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Relaxed Indistinguishability Relations and Relaxed Metrics: The Aggregation Problem

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Abstract: The main purpose of this paper is to study the relationship between those functions that aggregate relaxed indistinguishability fuzzy relations with respect to a collection of t -norms and those functions that merge relaxed pseudo-metrics, extending the classical approach explored for pseudo-metrics and indistinguishability fuzzy relations. Special attention is paid to the distinguished class of SSI -relaxed indistinguishability fuzzy relations showing that functions merging this special type of relaxed indistinguishability fuzzy relations can be expressed through functions aggregating SSD -relaxed pseudo-metrics. Outstanding differences between those functions aggregating indistinguishability fuzzy relations and those that aggregate their counterpart separating points are shown.

Keywords: aggregation; relaxed pseudo-metric; relaxed indistinguishability fuzzy relation; additive generator; continuous Archimedean t -norm

MSC: 03B50; 03B52; 54E35



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1. Introduction

In [1], E. Trillas introduced the concept of indistinguishability fuzzy relation with respect to T (T -equivalence in [2]), where T is a t -norm. From now on, we assume that the reader is versed in the fundamentals of t -norms. A reference where such a theory is treated exhaustively is [2].

On account of [1] (see also [3]), an indistinguishability fuzzy relation with respect to T on a non-empty set X is a fuzzy set $E : X \times X \rightarrow [0, 1]$ which satisfies, for each $x, y, z \in X$, the axioms below:

- (I1) $E(x, x) = 1$;
- (I2) $E(x, y) = E(y, x)$;
- (I3) $T(E(x, y), E(y, z)) \leq E(x, z)$.

Following [2,3], an indistinguishability fuzzy relation with respect to T is called an indistinguishability fuzzy relation with respect to T that separates points (or T -equality) whenever

- (I1') $E(x, y) = 1$ implies $x = y$.

Since Trillas introduced the concept of indistinguishability fuzzy relation, many works have focused their efforts both on the study of theoretical aspects and their applications. A few references devoted to the aforementioned aim are [3–25].

In [25], a “metric behaviour” of indistinguishability fuzzy relations was proved (see also [9,10]). Concretely, a technique for generating extended pseudo-metrics from indistinguishability fuzzy relations, and vice versa was introduced. In order to introduce such a technique, let us recall that, according to [26], an extended pseudo-metric on a (non-empty) set X is a function $d : X \times X \rightarrow [0, \infty]$ such that for all $x, y, z \in X$:

- (d1) $d(x, x) = 0$;
- (d2) $d(x, y) = d(y, x)$;
- (d3) $d(x, z) \leq d(x, y) + d(y, z)$.

An extended pseudo-metric d on X is said to be an extended metric when, in addition, it satisfies the following condition for all $x, y \in X$:

(d1') $d(x, y) = 0 \Leftrightarrow x = y$.

Following [9,10,25], given an extended pseudo-metric d on X and a continuous Archimedean t -norm T with an additive generator f_T , an indistinguishability fuzzy relation E_{d,f_T} with respect to T on X can be induced for all $x, y \in X$ as follows:

$$E_{d,f_T}(x, y) = f_T^{(-1)}(d(x, y)), \tag{1}$$

where $f_T^{(-1)}$ is the pseudo-inverse of f_T . Let us recall that the pseudo-inverse $f_T^{(-1)}$ is given as follows:

$$f_T^{(-1)}(y) = \begin{cases} f_T^{-1}(y) & \text{if } 0 \leq y < f_T(0) \\ 0 & \text{if } f_T(0) \leq y \leq \infty \end{cases}.$$

Reciprocally, given an indistinguishability fuzzy relation E with respect to T on X , an extended pseudo-metric d can be generated on X for all $x, y \in X$, by

$$d(x, y) = f_T(E(x, y)). \tag{2}$$

Based on the exposed duality relationship between indistinguishability fuzzy relations and extended pseudo-metrics, A. Pradera, E. Trillas and E. Castiñeira studied the so-called aggregation problem for indistinguishability fuzzy relations. Thus, they gave a characterization of those functions that merge a collection of indistinguishability fuzzy relations into a single one in [27,28]. The aforementioned characterization establishes that such functions can always be obtained by means of those functions that aggregate a collection of extended pseudo-metrics into a single one. With the aim of recalling such a characterization, we introduce a few necessary notions exposed in [27,28].

Given a collection of t -norms $\mathcal{T} = \{T_i\}_{i=1}^n$ ($n \in \mathbb{N}$) and a non-empty set X , a collection of indistinguishability fuzzy binary relations $\{E_i\}_{i=1}^n$ is said to be a collection of \mathcal{T} -indistinguishability fuzzy relations on X provided that each E_i is an indistinguishability fuzzy binary relation on X with respect to T_i for all $i = 1, \dots, n$. Moreover, given a t -norm T , a function $F: [0, 1]^n \rightarrow [0, 1]$ aggregates \mathcal{T} -indistinguishability fuzzy relations into a T -indistinguishability fuzzy relation provided that $F(E_1, \dots, E_n)$ is an indistinguishability fuzzy relation with respect to T on a non-empty set X when $\{E_i\}_{i=1}^n$ is a collection of \mathcal{T} -indistinguishability fuzzy relations on X . Furthermore, a function $G: [0, \infty]^n \rightarrow [0, \infty]$ aggregates extended pseudo-metrics into an extended pseudo-metric provided that $G(d_1, \dots, d_n)$ is an extended pseudo-metric on a non-empty set Y whenever $\{d_i\}_{i=1}^n$ is a collection of extended pseudo-metrics on Y , and the function $G(d_1, \dots, d_n)$ is given for all $x, y \in Y$, by

$$G(d_1, \dots, d_n)(x, y) = G(d_1(x, y), \dots, d_n(x, y)).$$

According to [29], an extended pseudo-metric d on X is called an s -bounded pseudo-metric whenever there exists $s \in \mathbb{R}_{++} \cup \{\infty\}$ such that $d(x, y) \leq s$ for all $x, y \in X$, where $\mathbb{R}_{++} = \{a \in \mathbb{R}_+ : a > 0\}$. Observe that s -bounded extended pseudo-metrics are exactly extended pseudo-metrics when $s = \infty$.

If we have a collection of $(s_i)_{i=1}^n$ -bounded pseudo-metrics $\{d_i\}_{i=1}^n$ on Y and $s \in \mathbb{R}_{++} \cup \{\infty\}$, then a function $H: \prod_{i=1}^n [0, s_i] \rightarrow [0, s]$ aggregates a collection of $(s_i)_{i=1}^n$ -bounded pseudo-metrics $\{d_i\}_{i=1}^n$ on Y provided that $H(d_1, \dots, d_n)$ is a s -bounded pseudo-metric on Y .

In the sequel, given a collection of continuous Archimedean t -norms $\mathcal{T} = \{T_i\}_{i=1}^n$, we will say that $\{f_{T_i}\}_{i=1}^n$ is a collection of additive generators of \mathcal{T} provided that each f_{T_i} is an additive generator of T_i for all $i = 1, \dots, n$.

In consideration of the exposed concepts, the aforementioned characterization of functions aggregating \mathcal{T} -indistinguishability fuzzy relations is provided by the result below (see [27]):

Theorem 1. *Let $n \in \mathbb{N}$, let $\mathcal{T} = \{T_i\}_{i=1}^n$ be a collection of continuous Archimedean t -norms, and let $\{f_{T_i}\}_{i=1}^n$ be a collection of additive generators of \mathcal{T} . If T is a continuous Archimedean t -norm and $F: [0, 1]^n \rightarrow [0, 1]$ is a function, then the following assertions are equivalent:*

- (1) *F aggregates \mathcal{T} -indistinguishability fuzzy relations into a T -indistinguishability fuzzy relation.*
- (2) *The function $H : \prod_{i=1}^n [0, f_{T_i}(0)] \rightarrow [0, f_T(0)]$ aggregates every collection $\{d_i\}_{i=1}^n$ of $(f_{T_i}(0))_{i=1}^n$ -bounded pseudo-metrics into a $f_T(0)$ -bounded pseudo-metric, where $H = f_T \circ F \circ (f_{T_1}^{-1} \times \dots \times f_{T_n}^{-1})$.*

Notice that all t -norms in the statement of Theorem 1 are continuous. This is due to the fact that such a condition cannot be removed in order to guarantee that the technique given by (1) generates T -indistinguishability fuzzy relations from extended pseudo-metrics. A counterexample where this fact is made explicit can be found in [9] [Example 3].

The following result can be deduced from the preceding one in the particular case in which all continuous Archimedean t -norms are assumed to be strict.

Corollary 1. *Let $n \in \mathbb{N}$, and let $\mathcal{T} = \{T_i\}_{i=1}^n$ be a collection of strict continuous Archimedean t -norms. If T is a strict continuous Archimedean t -norm and $F: [0, 1]^n \rightarrow [0, 1]$ is a function, then the following assertions are equivalent:*

- (1) *F aggregates \mathcal{T} -indistinguishability fuzzy relations into a T -indistinguishability fuzzy relation.*
- (2) *The function $H : [0, +\infty]^n \rightarrow [0, \infty]$ aggregates every collection $\{d_i\}_{i=1}^n$ of extended pseudo-metrics into an extended pseudo-metric, where $H = f_T \circ F \circ (f_{T_1}^{-1} \times \dots \times f_{T_n}^{-1})$.*

The study, on the one hand, of the aggregation of fuzzy relations in general, and indistinguishability fuzzy relations in particular, and, on the other hand, of the aggregation of distances plays a relevant role in the literature in such a way that these topics are treated in outstanding monographs such as [3,26,30–33].

Regarding the aggregation of distances, J. Borsik and J. Doboš provided a description of those functions that are able to aggregate a collection of pseudo-metrics into a new one [31,34]. The same study was developed by Pradera and Trillas when bounded pseudo-metrics are under consideration [35]. The aforementioned description was given in terms of the so-called triangular triplets. Let us recall that a triplet (a, b, c) , with $s \in]0, \infty]$ and $a, b, c \in [0, s]^n$ ($n \in \mathbb{N}$), is said to form an n -dimensional triangular triplet whenever, for all $i = 1, \dots, n$,

$$a_i \leq b_i + c_i, b_i \leq a_i + c_i \text{ and } c_i \leq b_i + a_i.$$

Notice that \leq stands for the usual order in the extended real line.

The next result states the description given in [34,35].

Theorem 2. *Let $s \in]0, \infty]$. Then the below assertions are equivalent:*

- (1) *The function $H : [0, s]^n \rightarrow [0, s]$ aggregates every collection $\{d_i\}_{i=1}^n$ of s -bounded pseudo-metrics into an s -bounded pseudo-metric.*
- (2) *The function $H : [0, s]^n \rightarrow [0, s]$ transforms n -dimensional triangular triplets in $[0, s]^n$ into a one-dimensional triangular triplet in $[0, s]$ and $H(0, \dots, 0) = 0$.*

Observe that the fact that $H : [0, s]^n \rightarrow [0, s]$ transforms n -dimensional triangular triplets in $[0, s]^n$ into a one-dimensional triangular triplet in $[0, s]$ must be understood as $(H(a), H(b), H(c))$ is a one-dimensional triangular triplet provided that (a, b, c) ,

with $a, b, c \in [0, s]^n$, is an n -dimensional triangular triplet. It must be stressed that the description yielded by Borsik and Doboš is retrieved from the preceding result when $s = \infty$.

Later on, G. Mayor and J. Recasens gave a description of those functions that aggregate indistinguishability fuzzy relations in [19]. The aforesaid description is based on a new notion that they called T -triangular triplet and that it is inspired by the triangular triplet concept of Borsik and Doboš. Concretely, given a t -norm T , a triplet (a, b, c) , with $a, b, c \in [0, 1]^n$, is said to be an n -dimensional T -triangular triplet provided, for all $i = 1, \dots, n$, that

$$T(a_i, b_i) \leq c_i, T(a_i, c_i) \leq b_i \text{ and } T(b_i, c_i) \leq a_i.$$

In the spirit of Theorem 2, the new description of those functions that merge indistinguishability fuzzy relations was given as follows.

Theorem 3. *Let $n \in \mathbb{N}$ and let T be a t -norm. If $F : [0, 1]^n \rightarrow [0, 1]$ is a function, then the following assertions are equivalent:*

- (1) F aggregates T -indistinguishability fuzzy relations.
- (2) F satisfies the following conditions:
 - (2.1) $F(1_n) = 1$, where $1_n \in [0, 1]^n$ with $1_n = (1, \dots, 1)$.
 - (2.2) F transforms n -dimensional T -triangular triplets into one-dimensional T -triangular triplets.

Stimulated by the equivalence stated, on the one hand, in Theorem 3 and, on the other hand, in Theorem 1, the concept of T -triangular triplet was extended to the context of collections of t -norms \mathcal{T} in [36]. Specifically, given a collection of t -norms $\mathcal{T} = \{T_i\}_{i=1}^n$, a triplet (a, b, c) , with $a, b, c \in [0, 1]^n$, is said to be an n -dimensional \mathcal{T} -triangular triplet provided, for all $i = 1, \dots, n$, that

$$T_i(a_i, b_i) \leq c_i, T_i(a_i, c_i) \leq b_i \text{ and } T_i(b_i, c_i) \leq a_i.$$

In light of the preceding notion, the next new equivalence was given in [36].

Theorem 4. *Let $n \in \mathbb{N}$ and let $\mathcal{T} = \{T_i\}_{i=1}^n$ be a collection of t -norms. If T is a t -norm and $F : [0, 1]^n \rightarrow [0, 1]$ is a function, then the following assertions are equivalent:*

- (1) F aggregates \mathcal{T} -indistinguishability fuzzy relations into a T -indistinguishability fuzzy relation.
- (2) F holds the following conditions:
 - (2.1) $F(1_n) = 1$, where $1_n \in [0, 1]^n$ with $1_n = (1, \dots, 1)$.
 - (2.2) F transforms n -dimensional \mathcal{T} -triangular triplets into a one-dimensional T -triangular triplet.

Observe that Theorem 3 is recovered as a particular case when the collection of t -norms \mathcal{T} in Theorem 4 fulfills that $T_i = T$ for all $i = 1, \dots, n$.

When the \mathcal{T} -indistinguishability fuzzy relations separate points, the next result, which differs significantly from Theorem 4, was also obtained in [36].

Theorem 5. *Let $n \in \mathbb{N}$ and let $\mathcal{T} = \{T_i\}_{i=1}^n$ be a collection of t -norms. If T is a t -norm and $F : [0, 1]^n \rightarrow [0, 1]$ is a function, then the following assertions are equivalent:*

- (1) F aggregates \mathcal{T} -indistinguishability fuzzy relations that separate points into a T -indistinguishability fuzzy relation that separates points.
- (2) F holds the following conditions:
 - (2.1) $F(1_n) = 1$.
 - (2.2) Let $a \in [0, 1]^n$. If $F(a) = 1$, then there exists $i \in \{1, \dots, n\}$ such that $a_i = 1$.
 - (2.3) If $a, b, c \in [0, 1]^n$ such that (a, b, c) is an n -dimensional \mathcal{T} -triangular triplet, then $(F(a), F(b), F(c))$ is a one-dimensional T -triangular triplet.

In order to make the relationship between Theorems 1 and 4 clear, new equivalences between functions that merge \mathcal{T} -indistinguishability fuzzy relations and those functions that are able to merge extended pseudo-metrics have been given recently in [37]. In particular, the following results were obtained.

Theorem 6. Let $n \in \mathbb{N}$, let $\mathcal{T} = \{T_i\}_{i=1}^n$ be a collection of continuous Archimedean t -norms, and let $\{f_{T_i}\}_{i=1}^n$ be a collection of additive generators of \mathcal{T} . If T is a continuous Archimedean t -norm and $F: [0, 1]^n \rightarrow [0, 1]$ is a function, then the following assertions are equivalent:

- (1) F aggregates \mathcal{T} -indistinguishability fuzzy relations into a T -indistinguishability fuzzy relation.
- (2) The function $G: [0, +\infty]^n \rightarrow [0, f_T(0)]$ transforms n -dimensional triangular triplets in $[0, \infty]^n$ into a one-dimensional triangular triplet in $[0, f_T(0)]$ and $G(0, \dots, 0) = 0$, where $G = f_T \circ F \circ (f_{T_1}^{(-1)} \times \dots \times f_{T_n}^{(-1)})$.
- (3) The function $G: [0, +\infty]^n \rightarrow [0, f_T(0)]$ aggregates every collection $\{d_i\}_{i=1}^n$ of extended pseudo-metrics into a $f_T(0)$ -bounded pseudo-metric, where $G = f_T \circ F \circ (f_{T_1}^{(-1)} \times \dots \times f_{T_n}^{(-1)})$.
- (4) The function $H: \prod_{i=1}^n [0, f_{T_i}(0)] \rightarrow [0, f_T(0)]$ transforms n -dimensional triangular triplets in $\prod_{i=1}^n [0, f_{T_i}(0)]$ into a one-dimensional triangular triplet in $[0, f_T(0)]$ and $H(0, \dots, 0) = 0$, where $H = f_T \circ F \circ (f_{T_1}^{-1} \times \dots \times f_{T_n}^{-1})$.

In the particular case in which the indistinguishability fuzzy relations separate points, the equivalences below can be obtained ([37]).

Theorem 7. Let $n \in \mathbb{N}$ and let $\mathcal{T} = \{T_i\}_{i=1}^n$ be a collection of continuous Archimedean t -norms. If T is a continuous Archimedean t -norm and $F: [0, 1]^n \rightarrow [0, 1]$ is a function, then the following assertions are equivalent:

- (1) F aggregates \mathcal{T} -indistinguishability fuzzy relations that separate points into a T -indistinguishability fuzzy relation that separates points.
- (2) The function $G: [0, +\infty]^n \rightarrow [0, f_T(0)]$, where $G = f_T \circ F \circ (f_{T_1}^{(-1)} \times \dots \times f_{T_n}^{(-1)})$, fulfills the following conditions:
 - (2.1) $G(0, \dots, 0) = 0$;
 - (2.2) Let $a \in [0, +\infty]^n$. If $G(a) = 0$, then there exists $i \in \{1, \dots, n\}$ such that $a_i = 0$;
 - (2.3) G transforms n -dimensional triangular triplets in $[0, +\infty]^n$ into a one-dimensional triangular triplet in $[0, f_T(0)]$.
- (3) The function $G: [0, +\infty]^n \rightarrow [0, f_T(0)]$ aggregates every collection $\{d_i\}_{i=1}^n$ of extended metrics into an $f_T(0)$ -bounded metric, where $G = f_T \circ F \circ (f_{T_1}^{(-1)} \times \dots \times f_{T_n}^{(-1)})$.
- (4) The function $H: \prod_{i=1}^n [0, f_{T_i}(0)] \rightarrow [0, f_T(0)]$ aggregates every collection $\{d_i\}_{i=1}^n$ of $(f_{T_i}(0))_{i=1}^n$ -bounded metrics into an $f_T(0)$ -bounded metric, where $H = f_T \circ F \circ (f_{T_1}^{-1} \times \dots \times f_{T_n}^{-1})$.
- (5) The function $H: \prod_{i=1}^n [0, f_{T_i}(0)] \rightarrow [0, f_T(0)]$, where $H = f_T \circ F \circ (f_{T_1}^{-1} \times \dots \times f_{T_n}^{-1})$, fulfills the following conditions:
 - (5.1) $H(0, \dots, 0) = 0$;
 - (5.2) Let $a \in \prod_{i=1}^n [0, f_{T_i}(0)]$. If $H(a) = 0$, then there exists $i \in \{1, \dots, n\}$ such that $a_i = 0$;
 - (5.3) H transforms n -dimensional triangular triplets in $\prod_{i=1}^n [0, f_{T_i}(0)]$ into a one-dimensional triangular triplet in $[0, f_T(0)]$.

Although extended (pseudo-)metrics occupy a central place in the literature, their axiomatics limit their use in certain applications. In [38,39] (see also [40,41]), a new type of distance notion is introduced in order to develop suitable quantitative mathematical models and metric tools in computer science, appropriate, for instance, for program verification and logic programming. The aforesaid notion is called relaxed pseudo-metric in [42] (d -metric

in [38–41]). Let us recall that, according to [42], a relaxed pseudo-metric on a (non-empty) set X is a function $d : X \times X \rightarrow [0, \infty]$ such that for all $x, y, z \in X$:

- (r1) $d(x, y) = d(y, x)$,
- (r2) $d(x, z) \leq d(x, y) + d(y, z)$.

A relaxed pseudo-metric d on X is said to be a relaxed metric when, in addition, it satisfies the following condition for all $x, y \in X$:

- (r3) $d(x, y) = d(x, x) = d(y, y) \Leftrightarrow x = y$.

Clearly, an extended pseudo-metric d on X is a relaxed pseudo-metric such that $d(x, x) = 0$ for all $x \in X$. However, there are relaxed pseudo-metrics which are not extended pseudo-metrics. Indeed, according to [43], the following is an instance of a relaxed pseudo-metric which is not an extended pseudo-metric. Let $X = \{1, 2, 3\}$ and define $d : X \times X \rightarrow [0, \infty]$ by

$$d(x, y) = \begin{cases} 1 & \text{if } (x, y) = (1, 2) \text{ or } (x, y) = (2, 1) \\ \frac{1}{2} & \text{otherwise} \end{cases} . \tag{3}$$

A relaxed pseudo-metric d on a non-empty set X satisfies the small self-distances (SSD for short) property in the spirit of [44] whenever $d(x, x) \leq d(x, y)$ for all $x, y \in X$. In this case, we say that d is an SSD-relaxed pseudo-metric on X . Observe that the preceding example is an instance of an SSD-relaxed pseudo-metric. A celebrate special class of SSD-relaxed pseudo-metrics is known as a partial metric, and they have been applied to many fields in computer science (see, for instance, [38–41,45–49]).

Inspired by the dual relationship between \mathcal{T} -indistinguishability fuzzy relations (\mathcal{T} -indistinguishability fuzzy relation that separates points) and extended pseudo-metrics (extended metrics), the techniques given by (1) and (2) were extended to the relaxed framework in [43]. To this end, it was necessary to introduce a new notion of indistinguishability which is known as relaxed indistinguishability fuzzy relation. Moreover, such indistinguishability fuzzy relations are suggested to be the logical counterpart of relaxed pseudo-metrics in [42] (see also [17,50]).

On account of [42] (see also [43]), the notion of relaxed indistinguishability fuzzy relation can be formulated as follows:

Let X be a non-empty set and let $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ be a t -norm. A relaxed T -indistinguishability fuzzy relation E on X is a fuzzy set $E : X \times X \rightarrow [0, 1]$ satisfying for all $x, y, z \in X$ the following:

- (R1) $E(x, y) = E(y, x)$,
- (R2) $T(E(x, z), E(z, y)) \leq E(x, y)$.

Moreover, a relaxed T -indistinguishability fuzzy relation E is said to separate points provided that the condition below is satisfied for all $x, y \in X$:

- (R3) $E(x, y) = E(x, x) = E(y, y) \Rightarrow x = y$.

When a relaxed T -indistinguishability fuzzy relation E on X fulfills, for all $x, y \in X$, the condition

- (R4) $E(x, y) \leq E(x, x)$,

then E is said to be an SSI-relaxed T -indistinguishability fuzzy relation. Notice that SSI stands for small self-indistinguishability.

Next, we give an easy, but illustrative, example of SSI-relaxed T -indistinguishability fuzzy relation which has been extracted from [43]. To this end, fix $k \in]0, 1[$. Define the fuzzy binary relation $E_k : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow [0, 1]$ by

$$E_k(x, y) = k$$

for $x, y \in \mathbb{R}^+$. Clearly, it is obvious that E_k is an SSI -relaxed T_{Min} -indistinguishability fuzzy relation that does not separate points and, in addition, which is not a T_{Min} -indistinguishability fuzzy relation because $E(x, x) = k \neq 1$ for each $x \in \mathbb{R}^+$. Observe that E_k is a SSI -relaxed T_{Min} -indistinguishability fuzzy relation provided that it is an SSI -relaxed T -indistinguishability fuzzy relation for all t -norm T . We refer the reader to [43] [Example 3] for an instance of SSI -relaxed T -indistinguishability fuzzy relation that separates points.

It is worth mentioning that the techniques for generating, one from the other, indistinguishability fuzzy relations and extended pseudo-metrics can be adapted to the relaxed context simply interchanging indistinguishability fuzzy relations and extended pseudo-metrics by their corresponding relaxed counterpart in expressions (1) and (2) ([43] [Theorems 3 and 5]). An example of SSI -relaxed T_p -indistinguishability fuzzy relation that does not separate points can be obtained through the technique given by (2) and using the relaxed pseudo-metric given by (3) (compare [43] [Example 7]).

Inspired by the characterizations exposed in terms of \mathcal{T} -triangular triplets in Theorems 4 and 5, a new equivalence was stated for those functions that aggregate relaxed indistinguishability fuzzy relations in [36]. In order to introduce such an equivalence, let us recall that a collection of fuzzy relations $\{E_i\}_{i=1}^n$ is a collection of \mathcal{T} -relaxed indistinguishability fuzzy relations when each E_i is a relaxed indistinguishability fuzzy relation with respect to T_i . Concretely, the next result was proved.

Theorem 8. *Let $n \in \mathbb{N}$ and let $\mathcal{T} = \{T_i\}_{i=1}^n$ be a collection of t -norms. If T is a t -norm and $F : [0, 1]^n \rightarrow [0, 1]$ is a function, then the following assertions are equivalent:*

- (1) *F aggregates \mathcal{T} -relaxed indistinguishability fuzzy relations into a T -relaxed indistinguishability fuzzy relation.*
- (2) *F transforms n -dimensional \mathcal{T} -triangular triplets into a one-dimensional T -triangular triplet.*

It must be stressed that a few properties of those functions that aggregate \mathcal{T} -relaxed indistinguishability fuzzy relations into a T -relaxed indistinguishability fuzzy relation have been explored in [36]. In particular, every function F transforming n -dimensional \mathcal{T} -triangular triplets into a one-dimensional T -triangular triplet must dominate T with respect to the collection \mathcal{T} . Notice that, according to [36], a function $F : [0, 1]^n \rightarrow [0, 1]$ dominates a t -norm T with respect to a collection $\mathcal{T} = \{T_i\}_{i=1}^n$ of t -norms when F satisfies, for all $a, b \in [0, 1]^n$, the following condition:

$$T(F(a), F(b)) \leq F(T_1(a_1, b_1), \dots, T_n(a_n, b_n)).$$

Motivated by the fact that, on the one hand, the exposed relationship of duality between relaxed indistinguishability fuzzy relations and relaxed pseudo-metrics and, on the other hand, the possibility of expressing the equivalence provided by Theorem 8 in terms of new ones for functions aggregating relaxed pseudo-metrics in the spirit of Theorems 6 and 7 is not explored yet, the main purpose of this paper is to study the connection between those functions that aggregate relaxed indistinguishability fuzzy relations with respect to a collection of t -norms and those functions that merge relaxed pseudo-metrics complementing the information provided by Theorem 8 and extending, in some sense, Theorem 1 to the relaxed framework.

The remainder of the paper is organized as follows. In Section 2, we provide a new characterization of those functions that aggregate relaxed indistinguishability fuzzy relations. Thus, we show that there is an equivalence between functions that aggregate \mathcal{T} -relaxed indistinguishability fuzzy relations and those functions aggregating relaxed pseudo-metrics. Moreover, such an equivalence is expressed in terms of triangular triplets in the spirit of Theorem 2. An interesting consequence that can be derived from the aforementioned equivalence is that the functions under consideration are in correspondence with those that are subadditive. Moreover, the separating points case is approached and characterizations of the class of functions merging \mathcal{T} -relaxed indistinguishability fuzzy relations that separate points are obtained in terms of \mathcal{T} -triangular triplets. It must be

pointed out that the aggregation of this type of relaxed indistinguishability fuzzy relations was not explored in [36]. Outstanding differences between those functions aggregating \mathcal{T} -relaxed indistinguishability fuzzy relations and those that aggregate their counterpart separating points are shown.

Special attention is paid to the distinguished class of $SSI\text{-}\mathcal{T}$ -relaxed indistinguishability fuzzy relations in Section 3. Thus, we show that functions merging this special type of relaxed indistinguishability fuzzy relations can be expressed through functions aggregating SSD -relaxed pseudo-metrics. It must be stressed that there is a notable difference between the class of functions that are able to aggregate a collection of relaxed indistinguishability fuzzy relations (relaxed pseudo-metrics) and SSI -relaxed indistinguishability fuzzy relations (SSD -relaxed pseudo-metric). In this case, the appropriate class of functions are those that satisfy monotony and subadditivity. Section 4 ends the paper, exposing the conclusions and future work.

2. Aggregation of \mathcal{T} -Relaxed Indistinguishability Relations

In this subsection, we focus our efforts on the study of the interlink between those functions that aggregate relaxed indistinguishability fuzzy relations with respect to a collection of t -norms \mathcal{T} and those functions that merge relaxed pseudo-metrics in such a way that the information provided by Theorem 8 is complemented, and an extension of Theorem 6 to the relaxed context is obtained. Moreover, the case in which the relaxed indistinguishability fuzzy relations separate points is also explored and a version of Theorems 5 and 7 in the new framework is proved.

With the aim of stating the aforesaid link, we prove the following result that will be useful later on.

Lemma 1. *Let $n \in \mathbb{N}$ and let $a, b, c \in [0, +\infty]^n$. If (a, b, c) is an n -dimensional triangular triplet and $X = \{x, y, z\}$ with $\text{card}(X) = 3$, then there exists a collection $\{d_i\}_{i=1}^n$ of relaxed pseudo-metrics on X such that $d_i(x, y) = a_i$, $d_i(x, z) = b_i$ and $d_i(z, y) = c_i$ for all $i = 1, \dots, n$.*

Proof. Define, for each $i \in \{1, \dots, n\}$, the function $d_i : X \times X \rightarrow [0, +\infty]$ by $d_i(x, y) = d_i(y, x) = a_i$, $d_i(x, z) = d_i(z, x) = b_i$ and $d_i(z, y) = d_i(y, z) = c_i$. Clearly $d_i(u, v) = d_i(v, u)$ for all $u, v \in X$ and for all $i \in \{1, \dots, n\}$. Moreover, $d_i(u, v) \leq d_i(u, w) + d_i(w, v)$ for all $u, v, w \in X$ and for all $i \in \{1, \dots, n\}$, since (a, b, c) is an n -dimensional triangular triplet. So $\{d_i\}_{i=1}^n$ is a collection of relaxed pseudo-metrics on X . \square

The next result provides a sequence of equivalences revealing the aforementioned interlink.

Theorem 9. *Let $n \in \mathbb{N}$, let $\mathcal{T} = \{T_i\}_{i=1}^n$ be a collection of continuous Archimedean t -norms, and let $\{f_{T_i}\}_{i=1}^n$ be a collection of additive generators of \mathcal{T} . If T is a continuous Archimedean t -norm and $F : [0, 1]^n \rightarrow [0, 1]$ is a function, then the following assertions are equivalent:*

- (1) F aggregates \mathcal{T} -relaxed indistinguishability fuzzy relations into a T -relaxed indistinguishability fuzzy relation.
- (2) F transforms n -dimensional \mathcal{T} -triangular triplets into a one-dimensional T -triangular triplet.
- (3) The function $G : [0, +\infty]^n \rightarrow [0, f_T(0)]$ transforms n -dimensional triangular triplets into a one-dimensional triangular triplet in $[0, f_T(0)]$, where $G = f_T \circ F \circ (f_{T_1}^{(-1)} \times \dots \times f_{T_n}^{(-1)})$.
- (4) The function $G : [0, +\infty]^n \rightarrow [0, f_T(0)]$ aggregates every collection $\{d_i\}_{i=1}^n$ of relaxed pseudo-metrics into an $f_T(0)$ -bounded relaxed pseudo-metric, where $G = f_T \circ F \circ (f_{T_1}^{(-1)} \times \dots \times f_{T_n}^{(-1)})$.
- (5) The function $H : \prod_{i=1}^n [0, f_{T_i}(0)] \rightarrow [0, f_T(0)]$ aggregates every collection $\{d_i\}_{i=1}^n$ of $(f_{T_i}(0))_{i=1}^n$ -bounded relaxed pseudo-metrics into an $f_T(0)$ -bounded relaxed pseudo-metric, where $H = f_T \circ F \circ (f_{T_1}^{-1} \times \dots \times f_{T_n}^{-1})$.

(6) The function $H : \prod_{i=1}^n [0, f_{T_i}(0)] \rightarrow [0, f_T(0)]$ transforms n -dimensional triangular triplets in $\prod_{i=1}^n [0, f_{T_i}(0)]$ into a one-dimensional triangular triplet in $[0, f_T(0)]$, where $H = f_T \circ F \circ (f_{T_1}^{-1} \times \dots \times f_{T_n}^{-1})$.

Proof. (1) \Rightarrow (2). This implication is warranted by Theorem 8.

(2) \Rightarrow (3). Let (a, b, c) , with $a, b, c \in [0, +\infty]^n$, be an n -dimensional triangular triplet. Then

$$f_{T_i}^{(-1)}(u_i) \geq f_{T_i}^{(-1)}(v_i + w_i) \geq T_i(f_{T_i}^{(-1)}(v_i), f_{T_i}^{(-1)}(w_i))$$

for all $i \in \{1, \dots, n\}$ and for all $u, v, w \in \{a, b, c\}$. So $(\alpha, \beta, \gamma) \in [0, 1]^n$ forms an n -dimensional \mathcal{T} -triangular triplet, where $\alpha = (f_{T_1}^{(-1)}(a_1), \dots, f_{T_n}^{(-1)}(a_n))$, $\beta = (f_{T_1}^{(-1)}(b_1), \dots, f_{T_n}^{(-1)}(b_n))$ and $\gamma = (f_{T_1}^{(-1)}(c_1), \dots, f_{T_n}^{(-1)}(c_n))$. Thus, $(F(\alpha), F(\beta), F(\gamma))$ is a one-dimensional T -triangular triplet. It follows that

$$f_T \circ F(u) \leq f_T \circ T(F(v), F(w)) = f_T \circ f_T^{(-1)}(f_T \circ F(v) + f_T \circ F(w))$$

for all $u, v, w \in \{\alpha, \beta, \gamma\}$. Then

$$f_T \circ F(u) \leq f_T \circ F(v) + f_T \circ F(w)$$

for all $u, v, w \in \{\alpha, \beta, \gamma\}$, since

$$f_T \circ f_T^{(-1)}(f_T \circ F(v) + f_T \circ F(w)) \leq f_T \circ F(v) + f_T \circ F(w)$$

for all $u, v, w \in \{\alpha, \beta, \gamma\}$. Therefore

$$\begin{aligned} f_T \circ F \circ (f_{T_1}^{(-1)}(u_1) \times \dots \times f_{T_n}^{(-1)}(u_n)) &\leq \\ f_T \circ F \circ (f_{T_1}^{(-1)}(v_1) \times \dots \times f_{T_n}^{(-1)}(v_n)) &+ f_T \circ F \circ (f_{T_1}^{(-1)}(w_1) \times \dots \times f_{T_n}^{(-1)}(w_n)) \end{aligned}$$

for all $u, v, w \in \{a, b, c\}$. Whence we conclude that $(G(a), G(b), G(c))$ is a one-dimensional triangular triplet in $[0, f_T(0)]$.

(3) \Rightarrow (4). Consider a collection $\{d_i\}_{i=1}^n$ of relaxed pseudo-metrics on a non-empty set X . Symmetry is clear, since $G(d_1, \dots, d_n)(x, y) = G(d_1, \dots, d_n)(y, x)$. The fact that $\{d_i\}_{i=1}^n$ are relaxed pseudo-metrics on X gives that, for every $x, y, z \in X$, we have that $(d, e, f) \in [0, \infty]^n$ is an n -dimensional triangular triplet, where $d = (d_1(x, y), \dots, d_n(x, y))$, $e = (d_1(x, z), \dots, d_n(x, z))$ and $f = (d_1(z, y), \dots, d_n(z, y))$. Then $(G(d), G(e), G(f))$ is a one-dimensional triangular triplet in $[0, f_T(0)]$. Whence we have that

$$G(d_1, \dots, d_n)(u, v) \leq G(d_1, \dots, d_n)(u, w) + G(d_1, \dots, d_n)(w, v)$$

for all $u, v, w \in \{x, y, z\}$. Moreover, $G(d_1, \dots, d_n)(u, v) \leq f_T(0)$ for all $u, v \in \{x, y, z\}$. Therefore, $G(d_1, \dots, d_n)$ is an $f_T(0)$ -bounded relaxed pseudo-metric on X .

(4) \Rightarrow (5). It is obvious.

(5) \Rightarrow (6). Consider an n -dimensional triangular triplet (a, b, c) in $\prod_{i=1}^n [0, f_{T_i}(0)]$. Set $X = \{x, y, z\}$ with $\text{card}(X) = 3$. By Lemma 1, which remains valid for bounded n -dimensional triangular triplets, we have that there exists a collection of $(f_{T_i}(0))_{i=1}^n$ -bounded relaxed pseudo-metrics on X such that $d_i(x, y) = a_i$, $d_i(x, z) = b_i$ and $d_i(z, y) = c_i$ for all $i = 1, \dots, n$. Then $H(d_1, \dots, d_n)$ is a $f_T(0)$ -bounded relaxed pseudo-metric on X . Thus,

$$H(d_1, \dots, d_n)(u, v) \leq H(d_1, \dots, d_n)(u, w) + H(d_1, \dots, d_n)(w, v)$$

for all $u, v, w \in X$. It follows that $H(a) \leq H(b) + H(c)$, $H(b) \leq H(a) + H(c)$ and $H(c) \leq H(a) + H(b)$, where $H(a) = H(d_1(x, y), \dots, d_n(x, y))$, $H(b) = H(d_1(x, z), \dots, d_n(x, z))$ and $H(c) = H(d_1(z, y), \dots, d_n(z, y))$. Whence we obtain that $(H(a), H(b), H(c))$ is a one-dimensional triangular triplet in $[0, f_T(0)]$.

(6) \Rightarrow (1). Consider a collection $\{E_i\}_{i=1}^n$ of \mathcal{T} -relaxed indistinguishability fuzzy relations on a non-empty set X .

Next, consider $x, y, z \in X$. We need to show that

$$F(E_1, \dots, E_n)(u, v) \geq T(F(E_1, \dots, E_n)(u, w), F(E_1, \dots, E_n)(w, v))$$

for all $u, v, w \in \{x, y, z\}$. To this end, set $a_i = f_{T_i}(E_i(x, y))$, $b_i = f_{T_i}(E_i(y, z))$, and $c_i = f_{T_i}(E_i(x, z))$. We know that $T_i(E_i(u, w), E_i(w, v)) \leq E_i(u, v)$ for all $u, v, w \in \{x, y, z\}$. From this fact, we can infer that $a_i + b_i \geq c_i$, $b_i + c_i \geq a_i$ and $c_i + a_i \geq b_i$ for all $i = 1, \dots, n$. Next, we only show that $a_i + b_i \geq c_i$ for all $i = 1, \dots, n$. The remainder inequalities can be derived following similar arguments. From the fact that $T_i(E_i(x, y), E_i(y, z)) \leq E_i(x, z)$ for all $i = 1, \dots, n$ we obtain that

$$f_{T_i}^{(-1)}(f_{T_i}(E_i(x, y)) + f_{T_i}(E_i(y, z))) \leq E_i(x, z)$$

for all $i = 1, \dots, n$. Then $f_{T_i}^{(-1)}(a_i + b_i) \leq f_{T_i}^{(-1)}(c_i) = f_{T_i}^{-1}(c_i)$ for all $i = 1, \dots, n$. It follows that $f_{T_i} \circ f_{T_i}^{(-1)}(a_i + b_i) \geq c_i$ for all $i = 1, \dots, n$. Since $a_i + b_i \geq f_{T_i} \circ f_{T_i}^{(-1)}(a_i + b_i)$, we deduce that $a_i + b_i \geq c_i$ for all $i = 1, \dots, n$.

Consequently, we have that (a, b, c) is an n -dimensional triangular triplet in $\prod_{i=1}^n [0, f_{T_i}(0)]$. So $(H(a), H(b), H(c))$ is a one-dimensional triangular triplet in $[0, f_T(0)]$. Thus, $H(a) \leq H(b) + H(c)$, $H(b) \leq H(a) + H(c)$ and $H(c) \leq H(a) + H(b)$. From the preceding inequalities, we can obtain that

$$F(E_1, \dots, E_n)(u, v) \geq T(F(E_1, \dots, E_n)(u, w), F(E_1, \dots, E_n)(w, v))$$

for all $u, v, w \in \{x, y, z\}$. We only show that

$$F(E_1, \dots, E_n)(x, y) \geq T(F(E_1, \dots, E_n)(x, z), F(E_1, \dots, E_n)(z, y))$$

because the the same reasoning can be applied to prove the two remainder cases. Since $H(a) \leq H(b) + H(c)$, we deduce that

$$\begin{aligned} f_T \circ F \circ (f_{T_1}^{-1}(a_1) \times \dots \times f_{T_n}^{-1}(a_n)) &\leq \\ f_T \circ F \circ (f_{T_1}^{-1}(b_1) \times \dots \times f_{T_n}^{-1}(b_n)) + f_T \circ F \circ (f_{T_1}^{-1}(c_1) \times \dots \times f_{T_n}^{-1}(c_n)). \end{aligned}$$

Hence we obtain that

$$\begin{aligned} F \circ (f_{T_1}^{-1}(a_1) \times \dots \times f_{T_n}^{-1}(a_n)) &= f_T^{(-1)} \circ f_T \circ F \circ (f_{T_1}^{-1}(a_1) \times \dots \times f_{T_n}^{-1}(a_n)) \geq \\ f_T^{(-1)} \left(f_T \circ F \circ (f_{T_1}^{-1}(b_1) \times \dots \times f_{T_n}^{-1}(b_n)) + f_T \circ F \circ (f_{T_1}^{-1}(c_1) \times \dots \times f_{T_n}^{-1}(c_n)) \right) &= \\ T(F \circ (f_{T_1}^{-1}(b_1) \times \dots \times f_{T_n}^{-1}(b_n)), F \circ (f_{T_1}^{-1}(c_1) \times \dots \times f_{T_n}^{-1}(c_n))). \end{aligned}$$

Hence we conclude that

$$F(E_1, \dots, E_n)(x, y) \geq T(F(E_1, \dots, E_n)(x, z), F(E_1, \dots, E_n)(z, y)) \quad .$$

Therefore, $F(E_1, \dots, E_n)$ fulfills condition (R2). Clearly, $F(E_1, \dots, E_n)(u, v) = F(E_1, \dots, E_n)(v, u)$ for all $u, v \in X$. Therefore, $F(E_1, \dots, E_n)$ is a \mathcal{T} -relaxed indistinguishability fuzzy relation on X as claimed. \square

For strict continuous Archimedean t -norms, we have from Theorem 9 the following result.

Corollary 2. *Let $n \in \mathbb{N}$ and let $\mathcal{T} = \{T_i\}_{i=1}^n$ be a collection of strict continuous Archimedean t -norms. If T is a strict continuous Archimedean t -norm and $F : [0, 1]^n \rightarrow [0, 1]$ is a function, then the following assertions are equivalent:*

- (1) F aggregates \mathcal{T} -relaxed indistinguishability fuzzy relations into a T -relaxed indistinguishability fuzzy relation.
- (2) F transforms n -dimensional \mathcal{T} -triangular triplets into a one-dimensional T -triangular triplet.
- (3) The function $G : [0, +\infty]^n \rightarrow [0, +\infty]$ transforms n -dimensional triangular triplets into a one-dimensional triangular triplet, where $G = f_T \circ F \circ (f_{T_1}^{-1} \times \dots \times f_{T_n}^{-1})$.
- (4) The function $G : [0, +\infty]^n \rightarrow [0, +\infty]$ aggregates every collection $\{d_i\}_{i=1}^n$ of relaxed pseudo-metrics into a relaxed pseudo-metric, where $G = f_T \circ F \circ (f_{T_1}^{-1} \times \dots \times f_{T_n}^{-1})$.

The next result, which will be useful in our subsequent discussion, gives information about those functions that transform n -dimensional triangular triplets into a one-dimensional triangular triplet (see [19] [Proposition 7]). In order to introduce it, let us recall that, given $s \in]0, \infty]$, a function $G : [0, +\infty]^n \rightarrow [0, s]$ is subadditive provided that $G(a + b) \leq G(a) + G(b)$ for all $a, b \in [0, +\infty]^n$.

Proposition 1. *Let $n \in \mathbb{N}$ and let T be a continuous Archimedean t -norm with an additive generator f_T . If a function $G : [0, \infty]^n \rightarrow [0, f_T(0)]$ transforms n -dimensional triangular triplets into a one-dimensional triangular triplet in $[0, f_T(0)]$, then it is subadditive.*

In light of Proposition 1 and Theorem 9, we obtain the next result.

Corollary 3. *Let $n \in \mathbb{N}$ and let $\mathcal{T} = \{T_i\}_{i=1}^n$ be a collection of continuous Archimedean t -norms, and let $\{f_{T_i}\}_{i=1}^n$ be a collection of additive generators of \mathcal{T} . If T is a continuous Archimedean t -norm and $F : [0, 1]^n \rightarrow [0, 1]$ aggregates \mathcal{T} -relaxed indistinguishability fuzzy relations into a T -relaxed indistinguishability fuzzy relation, then the function $G : [0, +\infty]^n \rightarrow [0, f_T(0)]$ is subadditive, where $G = f_T \circ F \circ f_{T_1}^{(-1)} \times \dots \times f_{T_n}^{(-1)}$.*

In [36], the aggregation of relaxed indistinguishability fuzzy relations that separate points was not explored. Taking this fact into account, we obtain a characterization of those functions that are able to aggregate this type of \mathcal{T} -relaxed indistinguishability fuzzy relations extending Theorem 7 to the relaxed context and expanding the information provided about relaxed indistinguishability fuzzy relations in the aforementioned reference, and in addition, we give relationships between these functions and those that transform \mathcal{T} -triangle triplets and those that aggregate relaxed metrics. Outstanding differences between this class of functions and the class of functions aggregating \mathcal{T} -relaxed indistinguishability fuzzy relations are shown.

Theorem 10. *Let $n \in \mathbb{N}$, let $\mathcal{T} = \{T_i\}_{i=1}^n$ be a collection of continuous Archimedean t -norms, and let $\{f_{T_i}\}_{i=1}^n$ be a collection of additive generators of \mathcal{T} . If T is a continuous Archimedean t -norm and $F : [0, 1]^n \rightarrow [0, 1]$ is a function, then the following assertions are equivalent:*

- (1) F aggregates \mathcal{T} -relaxed indistinguishability fuzzy relations that separate points into a T -relaxed indistinguishability fuzzy relation that separates points.
- (2) F satisfies the following conditions:
 - (2.1) If $a, b, c \in [0, 1]^n$ such that $T_i(a_i, a_i) \leq b_i$ and $T_i(a_i, a_i) \leq c_i$ for all $i = 1, \dots, n$ with $a_i \in [0, 1[$ for all $i = 1, \dots, n$ and $F(a) = F(b) = F(c)$, then there exists $i \in \{1, \dots, n\}$ such that $a_i = b_i = c_i$.
 - (2.2) If $a, b, c \in [0, 1]^n$ such that (a, b, c) is an n -dimensional \mathcal{T} -triangular triplet, then $(F(a), F(b), F(c))$ is a one-dimensional T -triangular triplet.
- (3) The function $G : [0, +\infty]^n \rightarrow [0, f_T(0)]$, where $G = f_T \circ F \circ (f_{T_1}^{(-1)} \times \dots \times f_{T_n}^{(-1)})$, satisfies the following assertions:
 - (3.1) If $a, b, c \in \prod_{i=1}^n [0, f_{T_i}(0)]$ such that $f_{T_i}^{(-1)}(a_i + a_i) \leq f_{T_i}^{(-1)}(b_i)$ and $f_{T_i}^{(-1)}(a_i + a_i) \leq f_{T_i}^{(-1)}(c_i)$ for all $i = 1, \dots, n$ with $a_i \in]0, f_{T_i}(0)]$ for all $i = 1, \dots, n$ and $G(a) = G(b) = G(c)$, then there exists $i \in \{1, \dots, n\}$ such that $a_i = b_i = c_i$.

- 3.2) G transforms n -dimensional triangular triplets in $]0, +\infty[^n$ into a one-dimensional triangular triplet in $[0, f_T(0)]$.
- (4) The function $G : [0, +\infty[^n \rightarrow [0, f_T(0)]$ aggregates every collection $\{d_i\}_{i=1}^n$ of $(f_{T_i}(0))_{i=1}^n$ -bounded relaxed metrics into an $f_T(0)$ -bounded relaxed metric, where $G = f_T \circ F \circ (f_{T_1}^{(-1)} \times \dots \times f_{T_n}^{(-1)})$.
- (5) The function $H : \prod_{i=1}^n [0, f_{T_i}(0)] \rightarrow [0, f_T(0)]$ aggregates every collection $\{d_i\}_{i=1}^n$ of $(f_{T_i}(0))_{i=1}^n$ -bounded relaxed metrics into an $f_T(0)$ -bounded relaxed metric, where $H = f_T \circ F \circ (f_{T_1}^{-1} \times \dots \times f_{T_n}^{-1})$.
- (6) The function $H : \prod_{i=1}^n [0, f_{T_i}(0)] \rightarrow [0, f_T(0)]$, where $H = f_T \circ F \circ (f_{T_1}^{-1} \times \dots \times f_{T_n}^{-1})$, satisfies the following conditions:
- (6.1) If $a, b, c \in \prod_{i=1}^n [0, f_{T_i}(0)]$ such that $b_i \leq a_i + a_i$ and $c_i \leq a_i + a_i$ with $a_i \in]0, f_{T_i}(0)[$ for all $i = 1, \dots, n$ and $H(a) = H(b) = H(c)$, then there exists $i \in \{1, \dots, n\}$ such that $a_i = b_i = c_i$.
- (6.2) H transforms n -dimensional triangular triplets in $\prod_{i=1}^n]0, f_{T_i}(0)[$ into a one-dimensional triangular triplet in $[0, f_T(0)]$.

Proof. (1) \Rightarrow (2). First we prove that 2.1) holds. Suppose that $a, b, c \in [0, 1]^n$ such that $T_i(a_i, a_i) \leq b_i$ and $T_i(a_i, a_i) \leq c_i$ for all $i = 1, \dots, n$ with $a_i \in [0, 1[$ for all $i = 1, \dots, n$ and $F(a) = F(b) = F(c)$. For the purpose of contradiction, we assume that for all $i \in \{1, \dots, n\}$ we have that a_i, b_i, c_i are not equal, since otherwise we have the desired conclusion. Then consider the non-empty set $X = \{x, y\}$ with x, y different. Define on X the collection $\{E_i\}_{i=1}^n$ of \mathcal{T} -relaxed indistinguishability fuzzy relations that separate points as follows: $E_i(x, y) = E_i(y, x) = a_i$ and $E_i(x, x) = b_i$ and $E_i(y, y) = c_i$ for all $i = 1, \dots, n$. Since F aggregates \mathcal{T} -relaxed indistinguishability fuzzy relations that separate points into a T -relaxed indistinguishability fuzzy relation that separates points, we find that $F(E_1, \dots, E_n)$ is a T -relaxed indistinguishability fuzzy relation that separates points. Moreover, $F(a) = F(b) = F(c)$ and, thus $F(E_1, \dots, E_n)(x, y) = F(E_1, \dots, E_n)(x, x) = F(E_1, \dots, E_n)(y, y)$. So $x = y$ which is a contradiction.

Next, we prove (2.2). With this aim, we assume that (a, b, c) , with $a, b, c \in [0, 1]^n$, is a \mathcal{T} -triangular triplet. Consider a set $X = \{x, y, z\}$ with $card(X) = 3$. Define the collection of fuzzy binary relations $\{E_i\}_{i=1}^n$ on X given by $a_i = E_i(x, y) = E_i(y, x)$, $b_i = E_i(y, z) = E_i(z, y)$, $c_i = E_i(x, z) = E_i(z, x)$ and $E_i(x, x) = E_i(y, y) = E_i(z, z) = 1$ for all $i = 1, \dots, n$. Then $\{E_i\}_{i=1}^n$ is a collection of \mathcal{T} -relaxed indistinguishability fuzzy relations that separate points on X . So $F(E_1, \dots, E_n)$ is a T -relaxed indistinguishability fuzzy relation that separates points on X . Whence we deduce, by R2) that $T(F(a), F(b)) \leq F(c)$, $T(F(a), F(c)) \leq F(b)$, and $T(F(a), F(b)) \leq F(c)$. Therefore, we conclude that $(F(a), F(b), F(c))$ is a one-dimensional T -triangular triplet.

(2) \Rightarrow (1). Let X be a non-empty set and let $\{E_i\}_{i=1}^n$ be a collection of \mathcal{T} -relaxed indistinguishability fuzzy relations that separate points on X . Set $x, y, z \in X$ and assume that $F(E_1, \dots, E_n)(x, y) = F(E_1, \dots, E_n)(x, x) = F(E_1, \dots, E_n)(y, y)$. Take $a_i = E_i(x, y)$, $b_i = E_i(x, x)$ and $c_i = E_i(y, y)$ for all $i = 1, \dots, n$. Then $a, b, c \in [0, 1]^n$, $T_i(a_i, a_i) \leq b_i$ and $T_i(a_i, a_i) \leq c_i$ for all $i = 1, \dots, n$. Observe that we can suppose that $a_i \in [0, 1[$ because if $a_i = 1$ for any $i \in \{1, \dots, n\}$ we obtain that $E_i(x, y) = 1$ and, hence, $E_i(x, x) = E_i(y, y) = 1$ and, as a consequence, that $x = y$. The fact that $F(E_1, \dots, E_n)(x, y) = F(E_1, \dots, E_n)(x, x) = F(E_1, \dots, E_n)(y, y)$ gives that $F(a) = F(b) = F(c)$. Whence we deduce, by 2.1), that there exists $i \in \{1, \dots, n\}$ such that $a_i = b_i = c_i$. So $E_i(x, y) = E_i(x, x) = E_i(y, y)$ and $x = y$.

In the following, we show that

$$T(F(E_1, \dots, E_n)(u, v), F(E_1, \dots, E_n)(v, w)) \leq F(E_1, \dots, E_n)(u, w)$$

for all $u, v, w \in \{x, y, z\}$. Of course, we can assume that x, y, z are all different elements, since otherwise the preceding inequality is held immediately. So $E_i(x, y), E_i(y, z), E_i(x, z) \in [0, 1[$. Now set $a_i = E_i(x, y)$, $b_i = E_i(y, z)$ and $c_i = E_i(x, z)$ for all $i = 1, \dots, n$. Then (a, b, c) is a

\mathcal{T} -triangular triplet. It follows, from (2.2), that $(F(a), F(b), F(c))$ is a T -triangular triplet, and the above inequality is satisfied.

(2) \Rightarrow (3). Assume that $a, b, c \in \prod_{i=1}^n [0, f_{T_i}(0)]$ such that $f_{T_i}^{(-1)}(a_i + a_i) \leq f_{T_i}^{(-1)}(b_i)$ and $f_{T_i}^{(-1)}(a_i + a_i) \leq f_{T_i}^{(-1)}(c_i)$ for all $i = 1, \dots, n$ with $a_i \in]0, f_{T_i}(0)[$ for all $i = 1, \dots, n$ and $G(a) = G(b) = G(c)$. Set $d = (f_{T_1}^{(-1)}(a_1), \dots, f_{T_1}^{(-1)}(a_1))$, $e = (f_{T_1}^{(-1)}(b_1), \dots, f_{T_1}^{(-1)}(b_1))$ and, in addition, $g = (f_{T_1}^{(-1)}(c_1), \dots, f_{T_1}^{(-1)}(c_1))$. Clearly, $d, e, g \in [0, 1]^n$. Since $a_i \in]0, f_{T_i}(0)[$ for all $i = 1, \dots, n$, we have that $d \in [0, 1]^n$. Moreover, $T_i(d_i, d_i) \leq e_i$ for all $i = 1, \dots, n$, since

$$T_i(d_i, d_i) = T_i(f_{T_i}^{(-1)}(a_i), f_{T_i}^{(-1)}(a_i)) = f_{T_i}^{(-1)}(f_{T_i} \circ (f_{T_i}^{(-1)}(a_i)) + f_{T_i} \circ (f_{T_i}^{(-1)}(a_i))) = f_{T_i}^{(-1)}(a_i + a_i) \leq f_{T_i}^{(-1)}(b_i) = e_i.$$

Following the same arguments, we show that $T_i(d_i, d_i) \leq g_i$ for all $i = 1, \dots, n$.

The fact that $G(a) = G(b) = G(c)$ gives that $f_T(F(d)) = f_T(F(e)) = f_T(F(g))$. Since f_T is injective, by (2.1), we deduce that $F(d) = F(e) = F(g)$. So there exists $i \in \{1, \dots, n\}$ such that $d_i = e_i = g_i$. Whence we deduce that $f_{T_i}^{(-1)}(a_i) = f_{T_i}^{(-1)}(b_i) = f_{T_i}^{(-1)}(c_i)$. Therefore, we obtain that $a_i = b_i = c_i$ because $f_{T_i}^{(-1)}|_{[0, f_{T_i}(0)]} = f_{T_i}^{-1}$ and $f_{T_i}^{-1}$ is injective on $f_{T_i}([0, 1])$ (notice that f_{T_i} is continuous).

Next, consider an n -dimensional triangular triplet (a, b, c) in $]0, +\infty[^n$. Then $u_i \leq v_i + w_i$ for all $i = 1, \dots, n$ and for all $u, v, w \in \{a, b, c\}$. Thus,

$$f_{T_i}^{(-1)}(u_i) \geq f_{T_i}^{(-1)}(v_i + w_i) \geq T_i(f_{T_i}^{(-1)}(v_i), f_{T_i}^{(-1)}(w_i))$$

for all $u, v, w \in \{a, b, c\}$. Whence we deduce that $(f_{T_1}^{(-1)}(a_1), f_{T_1}^{(-1)}(b_1), f_{T_1}^{(-1)}(c_1))$ is a T_i -triangular triplet. So $(d, e, f) \in [0, 1]^n$ is an n -dimensional \mathcal{T} -triangular triplet with $d = (f_{T_1}^{(-1)}(a_1), \dots, f_{T_n}^{(-1)}(a_n))$, $e = (f_{T_1}^{(-1)}(b_1), \dots, f_{T_n}^{(-1)}(b_n))$ and $f = (f_{T_1}^{(-1)}(c_1), \dots, f_{T_n}^{(-1)}(c_n))$. Then, by (2.2), $(F(d), F(e), F(f))$ is a one-dimensional T -triangular triplet. Hence we have that

$$T(F(u), F(w)) \leq F(v)$$

for all $u, v, w \in \{d, e, f\}$. Whence we deduce that

$$f_T^{(-1)}(f_T \circ F(u) + f_T \circ F(w)) = T(F(u), F(w)) \leq F(v)$$

for all $u, v, w \in \{d, e, f\}$. Thus, we obtain

$$G(u) + G(w) = f_T \circ F(u) + f_T \circ F(w) \geq f_T \circ (f_T^{(-1)}(f_T \circ F(u) + f_T \circ F(w))) \geq f_T \circ F(v) = G(v).$$

for all $u, v, w \in \{a, b, c\}$. Therefore, $(G(a), G(b), G(c))$ is a one-dimensional triangular triplet in $[0, f_T(0)]$.

(3) \Rightarrow (4). Consider a collection $\{d_i\}_{i=1}^n$ of $(f_{T_i}(0))_{i=1}^n$ -bounded relaxed metrics on a non-empty set X . We show that $G(d_1, \dots, d_n)$ is a $f_T(0)$ -bounded relaxed metric on X .

Suppose that

$$G(d_1, \dots, d_n)(x, y) = G(d_1, \dots, d_n)(x, x) = G(d_1, \dots, d_n)(y, y)$$

for any $x, y \in X$. We can assume that $d_i(x, y) \in]0, f_{T_i}(0)[$ because otherwise we have that $d_i(x, y) = d_i(x, x) = d_i(y, y) = 0$ and, thus, that $x = y$.

Since, for any $x, y \in X, d_i(x, y), d_i(x, x) \in [0, f_{T_i}(0)]$ for all $i = 1, \dots, n$ and, in addition, $d_i(x, x) \leq d_i(x, y) + d_i(y, x)$ we obtain

$$f_{T_i}^{(-1)}(d_i(x, y) + d_i(x, y)) \leq f_{T_i}^{(-1)}(d_i(x, x))$$

for all $i = 1, \dots, n$. Similarly, we obtain

$$f_{T_i}^{(-1)}(d_i(x, y) + d_i(x, y)) \leq f_{T_i}^{(-1)}(d_i(y, y))$$

for all $i = 1, \dots, n$. Set $a = (d_1(x, y), \dots, d_n(x, y)), b = (d_1(x, x), \dots, d_n(x, x))$ and $c = (d_1(y, y), \dots, d_n(y, y))$. Then $G(a) = G(b) = G(c)$. So there exists $i \in \{1, \dots, n\}$ such that $a_i = b_i = c_i$. Thus, $d_i(x, y) = d_i(x, x) = d_i(y, y)$, and hence, $x = y$.

Clearly, $G(d_1, \dots, d_n)(x, y) = G(d_1, \dots, d_n)(y, x)$ for all $x, y \in X$.

Finally, we prove that the triangle inequality

$$G(d_1, \dots, d_n)(x, z) \leq G(d_1, \dots, d_n)(x, y) + G(d_1, \dots, d_n)(y, z)$$

is held for each $x, y, z \in X$. Observe that without loss of generality we can check the triangle inequality only for elements $x, y, z \in X$ with $u \neq v$ for all $u, v \in \{x, y, z\}$, and thus, we can assume that the cardinality of X is at least three. Otherwise, the triangle inequality is clearly satisfied.

By hypothesis, we have that $d_i(u, v) \neq 0$ for all $u, v \in \{x, y, z\}$ and for all $i = 1, \dots, n$. Otherwise, if there exists $i \in \{1, \dots, n\}$ such that $d_i(u, v) = 0$ for some $u, v \in \{x, y, z\}$ with $u \neq v$, then $d_i(u, u) = d_i(v, v) = 0$ and $u = v$, which is not possible.

Define $a = (d_1(x, z), d_2(x, z), \dots, d_n(x, z)), b = (d_1(x, y), d_2(x, y), \dots, d_n(x, y))$ and $c = (d_1(y, z), d_2(y, z), \dots, d_n(y, z))$. By our assumptions, (a, b, c) , with $a, b, c \in \prod_{i=1}^n [0, f_{T_i}(0)] \subseteq]0, +\infty]^n$, is an n -dimensional triangular triplet. This gives, by 3.2), that $(G(a), G(b), G(c))$ is a one-dimensional triangular triplet in $[0, f_T(0)]$. It follows that

$$G(d_1, \dots, d_n)(u, v) \leq$$

$$G(d_1, \dots, d_n)(u, w) + G(d_1, \dots, d_n)(w, v)$$

for all $u, v, w \in \{x, y, z\}$.

(4) \Rightarrow (5). It is obvious.

(5) \Rightarrow (6). We first prove 6.1). Let $a, b, c \in \prod_{i=1}^n [0, f_{T_i}(0)]$ such that $b_i \leq a_i + a_i$ and $c_i \leq a_i + a_i$ with $a_i \in]0, f_{T_i}(0)]$ for all $i = 1, \dots, n$ and $H(a) = H(b) = H(c)$. For the purpose of contradiction, we assume that $\{a_i, b_i, c_i\}$ are not equal for all $i = 1, \dots, n$.

Set $X = \{x, y\}$ with x, y different. Define the function $d_i : X \times X \rightarrow [0, f_{T_i}(0)]$ by $d_i(x, y) = d_i(y, x) = a_i$ and $d_i(x, x) = b_i$ and $d_i(y, y) = c_i$ for all $i = 1, \dots, n$. Then $\{d_i\}_{i=1}^n$ is a collection of $(f_{T_i}(0))_{i=1}^n$ -bounded relaxed metrics on X . Thus, $H(d_1, \dots, d_n)$ is an $f_T(0)$ -bounded relaxed metric on X . Since $H(a) = H(b) = H(c)$, we have that

$$H(d_1, \dots, d_n)(x, y) = H(d_1, \dots, d_n)(x, x) = H(d_1, \dots, d_n)(y, y),$$

which is impossible because $x \neq y$. Therefore, there exists $i \in \{1, \dots, n\}$ such that $a_i = b_i = c_i$.

In the following, we prove 6.2). Consider an n -dimensional triangular triplet (a, b, c) with $a, b, c \in \prod_{i=1}^n [0, f_{T_i}(0)]$. Set $X = \{x, y, z\}$ with $card(X) = 3$. Define, for each $i \in \{1, \dots, n\}$, the function $d_i : X \times X \rightarrow [0, +\infty]$ by $d_i(x, y) = d_i(y, x) = a_i, d_i(x, z) = d_i(z, x) = b_i, d_i(z, y) = d_i(y, z) = c_i$ and $d_i(x, x) = d_i(y, y) = d_i(z, z) = 0$. It is not hard to check that $\{d_i\}_{i=1}^n$ is a collection of $(f_{T_i}(0))_{i=1}^n$ -bounded relaxed metrics on X . Then $H(d_1, \dots, d_n)$ is an $f_T(0)$ -bounded relaxed metric on X . Whence we deduce that

$$H(d_1, \dots, d_n)(u, v) \leq H(d_1, \dots, d_n)(u, w) + H(d_1, \dots, d_n)(w, v)$$

for all $u, v, w \in \{x, y, z\}$. It follows that $(H(a), H(b), H(c))$ is a one-dimensional triangular triplet in $[0, f_T(0)]$.

(6) \Rightarrow (2). Next, we prove (2.1). Let $a, b, c \in [0, 1]^n$ such that $T_i(a_i, a_i) \leq b_i$ and $T_i(a_i, a_i) \leq c_i$ for all $i = 1, \dots, n$ with $a_i \in [0, 1[$ for all $i = 1, \dots, n$. The fact that $T_i(a_i, a_i) \leq b_i$ gives that $f_{T_i}^{(-1)}(f_{T_i}(a_i) + f_{T_i}(a_i)) \leq b_i$. Thus, we obtain, for all $i = 1, \dots, n$, that

$$f_{T_i}(b_i) \leq f_{T_i} \circ f_{T_i}^{(-1)}(f_{T_i}(a_i) + f_{T_i}(a_i)) \leq f_{T_i}(a_i) + f_{T_i}(a_i).$$

Following the same arguments, we obtain that $f_{T_i}(c_i) \leq f_{T_i}(a_i) + f_{T_i}(a_i)$.

Put $g, s, r \in \prod_{i=1}^n [0, f_{T_i}(0)]$ given by

$$g_i = f_{T_i}(a_i), s_i = f_{T_i}(b_i), r_i = f_{T_i}(c_i)$$

for all $i = 1, \dots, n$. Clearly, $g_i \in]0, f_{T_i}(0)]$ because, for any $i \in \{1, \dots, n\}$, $g_i = 0 \Leftrightarrow a_i = 1$. Moreover, we have that $s_i \leq g_i + g_i$ and $r_i \leq g_i + g_i$ for all $i = 1, \dots, n$.

Observe that $H(g) = f_T \circ F(a)$, $H(s) = f_T \circ F(b)$ and $H(r) = f_T \circ F(c)$. Suppose that $F(a) = F(b) = F(c)$. Then $H(g) = H(r) = H(s)$, and thus, there exists $i \in \{1, \dots, n\}$ with $g_i = r_i = s_i$. It follows that $f_{T_i}(a_i) = f_{T_i}(b_i) = f_{T_i}(c_i)$. The injectivity of f_T provides that $a_i = b_i = c_i$, as claimed.

Finally, we prove (2.2). Let $a, b, c \in [0, 1]^n$ such that (a, b, c) is an n -dimensional \mathcal{T} -triangular triplet. Then (e, f, g) is an n -dimensional triangular triplet in $\prod_{i=1}^n]0, f_{T_i}(0)]$, where $e = (f_{T_1}(a_1), \dots, f_{T_n}(a_n))$, $f = (f_{T_1}(b_1), \dots, f_{T_n}(b_n))$ and $g = (f_{T_1}(c_1), \dots, f_{T_n}(c_n))$. Indeed, we have that $T_i(u_i, v_i) \leq w_i$ for all $i = 1, \dots, n$ and for all $u, v, w \in \{a, b, c\}$. Hence we obtain that

$$f_{T_i}^{(-1)}(f_{T_i}(u_i) + f_{T_i}(v_i)) \leq w_i$$

for all $i = 1, \dots, n$ and for all $u, v, w \in \{a, b, c\}$. Whence we deduce that

$$f_{T_i} \circ f_{T_i}^{(-1)}(f_{T_i}(u_i) + f_{T_i}(v_i)) \geq f_{T_i}(w_i)$$

for all $i = 1, \dots, n$ and for all $u, v, w \in \{a, b, c\}$. Since

$$f_{T_i}(u_i) + f_{T_i}(v_i) \geq f_{T_i} \circ f_{T_i}^{(-1)}(f_{T_i}(u_i) + f_{T_i}(v_i))$$

we deduce that

$$f_{T_i}(u_i) + f_{T_i}(v_i) \geq f_{T_i}(w_i)$$

for all $i = 1, \dots, n$ and for all $u, v, w \in \{a, b, c\}$. Consequently, $(H(e), H(f), H(g))$ is a one-dimensional triangular triplet in $]0, f_T(0)]$. Thus,

$$f_T \circ F(u_1, \dots, u_n) \leq f_T \circ F(v_1, \dots, v_n) + f_T \circ F(w_1, \dots, w_n)$$

for all $u, v, w \in \{a, b, c\}$, since $H(e) = f_T \circ F(a_1, \dots, a_n)$, $H(f) = f_T \circ F(b_1, \dots, b_n)$ and $H(g) = f_T \circ F(c_1, \dots, c_n)$. Thus, we find that

$$f_T^{(-1)} \circ f_T \circ F(u_1, \dots, u_n) \geq f_T^{(-1)}(f_T \circ F(v_1, \dots, v_n) \circ f_T \circ F(w_1, \dots, w_n))$$

for all $u, v, w \in \{a, b, c\}$. So we have that

$$F(u) = F(u_1, \dots, u_n) \geq f_T^{(-1)}(f_T \circ F(v_1, \dots, v_n) + f_T \circ F(w_1, \dots, w_n)) = T(F(v), F(w))$$

for all $u, v, w \in \{a, b, c\}$. Therefore, $(F(a), F(b), F(c))$ is a one-dimensional T -triangular triplet. \square

From the preceding result, we retrieve the following one when all t -norms under consideration are strict.

Corollary 4. Let $n \in \mathbb{N}$, let $\mathcal{T} = \{T_i\}_{i=1}^n$ be a collection of strict continuous Archimedean t -norms, and let $\{f_{T_i}\}_{i=1}^n$ be a collection of additive generators of \mathcal{T} . If T is a strict continuous Archimedean t -norm and $F : [0, 1]^n \rightarrow [0, 1]$ is a function, then the following assertions are equivalent:

- (1) F aggregates \mathcal{T} -relaxed indistinguishability fuzzy relations that separate points into a T -relaxed indistinguishability fuzzy relation that separates points.
- (2) F satisfies the following conditions:
 - (2.1) If $a, b, c \in [0, 1]^n$ such that $T_i(a_i, a_i) \leq b_i$ and $T_i(a_i, a_i) \leq c_i$ for all $i = 1, \dots, n$ with $a_i \in [0, 1[$ for all $i = 1, \dots, n$ and $F(a) = F(b) = F(c)$, then there exists $i \in \{1, \dots, n\}$ such that $a_i = b_i = c_i$.
 - (2.2) If $a, b, c \in [0, 1]^n$ such that (a, b, c) is an n -dimensional \mathcal{T} -triangular triplet, then $(F(a), F(b), F(c))$ is a one-dimensional T -triangular triplet.
- (3) The function $G : [0, +\infty]^n \rightarrow [0, +\infty]$, where $G = f_T \circ F \circ (f_{T_1}^{-1} \times \dots \times f_{T_n}^{-1})$, satisfies the following assertions:
 - (3.1) If $a, b, c \in [0, +\infty]$ such that $f_{T_i}^{-1}(a_i + a_i) \leq f_{T_i}^{-1}(b_i)$ and $f_{T_i}^{-1}(a_i + a_i) \leq f_{T_i}^{-1}(c_i)$ for all $i = 1, \dots, n$ with $a_i \in]0, +\infty]$ for all $i = 1, \dots, n$ and $G(a) = G(b) = G(c)$, then there exists $i \in \{1, \dots, n\}$ such that $a_i = b_i = c_i$.
 - (3.2) G transforms n -dimensional triangular triplets in $]0, +\infty]^n$ into a one-dimensional triangular triplet in $[0, +\infty]$.
- (4) The function $G : [0, +\infty]^n \rightarrow [0, +\infty]$ aggregates every collection $\{d_i\}_{i=1}^n$ of relaxed metrics into a relaxed metric, where $G = f_T \circ F \circ (f_{T_1}^{-1} \times \dots \times f_{T_n}^{-1})$.
- (5) The function $H : [0, +\infty]^n \rightarrow [0, +\infty]$, where $H = f_T \circ F \circ (f_{T_1}^{-1} \times \dots \times f_{T_n}^{-1})$, satisfies the following conditions:
 - (5.1) If $a, b, c \in [0, +\infty]^n$ such that $b_i \leq a_i + a_i$ and $c_i \leq a_i + a_i$ with $a_i \in]0, +\infty]$ for all $i = 1, \dots, n$ and $H(a) = H(b) = H(c)$, then there exists $i \in \{1, \dots, n\}$ such that $a_i = b_i = c_i$.
 - (5.2) H transforms n -dimensional triangular triplets in $]0, +\infty]^n$ into a one-dimensional triangular triplet in $[0, +\infty]$.

3. Aggregation of SSI- \mathcal{T} -Relaxed Indistinguishability Relations

In this section, we focus our attention on the study of the distinguished class of SSI- \mathcal{T} -relaxed indistinguishability fuzzy relations. Concretely, we show that functions merging this special type of relaxed indistinguishability fuzzy relations can be expressed through functions aggregating SSD-relaxed pseudo-metrics. It must be pointed out that there is a notable difference between the class of functions that are able to aggregate a collection of relaxed indistinguishability fuzzy relations (relaxed pseudo-metrics) and SSI-relaxed indistinguishability fuzzy relations (SSD-relaxed pseudo-metrics). It must be stressed that SSI-relaxed indistinguishability fuzzy relations are known as weakly reflexive fuzzy relations in [51].

The next characterization was obtained for functions that aggregate SSI-relaxed indistinguishability fuzzy relations in [36] [Theorem 32]. Let us recall that a function $G : [0, s]^n \rightarrow [0, s]$, $s \in]0, \infty]$, is monotone when $G(a) \leq G(b)$ for all $a, b \in [0, s]^n$ with $a_i \leq b_i$ for all $i = 1, \dots, n$.

Theorem 11. Let $n \in \mathbb{N}$ and let $\mathcal{T} = \{T_i\}_{i=1}^n$ be a collection of t -norms. If T is a t -norm and $F : [0, 1]^n \rightarrow [0, 1]$ is a function, then the following assertions are equivalent:

- (1) F aggregates SSI- \mathcal{T} -relaxed indistinguishability fuzzy relations into an SSI- T -relaxed indistinguishability fuzzy relation.
- (2) F holds the following conditions:
 - (2.1) F is monotone.
 - (2.2) F transforms n -dimensional \mathcal{T} -triangular triplets into a one-dimensional T -triangular triplet.

In view of the preceding result, we introduce a few equivalences expressed in terms of functions aggregating SSD-relaxed pseudo-metrics and transforming \mathcal{T} -triangle triplets. In order to state such equivalences, we need the next auxiliary result which was proved in [19] [Proposition 7].

Proposition 2. *Let $n \in \mathbb{N}$ and let T be a continuous Archimedean t -norm with an additive generator f_T . If a function $G : [0, \infty]^n \rightarrow [0, f_T(0)]$ is monotone and subadditive, then it transforms n -dimensional triangular triplets into a one-dimensional triangular triplet in $[0, f_T(0)]$.*

The promised equivalences are given in the result below.

Theorem 12. *Let $n \in \mathbb{N}$, let $\mathcal{T} = \{T_i\}_{i=1}^n$ be a collection of continuous Archimedean t -norms, and let $\{f_{T_i}\}_{i=1}^n$ be a collection of additive generators of \mathcal{T} . If T is a continuous Archimedean t -norm and $F : [0, 1]^n \rightarrow [0, 1]$ is a function, then the following assertions are equivalent:*

- (1) *F aggregates SSI- \mathcal{T} -relaxed indistinguishability fuzzy relations into a SSI- T -relaxed indistinguishability fuzzy relation.*
- (2) *F holds the following conditions:*
 - (2.1) *F is monotone.*
 - (2.2) *F transforms n -dimensional \mathcal{T} -triangular triplets into a one-dimensional T -triangular triplet.*
- (3) *The function $G : [0, +\infty]^n \rightarrow [0, f_T(0)]$ is monotone and subadditive, where $G = f_T \circ F \circ (f_{T_1}^{(-1)} \times \dots \times f_{T_n}^{(-1)})$.*
- (4) *The function $G : [0, +\infty]^n \rightarrow [0, f_T(0)]$ is monotone and transforms n -dimensional triangular triplets into a one-dimensional triangular triplet in $[0, f_T(0)]$, where $G = f_T \circ F \circ (f_{T_1}^{(-1)} \times \dots \times f_{T_n}^{(-1)})$.*
- (5) *The function $G : [0, +\infty]^n \rightarrow [0, f_T(0)]$ aggregates every collection $\{d_i\}_{i=1}^n$ of SSD-relaxed pseudo-metrics into an SSD- $f_T(0)$ -bounded relaxed pseudo-metric, where $G = f_T \circ F \circ (f_{T_1}^{(-1)} \times \dots \times f_{T_n}^{(-1)})$.*
- (6) *The function $H : \prod_{i=1}^n [0, f_{T_i}(0)] \rightarrow [0, f_T(0)]$ aggregates every collection $\{d_i\}_{i=1}^n$ of SSD- $(f_{T_i}(0))_{i=1}^n$ -bounded relaxed pseudo-metrics into an SSD- $f_T(0)$ -bounded relaxed pseudo-metric, where $H = f_T \circ F \circ (f_{T_1}^{-1} \times \dots \times f_{T_n}^{-1})$.*
- (7) *The function $H : \prod_{i=1}^n [0, f_{T_i}(0)] \rightarrow [0, f_T(0)]$ is monotone and transforms n -dimensional triangular triplets in $\prod_{i=1}^n [0, f_{T_i}(0)]$ into a one-dimensional triangular triplet in $[0, f_T(0)]$, where $H = f_T \circ F \circ (f_{T_1}^{-1} \times \dots \times f_{T_n}^{-1})$.*

Proof. (1) \Rightarrow (2). Theorem 11 guarantees such an implication.

(2) \Rightarrow (3). Theorem 9 gives that condition (2.2) implies that G transforms n -dimensional triangular triplets into a one-dimensional triangular triplet in $[0, f_T(0)]$. Thus, by Proposition 1, we have that G is subadditive.

Next, we show that G is monotone provided that F is monotone. Indeed, consider $a, b \in [0, +\infty]^n$ with $a_i \leq b_i$ for all $i = 1, \dots, n$. Then $f_{T_i}^{(-1)}(a_i) \geq f_{T_i}^{(-1)}(b_i)$ for all $i = 1, \dots, n$. Whence we have that

$$F \circ (f_{T_1}^{(-1)}(a_1) \times \dots \times f_{T_n}^{(-1)}(a_n)) \geq F \circ (f_{T_1}^{(-1)}(b_1) \times \dots \times f_{T_n}^{(-1)}(b_n)),$$

since F is monotone. Hence

$$G(a) = f_T \circ F \circ (f_{T_1}^{(-1)}(a_1) \times \dots \times f_{T_n}^{(-1)}(a_n)) \leq f_T \circ F \circ (f_{T_1}^{(-1)}(b_1) \times \dots \times f_{T_n}^{(-1)}(b_n)) = G(b).$$

So G is monotone.

(3) \Rightarrow (4). Proposition 2 gives that G transforms n -dimensional triangular triplets into a one-dimensional triangular triplet in $[0, f_T(0)]$.

(4) \Rightarrow (5). The implication (3) \Rightarrow (4) in Theorem 9 provides that the function $G : [0, +\infty]^n \rightarrow [0, f_T(0)]$ aggregates every collection $\{d_i\}_{i=1}^n$ of relaxed pseudo-metrics on X into an $f_T(0)$ -bounded relaxed pseudo-metric on X . We only need to show that G aggregates SSD-relaxed pseudo-metrics into a SSD- $f_T(0)$ -bounded relaxed pseudo-metric.

Suppose that $d_i(x, x) \leq d_i(x, y)$ for all $x, y \in X$ and for all $i = 1, \dots, n$. The monotony of G gives that

$$G(d_1(x, x), \dots, d_n(x, x)) \leq G(d_1(x, y), \dots, d_n(x, y))$$

for all $x, y \in X$. So $G(d_1, \dots, d_n)$ is an SSD- $f_T(0)$ -bounded relaxed pseudo-metric on X .

(5) \Rightarrow (6). It is obvious.

(6) \Rightarrow (7). The proof of the fact that function H transforms n -dimensional triangular triplets in $\prod_{i=1}^n [0, f_{T_i}(0)]$ into a one-dimensional triangular triplet in $[0, f_T(0)]$ is the same as that given for the implication (5) \Rightarrow (6) in Theorem 9.

It remains to prove that H is monotone. With this aim, let $a, b \in \prod_{i=1}^n [0, f_{T_i}(0)]$ such that $a_i \leq b_i$ for all $i = 1, \dots, n$. Set $X = \{x, y\}$ with $x \neq y$. Define the function $d_i : X \times X \rightarrow [0, f_{T_i}(0)]$ by $d_i(x, x) = d_i(y, y) = a_i$ and $d_i(x, y) = d_i(y, x) = b_i$ for all $i = 1, \dots, n$. It is clear that $\{d_i\}_{i=1}^n$ is a collection of SSD- $(f_{T_i}(0))_{i=1}^n$ -bounded relaxed pseudo-metrics on X . Then $H(d_1, \dots, d_n)$ is an SSD- $f_T(0)$ -relaxed pseudo-metric on X . Hence $H(d_1, \dots, d_n)(x, x) \leq H(d_1, \dots, d_n)(x, y)$. Then

$$\begin{aligned} H(a) &= H(d_1(x, x), \dots, d_n(x, x)) = H(d_1, \dots, d_n)(x, x) \leq \\ &H(d_1, \dots, d_n)(x, y) = H(d_1(x, y), \dots, d_n(x, y)) = H(b). \end{aligned}$$

Whence we deduce that H is monotone.

(7) \Rightarrow (1). Suppose that $\{E_i\}_{i=1}^n$ is a collection of SSI- \mathcal{T} -relaxed indistinguishability fuzzy relations on a non-empty set X . By (6) \Rightarrow (1) in Theorem 9, we have that $F(E_1, \dots, E_n)$ is a T -relaxed indistinguishability fuzzy relation on X . Next, we show that $F(E_1, \dots, E_n)(x, y) \leq F(E_1, \dots, E_n)(x, x)$ for all $x, y \in X$. Since H is monotone and $F = f_T^{-1} \circ H \circ (f_{T_1} \times \dots \times f_{T_n})$, we obtain that F is monotone. $F(E_1, \dots, E_n)(x, y) \leq F(E_1, \dots, E_n)(x, x)$, since $E_i(x, y) \leq E_i(x, x)$ for all $x, y \in X$ and for all $i = 1, \dots, n$. Hence we deduce that $F(E_1, \dots, E_n)$ is an SSI- \mathcal{T} -relaxed indistinguishability fuzzy relation on X , which completes the proof. \square

It is worth mentioning that, according to [36], the functions F that aggregate SSI- \mathcal{T} -relaxed indistinguishability fuzzy relations into an SSI- T -relaxed indistinguishability fuzzy relation are exactly those that dominate the t -norm T with respect to \mathcal{T} .

We can derive the next result from Theorem 12 for strict continuous Archimedean t -norms.

Corollary 5. Let $n \in \mathbb{N}$, and let $\mathcal{T} = \{T_i\}_{i=1}^n$ be a collection of strict continuous Archimedean t -norms. If T is a strict continuous Archimedean t -norm and $F : [0, 1]^n \rightarrow [0, 1]$ is a function, then the following assertions are equivalent:

- (1) F aggregates SSI- \mathcal{T} -relaxed indistinguishability fuzzy relations into an SSI- T -relaxed indistinguishability fuzzy relation.
- (2) F holds the following conditions:
 - (2.1) F is monotone.
 - (2.2) F transforms n -dimensional \mathcal{T} -triangular triplets into a one-dimensional T -triangular triplet.
- (3) The function $G : [0, +\infty]^n \rightarrow [0, +\infty]$ is monotone and subadditive, where $G = f_T \circ F \circ (f_{T_1}^{-1} \times \dots \times f_{T_n}^{-1})$.
- (4) The function $G : [0, +\infty]^n \rightarrow [0, +\infty]$ is monotone and transforms n -dimensional triangular triplets into a one-dimensional triangular triplet, where $G = f_T \circ F \circ (f_{T_1}^{-1} \times \dots \times f_{T_n}^{-1})$.

- (5) The function $G : [0, +\infty]^n \rightarrow [0, +\infty]$ aggregates every collection $\{d_i\}_{i=1}^n$ of SSD-relaxed pseudo-metrics into an SSD-relaxed pseudo-metric, where $G = f_T \circ F \circ (f_{T_1}^{-1} \times \dots \times f_{T_n}^{-1})$.

We end the section taking into consideration those functions that are able to aggregate SSI- \mathcal{T} -relaxed indistinguishability fuzzy relations that separate points. We refer the reader to [43] [Example 3] for a non-trivial instance of an SSI-relaxed T -indistinguishability fuzzy relation that separates points.

Theorem 13. Let $n \in \mathbb{N}$, let $\mathcal{T} = \{T_i\}_{i=1}^n$ be a collection of continuous Archimedean t -norms, and let $\{f_{T_i}\}_{i=1}^n$ be a collection of additive generators of \mathcal{T} . If T is a continuous Archimedean t -norm and $F : [0, 1]^n \rightarrow [0, 1]$ is a function, then the following assertions are equivalent:

- (1) F aggregates SSI- \mathcal{T} -relaxed indistinguishability fuzzy relations that separate points into an SSI- T -relaxed indistinguishability fuzzy relation that separates points.
- (2) F satisfies the following conditions:
 - (2.1) If $a, b, c \in [0, 1]^n$ such that $a_i \leq b_i$ and $a_i \leq c_i$ for all $i = 1, \dots, n$ and $a_i \in [0, 1[$ for all $i = 1, \dots, n$ and $F(a) = F(b) = F(c)$, then there exists $i \in \{1, \dots, n\}$ such that $a_i = b_i = c_i$. Moreover, if for each $i \in \{1, \dots, n\}$, either $a_i < b_i$ or $a_i < c_i$, then $F(a) \leq F(b)$ and $F(a) \leq F(c)$.
 - (2.2) If $a, b, c \in [0, 1]^n$ such that (a, b, c) is an n -dimensional \mathcal{T} -triangular triplet, then $(F(a), F(b), F(c))$ is a one-dimensional T -triangular triplet.
- (3) The function $G : [0, +\infty]^n \rightarrow [0, f_T(0)]$, where $G = f_T \circ F \circ (f_{T_1}^{(-1)} \times \dots \times f_{T_n}^{(-1)})$, satisfies the following assertions:
 - (3.1) Let $a, b, c \in \prod_{i=1}^n]0, f_{T_i}(0)]$ such that $a_i \geq b_i$ and $a_i \geq c_i$ for all $i = 1, \dots, n$ with $a_i \in]0, f_{T_i}(0)]$ for all $i = 1, \dots, n$. If $G(a) = G(b) = G(c)$, then there exists $i \in \{1, \dots, n\}$ such that $a_i = b_i = c_i$. Moreover, if for each $i \in \{1, \dots, n\}$, either $a_i > b_i$ or $a_i > c_i$, then $G(a) \geq G(b)$ and $G(a) \geq G(c)$.
 - (3.2) G transforms n -dimensional triangular triplets in $]0, +\infty]^n$ into a one-dimensional triangular triplet in $[0, f_T(0)]$.
- (4) The function $G : [0, +\infty]^n \rightarrow [0, f_T(0)]$ aggregates every collection $\{d_i\}_{i=1}^n$ of SSD- $(f_{T_i}(0))_{i=1}^n$ -bounded relaxed metrics into an SSI- $f_T(0)$ -bounded relaxed metric, where $G = f_T \circ F \circ (f_{T_1}^{(-1)} \times \dots \times f_{T_n}^{(-1)})$.
- (5) The function $H : \prod_{i=1}^n]0, f_{T_i}(0)] \rightarrow [0, f_T(0)]$ aggregates every collection $\{d_i\}_{i=1}^n$ of SSD- $(f_{T_i}(0))_{i=1}^n$ -bounded relaxed metrics into a SSD- $f_T(0)$ -bounded relaxed metric, where $H = f_T \circ F \circ (f_{T_1}^{-1} \times \dots \times f_{T_n}^{-1})$.
- (6) The function $H : \prod_{i=1}^n]0, f_{T_i}(0)] \rightarrow [0, f_T(0)]$, where $H = f_T \circ F \circ (f_{T_1}^{-1} \times \dots \times f_{T_n}^{-1})$, satisfies the following conditions:
 - (6.1) Let $a, b, c \in \prod_{i=1}^n]0, f_{T_i}(0)]$ such that $a_i \geq b_i$ and $a_i \geq c_i$ for all $i = 1, \dots, n$ and $a_i \in]0, f_{T_i}(0)]$ for all $i = 1, \dots, n$ with $a_i \in]0, f_{T_i}(0)]$ for all $i = 1, \dots, n$. If $H(a) = H(b) = H(c)$, then there exists $i \in \{1, \dots, n\}$ such that $a_i = b_i = c_i$. Moreover, if for each $i \in \{1, \dots, n\}$, either $a_i > b_i$ or $a_i > c_i$, then $H(a) \geq H(b)$ and $H(a) \geq H(c)$.
 - (6.2) H transforms n -dimensional triangular triplets in $\prod_{i=1}^n]0, f_{T_i}(0)]$ into a one-dimensional triangular triplet in $[0, f_T(0)]$.

Proof. (1) \Rightarrow (2). Next, we prove (2.1). Suppose that $a, b, c \in [0, 1]^n$ such that $a_i \leq b_i$ and $a_i \leq c_i$ for all $i = 1, \dots, n$ with $a_i \in [0, 1[$ for all $i = 1, \dots, n$ and $F(a) = F(b) = F(c)$. For the purpose of contradiction, we can assume that, for all $i \in \{1, \dots, n\}$, we have that a_i, b_i, c_i are not equal, since otherwise we have the desired conclusion. Consider the non-empty set $X = \{x, y\}$ with x, y different. Define on X the collection $\{E_i\}_{i=1}^n$ of SSI- \mathcal{T} -relaxed indistinguishability fuzzy relations that separate points as follows: $E_i(x, y) = E_i(y, x) = a_i, E_i(x, x) = b_i$ and $E_i(y, y) = c_i$ for all $i = 1, \dots, n$. Since F aggregates SSI- \mathcal{T} -relaxed indistinguishability fuzzy relations that separate points into an SSI- T -relaxed

indistinguishability fuzzy relation that separates points we find that $F(E_1, \dots, E_n)$ is an $SSI-T$ -relaxed indistinguishability fuzzy relation that separates points. Moreover, $F(a) = F(b) = F(c)$ and, thus $F(E_1, \dots, E_n)(x, y) = F(E_1, \dots, E_n)(x, x) = F(E_1, \dots, E_n)(y, y)$. So $x = y$ which is impossible.

Now assume, in addition, that, for each $i \in \{1, \dots, n\}$, either $a_i < b_i$ or $a_i < c_i$. Again the collection $\{E_i\}_{i=1}^n$ of fuzzy relations defined above is a collection of $SSI-T$ -relaxed indistinguishability fuzzy relations that separate points on the non-empty set $X = \{x, y\}$. The fact that F aggregates $SSI-T$ -relaxed indistinguishability fuzzy relations that separate points into an $SSI-T$ -relaxed indistinguishability fuzzy relation that separates points yields that $F(E_1, \dots, E_n)$ is a $SSI-T$ -relaxed indistinguishability fuzzy relation that separates points. So $F(a) = F(E_1, \dots, E_n)(x, y) \leq F(E_1, \dots, E_n)(x, x) = F(b)$ and $F(a) = F(E_1, \dots, E_n)(x, y) \leq F(E_1, \dots, E_n)(y, y) = F(c)$.

The same arguments to those given in the proof of (1) \Rightarrow (2) in Theorem 10 remain valid in order to show (2.2), i.e., $(F(a), F(b), F(c))$ is a one-dimensional T -triangular triplet provided that $a, b, c \in [0, 1]^n$ and (a, b, c) is an n -dimensional T -triangular triplet.

(2) \Rightarrow (3). First we prove (3.1). Assume that $a, b, c \in \prod_{i=1}^n [0, f_{T_i}(0)]$ such that $a_i \geq b_i$ and $a_i \geq c_i$ for all $i = 1, \dots, n$ with $a_i \in]0, f_{T_i}(0)]$ for all $i = 1, \dots, n$ and $G(a) = G(b) = G(c)$. Set $d = (f_{T_1}^{(-1)}(a_1), \dots, f_{T_1}^{(-1)}(a_n))$, $e = (f_{T_1}^{(-1)}(b_1), \dots, f_{T_1}^{(-1)}(b_n))$, $g = (f_{T_1}^{(-1)}(c_1), \dots, f_{T_1}^{(-1)}(c_n))$. It is clear that $d, e, g \in [0, 1]^n$. Since $a_i \in]0, f_{T_i}(0)]$ for all $i = 1, \dots, n$ we have that $d \in [0, 1]^n$. Moreover, $d_i \leq e_i$ and $d_i \leq g_i$ for all $i = 1, \dots, n$.

The fact that $G(a) = G(b) = G(c)$ gives that $f_T(F(d)) = f_T(F(e)) = f_T(F(g))$. Since f_T is injective we deduce that $F(d) = F(e) = F(g)$. So there exists $i \in \{1, \dots, n\}$ such that $d_i = e_i = g_i$. Whence we deduce that $f_{T_i}^{(-1)}(a_i) = f_{T_i}^{(-1)}(b_i) = f_{T_i}^{(-1)}(c_i)$. Since $f_{T_i}^{(-1)}|_{]0, f_{T_i}(0)]} = f_{T_i}^{-1}$ and $f_{T_i}^{-1}$ is injective on $f_{T_i}([0, 1])$ we conclude that $a_i = b_i = c_i$.

Suppose, in addition, that, for each $i \in \{1, \dots, n\}$, either $a_i > b_i$ or $a_i > c_i$. Then either $f_{T_i}^{(-1)}(a_i) = f_{T_i}^{-1}(a_i) < f_{T_i}^{-1}(b_i) = f_{T_i}^{(-1)}(b_i)$ or $f_{T_i}^{(-1)}(a_i) = f_{T_i}^{-1}(a_i) < f_{T_i}^{-1}(c_i) = f_{T_i}^{(-1)}(c_i)$ for each $i = 1, \dots, n$. It follows that

$$F(f_{T_1}^{-1}(a_1), \dots, f_{T_1}^{-1}(a_n)) \leq F(f_{T_1}^{-1}(b_1), \dots, f_{T_1}^{-1}(b_n))$$

and

$$F(f_{T_1}^{-1}(a_1), \dots, f_{T_1}^{-1}(a_n)) \leq F(f_{T_1}^{-1}(c_1), \dots, f_{T_1}^{-1}(c_n)).$$

So $G(a) = f_T \circ F(f_{T_1}^{-1}(a_1), \dots, f_{T_1}^{-1}(a_n)) \geq G(b) = f_T \circ F(f_{T_1}^{-1}(b_1), \dots, f_{T_1}^{-1}(b_n))$ and $G(a) = f_T \circ F(f_{T_1}^{-1}(a_1), \dots, f_{T_1}^{-1}(a_n)) \geq G(c) = f_T \circ F(f_{T_1}^{-1}(c_1), \dots, f_{T_1}^{-1}(c_n))$.

In order to prove (3.2), the same arguments to those given in the proof of Theorem 10 provide that G transforms n -dimensional triangular triplets in $]0, +\infty[^n$ into one-dimensional triangular triplet in $[0, f_T(0)]$.

(3) \Rightarrow (4). Consider a collection $\{d_i\}_{i=1}^n$ of $SSD-(f_{T_i}(0))_{i=1}^n$ -bounded relaxed metrics on a non-empty set X . The same arguments to those given in Theorem 10 remain valid for showing fact that $G(d_1, \dots, d_n)$ satisfies the triangle inequality. Clearly $G(d_1, \dots, d_n)(x, y) = G(d_1, \dots, d_n)(y, x)$ for all $x, y \in X$. Next, suppose that $G(d_1, \dots, d_n)(x, y) = G(d_1, \dots, d_n)(x, x) = G(d_1, \dots, d_n)(y, y)$ for any $x, y \in X$. Since $d_i(x, x) \leq d_i(x, y)$ and $d_i(y, y) \leq d_i(x, y)$ for all $i = 1, \dots, n$ we have that there exists $i \in \{1, \dots, n\}$ such that $d_i(x, y) = d_i(x, x) = d_i(y, y)$ and, thus, $x = y$. Finally, we can assume that, for each $i \in \{1, \dots, n\}$, either $d_i(x, x) < d_i(x, y)$ either $d_i(y, y) < d_i(x, y)$. Then

$$G(d_1, \dots, d_n)(x, x) \leq G(d_1, \dots, d_n)(x, y)$$

and

$$G(d_1, \dots, d_n)(y, y) \leq G(d_1, \dots, d_n)(x, y).$$

(4) \Rightarrow (5). It is obvious.

(5) \Rightarrow (6). We prove 6.1). Let $a, b, c \in \prod_{i=1}^n [0, f_{T_i}(0)]$ such that $a_i \geq c_i$ and $b_i \geq c_i$ for all $i = 1, \dots, n$ and $a_i \in]0, f_{T_i}(0)]$ for all $i = 1, \dots, n$ and $H(a) = H(b) = H(c)$. Next, for the purpose of contradiction assume that, for all $i \in \{1, \dots, n\}$, the equality $a_i = b_i = c_i$ does not hold. It follows that there exists $i \in \{1, \dots, n\}$ such that either $a_i > c_i$ or $b_i > c_i$.

Set $X = \{x, y\}$ with x, y different. Define the function $d_i : X \times X \rightarrow [0, f_{T_i}(0)]$ by $d_i(x, y) = d_i(y, x) = a_i, d_i(x, x) = b_i$ and $d_i(y, y) = c_i$ for all $i = 1, \dots, n$. Then $\{d_i\}_{i=1}^n$ is a collection of $SSD-(f_{T_i}(0))_{i=1}^n$ -bounded relaxed metrics on X . Thus, $H(d_1, \dots, d_n)$ is an $SSD-f_T(0)$ -bounded relaxed metric on X . Since $H(a) = H(b) = H(c)$, we have that

$$H(d_1, \dots, d_n)(x, y) = H(d_1, \dots, d_n)(x, x) = H(d_1, \dots, d_n)(y, y),$$

which is impossible because $x \neq y$. Therefore, there exists $i \in \{1, \dots, n\}$ such that $a_i = b_i = c_i$.

Moreover, assume for each $i \in \{1, \dots, n\}$ that either $a_i > b_i$ or $a_i > c_i$. Then the above construction gives that $\{d_i\}_{i=1}^n$ is a collection of $SSD-(f_{T_i}(0))_{i=1}^n$ -bounded relaxed metrics on X . So

$$H(b) = H(d_1, \dots, d_n)(x, x) \leq H(d_1, \dots, d_n)(x, y) = H(a)$$

and

$$H(c) = H(d_1, \dots, d_n)(y, y) \leq H(d_1, \dots, d_n)(x, y) = H(a).$$

The proof of (6.2) runs in the same way as the proof given in Theorem 10.

(6) \Rightarrow (1). Consider a collection $\{E_i\}_{i=1}^n$ of $SSI-\mathcal{T}$ -relaxed indistinguishability fuzzy relations that separate points on a non-empty set X .

In the same manner as in the proof of Theorem 10, we can prove that H transforms n -dimensional triangular triplets in $\prod_{i=1}^n [0, f_{T_i}(0)]$ into a one-dimensional triangular triplet in $[0, f_T(0)]$ implies that $F(E_1, \dots, E_n)$ satisfies that

$$T(F(E_1, \dots, E_n)(u, v), F(E_1, \dots, E_n)(v, w)) \leq F(E_1, \dots, E_n)(u, w)$$

for all $u, v, w \in \{x, y, z\}$ and for each $x, y, z \in X$.

It is obvious that $F(E_1, \dots, E_n)(x, y) = F(E_1, \dots, E_n)(y, x)$ for all $x, y \in X$.

Now let $x, y \in X$. Set

$$a = (f_{T_1}(E_1(x, y)), \dots, f_{T_n}(E_n(x, y))), b = (f_{T_1}(E_1(x, x)), \dots, f_{T_n}(E_n(x, x)))$$

and $c = (f_{T_1}(E_1(y, y)), \dots, f_{T_n}(E_n(y, y)))$. Clearly, $a, b, c \in \prod_{i=1}^n [0, f_{T_i}(0)]$. We can assume that $E_i(x, y) \in [0, 1[$ for all $i = 1, \dots, n$ because otherwise we have that $x = y$, and there is nothing to prove. Notice that $b_i \leq a_i$ and $c_i \leq b_i$ for all $i = 1, \dots, n$. Suppose that

$$F(E_1, \dots, E_n)(x, y) = F(E_1, \dots, E_n)(x, x) = F(E_1, \dots, E_n)(y, y).$$

Then $H(a) = f_T \circ F(E_1, \dots, E_n)(x, y) = f_T \circ F(E_1, \dots, E_n)(x, x) = H(b)$ and $H(a) = f_T \circ F(E_1, \dots, E_n)(y, y) = H(c)$. Whence we deduce that there exist $i \in \{1, \dots, n\}$ such that $a_i = b_i = c_i$. The injectivity of f_T gives that $E_i(x, y) = E_i(x, x) = E_i(y, y)$ and, thus, that $x = y$.

Finally, we show that $F(E_1, \dots, E_n)(x, y) \leq F(E_1, \dots, E_n)(x, x)$ for all $x, y \in X$. Indeed, we can assume that, for each $i \in \{1, \dots, n\}$, either $E_i(x, y) < E_i(x, x)$ or $E_i(x, y) < E_i(y, y)$. Otherwise, $x = y$, and hence, the preceding inequality is held trivially. Then we have that, for each $i \in \{1, \dots, n\}$, either $a_i > b_i$ or $a_i > c_i$. So $H(a) \geq H(b)$ and $H(a) \geq H(c)$. Hence

$$f_T \circ F(E_1, \dots, E_n)(x, y) \geq f_T \circ F(E_1, \dots, E_n)(x, x)$$

and

$$f_T \circ F(E_1, \dots, E_n)(x, y) \geq f_T \circ F(E_1, \dots, E_n)(y, y).$$

Since f_T is a strictly decreasing function, we find that $F(E_1, \dots, E_n)(x, y) \leq F(E_1, \dots, E_n)(x, x)$ and $F(E_1, \dots, E_n)(x, y) \leq F(E_1, \dots, E_n)(y, y)$. \square

In the particular case in which all t -norms are strict continuous Archimedean t -norms, we obtain from Theorem 13 the result below.

Corollary 6. *Let $n \in \mathbb{N}$, let $\mathcal{T} = \{T_i\}_{i=1}^n$ be a collection of continuous Archimedean t -norms, and let $\{f_{T_i}\}_{i=1}^n$ be a collection of additive generators of \mathcal{T} . If T is a continuous Archimedean t -norm and $F : [0, 1]^n \rightarrow [0, 1]$ is a function, then the following assertions are equivalent:*

- (1) F aggregates SSI- \mathcal{T} -relaxed indistinguishability fuzzy relations that separate points into an SSI- T -relaxed indistinguishability fuzzy relation that separates points.
- (2) F satisfies the following conditions:
 - (2.1) If $a, b, c \in [0, 1]^n$ such that $a_i \leq b_i$ and $a_i \leq c_i$ for all $i = 1, \dots, n$ and $a_i \in [0, 1[$ for all $i = 1, \dots, n$ and $F(a) = F(b) = F(c)$, then there exists $i \in \{1, \dots, n\}$ such that $a_i = b_i = c_i$. Moreover, if for each $i \in \{1, \dots, n\}$, either $a_i < b_i$ or $a_i < c_i$, then $F(a) \leq F(b)$ and $F(a) \leq F(c)$.
 - (2.2) If $a, b, c \in [0, 1]^n$ such that (a, b, c) is an n -dimensional \mathcal{T} -triangular triplet, then $(F(a), F(b), F(c))$ is a one-dimensional T -triangular triplet.
- (3) The function $G : [0, +\infty]^n \rightarrow [0, +\infty]$, where $G = f_T \circ F \circ (f_{T_1}^{-1} \times \dots \times f_{T_n}^{-1})$, satisfies the following assertions:
 - (3.1) Let $a, b, c \in [0, +\infty]^n$ such that $a_i \geq c_i$ and $b_i \geq c_i$ for all $i = 1, \dots, n$ with $a_i \in]0, +\infty]$ for all $i = 1, \dots, n$. If $G(a) = G(b) = G(c)$, then there exists $i \in \{1, \dots, n\}$ such that $a_i = b_i = c_i$. Moreover, if for each $i \in \{1, \dots, n\}$, either $a_i > b_i$ or $a_i > c_i$, then $G(a) \geq G(b)$ and $G(a) \geq G(c)$.
 - (3.2) G transforms n -dimensional triangular triplets in $]0, +\infty]^n$ into a one-dimensional triangular triplet in $[0, +\infty]$.
- (4) The function $G : [0, +\infty]^n \rightarrow [0, +\infty]$ aggregates every collection $\{d_i\}_{i=1}^n$ of SSD-relaxed metrics into an SSD-relaxed metric, where $G = f_T \circ F \circ (f_{T_1}^{-1} \times \dots \times f_{T_n}^{-1})$.

4. Conclusions

In [36], characterizations of those functions that aggregate a collection of \mathcal{T} -relaxed indistinguishability fuzzy relations were given in terms of triangular triplets in the spirit of Theorem 2. In this paper, we have complemented the aforementioned work. In particular, we have shown that there is an equivalence between those functions that aggregate \mathcal{T} -relaxed indistinguishability fuzzy relations and those functions aggregating relaxed pseudo-metrics. The aforesaid equivalence has also been expressed in terms of triangular triplets. An interesting consequence derived from the obtained equivalence is that the functions under consideration are in correspondence with those that are subadditive. Moreover, the distinguished class of SSI- \mathcal{T} -relaxed indistinguishability fuzzy relations has been also studied. We have proved that functions merging this special type of relaxed indistinguishability fuzzy relations can be expressed through functions aggregating SSD-relaxed pseudo-metrics. It must be stressed that a notable difference between the class of functions that are able to aggregate a collection of relaxed indistinguishability fuzzy relations (relaxed pseudo-metrics) and SSI-relaxed indistinguishability fuzzy relations (SSD-relaxed pseudo-metric) has been shown. Concretely, in this case, the appropriate class of functions are those that satisfy monotony and subadditivity. The separating points case has also been approached, and characterizations of the class of functions merging \mathcal{T} -relaxed indistinguishability fuzzy relations that separate points are obtained in terms of \mathcal{T} -triangular triplets. It must be pointed out that the aggregation of this type of relaxed indistinguishability fuzzy relations was not explored in [36]. Outstanding differences between those functions aggregating \mathcal{T} -relaxed indistinguishability fuzzy relations and those that aggregate their counterpart separating points are shown. Illustrative examples of those functions able to aggregate any type of relaxed indistinguishability fuzzy relations will be given as future work in order to complement their description provided by the characterizations exposed in the present paper.

Although the exposed theory is presented for functions that aggregate a collection of \mathcal{T} -relaxed indistinguishability fuzzy relations into a T -relaxed indistinguishability fuzzy relation, where T is continuous Archimedean t -norm, surprising results were discovered in [36] [Theorem 40] when the t -norm T is considered as the minimum t -norm T_M , and in addition, relationships between functions aggregating \mathcal{T} -relaxed indistinguishability fuzzy relations into a T_{Min} -relaxed indistinguishability fuzzy and those that aggregate SSI - \mathcal{T} -relaxed indistinguishability fuzzy relations were obtained. As a future work, it seems interesting to explore new equivalences of the aforementioned functions in terms of functions that aggregate any kind of generalized metrics.

Observe that associativity does not seem to play any relevant role when relaxed \mathcal{T} -relaxed indistinguishability fuzzy relations are generated and aggregated. In the literature, several works are devoted to exploring methodologies that generate, through generalized additive generators, new conjunctors which are not t -norms because they are not associative (see, for instance, [52,53]). The description of those functions that aggregate fuzzy relations induced via generalized additive generators and the study of the existing relationship with those functions that aggregate some type of associated generalized metric are natural research lines that could be addressed in the near future.

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