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# Fixed points results for various types of interpolative cyclic contraction

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Abstract

*In this paper, we introduce four new types of contractions called in this order Kannan type cyclic contraction via interpolation, interpolative Ćirić-Reich-Rus type cyclic contraction, and we prove the existence and uniqueness for a fixed point for each situation.*

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## 1. Introduction

In the study of cyclic mappings, fixed point theorems provide conditions under which the existence and uniqueness of fixed points can be guaranteed. These conditions can vary depending on the specific class of cyclic mappings being considered.

For example, the Banach fixed point theorem guarantees the existence and uniqueness of a fixed point for contraction mappings on a complete metric

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space. On the other hand, the Brouwer fixed point theorem only guarantees the existence of at least one fixed point for continuous mappings on a compact, convex set.

There are also other fixed point theorems that guarantee the existence of a fixed point under weaker conditions or even multiple fixed points under certain conditions. For example, the Schauder fixed point theorem guarantees the existence of at least one fixed point for a continuous mapping on a closed, convex subset of a Banach space. And the Kakutani fixed point theorem guarantees the existence of a fixed point for a continuous mapping on a non-empty, convex, and compact subset of a locally convex topological vector space.

It is important to note that even if a fixed point theorem guarantees the existence of a fixed point for a given class of mappings, it does not necessarily provide a method for finding the fixed point. In practice, finding fixed points for a specific mapping may be diffcult or even impossible.

Additionally, the uniqueness of fixed points may be essential in certain problems, in such cases, the fixed point theorems that guarantee the uniqueness of fixed points will be more useful, and were further studied by different authors (see e.g.[2, 3, 4, 6, 9, 14, 15]).

In 2003, In their paper, Kirk et al. [13] introduced the concept of cyclical contractive mappings and extended the Banach fixed point result [5] to the class of cyclic mappings. They generalized the notion of contractive mappings to cyclical contractive mappings and proved that these mappings also have a unique fixed point. This expanded the class of mappings for which a fixed point can be guaranteed, and provided a new tool for studying the existence and uniqueness of fixed points in various mathematical contexts.

**Definition 1.1.** Let $(X, d)$ be a metric space and let $A$ and $B$ be two nonempty subsets of $X$.

A mapping $T : A \cup B \to A \cup B$ is said to be a cyclic mapping provided that

$$T(A) \subseteq B,\ T(B) \subseteq A. \tag{1.1}$$

A point $x \in A \cup B$ is called a best proximity point if $d(x, Tx) = d(A, B)$ where $d(A, B) = \inf(d(x, y) : x \in A, y \in B)$ .In 2011, Erdal Karapınar and Inci M. Erhan [12] proved.

The following fixed point theorem for a cyclic map.

**Definition 1.2.** Let $A$ and $B$ be non-empty subsets of a metric space $(X, d)$. A cyclic map $T : A \cup B \to A \cup B$ is said to be a Kannan Type cyclic contraction if there exists $k \in \left(0, \frac{1}{2}\right)$ such that:

$$d(Tx, Ty) \leq k\left[d(Tx, x) + d(Ty, y)\right],\ \forall x \in A, \forall y \in B. \tag{1.2}$$

Recently Karapinar [8] proposed a new Kannan-type contractive mapping using the concept of interpolation and proved a fixed point theorem in metric space. This new type of mapping, called "interpolative Kannan-type contractive mapping" is a generalization of Kannan's fixed point theorem. The interpolative method has been used in other research to generalize other forms of contractions as well [1, 9, 10, 11, 17]. This method has been found to be a powerful tool in the study of fixed point theory, as it allows for the construction of new classes of contractive mappings and the discovery of new fixed point theorems.

## 2. Main results

The interpolation method has been used to generalize the definition of Kannan-type cyclic contraction by incorporating the notion of interpolation. This leads to a more general definition of a Kannan-type cyclic contraction that allows for the construction of new classes of contractive mappings and the discovery of new fixed point theorems. The idea is that, by incorporating interpolation, the definition of a Kannan-type cyclic contraction can be expanded and new properties can be discovered.

**Definition 2.1.** Let $(X, d)$ be a metric space and let $A$ and $B$ be nonempty subsets of $X$. A cyclic map $T : A \cup B \to A \cup B$ is said to be a interpolative Kannan Type cyclic contraction if there exists $k \in [0, 1)$ and $\alpha \in (0, 1)$ such that

$$d\,(Tx, Ty) \leq k\,[d\,(Tx, x)]^{\alpha} [d\,(Ty, y)]^{1-\alpha}\,,\ \forall x \in A, \forall y \in B \tag{2.1}$$

for all $(x, y) \in A \times B$ with $x, y \notin Fix\,(T)$

Next, Given the generalization of the definition of a Kannan-type cyclic contraction via interpolation, one would expect to be able to prove a fixed point theorem for this new class of mappings. The fixed point theorem for a interpolative Kannan Type cyclic contraction can be stated as:

In a complete metric space, if a mapping satisfies certain conditions such as being an interpolative Kannan-type cyclic contraction, then it has a unique fixed point.

This theorem can be proved by using the properties of interpolative Kannan-type cyclic contraction mappings and the Banach fixed point theorem. By using interpolation, we can construct a new class of contractive mappings with unique fixed point.

**Theorem 2.2.** *Let $(X, d)$ be a complete metric space and let $A$ and $B$ be nonempty subsets of $X$ and let $T : A \cup B \to A \cup B$ be interpolative Kannan Type cyclic contraction. Then $T$ has a unique fixed point in $A \cap B$.*

*Proof.* Fix $x \in A$, From (2.1) it follows that

$$d\left(T^2x, Tx\right) \leq k\left[d\left(T^2x, Tx\right)\right]^{\alpha} \left[d\,(Tx, x)\right]^{1-\alpha}$$

which yields that

$$d\left(T^2x, Tx\right)^{1-\alpha} \leq kd\,(Tx, x)^{1-\alpha}\,.$$

And so $d\left(T^2x, Tx\right) \leq td\left(Tx, x\right)$ , where $t = k^{\frac{1}{1-\alpha}}$ and clearly $t \in (0,1)$ .Thus we have
$d\left(T^{n+1}x, T^nx\right) \leq t^n d\left(Tx, x\right)$ . Consequently,

$$\sum_{n=1}^{\infty} d\left(T^{n+1}x, T^nx\right) \leq (\sum_{n=1}^{\infty} t^n) d\left(Tx, x\right).$$

Hence $\{T^nx\}$ is a Cauchy sequence. Then, there exists a $z \in A \cup B$ such that $T^nx \to z$.

Notice that $\left\{T^{2n}x\right\}$ is a sequence in $A$ and $\left\{T^{2n+1}x\right\}$ is a sequence in $B$ having the same limit $z$. As $A$ and $B$ are closed, we conclude $z \in A \cap B$, that is, $A \cap B$ s nonempty.

We claim that $Tz = z$. Observe that :

$$\begin{aligned} d\left(z; Tz\right) &= \lim_{n\to\infty} d\left(Tz, T^{2n}x\right) = \lim_{n\to\infty} d\left(Tz, TT^{2n-1}x\right) \\ &\leq \lim_{n\to\infty} k\left[d\left(Tz, z\right)\right]^{\alpha}.\left[d\left(T^{2n-1}x, T^{2n}x\right)\right]^{1-\alpha} \end{aligned}$$

Taking $n \to \infty$ in the inequality above, we derive that $d\left(z; Tz\right) = 0$ that is $Tz = z$.

To prove the uniqueness of the fixed point $z$, assume that there exists $w \in A \cap B$ such that $z \neq w$ and $Tw = w$ . Taking into account that $T$ is a cyclic, we get $w \in A \cap B$ we have

$$\begin{aligned} d\left(z, w\right) &= d\left(Tz, Tw\right) \leq k\left[d\left(Tz, z\right)\right]^{\alpha}\left[d\left(Tw, w\right)\right]^{1-\alpha} \\ &\leq k\left[d\left(z, z\right)\right]^{\alpha}\left[d\left(w, w\right)\right]^{1-\alpha} = 0 \end{aligned}$$

which yields that $d\left(z, w\right) = 0$. We conclude that $z = w$ and hence $z$ is the unique fixed point of $T$. □

We introduce a new interpolative Reich–Rus–Ćirić type cyclic contraction in the following way

**Definition 2.3.** Let $(X, d)$ be a metric space and let $A$ and $B$ be nonempty subsets of $X$. A cyclic map $T : A \cup B \to A \cup B$ is said to be a interpolative Reich–Rus–Ćirić type cyclic contraction if there exists $k \in [0,1)$ and positive reals $\alpha, \beta$ with $\alpha + \beta < 1$ such that

$$d\left(Tx, Ty\right) \leq k\left[d\left(x, y\right)\right]^{\beta}\left[d\left(Tx, x\right)\right]^{\alpha}\left[d\left(Ty, y\right)\right]^{1-\alpha-\beta}, \ \forall x \in A, \forall y \in B \quad (2.2)$$

for all $(x, y) \in A \times B$ with $x, y \notin Fix\left(T\right)$

**Theorem 2.4.** *Let $(X, d)$ be a complete metric space and let $A$ and $B$ be nonempty subsets of $X$ and let $T : A \cup B \to A \cup B$ be interpolative Reich–Rus–Ćirić type cyclic contraction. Then $T$ has a unique fixed point in $A \cap B$.*

*Proof.* Fix $x \in A$, From (2.2) it follows that which yields that

$$d\left(T^2x, Tx\right) \leq k\left[d\left(Tx, x\right)\right]^{\beta}\left[d\left(T^2x, Tx\right)\right]^{\alpha}\left[d\left(Tx, x\right)\right]^{1-\alpha-\beta}$$

which yields

$$d\left(T^2x,Tx\right)^{1-\alpha}\leq kd\left(Tx,x\right)^{1-\alpha}.$$

And so $d\left(T^2x,Tx\right)\leq td\left(Tx,x\right)$ , where $t=k^{\frac{1}{1-\alpha}}$ and clearly $t\in(0,1)$ .Thus we have

$d\left(T^{n+1}x,T^nx\right)\leq t^n d\left(Tx,x\right)$ . Consequently,

$$\sum_{n=1}^{\infty}d\left(T^{n+1}x,T^nx\right)\leq(\sum_{n=1}^{\infty}t^n)d\left(Tx,x\right)<\infty.$$

Obviously, $\{T^nx\}$ is a Cauchy sequence. Then, there exists a $z\in A\cup B$ such that $T^nx\to z$

Notice that $\left\{T^{2n}x\right\}$ is a sequence in $A$ and $\left\{T^{2n+1}x\right\}$ is a sequence in $B$ having the same limit $z$. As $A$ and $B$ are closed, we conclude $z\in A\cap B$, that is, $A\cap B$ s nonempty.

We claim that $Tz=z$ . Observe that

$$\begin{aligned} d\left(z;Tz\right) &= \lim_{n\to\infty}d\left(Tz,T^{2n}x\right)=\lim_{n\to\infty}d\left(Tz,TT^{2n-1}x\right)\\ &\leq \lim_{n\to\infty}k\left[d\left(Tz,T^{2n-1}z\right)\right]^{\beta}\left[d\left(Tz,z\right)\right]^{\alpha}.\left[d\left(T^{2n+1}x,T^{2n}x\right)\right]_{-}^{1-\alpha-\beta}\end{aligned}$$

Taking $n\to\infty$ in the inequality above, we derive that $d\left(z;Tz\right)=0$ that is $Tz=z$.

To prove the uniqueness of the fixed point $z$, assume that there exists $w\in A\cup B$ such that $z\neq w$ and $Tw=w$ . Taking into account that $T$ is a cyclic, we get $w\in A\cap B$

we have

$$\begin{aligned} d\left(z,w\right) &= d\left(Tz,Tw\right)\leq k\left[d\left(z,w\right)\right]^{\beta}\left[d\left(Tz,z\right)\right]^{\alpha}\left[d\left(Tw,w\right)\right]^{1-\alpha-\beta}\\ &\leq k\left[d\left(z,w\right)\right]^{\beta}\left[d\left(z,z\right)\right]^{\alpha}\left[d\left(w,w\right)\right]^{1-\alpha-\beta}=0\end{aligned}$$

which yields that $d\left(z,w\right)=0$. We conclude that $z=w$ and hence $z$ is the unique fixed point of $T$. □

## References


[1] M. Abbas, R. Anjum and S. Riast, Fixed point results of enriched interpolative Kannan type operators with applications, Appl. Gen. Topol. 23, no. 2 (2022), 391–404.

[2] M. Asadi, Discontinuity of control function in the $(\mathcal{F},\phi,\theta)$-contraction in metric spaces, Filomat 31, no. 17 (2017), 5427–5433.

[3] M. Asadi, S. M. Vaezpour, V. Rakočević and B. E. Rhoades, Fixed point theorems for contractive mapping in cone metric spaces, Mathematical Communications 16, no. 1 (2011), 147–155.

[4] R. K. Bisht and V. Rakočević, Discontinuity at fixed point and metric completeness, Appl. Gen. Topol. 21, no. 2 (2022), 349–362.

[5] M. Edraoui, M. Aamri and S. Lazaiz, Relatively cyclic and noncyclic P-contractions in locally K-convex space, Axioms 8 (2019), 96.

[6] M. Eshraghisamani, S. M. Vaezpour and M. Asadi, New fixed point result on Branciari metric space, Journal of Mathematical Analysis 8, no. 6 (2017), 132–141.

[7] R. Kannan, Some results on fixed points, Bull. Calcutta Math. Soc. 60 (1968), 71-76.

[8] E. Karapınar, Revisiting the Kannan type contractions via interpolation, Adv. Theory Nonlinear Anal. Appl. 2, no. 2 (2018), 85–87.

[9] E. Karapınar, Revisiting Ciric type nonunique fixed point theorems via interpolation, Appl. Gen. Topol. 22, no. 2 (2021), 483–496.

[10] E. Karapınar, R. Agarwal and H. Aydi, Interpolative Reich-Rus-Ćirić type contractions on partial metric spaces, Mathematics 6 (2018), 256.

[11] E. Karapınar, O. Alqahtani and H. Aydi, On interpolative Hardy-Rogers type contractions, Symmetry 11 (2019), 8.

[12] E. Karapınar and I. M. Erhan, Best proximity point on different type contractions, Applied Mathematics & Information Sciences 5, no. 3 (2011), 558–569.

[13] W. A. Kirk, P. S. Srinivasan and P. Veeramani, Fixed point fo mappings satisfying cyclical contractive conditions, Fixed Point Theory 4 (2003), 79–89.

[14] H. Monfared, M. Asadi and M. Azhini, $F(\psi,\varphi)$-contractions for $\alpha$–admissible mappings on metric spaces and related fixed point results, Commun. Nonlinear Anal. 2 (2016), 86–94.

[15] H. Monfared, M. Asadi and A. Farajzadeh, New generalization of Darbo's fixed point theorem via $\alpha$-admissible simulation functions with application, Sahand Communications in Mathematical Analysis 17, no. 2 (2020), 161–171.

[16] K. Roy and S. Panja, From interpolative contractive mappings to generalized Ciric-quasi contraction mappings, Appl. Gen. Topol. 22, no. 1 (2021), 109–120.

[17] N. Taş, Interpolative contractions and discontinuity at fixed point, Appl. Gen. Topol. 24, no. 1 (2023), 145–156.