline{w}^k}{\Gamma_k} g(w)d\mu(w).$$

Moreover, the *Bergman kernel* K_z satisfies $|K_z(w)| \le C_z,\ w \in \mathbb{D}$.

Let $f : \mathbb{D} \to \mathbb{C}$ be a function in L^1. The Toeplitz operator T_f with symbol f is defined on $H_v^\infty(\mathbb{D})$ by

$$T_f(h) = \int_{\mathbb{D}} f(w)h(w)\overline{K_z(w)}\, d\mu(w).$$

The integral converges for all $z \in \mathbb{D}$ and for all $h \in H_v^\infty(\mathbb{D})$, since $hv \in L^\infty$. Even if $T_f h$ is a well defined analytic function, it is not necessarily an element of $H_v^\infty(\mathbb{D})$ and T_f need not be a bounded operator.

If $h \in H_v^\infty(\mathbb{D})$ is such that $f \cdot h \in L_v^2$, we also have

$$(T_f h)(z) = \sum_{n=0}^{\infty} \langle f \cdot h, e_n \rangle e_n(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_{2n}} \int_{\mathbb{D}} f(w)h(w)\overline{w}^n v(w) dA \,, \tag{8}$$

where the series converges in L_v^2. However, the formula also holds for all $h \in H_v^\infty(\mathbb{D})$ and the sum converges uniformly for z in compact subsets of the disc. More details can be seen in the introduction of [60].

If the symbol f is analytic, the operator T_f is the multiplier M_f, and in this case M_f is bounded on $H_v^\infty(\mathbb{D})$ if and only if $f \in H^\infty$, see [50]. The situation for harmonic symbols is different.

Theorem 76 *There is a bounded harmonic function $f : \mathbb{D} \to \mathbb{C}$ such that $T_f : H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ is not a bounded operator for any weight v on $\mathbb{D}$.*

As a consequence, the Bergman projection P_v is not a bounded mapping $L_v^\infty \to L_v^\infty$ for any weight under consideration.

Dostanic [94] showed that for the exponential weight $v(z) = \exp(-1/(1 - |z|))$, the orthogonal projection $L_v^2 \to A_v^2$ is bounded in L_v^p if and only if $p = 2$. Recent research about the continuity of the Bergman projection can be seen in [62, 84, 114, 140].

A general, abstract necessary and sufficient condition for the boundedness and for the compactness of the Toeplitz operator $T_f : H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ was presented in [60, Theorem 3.6], when f is a radial symbol in L^1, when the weight v satisfies a certain condition (B), which was introduced by Lusky in [127]. This condition (B) is closely related to condition (b) described in Sect. 5 and it is satisfied by standard and exponential weights, but not by logarithmic weights. The proof of [60, Theorem 3.6] is based on the deep work of Lusky.

Since the conditions for boundedness of T_f might be hard to evaluate, in [61] we prove sufficient conditions, which are easier to formulate and control, for the boundedness and compactness of Toeplitz operators T_f on $H_v^\infty(\mathbb{D})$ when the weights v and symbol f are assumed to be radial functions on $\mathbb{D}$, and v has condition (B) of Lusky. Here are some results in this directions.

Theorem 77 *Let v satisfy condition (B). If the symbol f is radial and continuously differentiable on $[0, 1]$, then $T_f : H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ is bounded.*

Theorem 78 *Let v be a standard weight.*

(i) *If $f \in L^1$ is radial and satisfies*

$$\limsup_{r \to 1} |f(r) \log(1 - r)| < \infty,$$

then $T_f : H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ is a bounded operator.

(ii) *If f satisfies $\limsup_{r\to 1} |f(r) \log(1 - r)| = 0$, then T_f is compact on $H_v^\infty(\mathbb{D})$.*

Theorem 79 *Let $v(r) = \exp(-\alpha/(1 - r)^\beta)$ be an exponential weight.*

(i) *Assume that $f \in L^1$ is radial and*

$$\limsup_{r \to 1} |f(r)|(1 - r)^{-1/2-\beta/4} < \infty.$$

Then, $T_f : H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ is a bounded operator.

(ii) *If*

$$\limsup_{r\to 1} |f(r)|(1-r)^{-1/2-\beta/4} = 0,$$

then T_f *is compact on* $H_v^\infty(\mathbb{D})$.

14 Other classes of operators

14.1 Hilbert matrix

The Hilbert matrix $\mathcal{H}$ with entries $a_{i,j} = 1/(i+j+1),\ i, j \in \mathbb{N}$, induces an operator on sequences

$$\mathcal{H}((a_n)_n) := \left(\sum_{k=0}^{\infty} \frac{a_k}{n+k+1} \right)_n,$$

which is continuous on the sequence space $\ell_p, 1 < p < \infty$ by Hilbert's inequality, with exact norm $\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$. The Hilbert matrix can be also considered as an operator on spaces of analytic functions on the unit disc, acting on the Taylor coefficients of $f(z) = \sum_{k=0}^{\infty} a_k z^k$ as follows:

$$\mathcal{H}(f)(z) := \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{a_k}{n+k+1} \right) z^n.$$

This operator admits also the following integral representation, see [91]:

$$\mathcal{H}(f)(z) = \int_0^1 T_t(f)(z)dt, \quad z \in \mathbb{D},$$

where $T_t(f)(z) = \omega_t(z) f(\phi_t(z))$ for $0 < t < 1$, $\omega_t(z) = \frac{1}{(t-1)z+1}$ and $\phi_t(z) = \frac{t}{(t-1)z+1}$.

Let $\alpha > 0$. In this subsection it is useful to denote by $A^{-\alpha}$ and $A_0^{-\alpha}$ the Korenblum type growth spaces for the weight $v(r) = (1-r)^\alpha$ and by H_α^∞ and H_α^0 the weighted spaces defined for the weight $w(r) = (1-r^2)^\alpha$. Clearly $A^{-\alpha} = H_\alpha^\infty$, $A_0^{-\alpha} = H_\alpha^0$ with equivalent norms. Aleman, Montes-Rodríguez and Sarafoleaunu [12, Theorem 2.1] proved that the Hilbert matrix operator $\mathcal{H}$ is bounded on $A^{-\alpha}$ and $A_0^{-\alpha}$ if (and only if) $0 < \alpha < 1$. The exact calculation of the norm of an operator in these spaces depends on the norm, hence on the selected weight. In recent years there has been much interest in computing the exact norm of the Hilbert matrix operator on different spaces of analytic functions. We refer the reader to the informative introduction of [124]. This paper also mentions open questions.

Theorem 80 *Let* $0 < \alpha < 1$.

(i) [124] *The exact norm* $||\mathcal{H}||_{A^{-\alpha}}$ *of* $\mathcal{H} : A^{-\alpha} \to A^{-\alpha}$ *is* $\frac{\pi}{\sin(\alpha\pi)}$.
(ii) [123] *If* $0 < \alpha \leq 2/3$, *then the exact norm* $||\mathcal{H}||_{H_\alpha^\infty}$ *of* $\mathcal{H} : H_\alpha^\infty \to H_\alpha^\infty$ *is* $\frac{\pi}{\sin(\alpha\pi)}$.
(iii) [90] *There is an exact value* α_0 *with* $2/3 < \alpha_0 < 1$ *such that* $||\mathcal{H}||_{H_\alpha^\infty} = \frac{\pi}{\sin(\alpha\pi)}$ *if* $0 < \alpha \leq \alpha_0$ *and this norm is strictly greater than* $\frac{\pi}{\sin(\alpha\pi)}$ *if* $\alpha_0 < \alpha < 1$.

14.2 Libera operator

Let $H(\overline{\mathbb{D}})$ be the space of germs of analytic functions on the closed unit disc $\overline{\mathbb{D}}$, that is, the functions analytic g on an open neighbourhood of $\overline{\mathbb{D}}$, which depends on g. The *Libera operator* $\mathcal{L}$ is defined on $H(\overline{\mathbb{D}})$ by

$$\mathcal{L}g(z) := \frac{1}{1-z}\int_z^1 g(\zeta)d\zeta.$$

Equivalently, for $g(z) = \sum_{n=0}^{\infty} a_k z^k \in H(\overline{\mathbb{D}})$,

$$\mathcal{L}g(z) := \sum_{n=0}^{\infty}\left(\sum_{k=n}^{\infty}\frac{a_k}{k+1}\right)z^n.$$

The space $H(\overline{\mathbb{D}})$ can be identified with the topological dual of the Fréchet space $H(\mathbb{D})$. With this identification, the Libera operator coincides with the transpose of the Cesàro operator which was studied in Sect. 9.

The Libera operator cannot be extended to $H(\mathbb{D})$. Indeed, the sequence of polynomials $g_n(z) = \sum_{k=0}^{n} z^k, n \in \mathbb{N}$, converges to $g(z) = 1/(1-z)$ on $H(\mathbb{D})$, and the sequence $(\mathcal{L}g_n(0))_n$ diverges to ∞. In particular, this shows that the Libera operator is not defined on the growth spaces $A^{-\alpha}$ for $\alpha \geq 1$. On the other hand, Pavlovic [139] proved the following result.

Theorem 81 [139]

(i) *The Libera operator $\mathcal{L}$ acts as a bounded operator from $A^{-\alpha}$ into $A^{-\alpha}$ if and only if $0 < \alpha < 1$.*
(ii) *If $1/2 \leq \alpha < 1$, then there is $g(z) = \sum_{n=0}^{\infty} a_k z^k \in A^{-\alpha}$ such that $|a_n| \geq c(n+1)^{\alpha-1/2}$ for all n and some $c > 0$, and $\mathcal{L}$ acts as a bounded operator from $A^{-\alpha}$ into $A^{-\alpha}$.*

More results about the Libera operator on Banach spaces of analytic functions can be seen in [138, 139] and the references therein.

14.3 Hausdorff operators

We gave in [45] a few results about Hausdorff operators on weighted Banach spaces of holomorphic functions of type H^∞, both in the case of spaces of entire functions, in which the operator is defined as in [154], and in the case of the disc, where it is defined as in [136]. Hausdorff type operators have been investigated by many authors, especially in the one-dimensional case, starting with the work of Hardy and Littlewood. The question of what is the "correct" definition of Hausdorff operators on Euclidean spaces is in a certain sense open. Some clarification was presented in the recent paper by Karapetyants and Liflyand [119].

Let μ be a positive Radon measure on the unit disc $\mathbb{D}$ and let K be a μ-measurable function on $\mathbb{D}$. For $w \in \mathbb{D}$, we denote by φ_w the automorphism of the disc defined by

$$\varphi_w(z) = \frac{w-z}{1-\overline{w}z}, \quad z \in \mathbb{D}.$$

Mirotin [136] defines the *Hausdorff operator associated with μ and K* on the disc $\mathbb{D}$ by

$$\mathcal{H}_{K,\mu}(f)(z) := \int_{\mathbb{D}} K(w)f(\varphi_w(z))d\mu(w), \quad z \in \mathbb{D},$$

for an analytic function $f \in H(\mathbb{D})$ on the unit disc. He obtained conditions to ensure that the operator acts continuously on the Bloch, Bergman and Hardy spaces.

Theorem 82 *Let v be a radial weight on $\mathbb{D}$ satisfying condition $(L1)$. If the function $w \in \mathbb{D} \to K(w)\|C_{\varphi_w}\|$ belongs to $L^1(\mu)$, then the operators*

$$\mathcal{H}_{K,\mu} : H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D}),$$

and

$$\mathcal{H}_{K,\mu} : H_v^0(\mathbb{D}) \to H_v^0(\mathbb{D}),$$

are continuous. In this case, we have

$$\|\mathcal{H}_{K,\mu}\| \leq \int_{\mathbb{D}} |K(w)|\; \|C_{\varphi_w}\| d\mu(w).$$

Corollary 83 (1) *Let $v(r) = (1-r^2)^\gamma$ with $\gamma > 0$. If the function $w \in \mathbb{D} \to K(w)/(1-|w|)^\gamma$ belongs to $L^1(\mu)$, then $\mathcal{H}_{K,\mu} : H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ is continuous.*
(2) *Let $v(r) = (\log \frac{e}{1-r^2})^{-\alpha}$, $\alpha > 0$. If the function $w \in \mathbb{D} \to K(w)\log(1-|w|)$ belongs to $L^1(\mu)$, then $\mathcal{H}_{K,\mu} : H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ is continuous.*

We now turn our attention to the case of spaces of entire functions. Let μ be a positive measure on $(0, \infty)$. Stylogiannis and Galanopoulos [154] consider formally the *Hausdorff operator induced by the measure μ* defined by

$$\mathcal{H}_\mu(f)(z) := \int_0^\infty \frac{1}{t} f\left(\frac{z}{t}\right) d\mu(t), \quad z \in \mathsf{C},$$

where $f \in H(\mathbb{C})$ is an entire function. The operator $\mathcal{H}_\mu$ is studied in [154] on Fock spaces $\mathcal{F}_\alpha^p$, $1 \leq p \leq \infty, \alpha > 0$.

Proposition 84 *Let v be a weight on $\mathbb{C}$.*

(1) *If the operator $\mathcal{H}_\mu : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ is continuous, then*

$$\sup_{n\in\mathbb{N}_0} \int_0^\infty \frac{1}{t^{n+1}} d\mu(t) \leq \|\mathcal{H}_\mu\| < \infty, \tag{9}$$

and the operator $\mathcal{H}_\mu : H_v^0(\mathbb{C}) \to H_v^0(\mathbb{C})$ is also continuous.
(2) *If the operator $\mathcal{H}_\mu : H_v^0(\mathbb{C}) \to H_v^0(\mathbb{C})$ is continuous, then (9) holds.*
(3) *If the operator $\mathcal{H}_\mu : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ is compact, then*

$$\lim_{k\to\infty} \int_0^\infty \frac{1}{t^k} d\mu(t) = 0.$$

Theorem 85 *Let $\alpha > 0$ and $\beta > 0$. Let v be the weight on $\mathbb{C}$ defined by $v(r) = \exp(-\beta r^\alpha)$. The following conditions are equivalent.*

(i) *$\mathcal{H}_\mu : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ is continuous.*
(ii) *$\mathcal{H}_\mu : H_v^0(\mathbb{C}) \to H_v^0(\mathbb{C})$ is also continuous.*
(iii) $\sup_{n\in\mathbb{N}} \int_0^\infty \frac{1}{t^{n+1}} d\mu(t) < \infty$.

In this case, we have

$$\|\mathcal{H}_\mu\| \leq \sup_{n\in\mathbb{N}} \int_0^\infty \frac{1}{t^{n\alpha+1}} d\mu(t).$$

Proposition 86 *Let* $\alpha > 0$ *and* $\beta > 0$. *Let* v *be the weight on* $\mathbb{C}$ *defined by* $v(r) = \exp(-\beta r^{\alpha})$. *If the sequence* $\left(\int_0^{\infty} \frac{1}{t^k}\, d\mu(t)\right)_{k\in\mathbb{N}}$ *is in* ℓ_1, *then the operator* $\mathcal{H}_{\mu}$ *is compact on* $H_v^{\infty}(\mathbb{C})$ *and on* $H_v^0(\mathbb{C})$.

Some questions seems to be open concerning the operators mentioned in this last section.

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References

1. Abakumov, E., Doubtsov, E.: Reverse estimates in growth spaces. Math. Z. **271**(1–2), 399–413 (2012)
2. Abakumov, E., Doubtsov, E.: Moduli of holomorphic functions and logarithmically convex radial weights. Bull. Lond. Math. Soc. **47**(3), 519–532 (2015)
3. Abakumov, E., Doubtsov, E.: Approximation by proper holomorphic maps and tropical power series. Constr. Approx. **47**(2), 321–338 (2018)
4. Abakumov, E., Doubtsov, E.: Volterra type operators on growth Fock spaces. Arch. Math. (Basel) **108**(4), 383–393 (2017)
5. Abakumov, E., Doubtsov, E.: Univalent symbols of Volterra operators on growth spaces. Anal. Math. Phys. **9**(3), 911–917 (2019)
6. Abanin, A.V., Tien, P.T.: Differentiation and integration operators on weighted Banach spaces of holomorphic functions. Math. Nachr. **290**(8–9), 1144–1162 (2017)
7. Abanin, A.V., Tien, P.T.: Invariant subspaces for classical operators on weighted spaces of holomorphic functions. Integral Equ. Oper. Theory **89**(3), 409–438 (2017)
8. Abanin, A.V., Tien, P.T.: Compactness of classical operators on weighted Banach spaces of holomorphic functions. Collect. Math. **69**(1), 1–15 (2018)
9. Albanese, A.A., Bonet, J., Ricker, W.J.: The Cesàro operator in growth Banach spaces of analytic functions. Integral Equ. Oper. Theory **86**(1), 97–112 (2016)
10. Albanese, A.A., Bonet, J., Ricker, W.J.: The Cesàro operator on Korenblum type spaces of analytic functions. Collect. Math. **69**(2), 263–281 (2018)
11. Aleman, A., Constantin, O.: Spectra of integration operators on weighted Bergman spaces. J. Anal. Math. **109**, 199–231 (2009)
12. Aleman, A., Montes-Rodríguez, A., Sarafoleanu, A.: The eigenfunctions of the Hilbert matrix. Constr. Approx. **36**(3), 353–374 (2012)
13. Aleman, A., Peláez, J.A.: Spectra of integration operators and weighted square functions. Indiana Univ. Math. J. **61**, 1–19 (2012)
14. Aleman, A., Persson, A.-M.: Resolvent estimates and decomposable extensions of generalized Cesàro operators. J. Funct. Anal. **258**, 67–98 (2010)
15. Aleman, A., Siskakis, A.G.: An integral operator on H^p. Complex Var. Theory Appl. **28**, 149–158 (1995)
16. Aleman, A., Siskakis, A.G.: Integration operators on Bergman spaces. Indiana Univ. Math. J. **46**, 337–356 (1997)
17. Anderson, A., Jovovic, M., Smith, W.: Some integral operators acting on H^{∞}. Integral Equ. Oper. Theory **80**, 275–291 (2014)
18. Anderson, J.M., Shields, A.L.: Coefficient multipliers of Bloch functions. Trans. Am. Math. Soc. **224**, 255–265 (1976)

19. Arcozzi, N., Björn, A.: Dominating sets for analytic and harmonic functions and completeness of weighted Bergman spaces. Math. Proc. R. Ir. Acad. **102A**, 175–192 (2002)
20. Arendt, W., Célariès, B., Chalendar, I.: In Koenigs' footsteps: diagonalization of composition operators. J. Funct. Anal. **278**(2), 108313 (2002)
21. Arendt, W., Chalendar, I., Kumar, M., Srivastava, S.: Asymptotic behaviour of the powers of composition operators on Banach spaces of holomorphic functions. Indiana Univ. Math. J. **67**(4), 1571–1595 (2018)
22. Arendt, W., Chalendar, I., Kumar, M., Srivastava, S.: Powers of composition operators: asymptotic behaviour on Bergman, Dirichlet and Bloch spaces. J. Aust. Math. Soc. **108**(3), 289–320 (2020)
23. Aron, R., Lindström, M.: Spectra of weighted composition operators on weighted Banach spaces of analytic functions. Isr. J. Math. **141**, 263–276 (2004)
24. Atzmon, A., Brive, B.: Surjectivity and invariant subspaces of differential operators on weighted Bergman spaces of entire functions, Bergman spaces and related topics in complex analysis, pp. 27–39, Contemp. Math., vol. 404. Amer. Math. Soc., Providence (2006)
25. Basallote, M., Contreras, M.D., Hernández-Mancera, C., Martín, M.J., Paúl, P.J.: Volterra operators and semigroups in weighted Banach spaces of analytic functions. Collect. Math. **65**(2), 233–249 (2014)
26. Bayart, F., Matheron, É.: Dynamics of Linear Operators. Cambridge University Press, Cambridge (2009)
27. Beltrán-Meneu, M.J.: Dynamics of differentiation and integration operators on weighted spaces of entire functions. Studia Math. **221**(1), 35–60 (2014)
28. Beltrán-Meneu, M.J.: Dynamics of weighted composition operators on weighted Banach spaces of entire functions. J. Math. Anal. Appl. **492**(1), 124422 (2020)
29. Beltrán-Meneu, M.J., Jordá, E.: Dynamics of weighted composition operators on spaces of entire functions of exponential and infraexponential type. Mediterr. J. Math. **18**(5), Paper No. 212 (2021)
30. Beltrán-Meneu, M.J., Bonet, J., Fernández, C.: Classical operators on weighted Banach spaces of entire functions. Proc. Am. Math. Soc. **141**(12), 4293–4303 (2013)
31. Beltrán-Meneu, M.J., Bonet, J., Fernández, C.: Classical operators on the Hörmander algebras. Discrete Contin. Dyn. Syst. **35**(2), 637–652 (2015)
32. Beltrán-Meneu, M.J., Gómez-Collado, M.C., Jordá, E., Jornet, D.: Mean ergodic composition operators on Banach spaces of holomorphic functions. J. Funct. Anal. **270**(12), 4369–4385 (2016)
33. Beltrán-Meneu, M.J., Gómez-Collado, M.C., Jordá, E., Jornet, D.: Mean ergodicity of weighted composition operators on spaces of holomorphic functions. J. Math. Anal. Appl. **444**(2), 1640–1651 (2016)
34. Bennet, G., Stegenga, D.A., Timoney, R.M.: Coefficients of Bloch and Lipschitz functions. Ill. J. Math. **25**, 520–531 (1981)
35. Bierstedt, K.D., Bonet, J., Galbis, A.: Weighted spaces of holomorphic functions on bounded domains. Mich. Math. J. **40**, 271–297 (1993)
36. Bierstedt, K.D., Bonet, J., Taskinen, J.: Associated weights and spaces of holomorphic functions. Studia Math. **127**, 137–168 (1998)
37. Bierstedt, K.D., Summers, W.H.: Biduals of weighted Banach spaces of analytic functions. J. Aust. Math. Soc. **54**, 70–79 (1993)
38. Blasco, O.: Operators on Fock-type and weighted spaces of entire functions. Funct. Approx. Comment. Math. **59**(2), 175–189 (2018)
39. Blasco, O.: Boundedness of Volterra operators on spaces of entire functions. Ann. Acad. Sci. Fenn. Math. **43**(1), 89–107 (2018)
40. Bonet, J.: Weighted spaces of holomorphic functions and operators between them. Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003), pp. 117–138, Colecc. Abierta, vol. 64. Univ. Sevilla Secr. Publ., Seville (2003)
41. Bonet, J.: A note about the spectrum of composition operators induced by a rotation. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **114**(2), 63 (2020)
42. Bonet, J.: The spectrum of Volterra operators on Korenblum type spaces of analytic functions. Integral Equ. Oper. Theory **91**(5), Paper No. 46 (2019)
43. Bonet, J.: The spectrum of Volterra operators on weighted spaces of entire functions. Q. J. Math. **66**(3), 799–807 (2015)
44. Bonet, J.: Dynamics of the differentiation operator on weighted spaces of entire functions. Math. Z. **261**(3), 649–657 (2009)
45. Bonet, J.: Hausdorff operators on weighted Banach spaces of type H^∞. Complex Anal. Oper. Theory **16**(1), Paper No. 12 (2022)
46. Bonet, J., Bonilla, A.: Chaos of the differentiation operator on weighted Banach spaces of entire functions. Complex Anal. Oper. Theory **7**, 33–42 (2013)
47. Bonet, J., Domański, P.: A note on mean ergodic composition operators on spaces of holomorphic functions. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **105**(2), 389–396 (2011)

48. Bonet, J., Domański, P.: Power bounded composition operators on spaces of analytic functions. Collect. Math. **62**(1), 69–83 (2011)
49. Bonet, J., Domański, P., Lindström, M.: Essential norm and weak compactness of composition operators on weighted Banach spaces of analytic functions. Can. Math. Bull. **42**, 139–148 (1999)
50. Bonet, J., Domański, P., Lindström, M.: Pointwise multiplication operators on weighted Banach spaces of analytic functions. Studia Math. **137**, 177–194 (1999)
51. Bonet, J., Domański, P., Lindström, M.: Weakly compact composition operators on analytic vector-valued function spaces. Ann. Acad. Sci. Fenn. **26**, 233–248 (2001)
52. Bonet, J., Domański, P., Lindström, M., Taskinen, J.: Composition operators between weighted Banach spaces of analytic functions. J. Aust. Math. Soc. **64**, 101–118 (1998)
53. Bonet, J., Galbis, A.: Invariant subspaces of the integration operators on Hörmander algebras and Korenblum type spaces. Integral Equ. Oper. Theory **92**(4), Paper No. 34 (2020)
54. Bonet, J., Galindo, P., Lindström, M.: Spectra and essential spectral radii of composition operators on weighted Banach spaces of analytic functions. J. Math. Anal. Appl. **340**(2), 884–891 (2008)
55. Bonet, J., Gómez-Collado, M.C., Jordá, E., Jornet, D.: Nuclear weighted composition operators on weighted Banach spaces of analytic functions. Proc. Am. Math. Soc. **149**(1), 311–321 (2021)
56. Bonet, J., Lindström, M., Wolf, E.: Isometric weighted composition operators on weighted Banach spaces of type H^∞. Proc. Am. Math. Soc. **136**(12), 4267–4273 (2008)
57. Bonet, J., Lusky, W., Taskinen, J.: Distance formulas on weighted Banach spaces of analytic functions. Complex Anal. Oper. Theory **13**(3), 893–900 (2019)
58. Bonet, J., Lusky, W., Taskinen, J.: Solid hulls and cores of weighted H^∞-spaces. Rev. Mat. Complut. **31**(3), 781–804 (2019)
59. Bonet, J., Lusky, W., Taskinen, J.: Solid cores and solid hulls of weighted Bergman spaces. Banach J. Math. Anal. **13**(2), 468–485 (2019)
60. Bonet, J., Lusky, W., Taskinen, J.: On boundedness and compactness of Toeplitz operators in weighted H^∞-spaces. J. Funct. Anal. **278**(10), 108456 (2020)
61. Bonet, J., Lusky, W., Taskinen, J.: On the boundedness of Toeplitz operators with radial symbols over weighted sup-norm spaces of holomorphic functions. J. Math. Anal. Appl. **493**(1), 124515 (2021)
62. Bonet, J., Lusky, W., Taskinen, J.: Unbounded Bergman projections on weighted spaces with respect to exponential weights. Integral Equ. Oper. Theory **93**(6), 61 (2021)
63. Bonet, J., Mangino, E.: Associated weights for spaces of p-integrable entire functions. Quaest. Math. **43**(5–6), 747–760 (2020)
64. Bonet, J., Mengestie, T., Worku, M.: Dynamics of the Volterra-type integral and differentiation operators on generalized Fock spaces. Results Math. **74**(4), 197 (2019)
65. Bonet, J., Taskinen, J.: A note about Volterra operators on weighted Banach spaces of entire functions. Math. Nachr. **288**(11–12), 1216–1225 (2015)
66. Bonet, J., Taskinen, J.: Solid hulls of weighted Banach spaces of analytic functions on the unit disc with exponential weights. Ann. Acad. Sci. Fenn. **43**, 521–530 (2018)
67. Bonet, J., Taskinen, J.: Solid hulls of weighted Banach spaces of entire functions. Rev. Mat. Iberoam. **34**(2), 593–608 (2018)
68. Bonet, J., Vukotić, D.: Superposition operators between weighted Banach spaces of analytic functions of controlled growth. Monatsh. Math. **170**(3–4), 311–323 (2013)
69. Bonet, J., Vukotić, D.: A note on completeness of weighted normed spaces of analytic functions. Results Math. **72**(1–2), 263–279 (2017)
70. Bonet, J., Wolf, E.: A note on weighted Banach spaces of holomorphic functions. Arch. Math. **81**, 650–654 (2003)
71. Bourdon, P.S.: Essential angular derivatives and maximum growth of Koenigs eigenfunction. J. Funct. Anal. **160**, 561–580 (1998)
72. Bourdon, P.S.: Bellwethers for boundedness of composition operators on weighted Banach spaces of analytic functions. J. Aust. Math. Soc. **86**(3), 305–314 (2009)
73. Bourdon, P.S.: Invertible weighted composition operators. Proc. Am. Math. Soc. **142**(1), 289–299 (2014)
74. Boyd, C., Rueda, P.: Isometries between spaces of weighted holomorphic functions. Studia Math. **190**(3), 203–231 (2009)
75. Boyd, C., Rueda, P.: Isometries of weighted spaces of holomorphic functions on unbounded domains. Proc. R. Soc. Edinb. Sect. A **139**(2), 253–271 (2009)
76. Boyd, C., Rueda, P.: Holomorphic superposition operators between Banach function spaces. J. Aust. Math. Soc. **96**(2), 186–197 (2014)
77. Boyd, C., Rueda, P.: Superposition operators between weighted spaces of analytic functions. Quaest. Math. **36**(3), 411–419 (2013)

78. Boyd, C., Rueda, P.: Surjectivity of isometries between weighted spaces of holomorphic functions. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **113**(3), 2461–2477 (2019)
79. Bracci, F., Contreras, M.D., Díaz-Madrigal, S.: Continuous Semigroups of Holomorphic Self-maps of the Unit Disc. Springer, Berlin (2020)
80. Carroll, T., Gilmore, C.: Weighted composition operators on Fock spaces and their dynamics. J. Math. Anal. Appl. **502**(1), 125234 (2021)
81. Carswell, B.J., MacCluer, B.D., Schuster, A.: Composition operators on the Fock space. Acta Sci. Math. (Szeged) **69**, 871–887 (2003)
82. Colonna, F., Martínez-Avendaño, R.A.: Hypercyclicity of composition operators on Banach spaces of analytic functions. Complex Anal. Oper. Theory **12**(1), 305–323 (2018)
83. Constantin, O.: A Volterra-type integration operator on Fock spaces. Proc. Am. Math. Soc. **140**, 4247–4257 (2012)
84. Constantin, O., Peláez, J.A.: Integral operators, embedding theorems and a Littlewood-Paley formula on weighted Fock spaces. J. Geom. Anal. **26**(2), 1109–1154 (2016)
85. Constantin, O., Peláez, J.A.: Boundedness of the Bergman projection on L^p-spaces with exponential weights. Bull. Sci. Math. **139**, 245–268 (2015)
86. Constantin, O., Persson, A.-M.: The spectrum of Volterra-type integration operators on generalized Fock spaces. Bull. Lond. Math. Soc. **47**(6), 958–963 (2015)
87. Contreras, M.D., Peláez, J.A., Pommerenke, C., Rättyä, J.: Integral operators mapping into the space of bounded analytic functions. J. Funct. Anal. **271**, 2899–2943 (2016)
88. Contreras, M.D., Hernández-Díaz, A.G.: Weighted composition operators in weighted Banach spaces of analytic functions. J. Aust. Math. Soc. **69**, 41–60 (2000)
89. Cowen, C., MacCluer, B.: Composition Operators on Spaces of Analytic Functions. CRC Press, Boca Raton (1995)
90. Dai, J.: Norm of the Hilbert matrix operator on the Korenblum space. J. Math. Anal. Appl. **514**, 126270 (2022)
91. Diamantopoulos, E., Siskakis, A.G.: Composition operators and the Hilbert matrix. Studia Math. **140**(2), 191–198 (2000)
92. Domański, P., Lindström, M.: Sets of interpolation and sampling for weighted Banach spaces of holomorphic functions. Ann. Polon. Math. **79**, 233–264 (2002)
93. Domínguez, S., Girela, D.: Superposition operators between mixed norm spaces of analytic functions. Mediterr. J. Math. **18**(1), 18 (2021)
94. Dostanic, M.: Unboundedness of the Bergman projections on L^p spaces with exponential weights. Proc. Edinb. Math. Soc. **47**, 111–117 (2004)
95. Doubtsov, E.: Integration and differentiation in Hardy growth spaces. Complex Anal. Oper. Theory **13**(4), 1883–1893 (2019)
96. Dunford, N., Schwartz, J.T.: Linear Operators I: General Theory, 2nd Printing. Wiley Interscience Publ., New York (1964)
97. Duren, P.L.: Theory of H^p Spaces. Academic Press, New York (1970)
98. Eklund, T., Galindo, P., Lindström, M.: Königs eigenfunction for composition operators on Bloch and H^∞ type spaces. J. Math. Anal. Appl. **445**(2), 1300–1309 (2017)
99. Eklund, T., Galindo, P., Lindström, M., Nieminen, I.: Norm, essential norm and weak compactness of weighted composition operators between dual Banach spaces of analytic functions. J. Math. Anal. Appl. **451**(1), 1–13 (2017)
100. Eklund, T., Lindström, M., Pirasteh, M.M., Sanatpour, A.H., Wikman, N.: Generalized Volterra operators mapping between Banach spaces of analytic functions. Monatsh. Math. **189**(2), 243–261 (2019)
101. Fares, T., Lefèvre, P.: Nuclear composition operators on Bloch spaces. Proc. Am. Math. Soc. **148**(6), 2487–2496 (2020)
102. Galindo, P., Lindström, M.: Fredholm composition operators on analytic function spaces. Collect. Math. **63**(2), 139–145 (2012)
103. Galindo, P., Lindström, M.: Spectra of some weighted composition operators on dual Banach spaces of analytic functions. Integral Equ. Oper. Theory **90**(3), 31 (2018)
104. Galindo, P., Lindström, M., Wikman, N.: Spectra of weighted composition operators on analytic function spaces. Mediterr. J. Math. **17**(1), 34 (2020)
105. Gómez-Collado, M.C., Jornet, D.: Fredholm weighted composition operators on weighted Banach spaces of analytic functions of type H^∞. J. Funct. Spaces **2015**, 982135 (2015)
106. Grosse-Erdmann, K., Peris, A.: Linear Chaos. Springer, London (2011)
107. Guo, K., Izuchi, K.: Composition operators on Fock type space. Acta Sci. Math. (Szeged) **74**, 807–828 (2008)

108. Hai, P.V., Khoi, L.H.: Boundedness and compactness of weighted composition operators on Fock spaces $\mathcal{F}^p(\mathbb{C})$. Acta Math. Vietnam. **41**(3), 531–537 (2016)
109. Han, S.-A., Zhou, Z.-H.: Mean ergodicity of composition operators on Hardy space. Proc. Indian Acad. Sci. Math. Sci. **129**(4), 45 (2019)
110. Harutyunyan, A., Lusky, W.: On the boundedness of the differentiation operator between weighted spaces of holomorphic functions. Studia Math. **184**(3), 233–247 (2008)
111. Hedenmalm, H., Korenblum, B., Zhu, K.: Theory of Bergman Spaces. Grad. Texts in Math., vol. 199. Springer, New York (2000)
112. Heuser, H.R.: Functional Analysis. Wiley, New York (1982)
113. Hu, Z.: Extended Cesàro operators on mixed-norm spaces. Proc. Am. Math. Soc. **131**, 2171–2179 (2003)
114. Hu, Z., Lv, X., Schuster, A.: Bergman spaces with exponential weights. J. Funct. Anal. **276**, 1402–1429 (2019)
115. Hyvärinen, O., Lindström, M., Nieminen, I., Saukko, E.: Spectra of weighted composition operators with automorphic symbols. J. Funct. Anal. **265**(8), 1749–1777 (2013)
116. Hyvärinen, O., Kemppainen, M., Lindström, M., Rautio, A., Saukko, E.: The essential norm of weighted composition operators on weighted Banach spaces of analytic functions. Integral Equ. Oper. Theory **72**(2), 151–157 (2012)
117. Jevtić, M., Vukotić, D., Arsenović, M.: Taylor Coefficients and Coefficient Multipliers of Hardy and Bergman-Type Spaces, RSME Springer Series, vol. 2. Springer, Berlin (2016)
118. Jordá, E., Rodríguez-Arenas, A.: Ergodic properties of composition operators on Banach spaces of analytic functions. J. Math. Anal. Appl. **486**(1), 123891 (2020)
119. Karapetyants, A., Liflyand, E.: Defining Hausdorff operators on Euclidean spaces. Math. Methods Appl. Sci. **43**(16), 9487–9498 (2020)
120. Krengel, U.: Ergodic Theorems. de Gruyter Studies in Mathematics, vol. 6. Walter de Gruyter Co., Berlin (1985)
121. Liang, Y.-X., Zhou, Z.-H.: Hypercyclic behaviour of multiples of composition operators on weighted Banach spaces of holomorphic functions. Bull. Belg. Math. Soc. Simon Stevin **21**(3), 385–401 (2014)
122. Lin, Q.: Volterra type operators between Bloch type spaces and weighted Banach spaces. Integral Equ. Oper. Theory **91**(2), 13 (2019)
123. Lindström, M., Miihkinen, S., Wikman, N.: Norm estimates of weighted composition operators pertaining to the Hilbert matrix. Proc. Am. Math. Soc. **147**(6), 2425–2435 (2019)
124. Lindström, M., Miihkinen, S., Norrbo, D.: Exact essential norm of generalized Hilbert matrix operators on classical analytic functions spaces. Adv. Math. **408**, 108598 (2022)
125. Lusky, W.: On weighted spaces of harmonic and holomorphic functions. J. Lond. Math. Soc. **51**, 309–320 (1995)
126. Lusky, W.: On the Fourier series of unbounded harmonic functions. J. Lond. Math. Soc. **61**(2), 568–580 (2000)
127. Lusky, W.: On the isomorphism classes of weighted spaces of harmonic and holomorphic functions. Studia Math. **175**(1), 19–40 (2006)
128. Malman, B.: Spectra of generalized Cesàro operators acting on growth spaces. Integral Equ. Oper. Theory **90**(3), 26 (2018)
129. Martín, M.J., Vukotić, D.: Isometries of the Bloch space among the composition operators. Bull. Lond. Math. Soc. **39**(1), 151–155 (2007)
130. Mas, A., Vukotić, D.: Invertible and isometric composition operators. Maediterr. J. Math. **19**, 173 (2022)
131. Meise, R., Vogt, D.: Introduction to Functional Analysis. Clarendon Press, Oxford University Press, New York (1997)
132. Mengestie, T.: A note on the differential operator on generalized Fock spaces. J. Math. Anal. Appl. **458**(2), 937–948 (2018)
133. Mengestie, T.: On the spectrum of Volterra-type integral operators on Fock–Sobolev spaces. Complex Anal. Oper. Theory **11**(6), 1451–1461 (2017)
134. Mengestie, T., Ueki, S.-I.: Integral, differential and multiplication operators on generalized Fock spaces. Complex Anal. Oper. Theory **13**(3), 935–958 (2019)
135. Miralles, A., Wolf, E.: Hypercyclic composition operators on H0v-spaces. Math. Nachr. **286**(1), 34–41 (2013)
136. Mirotin, A.R.: Hausdorff operators on some spaces of holomorphic functions on the unit disc. Complex Anal. Oper. Theory **15**, 85 (2021)
137. Montes-Rodríguez, A.: Weighted composition operators on weighted Banach spaces of analytic functions. J. Lond. Math. Soc. **61**, 872–884 (2000)
138. Nowak, M., Pavlovic, M.: On the Libera operator. J. Math. Anal. Appl. **370**(2), 588–599 (2010)

139. Pavlovic, M.: Definition and Properties of the Libera operators on mixed norm spaces. Sci. World J. **2014**, 590656 (2014)
140. Peláez, J.A., Rättyä, J.: Bergman projection induced by radial weight. Adv. Math. **391**, 107950 (2021)
141. Perfekt, K.-M.: Duality and distance formulas in spaces defined by means of oscillation. Ark. Mat. **51**, 345–361 (2013)
142. Perfekt, K.-M.: On M-ideals and $o - O$ type spaces. Math. Scand. **121**(1), 151–160 (2017)
143. Persson, A.-M.: On the spectrum of the Cesàro operator on spaces of analytic functions. J. Math. Anal Appl. **340**, 1180–1203 (2008)
144. Pommerenke, Ch.: Schlichte Funktionen un analytische Functionen von beschrnkter mittlerer Oszilation. Comment. Math. Helv. **52**, 591–602 (1977)
145. Ramos Fernández, J.C.: Bounded superposition operators between weighted Banach spaces of analytic functions. Appl. Math. Comput. **219**(10), 4942–4949 (2013)
146. Rudin, W.: Real and Complex Analysis. Mc Grawn Hill, New York (1974)
147. Schindl, G.: Solid hulls and cores of classes of weighted entire functions defined in terms of associated weight functions. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **114**(4), 176 (2020)
148. Seyoum, W., Mengestie, T.: Spectrums and uniform mean ergodicity of weighted composition operators on Fock spaces. Bull. Malays. Math. Sci. Soc. **45**(1), 455–481 (2022)
149. Seyoum, W., Mengestie, T., Bonet, J.: Mean ergodic composition operators on generalized Fock spaces. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **114**(1), 6 (2020)
150. Shapiro, J.H.: Composition Operators and Classical Function Theory. Springer, New York (1993)
151. Shields, A.L., Williams, D.L.: Bounded projections and the growth of harmonic conjugates in the disc. Mich. Math. J. **29**, 3–25 (1982)
152. Smith, W., Stolyarov, D.M., Volberg, A.: Uniform approximation of Bloch functions and the boundedness of the integration operator on H^∞. Adv. Math. **314**, 185–202 (2017)
153. Stević, S.: Boundedness and compactness of an integral operator on a weighted space on the polydisc. Indian J. Pure Appl. Math. **37**, 343–355 (2006)
154. Stylogiannis, G., Galanopoulos, P.: Hausdorff operators on Fock spaces. arXiv:2008.06684v2
155. Taskinen, J.: Compact composition operators on general weighted spaces. Houst. J. Math. **27**, 203–218 (2001)
156. Taskinen, J., Virtanen, J.A.: Boundedness of Toeplitz operators in Bergman-type spaces. In: Aron, R.M., Moslehian, M.S., Spitkovsky, I.M., Woerdeman, H.J. (eds.) Operator and Norm Inequalities and Related Topics. Trends in Mathematics, pp. 461–490. Birkhäuser, Basel (2022)
157. Tien, P.T.: Translation operators on weighted spaces of entire functions. Proc. Am. Math. Soc. **145**(2), 805–815 (2017)
158. Tien, P.T.: The iterates of composition operators on Banach spaces of holomorphic functions. J. Math. Anal. Appl. **487**(1), 123945 (2020)
159. Tien, P.T.: Composition operators on weighted Banach spaces of entire functions. Complex Var. Ellipt. Equ. **66**(9), 1517–1533 (2021)
160. Tien, P.T., Khoi, L.H.: Weighted composition operators between different Fock spaces. Potential Anal. **50**(2), 171–195 (2019)
161. Ueki, S.: Weighted composition operator on the Fock space. Proc. Am. Math. Soc. **35**, 1405–1410 (2007)
162. Ueki, S.: Weighted composition operators on some function spaces of entire functions. Bull. Belg. Math. Soc. Simon Stevin **17**, 343–353 (2010)
163. Vukotić, D.: Integrability, Growth of Conformal Maps, and Superposition Operators, Technical Report 10. Department of Mathematics, Aristotle University of Thessaloniki, Thessaloniki (2004)
164. Wolf, E.: A note on spectra of weighted composition operators on weighted Banach spaces of holomorphic functions. Bull. Malays. Math. Sci. Soc. (2) **31**(2), 145–152 (2008)
165. Zhu, K.: Operator Theory on Function Spaces, Math. Surveys and Monographs, vol. 138. Amer. Math. Soc., Providence (2007)
166. Zhu, K.: Analysis on Fock Spaces, Graduate Texts in Mathematics, vol. 263. Springer, New York (2012)