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# Remarks on fixed point assertions in digital topology, 6

Laurence Boxer

Department of Computer and Information Sciences, Niagara University, NY 14109, USA;
Department of Computer Science and Engineering, State University of New York at Buffalo
(boxer@niagara.edu)

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Abstract

*This paper continues a series discussing flaws in published assertions concerning fixed points in digital metric spaces.*

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## 1. Introduction

As stated in [5]:

> The topic of fixed points in digital topology has drawn much attention in recent papers. The quality of discussion among these papers is uneven; while some assertions have been correct and interesting, others have been incorrect, incorrectly proven, or reducible to triviality.

Here, we continue the work of [9, 5, 6, 7, 8], discussing many shortcomings in earlier papers and offering corrections and improvements.

Quoting and paraphrasing [8]:

> Authors of many weak papers concerning fixed points in digital topology seek to obtain results in a "digital metric space" (see section 2.2 for its definition). This seems to be a bad idea. We slightly paraphrase [7]:

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- Nearly all correct nontrivial published assertions concerning digital metric spaces use the metric and do not use the adjacency. As a result, the digital metric space seems to be an artificial notion.
- If $X$ is finite (as in a "real world" digital image) or the metric $d$ is a common metric such as any $\ell_p$ metric, then $(X, d)$ is uniformly discrete as a topological space, hence not very interesting.
- Many published assertions concerning digital metric spaces mimic analogues for subsets of Euclidean $\mathbb{R}^n$. Often, the authors neglect important differences between the topological space $\mathbb{R}^n$ and digital images, resulting in assertions that are incorrect or incorrectly "proven," trivial, or trivial when restricted to conditions that many regard as essential. E.g., in many cases, functions that satisfy fixed point assertions must be constant or fail to be digitally continuous [9, 5, 6].

Since the publication of [8], additional highly flawed papers rooted in digital metric spaces have come to our attention. The current paper discusses shortcomings in [1, 10, 20, 21, 22, 23, 25, 26, 27, 29, 32].

## 2. Preliminaries

Much of the material in this section is quoted or paraphrased from [7].

We use $\mathbb{N}$ to represent the natural numbers, $\mathbb{Z}$ to represent the integers, and $\mathbb{R}$ to represent the reals.

A *digital image* is a pair $(X, \kappa)$, where $X \subset \mathbb{Z}^n$ for some positive integer $n$, and $\kappa$ is an adjacency relation on $X$. Thus, a digital image is a graph. In order to model the "real world," we usually take $X$ to be finite, although there are several papers that consider infinite digital images. The points of $X$ may be thought of as the "black points" or foreground of a binary, monochrome "digital picture," and the points of $\mathbb{Z}^n \setminus X$ as the "white points" or background of the digital picture.

### 2.1. **Adjacencies, continuity, fixed point.**

In a digital image $(X, \kappa)$, if $x, y \in X$, we use the notation $x \leftrightarrow_\kappa y$ to mean $x$ and $y$ are $\kappa$-adjacent; we may write $x \leftrightarrow y$ when $\kappa$ can be understood. We write $x \leftrightarroweq_\kappa y$, or $x \leftrightarroweq y$ when $\kappa$ can be understood, to mean $x \leftrightarrow_\kappa y$ or $x = y$.

The most commonly used adjacencies in the study of digital images are the $c_u$ adjacencies. These are defined as follows.

**Definition 2.1.** Let $X \subset \mathbb{Z}^n$. Let $u \in \mathbb{Z}$, $1 \leq u \leq n$. Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in X$. Then $x \leftrightarrow_{c_u} y$ if

- $x \neq y$,
- for at most $u$ distinct indices $i$, $|x_i - y_i| = 1$, and
- for all indices $j$ such that $|x_j - y_j| \neq 1$ we have $x_j = y_j$.

**Definition 2.2** ([30]). A digital image $(X, \kappa)$ is *$\kappa$-connected*, or just *connected* when $\kappa$ is understood, if given $x, y \in X$ there is a set $\{x_i\}_{i=0}^n \subset X$ such that $x = x_0$, $x_i \leftrightarrow_\kappa x_{i+1}$ for $0 \leq i < n$, and $x_n = y$.

**Definition 2.3** ([30, 3]). Let $(X, \kappa)$ and $(Y, \lambda)$ be digital images. A function $f : X \to Y$ is *$(\kappa, \lambda)$-continuous*, or *$\kappa$-continuous* if $(X, \kappa) = (Y, \lambda)$, or *digitally continuous* when $\kappa$ and $\lambda$ are understood, if for every $\kappa$-connected subset $X'$ of $X$, $f(X')$ is a $\lambda$-connected subset of $Y$.

**Theorem 2.4** ([3]). *A function $f : X \to Y$ between digital images $(X, \kappa)$ and $(Y, \lambda)$ is $(\kappa, \lambda)$-continuous if and only if for every $x, y \in X$, if $x \leftrightarrow_\kappa y$ then $f(x) \leftrightarroweq_\lambda f(y)$.*

We use $1_X$ to denote the identity function on $X$, and $C(X, \kappa)$ for the set of functions $f : X \to X$ that are $\kappa$-continuous.

A *fixed point* of a function $f : X \to X$ is a point $x \in X$ such that $f(x) = x$. We denote by $\text{Fix}(f)$ the set of fixed points of $f : X \to X$.

As a convenience, if $x$ is a point in the domain of a function $f$, we will often abbreviate "$f(x)$" as "$fx$".

2.2. **Digital metric spaces.** A *digital metric space* [16] is a triple $(X, d, \kappa)$, where $(X, \kappa)$ is a digital image and $d$ is a metric on $X$. The metric is usually taken to be the Euclidean metric or some other $\ell_p$ metric; alternately, $d$ might be taken to be the shortest path metric. These are defined as follows.

- Given $x = (x_1, \ldots, x_n) \in \mathbb{Z}^n$, $y = (y_1, \ldots, y_n) \in \mathbb{Z}^n$, $p > 0$, $d$ is the $\ell_p$ metric if
$$d(x, y) = \left( \sum_{i=1}^n \mid x_i - y_i \mid^p \right)^{1/p}.$$
Note the special cases: if $p = 1$ we have the *Manhattan metric*; if $p = 2$ we have the *Euclidean metric*.
- [17] If $(X, \kappa)$ is a connected digital image, $d$ is the *shortest path metric* if for $x, y \in X$, $d(x, y)$ is the length of a shortest $\kappa$-path in $X$ from $x$ to $y$.

Under conditions in which a digital image models a "real world" image, $X$ is finite or $d$ is (usually) an $\ell_p$ metric, so that $(X, d)$ is *uniformly discrete* as a topological space, i.e., there exists $\varepsilon > 0$ such that for $x, y \in X$, $d(x, y) < \varepsilon$ implies $x = y$.

We say a sequence $\{x_n\}_{n=0}^\infty$ is *eventually constant* if for some $m > 0$, $n > m$ implies $x_n = x_m$. The notions of convergent sequence and complete digital metric space are often trivial, e.g., if the digital image is uniformly discrete, as noted in the following, a minor generalization of results of [18, 9].

**Proposition 2.5** ([7])**.** *Let $(X, d)$ be a metric space. If $(X, d)$ is uniformly discrete, then any Cauchy sequence in $X$ is eventually constant, and $(X, d)$ is a complete metric space.*

Let $(X, d)$ be a metric space and $f : X \to X$. We say $f$ is a *contraction map* [15] if for some $k \in [0, 1)$ and all $x, y \in X$, $d(f(x), f(y)) \leq kd(x, y)$. Such a function is not be confused with a digital contraction [2], a homotopy between an identity map and a constant function.

## 3. On [1]

This paper has so many typos and unexplained symbols that it is extremely difficult to understand. A few examples:

- In Definitions 2.1, 2.3, 3.1, and 3.7, the "$\to$" character appears where it seems likely that the intended character is "$\times$" to indicate a Cartesian product.
- In Definition 2.1, it appears the "$S$" (appearing twice) is intended to be "$T$", since "$S$" is not defined.
- In Definition 3.1, "$X$" is undefined. It seems likely "$X$" is intended to be "$\tilde{X}$" or something related to the latter. If the former, then this definition seems to duplicate Definition 2.3.

The assertions stated as Theorems 4.1 and 4.2 of [1] promise existence of unique fixed points. The assertions and their respective arguments offered as proofs are very difficult to follow, but one can easily see that they fail to establish uniqueness of fixed points.

## 4. On [10]

4.1. **"Theorem" 3.1 of [10].** The following Definition 4.1 appears in [10], where it is incorrectly attributed to [4], where it does not appear. The inspiration for Definition 4.1 may be the rather different definition of compatible functions appearing in [24].

**Definition 4.1.** Suppose $(X, d, \rho)$ is a digital metric space. Suppose $P, Q : X \to X$. Then $P$ and $Q$ are *compatible* if

$$d(PQx, QPx) \leq d(Px, Qx) \text{ for all } x \in X.$$

The following is stated as Theorem 3.1 of [10].

*Assertion* 4.2. Let $P$, $Q$, $G$, and $H$ be quadruple mappings of a complete digital metric space $(X, d, \rho)$ satisfying the following.

(1) $G(X) \subset Q(X)$ and $H(X) \subset P(X)$.
(2) Let $0 < \alpha < 1$. For all $x, y \in X$,

$$d(x, y) = \alpha \max \left\{ \begin{array}{c} d(Gx, Hy), d(Gx, Px), d(Hy, Qy), d(Hx, Qx), \\ \frac{1}{2}[d(Gx, Qy) + d(Hy, Px)] \end{array} \right\}.$$

(3) One of $P$, $Q$, $G$, and $H$ is continuous.
(4) The pairs $(P, G)$ and $(Q, H)$ are compatible.

Then $P$, $Q$, $G$, and $H$ have a unique common fixed point in $X$.

Flaws in this argument given for this assertion in [10] include the following.

- The first line of the "proof" reverses the containments stated in item 1) of the hypotheses, stating that $Q(X) \subset G(X)$ and $P(X) \subset H(X)$.
- According to subsequent usage, the "=" in item 2) of the hypothesis should be "$\leq$".
- Sequences $\{x_n\}$ and $\{y_n\}$ are constructed, via the rules:
  - $x_0$ is an arbitrary point of $X$.
  - inductively, $y_{2n} = Gx_{2n} = Qx_{2n+1}$ and $y_{2n+1} = Hx_{2n+1} = Px_{2n+2}$.

  Then the following statement appears.

$$d(y_{2n}, y_{2n+1}) = d(Gx_{2n}, Hx_{2n+1}) \leq$$

$$\alpha \max \left\{ \begin{array}{c} d(Px_{2n}, Qx_{2n+1}), d(Px_{2n}, Gx_{2n}), d(Qx_{2n+1}, Hx_{2n+1}), \\ d(Gx_{2n}, Hx_{2n+1}), \frac{1}{2}[d(Gx_{2n}, Qx_{2n+1}) + d(Px_{2n}, Hx_{2n+1})] \end{array} \right\}. \quad (4.1)$$

  This statement is not given a justification, and we will show that it does not correspond to item 2) of the hypotheses. Direct substitution into item 2) of the hypotheses (with the substitution of "$\leq$" for "=") yields the following.

$$d(y_{2n}, y_{2n+1}) \leq \alpha \max \left\{ \begin{array}{c} d(Gy_{2n}, Hy_{2n+1}), d(Gy_{2n}, Py_{2n}), \\ d(Hy_{2n+1}, Qy_{2n+1}), d(Hy_{2n}, Qy_{2n}), \\ \frac{1}{2}[d(Gy_{2n}, Qy_{2n+1}) + d(Hy_{2n+1}, Py_{2n})] \end{array} \right\} \quad (4.2)$$

  The expression $Gy_{2n}$ appears 3 times in (4.2). No provision occurs in [10] for converting this expression into an expression of the sequence $\{x_n\}$, as in (4.1).

We must conclude that Assertion 4.2 is unproven.

4.2. **Example 3.2 of [10].** This example is based on $[0, \infty)$, which the authors want to consider as a digital metric space. It is not a digital metric space, as $[0, \infty)$ is not a subset of $\mathbb{Z}^n$ for any $n$. Nor is a conclusion stated for this example.

## 5. On [20, 21]

The papers [20, 21] introduce, for $f, g : X \to X$ on a digital metric space $(X, d, \kappa)$, notions of *compatible of type* $K$ and and *compatible of type* $R$, claiming that they yield fixed point results. In section 5.1 we discuss how these notions are related to other notions of compatibility that have appeared in the literature. In section 5.2 we show flaws in the fixed point assertions of [20, 21].

5.1. **Variants on compatibility.** In this section, we consider sequences $\{ x_n \} \subset X$ such that

$$lim_{n \to \infty} f x_n = lim_{n \to \infty} g x_n = t \in X \tag{5.1}$$

**Definition 5.1** ([11])**.** Suppose $f$ and $g$ are self-functions on a metric space $(X, d)$. If every sequence $\{ x_n \} \subset X$ satisfying (5.1) also satisfies

$$lim_{n \to \infty} d(f(g x_n), g(f x_n)) = 0,$$

then $f$ and $g$ are *compatible*.

**Definition 5.2** ([20])**.** Let $(X, d)$ be a metric space. Let $f, g : X \to X$. We say $f$ and $g$ are *digitally compatible of type K* if for every infinite sequence $\{ x_n \} \subset X$ satisfying (5.1) we have

$$\lim_{n \to \infty} d(f f x_n, g t) = 0 \quad \text{and} \quad \lim_{n \to \infty} d(g g x_n, f t) = 0. \tag{5.2}$$

**Definition 5.3** ([21])**.** Let $(X, d)$ be a metric space. Let $f, g : X \to X$. We say $f$ and $g$ are *digitally compatible of type R* if for every infinite sequence $\{ x_n \} \subset X$ satisfying (5.1) we have

$$\lim_{n \to \infty} d(f g x_n, g f x_n) = 0 = \lim_{n \to \infty} d(f f x_n, g g x_n).$$

We have the following.

**Proposition 5.4.** *Suppose $S$ and $T$ are self-functions on a metric space $(X, d)$ that is uniformly discrete. Then $S$ and $T$ are compatible of type $K$ if and only if $S$ and $T$ are compatible.*

*Proof.* Let $\{ x_n \} \subset X$ be a sequence satisfying (5.1) for the functions $S, T$. The uniform discreteness hypothesis implies for almost all $n$

$$S x_n = t = T x_n, \quad S S x_n = S t, \quad \text{and} \quad T T x_n = T t.$$

If $S$ and $T$ are compatible,

$$d(S S x_n, T t) = d(S T x_n, T S x_n) = 0 = d(T S x_n, S T x_n) = d(T T x_n, S t)$$

so $S$ and $T$ are compatible of type K.

Suppose $S$ and $T$ are compatible of type K. Then for almost all $n$,

$$0 = d(S S x_n, T t) = d(S T x_n, T S x_n)$$

so $S$ and $T$ are compatible. □

**Proposition 5.5.** *Suppose $S$ and $T$ are self-functions on a metric space $(X, d)$. If $S$ and $T$ are compatible of type $R$, then $S$ and $T$ are compatible. The converse is true if $(X, d)$ is uniformly discrete.*

*Proof.* It is clear from Definitions 5.1 and 5.3 that a type R pair is a compatible pair.

Suppose $S$ and $T$ are compatible and $(X, d)$ is uniformly discrete. Let $\{ x_n \} \subset X$ satisfy (5.1).

- By Definition 5.1, $\lim_{n \to \infty} d(f g x_n, g f x_n) = 0$.

- Since $(X,d)$ is uniformly discrete, for almost all $n$ we have $fx_n = gx_n = t$. Therefore, for almost all $n$,

$$ffx_n = fgx_n = gfx_n = ggx_n.$$

Therefore, $S$ and $T$ are compatible of type R. $\square$

Several variants of compatible functions have been defined in the literature. These include compatible of type A [11], compatible of type B [14], compatible of type C [14], and compatible of type P [11].

**Theorem 5.6.** *Let $(X,d)$ be a metric space that is uniformly discrete. Let $S,T : X \to X$. The following are equivalent.*

- *$S$ and $T$ are compatible.*
- *$S$ and $T$ are compatible of type A.*
- *$S$ and $T$ are compatible of type B.*
- *$S$ and $T$ are compatible of type C.*
- *$S$ and $T$ are compatible of type K.*
- *$S$ and $T$ are compatible of type P.*
- *$S$ and $T$ are compatible of type R.*

*Proof.* The equivalence of compatible, compatible of type A, compatible of type B, compatible of type C, and compatible of type P, is shown in Theorem 3.9 of [7]. The equivalence of compatible and compatible of type K is shown in Proposition 5.4. The equivalence of compatible and compatible of type R is shown in Proposition 5.5. $\square$

5.2. **On the fixed point assertion of [20].** The following is stated as Theorem 4.1 of [20].

*Assertion* 5.7. Let $A,B,S,T : X \to X$ where $(X,d,\kappa)$ is a digital metric space such that

(1) $S(X) \subset B(X)$ and $T(X) \subset A(X)$.
(2) For all $x,y \in X$ and some $\alpha \in (0,1)$,

$$d(Sx,Ty) =$$
$$\alpha \max\{\, d(Ax,By), d(Ax,Sx), d(By,Ty), d(Sx,By), d(Ax,Ty)\,\}.$$

*[Probably, the comparison operator in the latter statement should be "$\leq$" rather than "$=$".]*

(3) Pairs $(A,S)$ and $(B,T)$ are reciprocally continuous.
(4) $(A,S)$ and $(B,T)$ are pairs of compatible of type K functions.

Then $A,B,S$, and $T$ have a unique common fixed point in $X$.

We note the following flaw in the argument offered as proof of this assertion. A sequence $\{\, y_n \,\}$ is constructed. It is claimed that this is a Cauchy sequence, "From the proof of [*what the current paper will refer to as* [23]]". We find that [23] is listed in [20] as submitted to the *Journal of Mathematical Imaging and Vision.* At the current writing, 4 years after the publication of [20], neither a search of the *JMIV* website nor a general Google search succeeds in locating [23].

We conclude that Assertion 5.7 is unproven.

5.3. **On fixed point assertions of [21].**

**Proposition 5.8.** *Let $f, g : X \to X$ be compatible of type R on a uniformly discrete digital metric space $(X, d, \kappa)$. If*

$$ft = gt \text{ for some } t \in X, \tag{5.3}$$

*then $fgt = fft = ggt = gft$.*

*Proof.* We remark that this assertion is a modified version of Proposition 3.2 of [21]; the latter appeared in [21] with neither proof nor citation. The modification we have made is inclusion of the assumption of uniform discreteness.

Let $x_n = t$ for all $n$. From Definition 5.3, $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ and $\lim_{n\to\infty} d(ffx_n, ggx_n) = 0$. Since $(X, d)$ is uniformly discrete, for almost all $n$, $fgx_n = gfx_n$ and $ffx_n = ggx_n$, hence $fgt = gft$ and $fft = ggt$. We complete the proof by observing that (5.3) implies $fgt = fft$. □

**Proposition 5.9.** *Let $f, g : X \to X$ be compatible of type R on a uniformly discrete digital metric space $(X, d, \kappa)$. If $\{ x_n \}$ is a sequence in $X$ satisfying (5.1), then*

(1) *$ft = gt$;*
(2) *$\lim_{n\to\infty} gfx_n = ft$;*
(3) *$\lim_{n\to\infty} fgx_n = gt$; and*
(4) *$fgt = gft$.*

*Proof.* We remark that this proposition is a modified version of Proposition 3.3 of [21], which appears there with neither proof nor citation. Our modification is to replace Jain's assumptions of digital continuity (which Jain may have confused with metric continuity, since it seems unlikely that one can easily show that the desired outcome follows from digital continuity) with the assumption of "uniformly discrete".

(1) From (5.1) and the assumption of uniform discreteness, it follows that for almost all $n$, $fx_n = gx_n = t$. From compatibility of type R, it follows that
$$0 = lim_{n\to\infty} d(fgx_n, gfx_n) = d(ft, gt). \tag{5.4}$$
Thus, $ft = gt$.
(2) By the uniform discreteness property, $\lim_{n\to\infty} gfx_n = gt = ft$.
(3) Similarly, $\lim_{n\to\infty} fgx_n = ft = gt$.
(4) $fgt = gft$ by Proposition 5.8.

□

The following is stated as Theorem 4.1 of [21].

*Assertion* 5.10. Let $(X, d, \kappa)$ be a digital metric space. Let $A, B, S, T : X \to X$ such that

(1) $S(X) \subset B(X)$ and $T(X) \subset A(X)$;
(2) for $0 < \alpha < 1$ and all $x, y \in X$,

$$d(Sx, Ty) \le \alpha \max \left\{ \begin{array}{c} d(Ax, By), d(Ax, Sx), d(By, Ty), \\ d(Sx, By), d(Ax, Ty) \end{array} \right\};$$

(3) one of the functions $A, B, S, T$ is continuous; and
(4) $(A, S)$ and $(B, T)$ are pairs of functions that are compatible of type R.

Then $A, B, S$, and $T$ have a common fixed point in $X$.

The argument offered in [21] as proof of Assertion 5.10 is flawed as follows. Sequences $\{ y_n \} \subset X$ and $\{ d_n = d(y_n, y_{n+1}) \}$ are constructed, and it is shown that for all $n$,

$$d_{2n} \le \alpha \max\{ d_{2n-1}, d_{2n}, d_{2n-1} + d_{2n} \} \tag{5.5}$$

and $d_{2n} \le d_{2n-1}$, i.e.,

$$d(y_{2n}, y_{2n+1}) \le d(y_{2n-1}, y_{2n}). \tag{5.6}$$

Note for inequality (5.5) the index on the left side is even, and in (5.6), the smaller index on the left side is even. In an apparent attempt to obtain a geometric series from the inequalities (5.5) and (5.6), it is claimed these inequalities imply that for $m, n \in \mathbb{N}$ with $m > n$,

$$d(y_m, y_n) \le \alpha d(y_m, y_{m-1}) + \ldots + \alpha d(y_{n+1}, y_n). \tag{5.7}$$

But since there is no analogue of either of the inequalities (5.5) and (5.6) in which the index on the left side of (5.5) is odd or the smaller index on the left side of (5.6) is odd, it is not clear that one can obtain (5.7).

Thus, we must consider Assertion 5.10 as unproven.

## 6. On [22]

We will show that the assertions of [22] are, for the most part, trivial in that their hypotheses are impossible.

### 6.1. Some definitions and elementary properties.

**Definition 6.1** ([31]). $\Psi$ is the set of nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t \ge 0$.

*Remarks* 6.2. Remark 2.13(1) of [22] says if $\psi : [0, \infty) \to [0, \infty)$ is nondecreasing such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t \ge 0$, then $\psi \in \Psi$. This turns out to be trivial, because such a function must be constant, with value 0.

*Proof.* We have $\psi(x) \ge 0$ for all $x \in [0, \infty)$. Suppose there exists $t > 0$ such that $\psi(t) = q > 0$; let this be the base case of an induction to show $\psi^n(t) \ge q$ for all $n \in \mathbb{N}$.

Suppose we have $\psi^n(t) \ge q$ for $n \le k$. Since $\psi$ is nondecreasing,

$$\psi^{k+1}(t) = \psi(\psi^k(t)) \ge \psi(q) \ge q,$$

which completes the induction.

Therefore, $\sum_{i=1}^{n} \psi^n(t) \ge nq$ for all $n \in \mathbb{N}$, contrary to the assumption that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$. The contradiction establishes the assertion. □

6.2. **Remark 3.2 of [22].**

**Definition 6.3** ([22])**.** Let $S : X \to X$ for a digital metric space $(X, d, \kappa)$. Let $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that for all $x, y \in X$,

$$\psi(d(Sx, Sy)) \geq \alpha(x, y) d(x, y).$$

Then $S$ is a *digital* $\alpha - \psi$ *expansive* mapping.

**Definition 6.4.** A self-map $S$ on a metric space $(X, d)$ is *expansive* if for all $x, y \in X$, $d(Sx, Sy) \geq d(x, y)$.

Remark 3.2 of [22] claims the following.

*Assertion* 6.5. Any expansive mapping is a digital $\alpha - \psi$ expansive mapping with $\alpha(x, y) = 1$ for all $x, y \in X$ and $\psi(t) = kt$ for $0 < k < 1$.

*Remarks* 6.6. Assertion 6.5 is not generally true. It is clear that $\mathrm{id}_X$ is an expansive mapping. Let $S = \mathrm{id}_X$. Under the assumptions of Assertion 6.5, we would have from Definition 6.3 the inequality

$$kd(x, y) \geq d(x, y), \qquad \text{which is false for } x \neq y.$$

6.3. **"Theorems" 3.4 and 3.5 of [22].**

**Definition 6.7** ([22])**.** Let $S : X \to X$ and $\alpha : X \times X \to [0, \infty)$. We say $S$ is $\alpha$-admissible if for all $x, y \in X$,

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Sx, Sy) \geq 1.$$

The following is stated as Theorem 3.4 of [22].

*Assertion* 6.8. Let $(X, d, \kappa)$ be a complete digital metric space. Let $S : X \to X$ be a bijective, digital $\alpha - \psi$-expansive mapping such that

- $S^{-1}$ is $\alpha$-admissible;
- for some $x_0 \in X$, $\alpha(x_0, S^{-1}(x_0)) \geq 1$; and
- $S \in C(X, \kappa)$.

Then $S$ has a fixed point in $X$.

The following is stated as Theorem 3.5 of [22].

*Assertion* 6.9. Suppose we replace the continuity assumption in Assertion 6.8 by the assumption that

$$\text{if } \{ x_n \} \subset X \text{ and } \lim_{n\to\infty} x_n = x, \text{ then } \alpha(S^{-1}x_n, S^{-1}x) \geq 1 \text{ for all } n.$$

Then $S$ has a fixed point in $X$.

*Remarks* 6.10. Both of Assertion 6.8 and Assertion 6.9 are false, as shown by the example

$$S : \mathbb{Z} \to \mathbb{Z} \text{ given by } S(x) = x + 1; \quad \alpha(x, y) = 1.$$

6.4. **Example 3.7.** Example 3.7 of [22] wishes to consider a digital metric space $(X, d, \kappa)$ and a function $T : X \to X$ defined by

$$T(x) = \begin{cases} 2x - \frac{11}{6} & \text{if } x > 1; \\ \frac{x}{6} & \text{if } x \leq 1. \end{cases}$$

We are not told what $X$ is. From the definition of $T$, we must have $X \subset \mathbb{Z}$. But $T$ does not appear to be integer-valued, e.g., if $1 \in X$ then $T(1) \notin \mathbb{Z}$.

## 7. On [25]

We consider Theorems 3.1 and 3.2 of [25], stated below as Theorem 7.1 and Assertion 7.5, respectively. We show that the former reduces to triviality and the latter is false.

The following is stated as Theorem 3.1 of [25].

**Theorem 7.1.** *Let $(X, d, \kappa)$ be a complete digital metric space and suppose $T : (X, d, \kappa) \to (X, d, \kappa)$ satisfies $d(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$, where $\psi : [0, \infty) \to [0, \infty)$ is monotone non-decreasing and satisfies*

$$\lim_{n \to \infty} \psi^n(t) = 0 \text{ for all } t > 0. \tag{7.1}$$

*Then $T$ has a unique fixed point in $(X, d, \kappa)$.*

We show that Theorem 7.1 reduces to triviality.

**Proposition 7.2.** *Let $\psi$ be as in Theorem 7.1. Then*

- *$\psi$ is the constant function with value 0.*
- *$T$ is a constant function.*

*Proof.* Suppose we have some $t_0$ for which $\psi(t_0) > 0$. Then a simple induction shows that $\psi^{n+1}(t_0) \geq \psi(t_0) > 0$ for all $n \in \mathbb{N}$, contrary to (7.1). Thus $\psi$ must be the constant function with value 0.

It follows that $d(Tx, Ty) = 0$ for all $x, y \in X$, so $T$ is a constant function. $\square$

Theorem 3.2 of [25] depends on the following.

**Definition 7.3** ([13])**.** Let $(X, d, \kappa)$ be a digital metric space. Then $T : X \to X$ is a *weakly uniformly strict digital contraction* if given $\varepsilon > 0$ there exists $\delta > 0$ such that $\varepsilon \leq d(x, y) < \varepsilon + \delta$ implies $d(Tx, Ty) < \varepsilon$ for all $x, y \in X$.

But Definition 7.3 turns out to be a triviality in many important cases, as shown by the following.

*Proposition* 7.4. Let $(X, d)$ be a finite metric space. Then every weakly uniformly strict digital contraction on $X$ is a contraction map.

*Proof.* Let $x \neq y$ in $X$. Let $\varepsilon = d(x, y) > 0$. Let $T : X \to X$ be a weakly uniformly strict digital contraction on $X$. It follows from Definition 7.3 that $d(Tx, Ty) < \varepsilon = d(x, y)$.

Since $X$ is finite, there exists $k \in (0, 1)$ such that for all $x, y \in X$, $d(Tx, Ty) \leq kd(x, y)$. $\square$

The following is stated as Theorem 3.2 of [25].
The following is stated as Theorem 3.2 of [25].

*Assertion* 7.5. Let $(X, d, \kappa)$ be a complete digital metric space. Let $T : (X, d, \kappa) \to (X, d, \kappa)$ be a weakly uniformly strict digital contraction mapping. Then $T$ has a unique fixed point $z$. Moreover, for any $x \in X$, $lim_{n\to\infty} T^n x = z$.

*Remarks* 7.6. Assertion 7.5 is trivial if $X$ is finite and $c_1$-connected and $d$ is an $\ell_p$ metric or the shortest path metric.

*Proof.* By Proposition 7.4, $T$ is a contraction map. Therefore [9], our hypotheses imply $T$ is a constant map. □

## 8. On [26]

The main result of [26] is Theorem 8.1, below. This theorem depends on the notions of 1-*chainable* and *uniformly local contractive mapping*; it seems unnecessary to define these here.

**Theorem 8.1.** *Let $(X, d, \ell)$ be a 1-chainable complete digital metric space. Let $T : X \to X$ be a $(1, \ell)$-uniformly locally contractive mapping. Then $T$ has a unique fixed point in $X$.*

This assertion is correct. However, its publication is unfortunate, for the following reasons.

- The assertion turns out to be trivial, since such a map must be constant; indeed, the latter is shown in [26].
- Theorem 8.1 duplicates Theorem 5.1 of the earlier [19] (a result later shown to be trivial in [9]).

## 9. On [27, 28]

9.1. **Geraghty contraction.** The papers [27, 28] focus on digital Geraghty contraction maps.

Unfortunately, these papers use the symbol $S$ both to represent a single function and a certain set of functions. This leads to an unfortunate simultaneous use of both interpretations of this symbol. We will try to unravel this confusion by using the following, with $G$ as the symbol for the set of functions discussed.

**Definition 9.1** ([28])**.** $G = \{ \beta : [0, \infty) \to [0, 1) \mid \beta(t_n) \to 1 \Rightarrow t_n \to 0 \}$.

**Definition 9.2** ([28])**.** Let $(X, d, \kappa)$ be a digital metric space. The function $f : X \to X$ is a digital Geraghty contraction map if there exists $\beta \in G$ such that

$$d(f(x), f(y)) \leq \beta(d(x, y))d(x, y) \text{ for all } x, y \in X.$$

*Remark* 9.3. It is observed in [28] that a digital contraction map is a digital Geraghty contraction map, but the converse is not true. However, the converse is true for finite $X$, since in this case we can replace $\beta(d(x, y))$ by

$$\beta'(d(x, y)) = \max\{ \beta(d(x, y)) \mid x, y \in X \} < 1,$$

since

$$d(fx, fy) \le \beta(d(x,y))d(x,y) \le \beta'(d(x,y))d(x,y).$$

The following shows an important case in which Definition 9.2 reduces to triviality.

**Proposition 9.4.** *Suppose $(X, d, \kappa)$ is a digital metric space and $d$ is an $\ell_p$ metric or the shortest path metric. If $X$ is $c_1$-connected (whether or not $\kappa$ is $c_1$) then a digital Geraghty contraction map must be a constant function.*

*Proof.* Let $x \leftrightarrow_{c_1} y$ in $X$. Then $d(x,y) = 1$. From Definition 9.2,

$$d(fx, fy) < \beta(1) < 1, \qquad \text{so} \qquad d(fx, fy) = 0.$$

Since $(X, c_1)$ is connected, the assertion follows. □

9.2. **Theorem of [28].**

**Theorem 9.5** ([28]). *Let $(X, d, \kappa)$ be a complete digital metric space, where $d$ is the Euclidean metric. Let $f : X \to X$ be a digital Geraghty contraction map. Then $f$ has a fixed point in $X$.*

The assertion of this theorem is correct. Here, we discuss flaws in the argument offered for its proof in [28], resulting in a much longer argument than necessary.

- The authors seek to show a sequence $\{x_n\}$ is Cauchy by obtaining a contradiction to the assumption that it isn't, leading to a choice of $\varepsilon > 0$ and a subsequence of $\{ x_n = f(x_{n-1}) \}$ with members with arbitrarily large indices $u, v$ such that

  $$\varepsilon \le d(x_u, x_v) < \varepsilon.$$

  Rather than recognize that this gives the desired contradiction, the authors proceed with several more paragraphs before concluding that they have the desired contradiction.
- The authors neglect to complete the proof, failing to show the existence of a unique fixed point. This can be done as follows. We have

  $$u = \lim_{n \to \infty} x_n = \lim_{n \to \infty} f(x_n).$$

  Since $d$ is the Euclidean metric, $(X, d)$ is uniformly discrete, so for almost all $n$,

  $$u = x_n = x_{n+1} = f(x_n) = f(u).$$

  Thus, $u$ is a fixed point.

  The uniqueness of $u$ as a fixed point of $f$ is shown as follows. Suppose $u'$ is a fixed point. Then

  $$d(u, u') = d(fu, fu') \le \beta(u, u')d(u, u')$$

  which implies $\beta(u, u')d(u, u') = 0$ and thus $d(u, u') = 0$, so $u = u'$.

*Remark* 9.6. In light of Proposition 9.4, we see that there are many cases for which Theorem 9.5 is trivial.

9.3. **On** [27]**.** The paper [27] is concerned with pairs of Geraghty contraction maps in digital metric spaces.

**Definition 9.7** ([27])**.** Let $(X, d, \kappa)$ be a digital metric space. Let $S, T : X \to X$, If there exists $\beta \in G$ such that for all $x, y \in X$,

$$d(Sx, Sy) \leq \beta(d(Tx, Ty))d(Tx, Ty),$$

then $(T, S)$ is a pair of Geraghty contraction maps.

We note an important case in which Definition 9.7 reduces to triviality.

**Proposition 9.8.** *Let $(X, d, c_1)$ be a digital metric space and let $(T, S)$ be a pair of Geraghty contraction maps on $X$. If*

- *$d$ is any $\ell_p$ metric or the shortest path metric;*
- *$T \in C(S, c_1)$; and*
- *$(X, c_1)$ is connected,*

*then $S$ is a constant function.*

*Proof.* Let $x \leftrightarrow_{c_1} y$ in $X$. By Definition 9.7,

$$d(Sx, Sy) \leq \beta(d(Tx, Ty))(d(Tx, Ty)) < d(Tx, Ty).$$

Since $Tx \leftrightarroweq_{c_1} Ty$, we have $d(Sx, Sy) < 1$, so $d(Sx, Sy) = 0$. Since $(X, c_1)$ is connected, $S$ must be constant. □

The following is stated as Theorem 2.2 of [27].

*Assertion* 9.9 ([27]). Let $(T, S)$ be a pair of Geraghty contraction maps on $X$, where $(X, d, \kappa)$ is a digital metric space. Suppose

- $S(X) \subset T(X)$;
- $T$ is continuous; and
- $S$ and $T$ commute.

Then $T$ and $S$ have a common fixed point.

The argument given as proof of this assertion is flawed as follows. A sequence $\{ x_n \}$ is formed in $X$ such that $\{ d(Tx_{n+1}, Tx_n) \}$ is a decreasing sequence. It is derived that

$$\frac{d(Tx_{n+2}, Tx_{n+1})}{d(Tx_{n+1}, x_n)} \leq \beta(d(Tx_{n+1}, Tx_n)) < 1.$$

It is then claimed that the latter implies

$$\lim_{n \to \infty} \frac{d(Tx_{n+2}, Tx_{n+1})}{d(Tx_{n+1}, x_n)} = 1.$$

No justification is given for this claim. Therefore, Assertion 9.9 must be regarded as unproven.

We note in the following that there are important cases for which Assertion 9.9 reduces to triviality.

**Example 9.10.** Let $(T, S)$ be a pair of Geraghty contraction maps on $X$, where $(X, d, c_1)$ is a digital metric space. Suppose

- $T \in C(X, c_1)$;
- $d$ is any $\ell_p$ metric or the shortest path metric;
- $(X, c_1)$ is connected; and
- $S$ and $T$ commute.

Then $S$ is a constant function, and $S$ and $T$ have a common fixed point.

*Proof.* By Proposition 9.8, $S$ is constant; say, $S(x) = x_0$ for all $x \in X$. Since $S$ and $T$ commute,

$$Tx_0 = TSx_0 = STx_0 = x_0.$$

□

## 10. On [29]

### 10.1. Admissible functions.

**Definition 10.1** ([33])**.** Consider functions $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. We say $T$ is *$\alpha$-admissible* if

$$x, y \in X,\ \alpha(x, y) \geq 1 \ \Rightarrow\ \alpha(Tx, Ty) \geq 1.$$

**Definition 10.2** ([29])**.** Let $S, T : X \to X$, $\alpha : X \times X \to [0, \infty)$. We say $S$ is $\alpha - \beta$-admissible with respect to $T$ if for all $x, y \in X$ we have $\alpha(Tx, Ty) \geq 1$ and $\beta(Tx, Ty) \geq 1$ implies $\alpha(Sx, Sy) \geq 1$ and $\beta(Sx, Sy) \geq 1$.

Presumably, $\beta : X \times X \to [0, \infty)$ also, but this is not stated in the definition quoted above.

### 10.2. "Theorem" 3.1 of [29]. 

Let

$$\Phi = \left\{ \begin{array}{c} \varphi : [0, \infty) \to [0, \infty) \mid \varphi \text{ is increasing}, \\ t > 0 \Rightarrow \varphi(t) < t, \text{ and } \varphi(0) = 0 \end{array} \right\}$$

**Definition 10.3** ([29])**.** Let $(X, d, \rho)$ be a complete digital metric space. Let $T : X \to X$, $\alpha, \beta : [0, \infty) \to [0, \infty)$. $X$ is *$\alpha - \beta$ regular* if for every sequence $\{ x_n \}$ in $X$ such that $x_n \to x \in X$ and $\alpha(x_n, x_{n+1}) \geq 1$ and $\beta(x_n, x_{n+1}) \geq 1$ for all $n$, then there exists a subsequence $\{ x_{n_k} \}$ of $\{ x_n \}$ such that $\alpha(x_{n_k}, x_{n_{k+1}}) \geq 1$, $\beta(x_{n_k}, x_{n_{k+1}}) \geq 1$, $\alpha(x, Tx) \geq 1$, and $\beta(x, Tx) \geq 1$.

The following is stated as Theorem 3.1 of [29].

*Assertion* 10.4. Let $(X, d, \rho)$ be a complete connected digital metric space. Let $T : X \to X$ and let $\alpha, \beta : X \times X \to [0.\infty)$ be such that

1. $T$ is $\alpha - \beta$ admissible;
2. There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$;
3. Either $T$ is continuous or $X$ is $(\alpha - \beta)$ regular; and
4. For some $\psi, \varphi \in \Phi$ and all $x, y \in X$,

$$\alpha(x, Tx)\beta(y, Ty)\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y)). \tag{10.1}$$

Then $T$ has a fixed point. Further, if $u$ and $v$ are fixed points of $T$ such that $\alpha(u, Tu) \geq 1$, $\alpha(v, Tv) \geq 1$, $\beta(u, Tu) \geq 1$, and $\beta(v, Tv) \geq 1$, then $u = v$.

Assertion 10.4 and its "proof" in [29] are flawed as follows. Some flaws listed below are easily corrected but are possibly confusing.

- The statement of [29] labeled (1) has an "$x$" that should be "$x_0$" - the statement should be
$$x_{n+1} = Tx_n = T^{n+1}x_0 \text{ for all } n \geq 0.$$
- There are multiple references to hypothesis (3) that should refer to hypothesis (4).
- Statement (3) of [29] says
$$d(x_{n_k}, x_{m_k}) \geq \varepsilon. \qquad \text{(3) of [29]}$$
It is then claimed that this implies
$$d(x_{n_k-1}, x_{m_k}) < \varepsilon. \qquad \text{(4) of [29]}$$
There is no justification for this claim, although it could be justified by choosing $n_k$ as the minimal index of a member of the subsequence for a given $m_k$ to satisfy (3) of [29].
- Suppose (4) of [29] is valid. It leads to a 3-line chain of inequalities in which the second line is marked "(7)". The third line is missing "$\leq$" at its beginning.
- The above leads to (9) of [29]:
$$\psi(d(x_{n_k}, x_{m_k})) \leq \psi(d(x_{n_k-1}, x_{m_k-1})) - \phi(d(x_{n_k-1}, x_{m_k-1}))$$
which is then taken to a limit in order to obtain the contradiction $\varepsilon = 0$. But this limit depends on an unstated hypothesis that the functions $\psi$ and $\phi$ are continuous.
- The contradiction mentioned above resulted from assuming $\varepsilon > 0$. Therefore $\varepsilon = 0$. It follows that $\{x_n\}$ is a Cauchy sequence, and since $X$ is complete, $x_n \to x \in X$. The authors spend additional paragraphs to argue that $x$ is a fixed point of $T$; however, $(X, d, \rho)$ is uniformly discrete, so $x_n = x_{n+1} = Tx_n = x$ for almost all $n$.
- Since the authors' argument for the existence of a fixed point as a consequence of $\{x_n\}$ being Cauchy is their only use of hypothesis 3) (recall other references to hypothesis 3) should be references to hypothesis 4)), the above shows hypothesis 3) is unnecessary.

Thus, the assertion as written is unproven. By modifying [29] as discussed above, we get a proof of the following version of the assertion.

**Theorem 10.5.** *Let $(X, d, \rho)$ be a complete connected digital metric space. Let $T : X \to X$ and let $\alpha, \beta : X \times X \to [0.\infty)$ be such that*

1. *$T$ is $\alpha - \beta$ admissible;*
2. *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$; and*
3. *For some continuous functions $\psi, \varphi \in \Phi$ and all $x, y \in X$,*
$$\alpha(x, Tx)\beta(y, Ty)\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y)). \qquad (10.2)$$

*Then $T$ has a fixed point. Further, if $u$ and $v$ are fixed points of $T$ such that $\alpha(u, Tu) \geq 1$, $\alpha(v, Tv) \geq 1$, $\beta(u, Tu) \geq 1$, and $\beta(v, Tv) \geq 1$, then $u = v$.*

10.3. **"Theorem" 3.2 of [29].**

**Definition 10.6** ([29])**.** Let $(X, d, \rho)$ be a complete digital metric space and let $S, T : X \to X$ and $\alpha, \beta : X \times X \to [0, \infty)$. The pair $(S, T)$ is a pair of $\alpha - \beta - \psi - \varphi$ *contractive mappings*, where $\psi, \varphi \in \Phi$, if

$$\alpha(x, Tx)\beta(y, Ty)\psi(d(Sx, Sy)) \leq \psi(d(Tx, Ty)) - \varphi(d(Tx, Ty))$$

for all $x, y \in X$.

*Remark* 10.7. It seems likely that there should be further restrictions on $\alpha$ and $\beta$. E.g., given any pair $S, T : X \to X$ and any $\psi, \phi$ such that $\psi(t) \geq \phi(t)$ for all $t \geq 0$, if either of $\alpha$ or $\beta$ is the constant function with value 0, then according to Definition 10.6, $(S, T)$ is a pair of $\alpha - \beta - \psi - \varphi$ contractive mappings.

The following is stated as Theorem 3.2 of [29].

*Assertion* 10.8. Let $(X, d, \rho)$ be a complete digital metric space. Let $S, T : X \to X$ be be $\alpha - \beta - \psi - \varphi$ mappings for $\alpha, \beta : X \times X \to [0, \infty)$ such that

(1) $S(X) \subset T(X)$;
(2) $S$ is $\alpha - \beta$-admissible with respect to $T$;
(3) there exists $x_0 \in X$ such that $\alpha(Tx_0, Sx_0) \geq 1$ and $\beta(Tx_0, Sx_0) \geq 1$, and
(4) we have
$$\alpha(x, Tx)\beta(y, Ty)\psi(d(Sx, Sy)) \leq \psi(d(Tx, Ty)) - \varphi(d(Tx, Ty))$$
(5) If $\{ Tx_n \} \subset X$ such that $\alpha(Tx_n, Tx_{n+1}) \geq 1$ and $\beta(Tx_n, Tx_{n+1}) \geq 1$ for all $n$ and $Tx_n \to_{n\to\infty} Tx$, then there exists a subsequence $\{ Tx_{n(k)} \}$ of $\{ Tx_n \}$ and $z \in X$ such that $\alpha(Tx_{n(k)}, Tz) \geq 1$ and $\beta(Tx_{n(k)}, Tz) \geq 1$ for all $k$.
(6) $T(X)$ is closed.

Then $S$ and $T$ have a coincidence point.

Among the flaws of Assertion 10.8 as presented in [29] are the following.

- It is assumed in 10.8 (see that paper's Definition 2.3) that $d$ is the Euclidean metric, which is uniformly discrete on subsets of $\mathbb{Z}^n$. Therefore, the assumption of a discrete metric space need not be stated; the hypothesis that $T(X)$ is closed, is unnecessary; and $(X, d)$ may be more generally assumed to be uniformly discrete.
- At the line marked "(11)", there is no clear justification for the claim
$$x_n = Sx_n.$$
This error propagates to the lines marked "(14)".
- At the chain of inequalities marked "(14)" and its subsequent paragraph, the error noted at the line marked "(11)" results in the conclusion that $lim_{n\to\infty} d(Sx_n, Sx_{n+1})$ is 0, i.e., that the line marked "(12)" is valid.

Thus, the error noted at "(11)" propagates through the "proof." We must therefore regard Assertion 10.8 as unproven.

10.4. **"Theorem" 3.3 of [29].** The following is stated as Theorem 3.3 of [29].

*Assertion* 10.9. Let $(X, d, \rho)$ be a complete digital metric space and $S, T : X \to X$ and $\alpha, \beta : X \times X \to [0, \infty)$ be mappings such that

(1) $S(X) \subset T(X)$;
(2) $S$ is $\alpha - \beta$ admissible with respect to $T$;
(3) there exists $x_0 \in X$ such that $\alpha(Tx_0, Sx_0) \geq 1$ and $\beta(Tx_0, Sx_0) \geq 1$;
(4) there exist $\psi, \varphi \in \Phi$ such that

$$\alpha(Tx, Ty)\psi(d(Sx, Sy)) \leq \psi(M(x, y)) - \varphi(M(x, y))$$

for all $x, y \in X$, where

$$M(x, y) = \max\{\, d(Tx, Ty), d(Tx, Sx), d(Sy, Ty), d(Sx, Ty) \,\};$$

(5) If $\{ x_n \}$ is a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ and $\beta(x_n, x_{n+1}) \geq 1$ for all $n$ and $Tx_n \to_{n\to\infty} Tx \in T(X)$, then there is a subsequence $\{ x_{n(k)} \}$ of $\{ x_n \}$ such that $\alpha(Tx_{n(k)}, Tz) \geq 1$ and $\beta(Tx_{n(k)}, Tz) \geq 1$ for all $k$; and
(6) T(X) is closed.

Then $S$ and $T$ have a coincidence point.

The argument in [29] for Assertion 10.9 is flawed as follows. Statement (14) of [29] is used to justify the claim that

$$d(Sx_n, Sx_{n+1}) \to_{n\to\infty} 0.$$

However, as noted above, statement (14) is not correctly derived. Thus Assertion 10.9 is unproven.

10.5. **"Theorem" 3.4 of [29].** The following is stated as Theorem 3.4 of [29].

*Assertion* 10.10. Let $(X, d, \rho)$ be a complete digital metric space and $S, T : X \to X$ and $\alpha, \beta : X \times X \to [0, \infty)$ be mappings such that

(1) $S(X) \subset T(X)$;
(2) $S$ is $\alpha - \beta$ admissible with respect to $T$;
(3) there exists $x_0 \in X$ such that $\alpha(Tx_0, Sx_0) \geq 1$;
(4) there exist $\psi, \varphi \in \Phi$ such that

$$\alpha(x, Tx)\beta(y, Ty)\psi(d(Sx, Sy)) \leq \psi(M(x, y)) - \varphi(M(x, y))$$

for all $x, y \in X$, where

$$M(x, y) = \max\{\, d(Sx, Sy), d(Tx, Sx), d(Sy, Ty), \frac{d(Sx, Ty) + d(Sy, Tx)}{2} \,\};$$

(5) If $\{ x_n \}$ is a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n$ and $Tx_n \to_{n\to\infty} Tx \in T(X)$, then there is a subsequence $\{ x_{n(k)} \}$ of $\{ x_n \}$ such that $\alpha(Tx_{n(k)}, Tz) \geq 1$ for all $k$; and
(6) T(X) is closed.

Then $S$ and $T$ have a coincidence point.

The argument given in [29] as proof of Assertion 10.10 is that it follows immediately from Assertion 10.8. As we have shown above that the latter is unproven, it follows that Assertion 10.10 is unproven.

10.6. **"Theorem" 3.5 of [29].** The following is stated as Theorem 3.5 of [29].

*Assertion* 10.11. In addition to the hypotheses of Theorem 3.2 *[Assertion 10.8 of the current paper]*, suppose that for each pair $x, y$ of common fixed points of $S$ and $T$ there exists $z \in X$ such that

$$\alpha(Tx, Tz) \geq 1, \quad \beta(Tx, Tz) \geq 1, \quad \alpha(Ty, Tz) \geq 1, \quad \beta(Ty, Tz) \geq 1, \tag{10.3}$$

and $S$ and $T$ commute at coincidence points. Then $S$ and $T$ have a unique common fixed point.

Among the flaws of the argument offered as proof in [29] of Assertion 10.11, we find the following. The symbol "$z$" is introduced with two distinct meanings: as stated above among the hypotheses, and then in the claim "... there exists $z \in X$ such that $\lim_{n \to \infty} Tz_n = Tz$." No reason is given to believe that the latter point named "$z$" must be the same as the point of that name in hypothesis (10.3). The argument proceeds assuming the symbol represents both the first "$z$" and the second "$z$".

We conclude that Assertion 10.11 is unproven.

10.7. **"Theorems" 4.1 and 4.2 of [29].** The following are stated as Theorem 4.1 and Theorem 4.2 of [29].

*Assertion* 10.12. Let $(X, d, \rho)$ be a complete digital metric space. Let $A$ and $B$ be nonempty closed subsets of $X$. Suppose $\alpha : X \times X \to [0, \infty)$ and $T : A \cup B \to A \cup B$ are such that

(1) $T(A) \subset B$ and $T(B) \subset A$;
(2) if $\alpha(x, y) \geq 1$ then $\alpha(Tx, Ty) \geq 1$;
(3) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(4) $T$ is continuous or $X$ is $\alpha$-regular;
(5) for some $\psi, \varphi \in \Phi$, $\alpha(x, y)\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y))$ for all $x, y \in X$.

Then $T$ has a fixed point in $A \cap B$. Further if $u$ and $v$ are fixed points of $T$ such that $\alpha(u, Tu) \geq 1$, $\alpha(v, Tv) \geq 1$, $\beta(u, Tu) \geq 1$, and $\beta(v, Tv) \geq 1$, then $u = v$.

*Assertion* 10.13. Let $(X, d, \rho)$ be a complete digital metric space. Let $A$ and $B$ be nonempty closed subsets of $X$. Let $Y = A \cup B$ and let $S, T : Y \to Y$ satisfy

- $T(A)$ and $T(B)$ are closed;
- $S(A) \subset T(B)$ and $S(B) \subset T(A)$;
- $T$ is one-to-one;

- for some $\psi, \varphi \in \Phi$,

  $$\psi(d(Sx, Sy)) \leq \psi(M(x,y)) - \varphi(M(x,y)) \quad \text{for all } x, y \in A \times B;$$

  *[presumably, the latter qualification is meant to be $x, y \in A \cup B$]*, where

  $$M(x,y) = \max\{\, d(Tx,Ty), d(Tx,Sx), d(Sy,Ty), d(Sx,Ty)\,\}.$$

  Then $S$ and $T$ have a coincidence point in $A \cap B$. Further, if $S$ and $T$ commute at their coincidence point, then $S$ and $T$ have a unique common fixed point in $A \cap B$.

It seems likely that there should be a hypothesis that $A \cap B \neq \varnothing$ in both Assertion 10.12 and Assertion 10.13. More definitely, the presentations of these assertions are flawed as follows. The argument offered in [29] for Assertion 10.12 depends on Assertion 10.4, and the argument offered in [29] for Assertion 10.13 depends on Assertion 10.8. Since we have shown above that Assertions 10.4 and 10.8 are unproven, it follows that Assertions 10.12 and 10.13 are unproven.

10.8. **Corollaries 4.3 and 4.4 of** [29]**.** The following Assertions 10.14 and 10.15 are stated in [29] as Corollaries 4.3 and 4.4, respectively, as immediate consequences of Assertion 10.13. Since we have shown above that Assertion 10.13 is unproven, it follows that Assertions 10.14 and 10.15 are unproven.

*Assertion* 10.14. Let $(X, d, \rho)$ be a complete digital metric space. Let $A$ and $B$ be nonempty closed subsets of $X$. Let $Y = A \cup B$ and $S, T : Y \to Y$, satisfying the following.

- $T(A)$ and $T(B)$ are closed.
- $S(A) \subset T(B)$ and $S(B) \subset T(A)$.
- $T$ is one-to-one.
- There exist $\psi, \varphi \in \Phi$ such that

  $$\psi(d(Sx, Sy)) \leq \psi(M(x,y)) - \phi(M(x,y)) \quad \text{for all } x, y \in Y,$$

  where

$$M(x,y) = \max\{\, d(Sx,Sy), d(Tx,Sx), d(Sy,Ty), \frac{d(Sx,Ty) + d(Sy,Tx)}{2}\,\}.$$

  Then $S$ and $T$ have a coincidence point in $A \cap B$. Further, if $S$ and $T$ commute at their coincidence point, then $S$ and $T$ have a unique common fixed point in $A \cap B$.

*Assertion* 10.15. Let $(X, d, \rho)$ be a complete digital metric space. Let $A$ and $B$ be nonempty closed subsets of $X$. Let $Y = A \cup B$ and $S, T : Y \to Y$, satisfying the following.

- $T(A)$ and $T(B)$ are closed.
- $S(A) \subset T(B)$ and $S(B) \subset T(A)$.
- $T$ is one-to-one.

- There exist $\psi, \varphi \in \Phi$ such that

  $\psi(d(Sx, Sy)) \leq \psi(d(Tx, Ty)) - \varphi(d(Tx, Ty)) \quad$ for all $x, y \in Y$.

  Then $S$ and $T$ have a coincidence point in $A \cap B$. Further, if $S$ and $T$ commute at their coincidence point, then $S$ and $T$ have a unique common fixed point in $A \cap B$.

## 11. On [32]

11.1. **"Theorem" 3.1.** The following is stated as Theorem 3.1 of [32].

*Assertion* 11.1. Let $(X, d, \kappa)$ be a digital metric space and $S, T$ be self maps on $X$ satisfying

$$d(Sx, Ty) \leq \alpha d(x, Sx) + d(y, Ty) \text{ for all } x, y \in X \text{ and } 0 < \alpha < 1/2.$$

Then $S$ and $T$ have a unique common fixed point in $X$.

That this assertion is incorrect is shown by the following.

**Example 11.2.** Let $D$ be the "diamond,"

$$D = \{ (1,0), (0,1), (-1,0), (0,-1) \},$$

a $c_2$-digital simple closed curve.

Let $S = \mathrm{id}_D$ and let $T(x) = -x$. If $d$ is the Euclidean metric, we have

$$d(Sx, Ty) \leq \mathit{diameter}(D) = 2 = \alpha(0) + 2 = \alpha d(x, Sx) + d(y, Ty).$$

Since $T$ has no fixed point in $D$, we have established that Assertion 11.1 is not generally valid.

11.2. **"Theorem" 3.2.** The following is stated as Theorem 3.2 of [32].

*Assertion* 11.3. Let $(X, d, \kappa)$ be a digital metric space and let $S, T : X \to X$ such that

$$d(Sx, Ty) \leq \alpha d(x, Ty) + d(y, Sx)$$

for all $x, y \in X$ and $0 < \alpha < 1/2$. Then, $S$ and $T$ have a unique common fixed point in $X$.

There are multiple errors in the argument of [32] offered as proof of Assertion 11.1. We quote one section of this argument, with labeled lines.

Begin quote:

Consider

$$d(x_1, x_2) = d(Sx_0, Tx_1) \tag{11.1}$$

Now

$$d(x_1, x_2) \leq \alpha d(x_0, Tx_1) + d(x_1, Sx_0) \tag{11.2}$$

$$d(x_1, x_2) \leq \alpha d(x_0, x_2) + d(x_1, x_1) \tag{11.3}$$

$$d(x_1, x_2) \leq \alpha d(x_0, x_1) + d(x_1, x_2) \tag{11.4}$$

$$d(x_1, x_2) - \alpha d(x_1, x_2) \leq \alpha d(x_0, x_1) \tag{11.5}$$

$$(1 - \alpha) d(x_1, x_2) \leq \alpha d(x_0, x_1) \tag{11.6}$$

End quote

Statement (11.4) does not appear correctly derived from (11.3), and is equivalent to the unhelpful

$$0 \leq \alpha d(x_0, x_1)$$

Clearly, (11.5) is not correctly derived from (11.4).

Near the bottom of page 229 of [32], we see

> ... there is a point $u \in X$ such that $x_n \to u$. Therefore, subsequence $< Sx_{2n} > \to u \ldots$ and $< T_{2n+1} > \to u$ since $S$ and $T$ are $(\kappa, \kappa)$-continuous ... we have, $Su = u$ ....

This is incorrect reasoning. The author does not say if the undefined term "map" is assumed to include a hypothesis of digital continuity; nor is continuity proven. Further, Example 11.4, given below, provides a counterexample to the claim that such subsequences imply the existence of a fixed point, even if $S$ is digitally continuous. Thus, the existence of a fixed point is not established.

In statements (1) and (2) on page 230, there are instances of "=" that should be "≤".

Even if we assume common fixed points $u, v$ of $S$ and $T$, the argument provided on page 230 to show $u$ and $v$ coincide is incorrect. The author claims the inequality $d(u, v) - d(v, u) \leq 0$ implies $d(u, v) = 0$, but clearly it does not.

Thus, we conclude that Assertion 11.1 is unproven.

**Example 11.4.** Let $S = T : \mathbb{N} \to \mathbb{N}$ be the function

$$S(n) = T(n) = \begin{cases} 0 & \text{if } n = 1; \\ 1 & \text{if } n \neq 1. \end{cases}$$

Let $x_n = n$. $S$ and $T$ are both $c_1$-continuous. Also, $Sx_{2n} \to 1$ and $Tx_{2n+1} \to 1$ but $S(1) \neq 1$. Further, neither $S$ nor $T$ has a fixed point.

11.3. **"Theorem" 3.3.** The following is stated as Theorem 3.3 in [32].

*Assertion* 11.5. Let $(X, d, \kappa)$ be a digital metric space and let $S, T$ be self-maps on $X$ such that

$$d(Sx, Ty) \leq ad(x, Sx) + bd(y, Ty) + cd(x, y) \text{ for all } x, y \in X,$$

where $a, b, c$ are nonnegative real numbers such that $a + b + c < 1$. Then $S$ and $T$ have a unique common fixed point in $X$.

The argument offered in [32] as proof of this assertion is flawed as follows.

- Errors similar to above for Assertion 11.3: it is incorrectly claimed that (unestablished digital) continuity and $x_n \to u$ imply $Sx_{2n} \to u$ and $Tx_{2n+1} \to u$. From this is wrongly (see Example 11.4) concluded that $u$ is a common fixed point of $S$ and $T$.

- The argument for uniqueness of any fixed point has a correctible error:

  $$\text{“}d(u,v) = cd(u,v)\text{” should be “}d(u,v) \leq cd(u,v)\text{”},$$

  which does imply the desired conclusion, $d(u,v) = 0$.

Due to the errors discussed in the first bullet, Assertion 11.5 is unproven.

11.4. **"Theorems" 3.4 and 3.5.** We discuss two assertions of [32] whose arguments given as proofs contain the same errors.

The following is stated in [32] as Theorem 3.4.

*Assertion* 11.6. Let $(X, d, \kappa)$ be a digital metric space and let $S$ be a self-map on $X$ such that

$$d(Sx, Sy) \leq \frac{[ad(y, Sy)][1 + d(x, Sx)]}{1 + d(x, y)} + bd(x, y)$$

for all $x, y \in X$, where $a, b \geq 0$ and $a + b < 1$. Then $S$ has a unique fixed point in $X$.

The following is stated in [32] as Theorem 3.5.

*Assertion* 11.7. Let $(X, d, \kappa)$ be a digital metric space and let $S$ be a self-map on $X$ such that

$$d(Sx, Sy) \leq \frac{[ad(y, Sy)][1 + d(x, Sx)]}{1 + d(x, y)} + bd(x, y) + c\frac{d(y, Sy) + d(y, Sx)}{1 + d(y, Sy)d(y, Sx)}$$

for all $x, y \in X$, where $a, b, c$ are nonnegative and $a + b + c < 1$. Then $S$ has a unique fixed point in $X$.

The arguments offered in [32] for these assertions are both flawed as follows.

- Errors similar to above for Assertions 11.3 and 11.5: it is incorrectly claimed that (questionable digital) continuity and $x_n \to u$ imply $Sx_{2n} \to u$ and $Sx_{2n+1} \to u$. From this is wrongly (see Example 11.4) concluded that $u$ is a fixed point of $S$.
- Despite the author's claim, the statement $d(u, u) = 0$ fails to establish uniqueness of a fixed point of $S$; one must show $d(u, v) = 0$ for $u, v \in \mathrm{Fix}(S)$.

Thus, Assertions 11.6 and 11.7 are unproven.

## 12. Further remarks

We have continued the work of [9, 5, 6, 7, 8] in discussing flaws in papers rooted in the notion of a digital metric space. The papers we have considered have many errors and assertions that turn out to be trivial.

Although authors are responsible for their errors and other shortcomings, it is clear that many of the papers studied in the current paper were given inadequate review.

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