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Elements of the parabola in the three-dimensional space and applications in teaching through a project based on reflection of light

Elementos de la parábola en el espacio tridimensional y aplicaciones en la enseñanza a través de un proyecto sobre la reflexión de la luz

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Abstract

This work, structured in two main parts, is devoted to the parabola topic. In the first, the elements of a parabola are reviewed by linking the definition of intersection of a plane and a cone with a locus of the plane. The necessary transformations for the calculation of the elements as a curve in space are pointed out as well as historical notes and properties of light are included. In the second part, the applications in teaching at a high school level or university courses are explained. A project based on the reflection of light is proposed that seeks to connect various subjects in line with the new educational paradigm of development of key competencies, joining different fields of knowledge. For university courses, applications of linear algebra are obtained in the establishment of relationships between analytical geometry in the rotation and translation of planes, and the dihedral system that is taught in technical drawing subjects. As a conclusion, it is obtained that the teaching of the parabola from different approaches allows a complete learning of diverse fields of knowledge, even of different topics of mathematics, an essential factor in the development of thought at any educational level.

Este trabajo, estructurado en dos partes principales, está dedicado al tópico de la parábola. Primeramente, se revisan los elementos de una parábola relacionando la definición de intersección de un plano y un cono con un lugar geométrico del plano. Se señalan las transformaciones necesarias para el cálculo de los elementos como curva en el espacio, se incluyen reseñas históricas y propiedades de la luz. En la segunda parte se explican las aplicaciones en la docencia a nivel bachillerato o cursos universitarios. Se propone un proyecto basado en la reflexión de la luz que busca conectar diversas materias en consonancia con el nuevo paradigma educativo de desarrollo de competencias clave, uniendo diferentes campos del conocimiento. Para los cursos universitarios se obtienen aplicaciones del álgebra lineal en el establecimiento de relaciones entre la geometría analítica en la rotación y traslación de planos, y el sistema diedro que se enseña en las asignaturas de dibujo técnico. Como conclusión se obtiene que la enseñanza de la parábola desde diferentes enfoques permite un aprendizaje completo de diversos campos del conocimiento, incluso de diferentes temas de las matemáticas, factor fundamental en el desarrollo del pensamiento en cualquier nivel educativo.

Keywords: Conic, reflection of light, project-based learning, parabola, locus
Palabras clave: Cónica, reflexión de la luz, aprendizaje basado en proyectos, parábola, lugar geométrico

1. Introduction

Conics involve mathematical objects such as conic sections, loci, functions, equations, systems, lines, planes, surfaces of revolution, ruled surfaces, matrices, etc., highlighting elements of linear algebra and projective geometry for their classification. There are works on didactic proposals around these concepts (see, for example, Teófilo de Sousa, R., & Vieira Alves, F. R., 2022 and Florio, E., 2022) where the geometric point of view is reinforced with the support of GeoGebra. Another approach is the proposal for engineers by Teófilo de Sousa, R., & Vieira Alves, F. R. (2022), where the parabola, a particular case of conic, is studied as a quadratic function to create real geometric constructions like bridges in GeoGebra 3D. Regarding the classification, Shonoda, E. N. (2018), made an extension for curves in the Minkowski spacetime of the plane.

Conics have many applications, even in emerging fields such as datasets (Cevíkalp, H., Címen, E., & Ozturk, G., 2021) and also, they are related to other disciplines such as physics and technical drawing. In the case of physics, they appear in the application of Newton's Law of Universal Gravitation from which it is obtained as a consequence that the orbits of celestial bodies can be ellipses, hyperbolas or parabolas as it is well known. The trajectory followed by a mobile with uniformly accelerated motion is a parabola, and of course in the phenomenon treated in this present paper about reflection of light. In technical drawing it is possible to trace the conic curve in the plane from Dandelin's theorem and strategies of the dihedral system.

For these reasons, the inclusion of conics in current curricula of mathematics is fully justified both in high school and in subjects related to linear algebra and projective geometry in mathematics, physics, and engineering degrees, as well as, in the subject of technical drawing and also in some way as part of the Physics subject. Proposing learning situations that connect various disciplines could lead students to an optimal development of mathematical competence, while working on other key competencies in coherence with the European Parliament and Council Recommendation $C189-1$ (Consejo, U., 2018).

The point of view proposed here emphasizes the objective in analytical geometry in three dimensions. One of the main problems that arise is how to analytically calculate the elements of a parabola (directrix, vertex and focus) that is determined by the intersection of a cone and a plane in space. It is also intended that the operation of technical drawing techniques is based on mathematical arguments.

Specifically, this article focuses on the parabola. In section 1, the concept of parabola is briefly reviewed in its initial history, going on to the parabola as the locus in the plane, then as a conic section, and finally Fermat's principle and the laws of reflection of light that will be used for its completeness in order to justify the project that is proposed in the third section in which the project of the parabola is shown, describing the necessary elements of the construction of a structure for the realization of the experiment and the mathematical developments that could be asked of the students in the different educational stages and the results of the implementation with 34 students. Finally, the conclusions reached are explained. The standard notation of linear algebra and geometry is used (see Burgos, 2006 and Anzola et al., 1982).

2. Review of parabola concept

In order to establish the fundamentals of the questions that underlie the conics, we begin with the historical reviews that motivated the initial study, associating the relationship with later problems.

2.1. History of the parabola

Apollonius of Perga (3rd century BC and 2nd century BC) studied conics, demonstrating these curves are obtained by sections through a plane to any cone, not necessarily with an angle at the vertex of $\frac{\pi}{2}$ *rad*. It is not clear that Apollonius knew the importance of the directrix of the parabola (Boyer, 1987). Apollonius also gave the definition of a conical surface as that which is generated when a straight line remains fixed at one point while, through another point, it moves along a fixed circumference. The point that remains fixed is called the vertex and the line that passes through the vertex and the center of the circle is called the axis of the cone. Menecmus (4th century BC) worked on parabolic curves as a tool to solve the problem of doubling the cube (Boyer, 1987). In this problem, given a real number $p > 0$, there is a hexahedron of volume p^3, the question is what the length of the edge will be, if we want to build a hexahedron with twice the volume, that is, $2p^3$. Clearly, the length is $\sqrt[3]{2}p$, but it is intended that this value can be constructed by straightedge and compass with a finite number of steps. As a consequence of Galois theory, it is known that the cube cannot be duplicated using straightedge and compass constructions (Theorem 5.3, Stewartz, I., 1989). However, Menecmus studied this problem considering the proportions:

$$\frac{x}{2p} = \frac{y}{x} = \frac{2p}{y}.$$

The solution (x_0, y_0) is the intersection point of the curves $y^2 = 2px, x^2 = 2py$. Even more, $x_0 = \sqrt[3]{2}p$. Therefore, finding $\sqrt[3]{2}p$ is possible if both curves can be traced. Arquimedes de Syracuse (287 BC-212 BC) took a keen interest in the parabola and dedicated the book entitled *The quadrature of the parabola* to it (Stewartz, I., 2018).

2.2. The parabola as a locus

The parabola in the plane is defined as the set of points P such that they are equidistant from a fixed point F called the focus and a line r called the directrix. In this way, taking a suitable reference system where the focus is made to coincide with the point $(p/2, 0)$, $p > 0$ and the directrix with the line of equation $x = -p/2$, we obtain the reduced equation $y^2 = 2px$. It is important to highlight the property that for any point $P = (x, y)$ of the parabola the square of side y has an area equal to the rectangle with basis x and height $2p$ (*latus rectum*). In figure 1, it can be seen the rectangle with basis $p/2$ and height $2p$ has an area equal to the area of the square (p^2). If we take the point $(0, p/2)$ as the focus and the line $y = -p/2$ as the directrix, then we have the equation $x^2 = 2py$. In the general case for the directrix r of the equation $Ax + By + C = 0$ and the focus $F = (a, b)$ with $Aa + Bb + C \neq 0$, the axis of symmetry s is the line perpendicular to the directrix that passes through the focus that has equation $-Bx + Ay + Ba - Ab = 0$. The intersection point of r and s, say Q, is obtained by solving the system:

$$\begin{cases} Ax + By + C = 0 \\ -Bx + Ay + Ba - Ab = 0 \end{cases}$$

Thus,

$$Q = \frac{1}{A^2 + B^2} \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} -C \\ -Ba + Ab \end{pmatrix},$$

$$Q = \left(\frac{-AC + B^2 a - ABb}{A^2 + B^2}, \frac{-ABa + A^2 b - BC}{A^2 + B^2} \right).$$

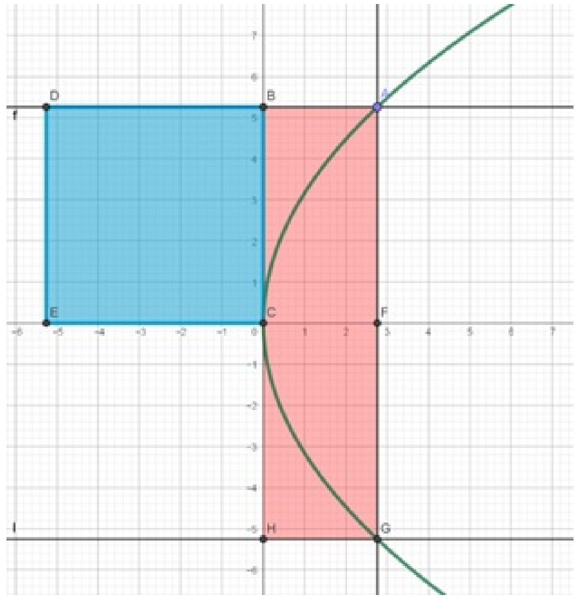

Figure 1: Latus rectum. Source: Created by the authors with GeoGebra.

The vertex is the midpoint of the segment $\overline{QF}$, i.e.,

$$V=(v_1,v_2)=\frac{1}{2}\left(a+\frac{-AC+B^2a-ABb}{A^2+B^2},b+\frac{-ABa+A^2b-BC}{A^2+B^2}\right).$$

Consider the orthonormal reference system $\mathcal{R}=\{V;u_1,u_2\}$, being

$$u_1=\frac{1}{\sqrt{A^2+B^2}}\left(A,B\right),$$

$$u_2=\frac{1}{\sqrt{A^2+B^2}}\left(-B,A\right),$$

corresponding to vector directors of lines s and r respectively. The vector $\overrightarrow{QF}$ is parallel to u_1. But, it could be opposite vectors. In fact, $\overrightarrow{QF}$ and u_1 have the same direction if and only if $Aa+Bb+C>0$. Now, let P be a point of the parabola with coordinates x',y' respect of system $\mathcal{R}$. If it is assumed that $Aa+Bb+C>0$, then $y'^2=2px'$ where p is the distance from the focus to the directrix, i.e., $p=\frac{|Aa+Bb+C|}{\sqrt{A^2+B^2}}$. The condition $Aa+Bb+C>0$ can be assumed without loss of generality, since $-Ax-By-C=0$ defines the same line r. By application of changing coordinates matrix from the basis u_1,u_2 to the canonical basis,

$$\begin{pmatrix}x'\\y'\end{pmatrix}=\frac{1}{\sqrt{A^2+B^2}}\begin{pmatrix}A & B\\-B & A\end{pmatrix}\begin{pmatrix}x-v_1\\y-v_2\end{pmatrix}.$$

And the formula stays,

$$\left(-B\left(x-v_1\right)+A\left(y-v_2\right)\right)^2=2\left|Aa+Bb+C\right|\left(A\left(x-v_1\right)+B\left(x-v_2\right)\right).$$

As it can be seen, a translation and a rotation intervene in the change of coordinates.

2.3. Parabola in the conic classification with the projective plane

Consider an equation of the type $f\left(x,y\right)=0$, where $f\left(x,y\right)$ is a polynomial of the second degree and the question consists of determining if it is a parabola or not, finding the equation of the directrix and the focus. This problem has been studied from the point of view of projective geometry and multiple examples can be found in Anzola et al. (1982).

If $f\,(x,y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2b_1x + 2b_2y + c = 0$, Then, it is a parabola if and only if $det\begin{pmatrix} c & b_1 & b_2 \\ b_1 & a_{11} & a_{12} \\ b_2 & a_{12} & a_{22}\end{pmatrix} \neq 0$, $\det\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22}\end{pmatrix} = 0$ (Burgos, 2006). In the case of the parabola, there is no translation that cancels the terms of degree 1. For this reason, it is said that it does not have a centre. The axis of the parabola is given by the polar of $(0, a_{11}, a_{12})$ or $(0, a_{21}, a_{22})$. The vertex is the intersection of the conic with the conic axis, and, also, $p = \sqrt{\frac{-det(A)}{(a_{11}+a_{22})^3}}$. The directrix is the focus polar. The focus is calculated by solving the system: $\begin{cases} -2A_{01}x + 2A_{02}y + A_{11} - A_{22} = 0 \\ -A_{02}y - A_{01}y + A_{12} = 0 \end{cases}$

where $\begin{pmatrix} A_{00} & A_{01} & A_{02} \\ A_{01} & A_{11} & A_{12} \\ A_{20} & A_{12} & A_{22}\end{pmatrix}$ is the adjoint matrix of $\begin{pmatrix} c & b_1 & b_2 \\ b_1 & a_{11} & a_{12} \\ b_2 & a_{12} & a_{22}\end{pmatrix}$.

Another way to deal with this problem is to determine the rotation required to make the axis of symmetry of the parabola parallel to the x-axis. If one starts from a parabola $y'^2 = 2px'$, $p > 0$ and applies a rotation of θ and a translation of vector (v_1, v_2), it is followed,

$$x - v_1 = x'\cos\theta - y'\sin\theta, y - v_2 = x'\sin\theta + y'\cos\theta \Leftrightarrow$$

$$x' = (x - v_1)\cos\theta + (y - v_2)\sin\theta, y' = (x - v_1)\sin\theta - (y - v_2)\cos\theta.$$

From here, the equation follows,

$$((x - v_1)\sin\theta - (y - v_2)\,cos\theta)^2 = 2p((x - v_1)\,cos\theta + (y - v_2)\,sin\theta).$$

Comparing with the coefficients of the classical classification approach, we obtain: $a_{11} = \lambda sin^2\theta$, $a_{22} = \lambda cos^2\theta$, $a_{12} = -\lambda\cos\theta\sin\theta$, for certain $\lambda \in \mathbb{R}\setminus\{0\}$. Hence, $cos^2\theta = \frac{a_{22}}{a_{11}+a_{22}}$, $\sin^2\theta = \frac{a_{11}}{a_{11}+a_{22}}$, $\tan^2\theta = \frac{a_{11}}{a_{22}}$. Moreover, the slope of the axis of symmetry of the parabola is obtained from the polar $(0, a_{11}, a_{12})$ is $\frac{-a_{11}}{a_{12}}$, therefore, the rotation of angle θ satisfies $\tan\theta = \frac{-a_{11}}{a_{12}}$. If one starts with the parabola $x'^2 = 2py'$, $p > 0$ and applies a rotation and a translation, it is gotten,

$$x - v_1 = x'\cos\theta - y'\sin\theta, y - v_2 = x'\sin\theta + y'\cos\theta \Leftrightarrow$$

$$x' = (x - v_1)\cos\theta + (y - v_2)\sin\theta, y' = (x - v_1)\sin\theta - (y - v_2)\cos\theta.$$

From here, the same way as before,

$$((x - v_1)\,cos\theta + (y - v_2)\,sin\theta)^2 = 2p((x - v_1)\,cos\theta - (y - v_2)\,sin\theta).$$

Comparing with the coefficients of the classical classification approach, we obtain: $a_{11} = \lambda cos^2\theta$, $a_{22} = \lambda sin^2\theta$, $a_{12} = \lambda cos\theta sin\theta$, for certain $\lambda \in \mathbb{R}\setminus\{0\}$. Hence, $cos^2\theta = \frac{a_{11}}{a_{11}+a_{22}}$, $\sin^2\theta = \frac{a_{22}}{a_{11}+a_{22}}$, $\tan^2\theta = \frac{a_{22}}{a_{11}}$ In this case, the axis of symmetry of the parabola is reached from the polar of $(0, a_{12}, a_{22})$, then the slope is $\frac{-a_{12}}{a_{22}} = \frac{-a_{11}}{a_{12}}$, $\tan\left(\frac{\pi}{2} + \theta\right) = \frac{-a_{12}}{a_{22}}$. The results are consistent, since $det\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22}\end{pmatrix} = 0$. With all these calculations we arrive at the statement of the following proposition.

Proposición 7.1 *Consider the parabola given by $f(x,y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2b_1x + 2b_2y + c = 0$. Then the coordinates changing defined by*

$$\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix} = R_{-\theta}\begin{pmatrix}x\\ y\end{pmatrix},$$

with $R_\theta = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$, with θ one of the two solution of $\tan\theta = \frac{-a_{11}}{a_{12}}$, $\theta \in [0, 2\pi[$ transforms the equation in $\hat{y}^2 = 2p(\hat{x} - \hat{v}_1)$. Moreover, the vertex $V = R_\theta \begin{pmatrix}\hat{v}_1\\ 0\end{pmatrix}$ and the focus is $F = R_\theta\begin{pmatrix}\hat{v}_1 + p/2\\ 0\end{pmatrix}$.

We note that $\tan\theta = \frac{-a_{11}}{a_{12}}$, $\theta \in [0, 2\pi[$ has two solutions whose difference is π. One solution transforms the equation into $\hat{y}^2 = 2p(\hat{x} - \hat{v}_1)$ and the other one into $\hat{y}^2 = -2p(\hat{x} - \hat{v}_1)$.

2.4. Parabola as a section of a conical surface

Dandelin's Theorem is well known, which establishes that if we cut a cone by a plane that forms an angle α with the axis, the same angle α that forms the axis with the generatrices, then the focus of the parabola is the intersection of an inscribed sphere in the cone with the plane. With technical drawing techniques belonging to the dihedral theory, it is possible to graphically determine the characteristics. The calculation of these values is proposed from the point of approach of analytical geometry. We begin with the general results attained.

Proposición 7.2 *Let a plane be of equation $Ax + By + Cz + D = 0$ and a double cone given by $z^2 = a^2(x^2 + y^2)$ such that $a > 0$. Then the plane forms an angle φ with the cone axis equal to the angle that the axis forms with the generatrices if and only if $C^2 = \frac{(A^2+B^2)}{a^2}$. Moreover,* $\cot(\varphi) = a$, $\varphi \in]0, \pi/2[$.

Proof. Consider the axis z (cone axis) and the generatrix $r \equiv \{z = ay, x = 0\}$ and their vector directors $u = (0,0,1)$, $u_g = (0,1,a)$, respectively. Hence, the angle that forms the cone axis with each generatrix satisfies $\cot(\varphi) = a$, $\varphi \in]0, \pi/2[$. The vector $n = (A, B, C)$ is normal to the plane. Then,

$$\sin(\varphi) = cos\,(\pi/2 - \varphi) = \frac{|u \cdot n|}{|u| \cdot |n|} = \frac{|C|}{\sqrt{A^2 + B^2 + C^2}}.$$

Now, $1 + a^2 = 1 + \cot^2(\varphi) = \frac{1}{sin^2(\varphi)} = \frac{A^2+B^2+C^2}{C^2} \Leftrightarrow C^2 = \frac{(A^2+B^2)}{a^2}$. ∎

Remarca 7.3 *In the conditions of previous proposition,* $\sin(\varphi) = \frac{1}{\sqrt{1+a^2}}$, $cos\,(\varphi) = \frac{a}{\sqrt{1+a^2}}$.

Remarca 7.4 *A quadric defined by $a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz + 2b_1x + 2b_2y + 2b_3z + c = 0$ can be written in the way:*

$$(1, x, y, z)\, Q \begin{pmatrix}1\\ x\\ y\\ z\end{pmatrix} = 0$$

where

$$Q = \begin{pmatrix} c & b_1 & b_2 & b_3 \\ b_1 & a_{11} & a_{12} & a_{13} \\ b_2 & a_{12} & a_{22} & a_{23} \\ b_3 & a_{13} & a_{23} & a_{33} \end{pmatrix}.$$

The intersection of the plane $Ax + By + Cz + D = 0$ with the plane $z = 0$ is a line with vector director $(-B, A, 0)$.

Teorema 7.5 *Let the plane be of equation $Ax + By + Cz + D = 0$ and the cone given by $z^2 = a^2(x^2 + y^2)$ such that $a > 0$, $C = \frac{\sqrt{A^2+B^2}}{a}$, and $\cot(\varphi) = a$, $\varphi \in]0, \pi/2[$. If a plane rotation of $\frac{\pi}{2} - \varphi$ is applied in the direction of $(-B, A, 0)$, then the curve defined by the intersection between the plane and the cone is a parabola with equation*

$$(1, x'', y'', 0)\, P \begin{pmatrix} 1 \\ x'' \\ y'' \\ 0 \end{pmatrix} = 0,$$

being

$$P = \begin{pmatrix} 1 & 0 \\ 0 & MGM^t \end{pmatrix} Q_V \begin{pmatrix} 1 & 0 \\ 0 & MG^tM^t \end{pmatrix},$$

$$G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin(\varphi) & -\cos(\varphi) \\ 0 & \cos(\varphi) & \sin(\varphi) \end{pmatrix},$$

$$M = \begin{pmatrix} -B/\sqrt{A^2+B^2} & A/\sqrt{A^2+B^2} & 0 \\ A/\sqrt{A^2+B^2} & B/\sqrt{A^2+B^2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

Q_V the matrix of the cone obtained by applying the translation

$$(x', y', z') = (x, y, z) + \frac{D}{A^2 + B^2 + C^2}(A, B, C).$$

Proof. The cone equation can be written as $(1, x, y, z)\, Q \begin{pmatrix} 1 \\ x \\ y \\ z \end{pmatrix} = 0$, with

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1/a^2 \end{pmatrix}.$$

Introducing a coordinates changing,

$$(x', y', z') = (x, y, z) + \frac{D}{A^2 + B^2 + C^2}(A, B, C).$$

the initial plane is carried to $Ax' + By' + Cz' = 0$, which passes through the origin of coordinates. Let $V = \frac{D}{A^2+B^2+C^2}\begin{pmatrix} 0 \\ A \\ B \\ C \end{pmatrix}$ and $X' = \begin{pmatrix} 1 \\ x' \\ y' \\ z' \end{pmatrix}$. Then, the cone equation becomes: $(X' - V)^t Q (X' -$

$V) = 0 \Leftrightarrow X''^t Q_V X' = 0$, where X''^t stands for the transpose matrix of X', and

$$Q_V = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \lambda_2 & 1 & 0 & 0 \\ \lambda_3 & 0 & 1 & 0 \\ \lambda_4 & 0 & 0 & -1/a^2 \end{pmatrix},$$

being, $\lambda_1 = V^t Q V$, $(\lambda_2, \lambda_3, \lambda_4) = -\frac{D}{A^2+B^2+C^2}\left(A, B, \frac{-1}{a^2}C\right)$.

Now, we apply the changing coordinates,

$$\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = MGM^t \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}.$$

From here, it follows,using block matrix notation,

$$(1, x'', y'', z'') \begin{pmatrix} 1 & 0 \\ 0 & MGM^t \end{pmatrix} Q_V \begin{pmatrix} 1 & 0 \\ 0 & MG^tM^t \end{pmatrix} \begin{pmatrix} 1 \\ x'' \\ y'' \\ z'' \end{pmatrix} = 0.$$

Now, make the intersection with the plane $z'' = 0$. It is easily checked that when applying this new change of coordinates to the plane, this is transformed in $z'' = 0$. Indeed, consider $0 = (A, B, C) \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \Leftrightarrow 0 = (A, B, C)\, MG^tM^t \begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix}$ and introduce $C = \frac{\sqrt{A^2+B^2}}{a}$, $\cot(\varphi) = a$, then $z'' = 0$.

■

Ejemplo 7.6 *Consider the parabola defined by the equation cone $0.5\,(x^2 + y^2) = z^2$ and the plane $x + y + 2z = 3$. (See figure 2). Find out the focus coordinates. In this case,*

$$Q_V = \begin{pmatrix} -3/2 & 1/2 & 1/2 & -2 \\ 1/2 & 1 & 0 & 0 \\ 1/2 & 0 & 1 & 0 \\ -2 & 0 & 0 & -2 \end{pmatrix},$$

$$M = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2/3} & -\sqrt{1/3} \\ 0 & \sqrt{1/3} & \sqrt{2/3} \end{pmatrix},$$

$$P = \begin{pmatrix} -3/2 & \sqrt{3/2} & \sqrt{3/2} & -\sqrt{3/2} \\ \sqrt{3/2} & 1/2 & -1/2 & 1 \\ \sqrt{3/2} & -1/2 & 1/2 & 1 \\ -\sqrt{3/2} & 1 & 1 & -1 \end{pmatrix},$$

$$(1, x'', y'', 0)\, P \begin{pmatrix} 1 \\ x'' \\ y'' \\ 0 \end{pmatrix} = 0.$$

Then, the parabola equation in the plane $z''=0$ is

$$\frac{1}{2}x''^2+\frac{1}{2}y''^2-x''y''+2\sqrt{\frac{3}{2}}x''+2\sqrt{\frac{3}{2}}y''-\frac{3}{2}=0 \Leftrightarrow$$

$$(1,x'',y'')\begin{pmatrix}-3/2 & \sqrt{3/2} & \sqrt{3/2}\\ \sqrt{3/2} & 1/2 & -1/2\\ \sqrt{3/2} & -1/2 & 1/2\end{pmatrix}\begin{pmatrix}1\\ x''\\ y''\end{pmatrix}=0.$$

We note that $\det\begin{pmatrix}-3/2 & \sqrt{3/2} & \sqrt{3/2}\\ \sqrt{3/2} & 1/2 & -1/2\\ \sqrt{3/2} & -1/2 & 1/2\end{pmatrix}\neq 0$, $\det\begin{pmatrix}1/2 & -1/2\\ -1/2 & 1/2\end{pmatrix}=0$, *the eigenvalues of the last matrix are* 1 *and* 0*. According to the classification of conics, it is effectively a parabola.* $p=\sqrt{-det\,(A)\,/\,(a_{11}+a_{22})^3}=\sqrt{-\frac{-3}{1^3}}=\sqrt{3}$

The parabola axis is the polar of the point $(0,1/2,-1/2)$*:*

$$(0,1/2,-1/2)\begin{pmatrix}-3/2 & \sqrt{3/2} & \sqrt{3/2}\\ \sqrt{3/2} & 1/2 & -1/2\\ \sqrt{3/2} & -1/2 & 1/2\end{pmatrix}\begin{pmatrix}1\\ x''\\ y''\end{pmatrix}=0.$$

Thus, $y''=x''$*. The vertex is the intersection between the parabola and the line* $y''=x''$*. So,*

$$\frac{1}{2}x''^2+\frac{1}{2}y''^2-x''y''+2\sqrt{\frac{3}{2}}x''+2\sqrt{\frac{3}{2}}y''-\frac{3}{2}=0, y''=x''.$$

The solution is $\left(\frac{\sqrt{6}}{8},\frac{\sqrt{6}}{8}\right)$.

The focus is in the line $y''=x''$ *at a distance* $p/2$ *from the vertex. Then, the focus is* $\left(\frac{-\sqrt{6}}{8},\frac{-\sqrt{6}}{8}\right)$.

$$Adj\begin{pmatrix}-3/2 & \sqrt{3/2} & \sqrt{3/2}\\ \sqrt{3/2} & 1/2 & -1/2\\ \sqrt{3/2} & -1/2 & 1/2\end{pmatrix}=\begin{pmatrix}A_{00} & A_{01} & A_{02}\\ A_{01} & A_{11} & A_{12}\\ A_{20} & A_{12} & A_{22}\end{pmatrix}=\begin{pmatrix}0 & -\sqrt{\frac{3}{2}} & -\sqrt{\frac{3}{2}}\\ -\sqrt{\frac{3}{2}} & -\frac{9}{4} & \frac{3}{4}\\ -\sqrt{\frac{3}{2}} & \frac{3}{4} & -\frac{9}{4}\end{pmatrix}.$$

As previously indicated in subsection 2.3, the system is solved:

$$\left\{\begin{matrix}2\sqrt{3/2}x-2\sqrt{3/2}y=0\\ \sqrt{3/2}x+\sqrt{3/2}y+3/4=0\end{matrix}\right\},$$

and it is observed that it provides the same solution. The directrix can be calculated as the polar of the focus or by finding the line perpendicular to the axis that passes through the symmetrical point of the focus with respect to the vertex. As the polar of the focus, we have

$$\left(1,\frac{-\sqrt{6}}{8},\frac{-\sqrt{6}}{8}\right)\begin{pmatrix}-3/2 & \sqrt{3/2} & \sqrt{3/2}\\ \sqrt{3/2} & 1/2 & -1/2\\ \sqrt{3/2} & -1/2 & 1/2\end{pmatrix}\begin{pmatrix}1\\ x''\\ y''\end{pmatrix}=0.$$

So that, $\sqrt{3/2}x''+\sqrt{3/2}y''-9/4=0$ *The purpose is to compute the focus coordinates in the plane* $Ax+By+Cz+D=0$*. Consequently, now all the changing coordinates have to be undone.*

$F'' = (x'', y'', z'') = (-\sqrt{6}/8, -\sqrt{6}/8, 0)$,

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = MG^tM^t \begin{pmatrix} \frac{-\sqrt{6}}{8} \\ \frac{-\sqrt{6}}{8} \\ 0 \end{pmatrix} = \begin{pmatrix} -1/4 \\ -1/4 \\ 1/4 \end{pmatrix},$$

$$F' = (x', y', z') = (x, y, z) + \frac{D}{A^2 + B^2 + C^2}(A, B, C),$$
$$F = (-1/4, -1/4, 1/4) - (-3)/6\,(1, 1, 2) = (1/4, 1/4, 5/4).$$

Similarly, for the vertex,

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = MG^tM^t \begin{pmatrix} \frac{\sqrt{6}}{8} \\ \frac{\sqrt{6}}{8} \\ 0 \end{pmatrix} = \begin{pmatrix} 1/4 \\ 1/4 \\ -1/4 \end{pmatrix},$$

$$V = (3/4, 3/4, 3/4),$$

since the translation was applied.

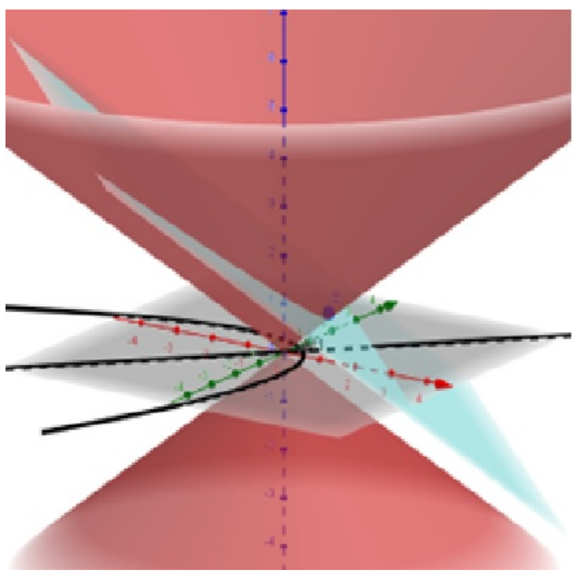

Figure 2: Cone and plane of the example. Source: Created by the authors with GeoGebra.

Ejemplo 7.7 *Here we present the parabola resulting from the section of the plane to which the rotation described in the preceding section is applied.*

$$\frac{1}{2}x''^2 + \frac{1}{2}y''^2 - x''y'' + 2\sqrt{\frac{3}{2}}x'' + 2\sqrt{\frac{3}{2}}y'' - \frac{3}{2} = 0.$$

Applying a rotation of angle θ such that

$$\tan\theta = \frac{-a_{12}}{a_{22}} = 1, \theta \in [0, 2\pi[.$$

There are two solutions: $\frac{\pi}{4}$, $\frac{5\pi}{4}$. For $\frac{\pi}{4}$, since Proposition 1,

$$\begin{pmatrix} x''' \\ y''' \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} x'' \\ y'' \end{pmatrix},$$

then,

$$\frac{1}{2}\left(1/\sqrt{2}x''' - 1/\sqrt{2}y'''\right)^2 + \frac{1}{2}\left(1/\sqrt{2}x''' + 1/\sqrt{2}y'''\right)^2$$
$$-\left(1/\sqrt{2}x''' - 1/\sqrt{2}y'''\right)\cdot\left(1/\sqrt{2}x''' + 1/\sqrt{2}y'''\right) + 2\sqrt{\frac{3}{2}}\left(1/\sqrt{2}x''' - 1/\sqrt{2}y'''\right)$$
$$+2\sqrt{\frac{3}{2}}\left(1/\sqrt{2}x''' + 1/\sqrt{2}y'''\right) - \frac{3}{2} = 0 \Leftrightarrow$$
$$y'''^2 = -2\sqrt{3}\left(x''' - \sqrt{3}/4\right).$$

From here, it follows that $|p| = \sqrt{3}$. *Undoing the variable change,*

$$F'' = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} -\sqrt{3}/4 \\ 0 \end{pmatrix} = \begin{pmatrix} -\sqrt{6}/8 \\ -\sqrt{6}/8 \end{pmatrix}.$$

For $\theta = \frac{5\pi}{4}$, *the equation becomes*

$$y'''^2 = 2\sqrt{3}\left(x''' + \sqrt{3}/4\right).$$

At this point, it is pending to explain the mathematical elements that influence in the behaviour of light when it hits a surface.

2.5. The reflection of light

The phenomenon of incident rays at a point on a surface where total reflection occurs has been extensively studied. For background, see Alonso, M., & Finn, E. J. (1967). By Fermat's principle, light always travels along the path that takes the least amount of time. According to the principles of optics, the incident ray makes an angle on the line perpendicular to the surface equal to the angle that the reflected ray makes on that perpendicular. Furthermore, the line perpendicular to the surface that passes through the point of incidence, the incident ray at that point, and the reflected ray are in the same plane. In the experiment that is proposed, the ray falls on a surface that is not flat, so at that point the tangent plane to the surface is considered. The surface with parameters $s, t \in \mathbb{R}$ has equations:

$$\begin{cases} x = \frac{1}{2p}s^2 \\ y = s \\ z = t \end{cases}$$

The incident rays in the experiment are in the plane $z = 0$.

3. Parabola Project

The previous facts suggest the development of class activities related to the topic of the parabola. The experiment should be carried out when the students have already assumed certain knowledge about conics such as knowledge of its elements: directrix and focus, and the relationship between any point on the parabola, the focus, and the directrix. Another previous knowledge required are:

- The determination of the equation of the straight line with known two points.
- The determination of the equation of the straigt line with known the slope and one point.

- Knowledge of the relationship between the slope and the tangent as a trigonometric ratio.
- Solve equations and systems.

The historical reviews on the proportions that Menecmus studied in relation to the parabola can be introduced at the beginning of its study or in the project itself to justify the technical drawing method used for its construction.

A problem in a real context related to the parabola and light reflection could be to explain why a specific type of solar panel, called a parabolic trough collector, is manufactured (see figure 3). This is a question related to the 2030 Sustainable Development Goals on the issue of using renewable energy with more efficient systems. The question would be: What happens to the sun's rays incident on the parabolic trough collector? Can we make a prototype for experiments? With the construction of the prototype, a simplification of the problem is intended. Once the experiment has been carried out and all calculations have been made, students are expected to be able to transfer the conclusion to the real context to explain that the tube shown in the picture, called absorber tube of the collector, is positioned so that it passes through all the foci of the generating parabolas.

Figure 3: Parabolic trough collector.
Source: Taken of Villasante2010.

3.1. Construction and experiment

The first step to execute the experiment consists of the construction of the structure that can be seen in figure 4. The following materials are required:

- 1 White paperboard, A2 size
- 2 White cardboard sheets, A2 size
- Silver paper
- Laser
- Technical Drawing Instruments
- Heat sealer
- Cutter
- Pair of scissors

On the white paperboard, a parabola with the equation $y^2 = 20x$ has been drawn, which is traced on two cardboards.

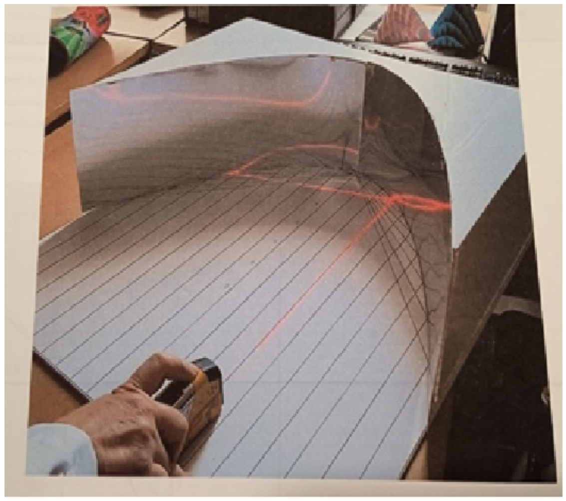

Figure 4: Structure for the experiment. Source: Self-made.

The projective beam technique is used to draw the parabola. The technique can be found in any first-year high school textbook, although the mathematical demonstration of why the method works is not explained there. The authors have not found a reference to the mathematical proof of its operation which is included for completeness. Consider the parabola of equation $y^2 = 2px$. From the point $P_{n+1} = (x_{n+1}, y_{n+1})$, with $y_{n+1} = \sqrt{2px_{n+1}}$ it is possible to get n points $P_1, \cdots, P_n$, by beam technique that implies the traced of the lines $r_1, \cdots, r_n$ which join the origin of coordinates with the points $Q_1 = \left(\frac{x_0}{n+1}, \sqrt{2px_0}\right), \cdots, Q_i = \left(\frac{x_0}{n+1}i, \sqrt{2px_0}\right), \cdots, Q_n = \left(\frac{x_0}{n+1}n, \sqrt{2px_0}\right)$, respectively. Now, horizontal lines $s_1, \cdots, s_i, \cdots, s_n$ with equations $y = \frac{\sqrt{2px_0}}{n+1}, \cdots, y = \frac{\sqrt{2px_0}}{n+1}i, \cdots, y = \frac{\sqrt{2px_0}}{n+1}n$, respectively, are traced. For each $i = 1, \ldots, n$ the point P_i is achieved by the intersection of lines r_i and s_i, i.e., solving the system,

$$\begin{cases} y = \frac{(n+1)\sqrt{2px_0}}{x_0 i}x \\ y = \frac{\sqrt{2px_0}}{n+1}i \end{cases}$$

Hence, for each $i = 1, \ldots, n$, $P_i = (x_i, y_i) = \left(\frac{x_0}{(n+1)^2}i^2, \frac{\sqrt{2px_0}}{n+1}i\right)$, $y_i^2 = 2px_i$. So that, for each $i = 1, \ldots, n$, P_i is a point of the parabola.

As it can be seen in figure 4, the reflected ray falls on the focus. The next subsection contains the mathematical proof. This general verification could exceed the objectives in a first year of high school if the geometric interpretation of the derivative has not yet been taught, but not in a first year of university. In a first year of baccalaureate, the demonstration can be requested at a given specific point (x_0, y_0).

3.2. Tangent, incident and reflected ray

Consider the parabola of equation $y^2 = 2px$. For $\left(x_0, \sqrt{2px_0}\,\right)$ with $x_0 > 0$, the equation of the tangent line is $y - \sqrt{2px_0} = \sqrt{\frac{p}{2x_0}}\,(x - x_0)$, i.e., $tan\,(\alpha) = \sqrt{\frac{p}{2x_0}}$ (see figure 5). The incident ray in this point has as equation $y = \sqrt{2px_0}$, while the slope of the reflected ray is

$$\tan(2\alpha) = tan\,(2\alpha) = \frac{2tan\,(\alpha)}{1 - tan^2\alpha} = \frac{2\sqrt{\frac{p}{2x_0}}}{1 - \left(\sqrt{\frac{p}{2x_0}}\right)^2} = \frac{2\sqrt{2px_0}}{2x_0 - p}.$$

Thus, the equation of the reflected ray stays, $y-\sqrt{2px_0}=\frac{2\sqrt{2px_0}}{2x_0-p}\left(x-x_0\right)$. Now, taking $y=0$, we get $0-\sqrt{2px_0}=\frac{2\sqrt{2px_0}}{2x_0-p}\left(x-x_0\right)\Leftrightarrow x=\frac{p}{2}$.

Therefore, the reflected ray from $\left(x_0,\sqrt{2px_0}\right)$ passed for the focus of the parabola and its equation can be rewritten in the way

$$y=\frac{\sqrt{2px_0}}{2x_0-p}\left(2x-p\right).$$

Moreover, if the system

$$\begin{cases} y=\frac{\sqrt{2px_0}}{2x_0-p}\left(2x-p\right) \\ y^2=2px \end{cases}$$

is solved, then the points $\left(x_0,\sqrt{2px_0}\right)$, $\left(\frac{p^2}{4x_0},-\sqrt{\frac{p^3}{2x_0}}\right)$ are obtained. If $x_0=p/2$ then the focus is the midpoint of the segment defined by the previous points.

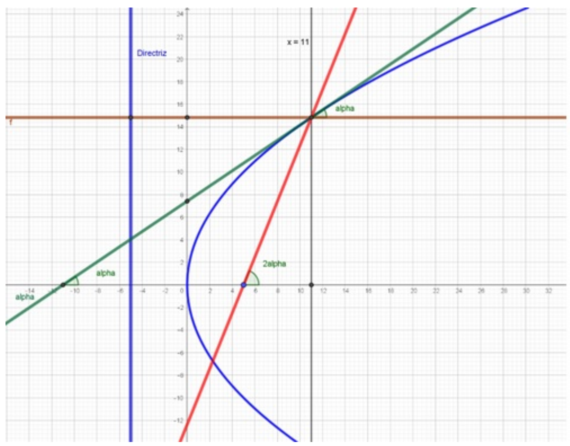

Figure 5: Structure for the experiment. Source: Generated by the authors with GeoGebra.

At the Baccalaureate level, the following contents are worked on:

- Concept of slope of a straight line.
- Meaning of the derivative of a function at a point.
- Trigonometry.
- Solving systems of equations.
- Mathematical work with digital resources.

In this way, there will be mathematical activities related to a real phenomenon, so that there is a modelling task. In the case of not having taught the concept of derivative, they can be asked to calculate the tangent line as the perpendicular bisector of the segment of extremes $\left(-\frac{p}{2},y_0\right)$ (a point in the directrix) and $\left(\frac{p}{2},0\right)$ (the focus), for some point (x_0,y_0), since the equation bisector of the segment is $\left(x+\frac{p}{2}\right)^2+\left(y-y_0\right)^2=\left(x-\frac{p}{2}\right)^2+y^2\Leftrightarrow 2px-2yy_0+{y_0}^2=0$, and, in addition, the system

$$\begin{cases} 2px-2yy_0+{y_0}^2=0 \\ y^2=2px \end{cases}$$

has a unique solution which is exactly the point (x_0,y_0). The advantage of this way is that students work locus at the same time. For university students, a parabola can be built in a plane of space with a specific equation and as an application of the isometries of space that make the respective coordinate changes to finally determine the coordinates of the focus in space as it has been done in the example in the previous section. As a final product, students will be asked to write a report that includes all the calculations, graphs made with GeoGebra and photographs of the construction with an academic work format, showing a summary of the elements of a parabola, the drawing of the parabola by means of projective beams, photographs of the experiment, explanation of the reflection of light, calculation of the equation of the tangent line at a specific point chosen by the students, as well as the incident and reflected ray, checking that the reflected ray passes through the focus, drawing the parabola, the tangent line at a point and the incident and reflected rays with GeoGebra. For university students, they can be asked for a Matlab program with all the calculations, thus it is a way of giving practical sense to the calculation of eigenvalues, coordinate changes, etc.

3.3. Results of its implementation

For the evaluation, a checklist was used (see figure 6) to verify the contents to be included with a table of remarks with comments and proposals for improvement. A grading scale was established for each item, evaluating the explanations in detail. Apart from the reflections or other aspects that can be included in the written work, it was emphasised that the written work should include the items to which the indicators refer. It was thought that the errors made would be of different types in each student's report, and therefore, an instrument was needed to allow remarks for each indicator in order to carry out a formative evaluation. In the remarks part, specific feedback was given to each report on that item. In fact, a space for personalised comments was left at the end of the check-list.

The strategies for solving the problem were:

- Brainstorming to understand the problem and formulate the most appropriate mathematical model and formulate conjectures about the phenomena involved.
- Searching for information about the physical phenomenon.
- Review of the concepts of conics and resolution of doubts.
- Search for information on the drawing technique for drawing a parabola.
- Construction of the prototype and experimentation.
- Development of conjectures.
- Group assignment of two or three people.
- Representation of the parabola and rays in GeoGebra.
- Determination of the intersection points of the parabola and the rays using intersection tool of GeoGebra.
- Performing the calculations of the mathematical model.
- Contrast the calculations made with those obtained in GeoGebra and with the experiment.
- Interpretation of the results obtained in relation to the initial problem.

The majority of student groups included all the items of the check-list except two groups. One of them did not do the calculations of the equation of the incident ray and reflected in

Checklist. Parabola work

Is it included?	Yes	No	Remark
Explanation of the experiment			
Information about light reflection			
Summary of the elements of the parabola			
Drawings with GeoGebra			
Calculations of the equation of the incident ray and the reflected ray for x=5			
Checking that the reflected ray for x=5 passes through (5,0).			
Calculations of the equation of the incident ray and the reflected ray for another value of x			
Checking that the reflected ray for another value of x passes through (5,0)			
Format, presentation, spelling and grammar			

Figure 6: Evaluation instrument. Source: Self-made.

another x-value. Only a group with negative marks delivered a very incomplete report, most of the sections were missing.

In some works of students, there were some errors in the simplification calculations with radicals. They had the difficulty of finding the slope of the tangent line at a point of the parabola because the topic of derivatives had not been given and they did not know a formula for it either, so they had to calculate the equation based on the definition of the tangent line to a parabola at a point, as the line that cuts the parabola at a single point. The main difficulties were:

- Understanding the tasks they had to perform. Some groups had to be guided and given concrete guidelines.
- Lack of basic knowledge about the parabola in students who had not assimilated these new concepts.
- Large number of students and limited space in the classroom to do the experiment.
- On the simplification of calculations with radicals.

Some students confused the value of the concrete abscissa with the variable x in the equation of the line. In order to solve these difficulties, it was necessary to review concepts in small groups, in large groups and individually, and share photographs of the experiment. Some students felt low confidence with open-ended questions where they must take initiative and they asked for more help.

With reference to the results obtained, 31 students out of 34, (91.18 %), delivered the work on time and properly. Of the students who delivered, 96.875 % obtained a positive rating.

A part of a student's work is shown in figure 7, in which she calculates the slope of the reflected ray from the slope of the tangent line.

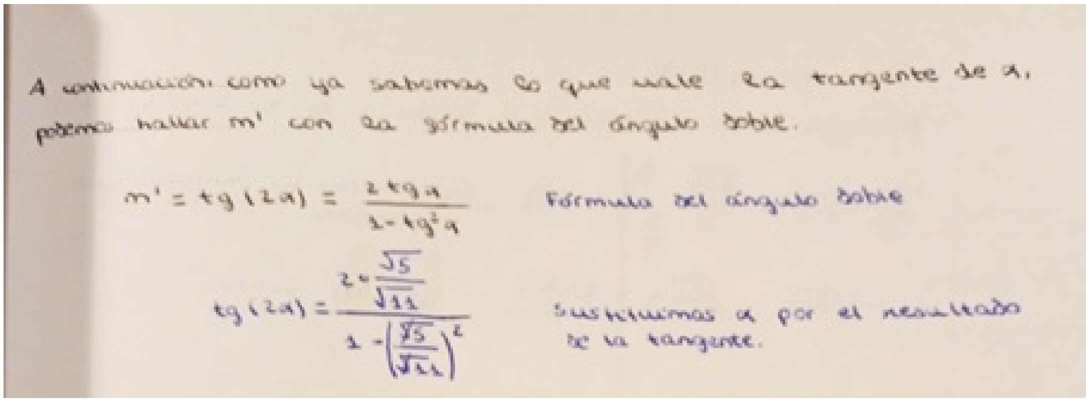

Figure 7: Calculation of the slope of the reflected ray per student. Source: Taken by the authors of a student's work.

4. Conclusions

The elements of the parabola in space can be determined analytically by applying an abatement of the plane using matrix techniques for calculating eigenvalues, applying rotations and translations. Practical experiments to verify the laws of total reflection can be proposed together with developments with applications such as GeoGebra and Matlab, interconnecting different mathematical topics. The establishment of intra-curricular connections with extra-mathematical knowledge is in line with the new teaching paradigm where the use of ICT tools provides added value, promoting the development of thought. It is conjectured that the project can be accepted by other high school groups, given the good results that have been obtained in the experience referred to in the previous section. Other models of structures could be set up to further investigate the incident and reflected ray equations, checking the effects when the surface is changed.

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