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Research Article

Analytic Solution for the Strongly Nonlinear Multi-Order Fractional Version of a BVP Occurring in Chemical Reactor Theory

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This study is devoted to constructing an approximate analytic solution of the fractional form of a strongly nonlinear boundary value problem with multi-fractional derivatives that comes in chemical reactor theory. We construct the solution algorithm based on the generalized differential transform technique in four simple steps. The fractional derivative is defined in the sense of Caputo. We also mathematically prove the convergence of the algorithm. The applicability and effectiveness of the given scheme are justified by simulating the equation for given parameter values presented in the system and compared with existing published results in the case of standard derivatives. In addition, residual error computation is used to check the algorithm's correctness. The results are presented in several tables and figures. The goal of this study is to justify the effects and importance of the proposed fractional derivative on the given nonlinear problem. The generalization of the adopted integer-order problem into a fractional-order sense which includes the memory in the system is the main novelty of this research.

1. Introduction

Chemical reactors are containers used in chemical engineering to contain chemical processes. Because of their numerous industrial uses, these reactors are crucial. Biological treatment, algae production, and gasoline production are some of the applications for tubular reactors. The mathematical model for an adiabatic tubular chemical reactor that performs an irreversible exothermic chemical reaction is examined in this work. The model may be simplified into the following nonlinear ordinary differential equation for steady-state solutions [1]:

$$\frac{d^2u}{dx^2} - \lambda\frac{du}{dx} + \lambda\mu(\beta - u)\exp(u) = 0, \quad (1)$$

where λ, μ, and β are the Péclet number, Damköhler number, and adiabatic temperature rise, respectively. The relative boundary conditions are given by

$$u'(0) = \lambda u(0), u'(1) = 0. \quad (2)$$

In references [2, 3], the authors investigated the existence of a solution to equations (1) and (2). In [1], the researchers established the existence of numerous solutions. To solve the problem under specific evaluations, certain numerical approaches were used. Green's function, for example, is used to turn the issue into a Hammerstein integral equation in [4]. After that, Adomian's decomposition approach was used to solve the resultant equation. The problem was solved using the Chebyshev finite difference approach in [5]. The authors in [6] used a solution aligned on embedding Green's function inside Krasnoselskii–Mann fixed point iteration method to solve the problem.

In this article, we incorporate fractional order into (1). The following noninteger-order differential equation describes the new equation:

$$\frac{d^q u}{dx^q} - \lambda \frac{d^\gamma u}{dx^\gamma} + \lambda\mu(\beta - u)\exp(u) = 0, \quad (3)$$

where d^q/dx^q and d^γ/dx^γ are the Caputo derivative operators along with fractional orders $q \in (1,2]$, $\gamma \in (0,1]$ subject to the boundary conditions. (1) can be considered as a particular form of (3) by fixing the orders $q = 2$ and $\gamma = 1$. This means the final solution of the fractional-order system must converge to the solution of the integer-order counterpart of the equation. Some recent results on boundary value problems in fractional-order sense can be seen in references [7, 8]. Mostly, the nature of complex dynamics cannot be better stated by integer-order differential equations. In the present case, the fractional-order model may strongly define the nature of the given system. In some circumstances, fractional models yield superior approximation results, according to Abbas et al. [9]. Iyiolaa et al. [10] have also demonstrated that for cancer tumours, the fractional model delivers a better approximation outcome than the integer-order one. Some recent studies related to the modeling in terms of fractional-order boundary value problems can be learned from references [11, 12]. Recently, a number of nonlinear fractional-order models have been proposed by the researchers to describe the dynamics of various real-life problems like AH1N1/09 influenza [13], childhood diseases [14], human liver dynamics [15], greenhouse gas effects on the population of aquatic animals [16], mosaic diseases in plants [17], maize streak virus [18], and so on.

Fractional calculus has more than three century history and has progressed steadily to the present day. Rieman and Liouville were the first ones who defined the fractional-order differentiation notion in the nineteenth century. Fractional differential equations (FDEs) have been shown to be a valuable tool for representing a wide range of scientific and engineering phenomena. Many FDEs that describe any phenomenon have lack of analytic solutions. As a result of the absence of analytic solutions, a significant variety of techniques for solving FDEs have been devised [19]. FDEs have attracted much more attention as a part of fractional calculus. It is worth noting that a general solution strategy for fractional differential equations is yet to be developed. The majority of problem-solving strategies in this field have been created for certain categories of challenges. For this reason, a single standard technique for solving problems related to fractional calculus has not been found. As a result, identifying compelling and beneficial solution strategies in combination with quick application techniques is valuable and worthy of further investigation [20] (see [21–27] for further information).

Under the best of our investigations, our study introduces the firstly produced numerical solution of (3). For this target, we are directed to find the approximate solution of (3) via generalized differential transform method (GDTM) [28–31].

This paper is organized as follows. In Section 2, a review of the GDTM [28] is given and some important preliminaries are given. In Section 3, the solution procedure is presented. Convergence theorem of the present solution is proved in Section 4. The solution approximations for equations (2) AND (3) are established in Section 5. Some conclusions are mentioned at the end.

2. The Generalized Differential Transform Method

For reader's facility, this part covers a review of the generalized differential transform [28] as well as some fundamental fractional calculus ideas and terminology.

The generalized differential transform of the kth derivative of the analytic function $f(x)$ is given by

$$F_\alpha(k) = \frac{1}{\Gamma(\alpha k + 1)}\left[\left(D_{x_0}^\alpha\right)^k f(x)\right]_{x=x_0}, \quad (4)$$

where $0 < \alpha \le 1$, $(D_{x_0}^\alpha)^k = D_{x_0}^\alpha . D_{x_0}^\alpha \ldots D_{x_0}^\alpha$, k-times, and $D_{x_0}^\alpha$ denotes the Caputo fractional differential operator of order α given by

$$D_a^\alpha f(x) = J_a^{m-\alpha} D^m f(x). \quad (5)$$

Here D^m is the integer-order differential operator of order m and J^m is the Riemann–Liouville integral operator of order μ with $\mu > 0$, which is given by

$$J_a^\mu f(x) = \frac{1}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1} f(t)\, dt, x > 0. \quad (6)$$

The generalized differential inverse transform of $F_\alpha(k)$ is defined as

$$f(x) = \sum_{k=0}^{\infty} F_\alpha(k)(x - x_0)^{\alpha k}, \quad (7)$$

which practically can be approximated by the following finite series:

$$f(x) \cong \sum_{k=0}^{M} F_\alpha(k)(x - x_0)^{\alpha k}. \quad (8)$$

Because the initial conditions are represented as integer-order derivatives, the GDTM defines the transformation of the initial conditions as follows:

$$F_\alpha(k) = \begin{cases} \dfrac{1}{(\alpha k)!} \dfrac{d^{(\alpha k)} f(x)}{dx^{(\alpha k)}}\Big|_{x=x_0}, & \text{if} \alpha k \in \mathbb{Z}^+, \\ 0, & \text{if} \alpha k \notin \mathbb{Z}, \end{cases} \quad (9)$$

$$\text{for}\, k = 0, 1, \ldots, \frac{q}{\alpha} - 1,$$

where q is the order of considered FDEs.

Putting (4) in (7) yields

$$f(x) = \sum_{k=0}^{\infty} F_\alpha(k)(x - x_0)^{\alpha k}. \quad (10)$$

The idea of a generalized differential transform is obtained from generalized Taylor's formula [28]. It is worth noting that the extended differential transform technique simplifies to the conventional differential transform method when $\alpha = 1$ [32]. Table 1 lists some of the essential features of GDTM derived from equations (4) and (5).

3. Solution Procedure

For solving (2) and (3), we proceed with the following algorithm steps.

(1) Choose proper value of α that satisfies $q/\alpha, \alpha/\alpha \in \mathbb{Z}^+$.

(2) Using the generalized differential transform on both sides of equation (3) and the characteristics mentioned in Table 1, we obtain the following recurrence relation:

$$U_\alpha\left(k + \frac{q}{\alpha}\right) = \beta\left[\lambda \frac{\Gamma(\alpha k + \gamma + 1)}{\Gamma(\alpha k + 1)} U_\alpha\left(k + \frac{\gamma}{\alpha}\right) - \lambda\mu \sum_{m=0}^{k} W_\alpha(m)\left[\beta\delta(k-m) - U_\alpha(k-m)\right]\right], \quad (11)$$

where $k = 0, 1, 2, \ldots,$ $\beta = \Gamma(\alpha k + 1)/\Gamma(\alpha k + q + 1)$, and $W_\alpha(k)$ is the generalized differential transform of $e^{u(x)}$ which is given by

$$W_\alpha(j) = \frac{1}{j}\sum_{m=0}^{j-1}(m+1)U_\alpha(m+1)W_\alpha(j-m-1), \quad j = \frac{q}{\alpha}, \frac{q}{\alpha}+1, \ldots \quad (12)$$

(3) The boundary conditions given in (2) for $x = 0$ are transformed by employing (9) as follows:

$$\begin{aligned} &U_\alpha(0) = A, \\ &U_\alpha\left(\frac{1}{\alpha}\right) = \lambda A, \\ &U_\alpha(i) = 0, \\ &\text{for} \frac{i}{\alpha} \notin \mathbb{Z}^+, \\ &\quad i = 1, \ldots, \frac{q}{\alpha} - 1, \end{aligned} \quad (13)$$

where $A = u(0)$ is the initial condition. We can define

$$\begin{aligned} &W_\alpha(0) = e^A, \\ &W_\alpha\left(\frac{1}{\alpha}\right) = \lambda A e^A, \\ &W_\alpha(i) = 0, \\ &\quad i = 1, 2\ldots, \frac{1}{\alpha} - 1, \frac{1}{\alpha} - 1, \ldots, \frac{q}{\alpha} - 1. \end{aligned} \quad (14)$$

The conditions in (2) for $x = 1$ are transformed by employing (7) as follows:

$$\sum_{k=0}^{N}(\alpha k)U_\alpha(k) = 0. \quad (15)$$

(4) Equations (11) and (13) are utilized to find $U_\alpha(k)$ up to any N-terms. Then, by using (15), the value of A is evaluated. Also, the Nth order of approximation is

$$u_N(x) = \sum_{k=0}^{N} U_\alpha(k)x^{\alpha k}. \quad (16)$$

For simplicity, we can generate the solution for $q = 1.9$, $\gamma = 0.9$ by assuming $\alpha = 0.1$. Then, applying (11) for $k = 0, 1, 2, \ldots, 15$, we have

$$\begin{aligned} u_{34} &= \sum_{i=0}^{34} U_{0.1}(i)x^{i/10} \\ &= A + \frac{1}{6}Ax\lambda(6 + x\lambda(3 + x\lambda)) + \frac{e^A x^{19/10}\lambda(29(A-\beta) + 10x(A(2+A) - (1+A)\beta)\lambda)\mu}{29\Gamma(29/10)}. \end{aligned} \quad (17)$$

Table 1: Basic properties of GDTM [28].

Original function	Transformed function
$f(x) = g(x) \pm h(x)$	$F_\alpha(k) = G_\alpha(k) \pm H_\alpha(k)$
$f(x) = ag(x)$	$F_\alpha(k) = aG_\alpha(k)$
$f(x) = g(x)h(x)$	$F_\alpha(k) = \sum_{l=0}^{k} G_\alpha(l)H_\alpha(k-l)$
$f(x) = D_{x_0}^{\alpha} g(x),\ 0<\alpha\leq 1$	$F_\alpha(k) = \Gamma(\alpha(k+1)+1)/\Gamma(\alpha k+1)G_\alpha(k+1)$
$f(x) = (x-x_0)^\gamma$	$F_\alpha(k) = \delta(k-\gamma/\alpha),\ \delta(k) = \begin{cases} 1, \text{if} k=0 \\ 0, \text{if} k\neq 0 \end{cases}$
$f(x) = D_{x_0}^{\beta} g(x), m-1<\beta\leq m, m\in\mathbb{Z}^+$	$F_\alpha(k) = \Gamma(\alpha k+\beta+1)/\Gamma(\alpha k+1)G_\alpha(k+\beta/\alpha)$
$f(x) = \exp(g(x))$	$F_\alpha(k) = \sum_{i=0}^{k-1} i+1/kG_\alpha(i+1)F_\alpha(k-i-1)$, where $F_\alpha(0) = \exp(G_\alpha(0))$

The condition in (15) gives

$$A\lambda + A\lambda^2 + \frac{A\lambda^3}{2} + \frac{19\left(Ae^A\lambda\mu - e^A\beta\lambda\mu\right)}{10\Gamma(29/10)} + \frac{29\left(2Ae^A\lambda^2\mu + A^2e^A\lambda^2\mu - e^A\beta\lambda^2\mu - Ae^A\beta\lambda^2\mu\right)}{10\Gamma(39/10)} = 0. \quad (18)$$

By fixing the values of λ, μ, and β, it is easy to solve the equation via the Newton–Raphson method.

4. Convergence Analysis

Lemma 1 (see [33]). *The standard power series* $\sum_{i=0}^{\infty} U_i r^i$, $r\in\mathbb{R}$, *has a radius of convergence R if and only if the fractional one* $\sum_{i=0}^{\infty} U_\alpha(i) r^{\alpha i}, r\geq 0$, *has a radius of* $R^{1/\mu}$.

Theorem 1. *The fractional power series:*

$$\sum_{i=0}^{\infty} U_\alpha(i) r^{\alpha i}, \quad (19)$$

where the coefficients are mentioned in (11), has a positive radius of convergence.

Proof. From equation (8), we have

$$U_\alpha(k) \leq \theta_1\lambda\mu \sum_{m=0}^{n-q/\alpha} W(m)\left(U_\alpha\left(n-m-\frac{q}{\alpha}\right) - \beta\delta\left(n-m-\frac{q}{\alpha}\right)\right) + \theta_2\lambda\frac{\Gamma(n\alpha+\gamma+1)}{\Gamma(n\alpha+1)}U_\alpha\left(\frac{\gamma}{\alpha}+n-\frac{q}{\alpha}\right), \quad (20)$$

where

$$\theta_1 = \left|\frac{\Gamma((n-q/\alpha)\alpha+1)}{\Gamma(q+(n-q/\alpha)\alpha+1)}\right|, \quad (21)$$

$$\theta_2 = \left|\frac{\Gamma((n-q/\alpha)\alpha+\gamma+1)}{\Gamma(q+(n-q/\alpha)\alpha+1)}\frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha+\gamma+1)}\right|. \quad (22)$$

Now, consider the series

$$z(r) = \sum_{n=0}^{\infty} a_n r^n, \quad (23)$$

for which $a_0 = |U_\alpha(0)|$, $a_1 = |U_\alpha(1/\alpha)|$, and

$$a_n = \left|\theta_1\lambda\mu \sum_{m=0}^{n-q/\alpha} W(m)\left(U_\alpha\left(n-m-\frac{q}{\alpha}\right) - \beta\delta\left(n-m-\frac{q}{\alpha}\right)\right) + \theta_2\lambda\frac{\Gamma(n\alpha+\gamma+1)}{\Gamma(n\alpha+1)}U_\alpha\left(\frac{\gamma}{\alpha}+n-\frac{q}{\alpha}\right)\right|, \quad (24)$$

Table 2: Numerical solutions for $\mu = 0.7, \lambda = 5, \beta = 0.8$.

x	GDTM	Method in [4]
0.0	0.10164623151473825557	0.10164623106412275851
0.1	0.15160800347538405664	0.15160800244418513970
0.2	0.19969082693869014133	0.19969082598774598630
0.3	0.24564764071345721042	0.24564763919882120986
0.4	0.28919552846307056138	0.28919552652683175159
0.5	0.32997387678814665835	0.32997387500728383184
0.6	0.36746668503341112233	0.36746668342588013969
0.7	0.40086147483935807845	0.40086147462729197629
0.8	0.42879563627198713505	0.42879563852127257524
0.9	0.44890276956983110601	0.44890277089707031201
1.0	0.45700543796118281437	0.45700543763742257810

Table 3: GDTM solution and numerical solution with its absolute and residual errors for $\lambda = 5, \mu = 0.7, \beta = 0.8$.

x	GDTM	Numerical solution	Absolute error	Residual error
0.0	0.1016462307	0.1016462315	$8.085193209 \times 10^{-10}$	$1.665334537 \times 10^{-16}$
0.1	0.1516080019	0.1516080035	$1.624451518 \times 10^{-9}$	$6.106226635 \times 10^{-16}$
0.2	0.199690825	0.1996908269	$1.943355393 \times 10^{-9}$	$4.996003611 \times 10^{-16}$
0.3	0.2456476375	0.2456476407	$3.187346082 \times 10^{-9}$	$5.828670879 \times 10^{-16}$
0.4	0.2891955237	0.2891955285	$4.774220019 \times 10^{-9}$	$1.276756478 \times 10^{-15}$
0.5	0.3299738702	0.3299738768	$6.62490951 \times 10^{-9}$	$8.398837181 \times 10^{-14}$
0.6	0.3674666751	0.367466685	$9.92236443 \times 10^{-9}$	$1.042421705 \times 10^{-11}$
0.7	0.4008614603	0.4008614748	$1.456031457 \times 10^{-8}$	$6.394951235 \times 10^{-10}$
0.8	0.4287956137	0.4287956363	$2.26142593 \times 10^{-8}$	$2.315349923 \times 10^{-8}$
0.9	0.4489027282	0.4489027696	$4.136782861 \times 10^{-8}$	$5.61353519 \times 10^{-7}$
1.0	0.4570053763	0.457005438	$6.16973152 \times 10^{-8}$	$9.929563344 \times 10^{-6}$

for $n = q/\alpha, q/\alpha + 1, \ldots$ and $a_n = 0$ for $1 < n < q/\alpha$.

Then, define

$$\begin{aligned}\Psi = z(r) &= a_0 + a_1 r + \left(\sum_{n=0}^{\infty} a_{n+2} r^n\right) r^2 \\ &= a_0 + a_1 r + r^2 \sum_{n=0}^{\infty} \left\{ \theta_1 \lambda \mu \sum_{m=0}^{n-q/\alpha} W(m)\left(Y\left(n - m - \frac{q}{\alpha}\right) - \beta\delta\left(n - m - \frac{q}{\alpha}\right)\right) + \theta_2 \lambda \frac{\Gamma(n\alpha + \gamma + 1)}{\Gamma(n\alpha + 1)} Y\left(\frac{\gamma}{\alpha} + n - \frac{q}{\alpha}\right)\right\} r^n.\end{aligned} \tag{25}$$

Now, we have the function of two variables:

$$\Theta(r, \Psi) = \Psi - a_0 - a_1 r - r^2\left(\theta_1 \lambda \mu e^{\Psi}(\Psi - \beta) r^q + \lambda \theta_2 (D^{\gamma}\Psi) r^q\right), \tag{26}$$

which is analytic in the plane (r, Ψ) with the characteristics $\Theta(0, a_0) = 0$ and $\Theta_{\Psi}(0, a_0) = 1 \neq 0$. Since $z(r)$ is an analytic function in a neighborhood of the point $(0, a_0)$ of the (r, Ψ)-plane with a positive radius of convergence, then by implicit function theorem, the series in (10) is convergent by Lemma 1. □

5. Numerical Experiments

This section derives the numerical experiments of the given procedure of Section 3 for solving equations (2) and (3). Because the exact solution to the given problem is not known, we instead find the absolute residual error function, which justifies how accurately the numerical solution agrees to the solution of main problems (2) and (3). So, the absolute residual error function is

$$\left|\mathrm{ER}_N(x)\right| = \left|\frac{d^q u_N(x)}{dx^q} - \lambda \frac{d^{\gamma} u_N(x)}{dx^{\gamma}} + \lambda\mu(\beta - u_N(x))\exp(u_N(x))\right|, 0 \leq x \leq 1. \tag{27}$$

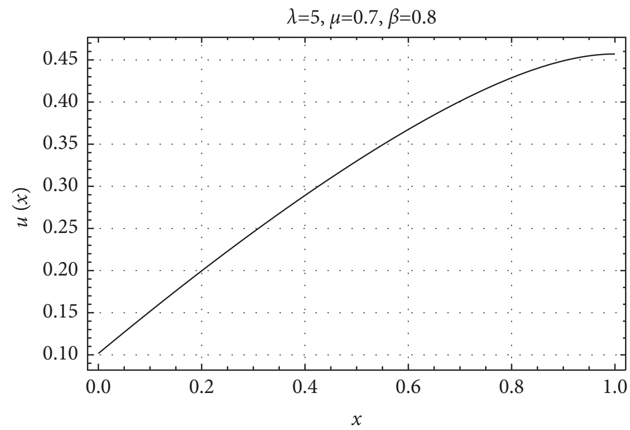

Figure 1: Graph of numerical outputs for $\lambda = 5, \mu = 0.7$, and $\beta = 0.8$.

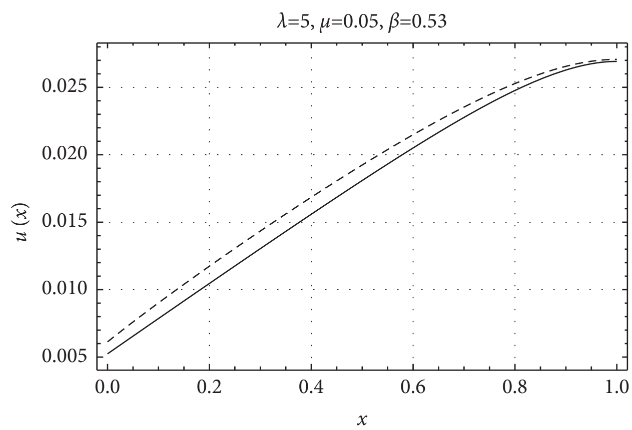

Figure 2: GDTM solution: $q = 1.9$ and $\gamma = 1$ (line); $q = 1.9$ and $\gamma = 0.9$ (dashed).

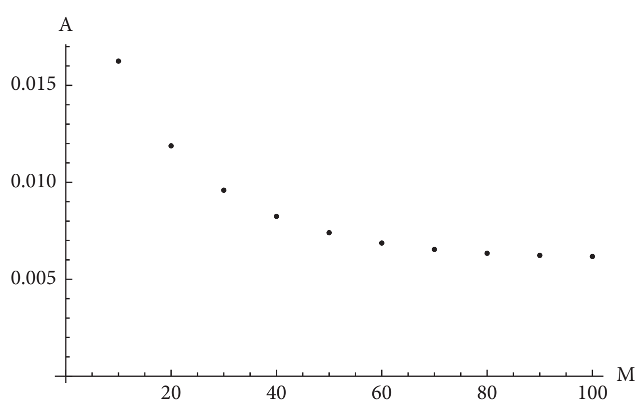

Figure 3: Variation of A with N for the case $\lambda = 5, \mu = 0.05, \beta = 0.53$.

Firstly, we start to generate the results for standard fractional derivative $q = 2, \gamma = 1$. By fixing $\alpha = 1, N = 21$, Table 2 displays our numerical results and matches them with those of the mentioned outputs in [4] for the same iteration number. Numerical solution using default Mathematica package and the GDTM solution and its absolute error and residual error are given in Table 3. Moreover, Figure 1 shows the approximate solutions for $\mu = 0.7, \lambda = 5$, and $\beta = 0.8$. Figure 1 is in good agreement with Figure 2 given in [5].

Now, we explore the impact of the fractional derivative on the solution of the model. In Figure 3, we show the convergence of obtained missing condition A with increasing N when $q = 1.9$ and $\gamma = 0.9$. It is clear that the value of A starts to be fixed when $N > 80$.

Tables 4 and 5 introduce the approximate solutions and the residual errors of problems (2) and (3) for different values of q, γ, λ, μ, and β. The residual error indicator demonstrates that the results are accurate for at least 6×10^{-5}.

Table 4: Numerical outputs for $\lambda = 5, \mu = 0.05, \beta = 0.53$.

x	$q = 1.9, \gamma = 1.0$		$q = 1.9, \gamma = 0.9$	
	GDTM	Residual error	GDTM	Residual error
0.0	0.00524258	5.55112×10^{-17}	0.00612891	1.69136×10^{-17}
0.1	0.00785749	2.77556×10^{-17}	0.00905165	9.54098×10^{-18}
0.2	0.0104581	1.5786×10^{-16}	0.0119341	4.55365×10^{-17}
0.3	0.013039	2.43347×10^{-14}	0.0148969	1.82536×10^{-15}
0.4	0.0155894	4.46358×10^{-12}	0.0179996	2.99132×10^{-13}
0.5	0.0180901	2.4929×10^{-10}	0.0212823	1.55918×10^{-11}
0.6	0.0205054	6.57931×10^{-9}	0.0247751	3.86786×10^{-10}
0.7	0.0227707	1.03475×10^{-7}	0.0285021	5.73491×10^{-9}
0.8	0.024768	1.11374×10^{-6}	0.0324832	5.82065×10^{-8}
0.9	0.0262822	8.96982×10^{-6}	0.0367354	4.41167×10^{-7}
1.0	0.0269209	5.7448×10^{-5}	0.0412736	2.64861×10^{-6}

Table 5: Numerical solutions for $\lambda = 0.05, \mu = 0.5, \beta = 0.6$.

x	$q = 1.9, \gamma = 0.9$		$q = 1.9, \gamma = 1.0$	
	GDTM	Residual error	GDTM	Residual error
0.0	0.233482	3.46945×10^{-18}	0.233528	0.
0.1	0.234572	3.46945×10^{-18}	0.23462	3.46945×10^{-18}
0.2	0.23553	3.46945×10^{-18}	0.23558	0.
0.3	0.236365	5.20417×10^{-18}	0.236418	1.73472×10^{-18}
0.4	0.237083	5.20417×10^{-18}	0.237138	1.73472×10^{-18}
0.5	0.237686	1.73472×10^{-18}	0.237744	0.
0.6	0.238176	0.	0.238236	0.
0.7	0.238556	5.20417×10^{-18}	0.238618	0.
0.8	0.238826	3.46945×10^{-18}	0.238889	1.73472×10^{-18}
0.9	0.238987	1.73472×10^{-18}	0.239051	1.73472×10^{-18}
1.0	0.239041	5.20417×10^{-18}	0.239105	0.

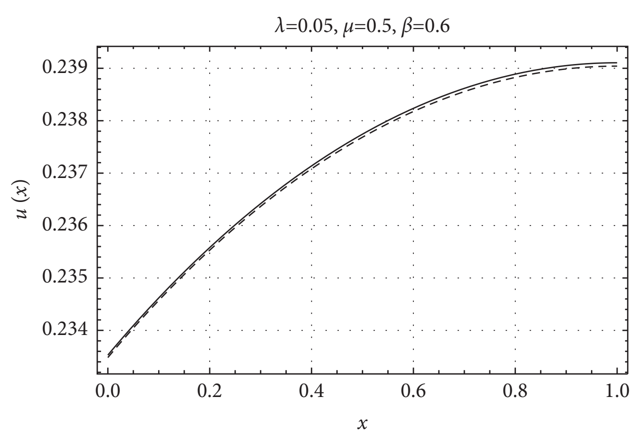

Figure 4: GDTM solution: $q = 1.9$ and $\gamma = 1$ (line); $q = 1.9$ and $\gamma = 0.9$ (dashed).

To study the solution behavior, we plot the present solution for the case of $\lambda = 5, \mu = 0.05, \beta = 0.53$ when $q = 1.9, \gamma = 0.9$ and $q = 1.9, \gamma = 1$ in Figure 2. The solution for the case of $\lambda = 0.05, \mu = 0.5, \beta = 0.6$ when $q = 1.9, \gamma = 0.9$ and $q = 1.9, \gamma = 1$ is presented in Figure 4. It is clear that the fractional derivatives can change the solution behavior.

6. Conclusions

Strongly nonlinear boundary value problem with multi-order fractional derivative that occurred in the chemical reaction theory has been successfully solved via the new algorithm based on GDTM. The constructed solution has been given in terms of convergent infinite series as seen in the provided theorem. The method was easy to apply, and the results have enough good accuracy as shown in the experimental results. The obtained solution was directly generated without any linearization or discretization of the domain. The given method is very powerful and can be easily applied for several kinds of fractional nonlinear boundary value problems in future.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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