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Partial actions on limit spaces

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Abstract

G-compactifications of continuous partial actions in the category of limit spaces are considered. In particular, sufficient conditions are given to ensure that (G, X, α) has a largest regular G-compactification.

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1. Introduction

The work presented here is a continuation of that given in [2]. Objects of the form (G, X, α) are studied, where α is a continuous partial action of the limit group G on the limit space X. If Y is a Hausdorff compactification of X in the category **LS** of limit spaces, requirements are given to ensure that (G, Y, β) is a Hausdorff G-compactification of (G, X, α). In particular, if X possesses a largest regular (including Hausdorff) compactification in **LS**, then (G, X, α) has a largest regular G-compactification whenever α is Cauchy continuous. Finally, an additional assumption is needed in the proof of Lemma 5.1 [2]. This additional assumption should also be added to Theorem 5.2 [2] and Theorem 5.4 [2].

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2. Preliminaries

The reader is asked to refer to [2] for definitions and notations not listed here. One variation is that Cauchy spaces are needed here and hence limit spaces replace convergence spaces of [2]. Let $F(X)$ denote the set of all filters on X. If $\mathcal{F}, \mathcal{G} \in F(X)$ and $F \cap G \neq \varnothing$ for each $F \in \mathcal{F}$ and $G \in \mathcal{G}$, then $\{F \cap G \mid F \in \mathcal{F}, G \in \mathcal{G}\}$ is a base for the smallest filter containing $\mathcal{F}$ and $\mathcal{G}$, denoted by $\mathcal{F} \vee \mathcal{G}$. We call $\mathcal{D} \subseteq F(X)$ a *Cauchy structure* on X if it satisfies:

(CS1) $x^\bullet \in \mathcal{D}$ for all $x \in X$,
(CS2) $\mathcal{G} \geq \mathcal{F} \in \mathcal{D}$ implies $\mathcal{G} \in \mathcal{D}$,
(CS3) $\mathcal{F}, \mathcal{G} \in \mathcal{D}$ and $\mathcal{F} \vee \mathcal{G}$ exists imples $\mathcal{F} \cap \mathcal{G} \in \mathcal{D}$.

The pair $(X, \mathcal{D})$ is called a *Cauchy space* whenever $\mathcal{D}$ is a Cauchy structure. A map $f : (X, \mathcal{D}) \to (Y, \mathcal{E})$ between two Cauchy spaces is *Cauchy continuous* if $f^{\to}\mathcal{F} \in \mathcal{E}$ whenever $\mathcal{F} \in \mathcal{D}$. Let **CHY** denote the category of Cauchy spaces and Cauchy continuous maps. Objects in **CHY** induce limit spaces. A pair (X, q) is a *limit space* provided:

(LS1) $x^\bullet \xrightarrow{q} x$ for each $x \in X$,
(LS2) $\mathcal{G} \geq \mathcal{F} \xrightarrow{q} x$ implies $\mathcal{G} \xrightarrow{q} x$,
(LS3) $\mathcal{F}, \mathcal{G} \xrightarrow{q} x$ implies $\mathcal{F} \cap \mathcal{G} \xrightarrow{q} x$

Note that every limit space is a convergence space. Let **LS** denote the full subcategory of the category **CS** of convergence spaces whose objects are all the limit spaces. Every $(X, \mathcal{D}) \in |\mathbf{CHY}|$ determines a limit space (X, q) by defining $\mathcal{F} \xrightarrow{q} x$ to mean $\mathcal{F} \cap x^\bullet \in \mathcal{D}$. Keller [3] characterized the limit spaces that are induced by Cauchy spaces as follows: if $x \neq y$, either x and y have no common convergent filters or $\mathcal{F} \to x$ if and only if $\mathcal{F} \to y$. In particular, Hausdorff limit spaces are induced by Cauchy spaces. The reader is referred to Lowen-Colebunders [4] and Preuss [5] for more details concerning Cauchy spaces.

Let $\mathbf{C}$ be the category whose objects are of the form (G, X, α), where G is a limit group, X is a limit space, and $\alpha : \Gamma_\alpha \to X$ is a continuous partial action. Here, $(g, x) \in \Gamma_\alpha$ if and only if $x \in X_{g^{-1}} \subseteq X$, $\alpha_g : X_{g^{-1}} \to X_g$ is a homeomorphism, and $\alpha_g(x) = \alpha(g, x)$. Morphisms in $\mathbf{C}$ are of the form $(k, f) : (G, X, \alpha) \to (H, Y, \beta)$, where $k : G \to H$ is a continuous homomorphism, $f : X \to Y$ is a continuous map, and the following diagram commutes:

$$\begin{array}{ccc} \Gamma_\alpha & \xrightarrow{k \times f} & \Gamma_\beta \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ X & \xrightarrow{f} & Y \end{array}$$

It is shown in [2] that if $(G, X, \alpha) \in |\mathbf{C}|$, then there exists an enveloping action $\alpha^e : G \times X^e \to X^e$ that is continuous and, moreover, $(\mathrm{id}_G, j) : (G, X, \alpha) \to (G, X^e, \alpha^e)$ is a morphism in $\mathbf{C}$ and $j : X \to X^e$ is a homeomorphism onto $j(X)$. Here, $j(x) = \langle (1_G, x) \rangle$ and $X^e = \{\langle (g, x) \rangle \mid g \in G, x \in X\}$, where

$(g,x) \sim (h,y)$ on $G \times X$ if and only if $x \in X_{g^{-1}h}$ and $\alpha_{h^{-1}g}(x) = y$. Moreover, $\alpha^e : G \times X^e \to X^e$ is defined by $\alpha^e(g, \langle (h,x) \rangle) = \langle (gh, x) \rangle$.

Assume that $(G, X, \alpha) \in |\mathbf{C}|$ and (X, q) is Hausdorff but non-compact. Let $X^* = X \cup \{\omega\}$ and define $k : X \to X^*$ by $k(x) = x$. Then $((X^*, q^*), k)$ is a Hausdorff limit-space compactification of (X, q), where q^* is defined by

$$\mathcal{H} \xrightarrow{q^*} k(x) \iff \mathcal{H} \geq k^{\to}\mathcal{F} \text{ for some } \mathcal{F} \xrightarrow{q} x$$
$$\mathcal{H} \xrightarrow{q^*} \omega \iff \mathcal{H} \geq k^{\to}\mathcal{F} \cap \omega^{\bullet} \text{ for some } \operatorname{adh}_X \mathcal{F} = \varnothing$$

Define:

$$\begin{aligned} X_g^* &= k(X_g) \cup \{\omega\}, g \neq 1_G \\ X_{1_G}^* &= X^* \\ \alpha_g^*(k(x)) &= k(\alpha_g(x)), x \in X_{g^{-1}} \\ \alpha_g^*(\omega) &= \omega \end{aligned}$$

Then $((G, X^*, \alpha^*), k)$ is called a *one-point Hausdorff G-compactification* of (G, X, α) in $\mathbf{C}$ whenever $(\mathrm{id}_G, k) : (G, X, \alpha) \to (G, X^*, \alpha^*)$ is a morphism in $\mathbf{C}$.

Definition 2.1. Let $(G, X, \alpha) \in |\mathbf{C}|$. Then X is said to be *weakly adherence restrictive* if for each $\mathcal{F} \in F(X)$ with $\operatorname{adh} j^{\to}\mathcal{F} = \varnothing$ and each $\mathcal{G} \to g$ on G, if $(\mathcal{G} \times \mathcal{F}) \vee \Gamma_\alpha^{\bullet}$ exists, then $\operatorname{adh} \alpha^{\to}((\mathcal{G} \times \mathcal{F}) \vee \Gamma_\alpha^{\bullet}) = \varnothing$.

The definition above is called *adherence restrictive* as defined in [2] whenever $\operatorname{adh} j^{\to}\mathcal{F} = \varnothing$ is replaced by $\operatorname{adh} \mathcal{F} = \varnothing$. It follows that if X is adherence restrictive, then it is weakly adherence restrictive.

3. One-point compactification

It is incorrectly stated in Lemma 5.1 [2] that if $(G, X, \alpha) \in |\mathbf{C}|$, then X is adherence restrictive. The error in the proof occurs near the end since α is defined only on Γ_α. This difficulty is overcome by passing to the enveloping action α^e. The related result is given below.

Lemma 3.1. *If $(G, X, \alpha) \in |\boldsymbol{C}|$, then X is weakly adherence restrictive.*

Proof. Assume that $\mathcal{F} \in F(X)$ and $\mathcal{G} \to g$ on G such that $(\mathcal{G} \times \mathcal{F}) \vee \Gamma_\alpha^{\bullet}$ exists. It must be shown that $\operatorname{adh} j^{\to}\mathcal{F} = \varnothing$ implies that $\operatorname{adh} \alpha^{\to}((\mathcal{G} \times \mathcal{F}) \vee \Gamma_\alpha^{\bullet}) = \varnothing$. Equivalently, using the contrapositive implication, $\operatorname{adh} \alpha^{\to}((\mathcal{G} \times \mathcal{F}) \vee \Gamma_\alpha^{\bullet}) \neq \varnothing$ implies that $\operatorname{adh} j^{\to}(\mathcal{F}) \neq \varnothing$. Suppose that $x \in \operatorname{adh} \alpha^{\to}((\mathcal{G} \times \mathcal{F}) \vee \Gamma_\alpha^{\bullet})$. Then there exists an ultrafilter $\mathcal{H} \to x$ such that $\mathcal{H} \geq \alpha^{\to}((\mathcal{G} \times \mathcal{F}) \vee \Gamma_\alpha^{\bullet})$. Since j and α^e are continuous, $\alpha^{e\to}(\mathcal{G}^{-1} \times j^{\to}\mathcal{H}) \to \alpha^e(g^{-1}, j(x)) = \alpha^e(g^{-1}, \langle (1_G, x) \rangle) = \langle (g^{-1}, x) \rangle$. It suffices to prove that $\langle (g^{-1}, x) \rangle \in \operatorname{adh} j^{\to}\mathcal{F}$.

Let us show that $\alpha^{e\to}(\mathcal{G}^{-1} \times j^{\to}\mathcal{H}) \vee j^{\to}\mathcal{F}$ exists. Assume that $A \in \mathcal{G}$, $H \in \mathcal{H}$ and $F \in \mathcal{F}$. Since $\mathcal{H} \geq \alpha^{\to}((\mathcal{G} \times \mathcal{F}) \vee \Gamma_\alpha^{\bullet})$, there exists $H_1 \in \mathcal{H}$, $H_1 \subseteq H$ such that $H_1 \subseteq \alpha((A \times F) \cap \Gamma_\alpha)$. Let $h_1 \in H_1$. Then there exists $g_1 \in A$,

$x_1 \in F$ such that $h_1 = \alpha(g_1, x_1)$ and $(g_1, x_1) \in \Gamma_\alpha$. Hence $\alpha^e(g_1^{-1}, j(h_1)) = \alpha^e(g_1^{-1}, \langle(1_G, h_1)\rangle) = \langle(g_1^{-1}, h_1)\rangle = \langle(g_1^{-1}, \alpha(g_1, x_1))\rangle = \langle(1_G, x_1)\rangle$ since $(g_1^{-1}, \alpha(g_1, x_1)) \sim (1_G, x_1)$. It follows that $\alpha^e(A^{-1} \times j(H)) \cap j(F) \neq \varnothing$ and hence $\alpha^{e\rightarrow}(\mathcal{G}^{-1} \times j^{\rightarrow}\mathcal{H}) \vee j^{\rightarrow}\mathcal{F}$ exists. Since $\alpha^{e\rightarrow}(\mathcal{G}^{-1} \times j^{\rightarrow}\mathcal{H}) \vee j^{\rightarrow}\mathcal{F} \rightarrow \langle(g^{-1}, x)\rangle$ on X^e, $\langle(g^{-1}, x)\rangle \in \operatorname{adh} j^{\rightarrow}\mathcal{F}$. □

Theorem 3.2. *Let $(G, X, \alpha) \in |\boldsymbol{C}|$ and assume that X is Hausdorff but not compact. Then $((G, X^*, \alpha^*), k)$ is a one-point Hausdorff G-compactification of (G, X, α) in $\boldsymbol{C}$ if and only if X is adherence restrictive.*

Proof. Under the assumption that X is adherence restrictive, proof of the "if" part follows that given in Theorem 5.2 [2]. Conversely, it must be shown that X is adherence restrictive. Assume that $\mathcal{F} \in F(X)$, $\operatorname{adh} \mathcal{F} = \varnothing$, $\mathcal{G} \rightarrow g$ on G and $(\mathcal{G} \times \mathcal{F}) \vee \Gamma_\alpha^\bullet$ exists. It follows that $k^{\rightarrow}\mathcal{F} \rightarrow \omega$ on X^* and thus $(\mathcal{G} \times k^{\rightarrow}\mathcal{F}) \vee \Gamma_{\alpha^*}^\bullet \rightarrow (g, \omega)$ on $G \times X^*$. Since $(\mathrm{id}_G, k) : (G, X, \alpha) \rightarrow (G, X^*, \alpha^*)$, is a morphism the diagram

$$\begin{array}{ccc} \Gamma_\alpha & \xrightarrow{\mathrm{id}_G \times k} & \Gamma_{\alpha^*} \\ \Big\downarrow \alpha & & \Big\downarrow \alpha^* \\ X & \xrightarrow{\quad k \quad} & X^* \end{array}$$

commutes. It follows that $k^{\rightarrow}(\alpha^{\rightarrow}(\mathcal{G} \times \mathcal{F}) \vee \Gamma_\alpha^\bullet) = \alpha^{*\rightarrow}((\mathrm{id}_G \times k)^{\rightarrow}((\mathcal{G} \times \mathcal{F}) \vee \Gamma_\alpha^\bullet)) = \alpha^{*\rightarrow}((\mathcal{G} \times k^{\rightarrow}\mathcal{F}) \vee \Gamma_{\alpha^*}^\bullet) \rightarrow \alpha^*(g, \omega) = \omega$ on X^*. Hence $\operatorname{adh} \alpha^{\rightarrow}((\mathcal{G} \times \mathcal{F}) \vee \Gamma_\alpha^\bullet) = \varnothing$ and X is adherence restrictive. □

An example is given of an object $(G, X, \alpha) \in |\mathbf{C}|$ for which X is not adherence restrictive. First, the following result by Abadie [1] is needed.

Theorem 3.3. *Assume that G is a topological group, Y is a topological space, $\lambda : G \times Y \rightarrow Y$ is a continuous action and X is an open subset of Y. Then λ induces a continuous partial action α of G on X in the topological sense as follows: $X_g = X \cap \lambda_g(X)$ and $\alpha_g : X_{g^{-1}} \rightarrow X_g$ is defined by $\alpha_g(x) = \lambda_g(x), x \in X_{g^{-1}}$, $g \in G$.*

Example 3.4. Let $G = (\mathbb{R}, +)$, $Y = \mathbb{R}$, each equipped with the usual topology, and let $\lambda : G \times Y \rightarrow Y$ denote the continuous action $\lambda(g, y) = g + y$ of G on Y. As mentioned in Theorem 3.3 above, (G, Y, λ) induces a continuous partial action on $X = (0, 1)$ as follows: for each $g \in G$, $X_g = (0, 1) \cap \lambda_g(0, 1) = (0, 1) \cap (g, 1 + g)$ and $\alpha_g : X_{-g} \rightarrow X_g$ is defined by $\alpha_g(x) = g + x$, $g \in G$. Then $(G, X, \alpha) \in |\mathbf{C}|$ and α is a continuous partial action of G on X. Observe that

$$X_g = \begin{cases} (g, 1), & 0 \leq g < 1 \\ (0, 1 + g), & -1 < g < 0 \\ \varnothing, & \text{otherwise} \end{cases}, \qquad g \in G$$

Define $\mathcal{G}$ to be the neighborhood filter on G at $g = \frac{1}{4}$ and let $\mathcal{F}$ denote the restriction to X of the neighborhood filter on Y at $y = 0$. Then $\mathcal{G} \rightarrow \frac{1}{4}$ on G and $\operatorname{adh} \mathcal{F} = \varnothing$. Choose $A = (0, \frac{1}{2}) \in \mathcal{G}$ and $B = (0, \frac{1}{2}) \in \mathcal{F}$. Observe that if

$0 < g < \frac{1}{2}$, then from above, $X_{-g} = (0, 1-g)$ and thus $B \subseteq X_{-g}$. It follows that $A \times B \subseteq \Gamma_\alpha$ and thus $(\mathcal{G} \times \mathcal{F}) \vee \Gamma_\alpha^\bullet$ exists. Hence $\alpha^{\rightarrow}((\mathcal{G} \times \mathcal{F}) \vee \Gamma_\alpha^\bullet) \to \frac{1}{4}$ on X and this implies that X is not adherence restrictive.

4. G-compactifications

Given $(G, X, \alpha) \in |\mathbf{C}|$, assume that G is a Hausdorff limit group and (Y, f) is any Hausdorff compactification of X in **LS**. Unlike section 3, Y is not restricted to be a one-point compactification. Since G, X and Y are Hausdorff limit spaces, each is induced by a Cauchy structure. The following notations are used:

$$\begin{aligned}
\Delta &= \{\mathcal{G} \in F(G) \mid \mathcal{G} \text{ converges on } G\}\\
\mathcal{D} &= \{\mathcal{F} \in F(X) \mid \mathcal{F} \text{ converges on } X\}\\
\mathcal{E} &= \{\mathcal{F} \in F(X) \mid f^{\rightarrow}\mathcal{F} \text{ converges on } Y\}\\
\Gamma_\alpha &= \{(g, x) \mid x \in X_{g^{-1}}\}\\
\Gamma_\alpha^* &= \{(g, f(x)) \mid (g, x) \in \Gamma_\alpha\}\\
\Gamma &= \Gamma_\alpha^* \cup (\{1_G\} \times Y)\\
\Sigma &= \{\mathcal{K} \in F(Y) \mid \mathcal{K} \text{ converges on } Y\}
\end{aligned}$$

Note that (G, Δ), $(X, \mathcal{D})$, $(X, \mathcal{E})$, and (Y, Σ) are Cauchy spaces.

The following lemma suggests that objects from **CHY** provide a natural setting for the study of G-compactifications.

Lemma 4.1. *Assume that $(G, X, \alpha) \in |\boldsymbol{C}|$, $G \in |\boldsymbol{LS}|$ is Hausdorff, and (Y, f) is a Hausdorff compactification of X in $\boldsymbol{LS}$. Define $\beta : \Gamma \to Y$ by $\beta(g, f(x)) = f(\alpha(g, x))$ when $g \neq 1_G$ and $\beta(1_G, y) = y$, $y \in Y$. Then the diagram below commutes and β is Cauchy continuous whenever α is Cauchy continuous.*

$$\begin{array}{ccc}
(\Gamma_\alpha, \Delta \times \mathcal{E}) & \xrightarrow{\mathrm{id}_G \times f} & (\Gamma, \Delta \times \Sigma)\\
\Big\downarrow \alpha & & \Big\downarrow \beta\\
(X, \mathcal{E}) & \xrightarrow{\quad f \quad} & (Y, \Sigma)
\end{array}$$

Proof. Let $\mathcal{H} \in \Delta \times \Sigma$ and $\Gamma \in \mathcal{H}$. Since G,Y are both complete, $\pi_1{}^{\rightarrow}\mathcal{H} \to g$ and $\pi_2{}^{\rightarrow}\mathcal{H} \to y$ for some $g \in G$, $y \in Y$.

Case 1. Assume that $\Gamma_\alpha^* \in \mathcal{H}$ and let $\mathcal{K} = (\mathrm{id}_G \times f)^{\leftarrow}\mathcal{H}$. Then $(\mathrm{id}_G \times f)^{\rightarrow}\mathcal{K} = \mathcal{H}$ and $\pi_1{}^{\rightarrow}\mathcal{K} = \pi_1{}^{\rightarrow}\mathcal{H} \to g$. Also, $f^{\rightarrow}(\pi_2{}^{\rightarrow}\mathcal{K}) = \pi_2{}^{\rightarrow}\mathcal{H} \to y$ and then $\pi_2{}^{\rightarrow}\mathcal{K} \in \mathcal{E}$. Then $\mathcal{K} \in \Delta \times \mathcal{E}$ and $\Gamma_\alpha \in \mathcal{K}$. Since $f : (X, \mathcal{E}) \to (Y, \Sigma)$ is Cauchy continuous, $\beta^{\rightarrow}\mathcal{H} = (\beta \circ (\mathrm{id}_G \times f))^{\rightarrow}\mathcal{K} = (f \circ \alpha)^{\rightarrow}\mathcal{K} \in \Sigma$.

Case 2. Suppose that $\{1_G\} \times Y \in \mathcal{H}$. Then $\beta^{\rightarrow}\mathcal{H} = \pi_2{}^{\rightarrow}\mathcal{H} \to y$ and thus $\beta^{\rightarrow}\mathcal{H} \in \Sigma$.

Case 3. Finally, assume that for each $H \in \mathcal{H}$, $H \cap \Gamma_\alpha^*$ and $H \cap (\{1_G\} \times Y)$ are each nonempty. Let $\mathcal{K} = (\mathrm{id}_G \times f)^{\leftarrow}\mathcal{H}$ and let $\mathcal{L}$ denote the filter on $G \times Y$ whose base is $\{H \cap (\{1_G\} \times Y) \mid H \in \mathcal{H}\}$. Then $\Gamma_\alpha \in \mathcal{K}$, $\Gamma \in \mathcal{H}$, $\pi_1{}^{\rightarrow}\mathcal{K} \geq \pi_1{}^{\rightarrow}\mathcal{H} \to 1_G$ and $f^{\rightarrow}(\pi_2{}^{\rightarrow}\mathcal{K}) \geq \pi_2{}^{\rightarrow}\mathcal{H} \to y$. It follows that $\mathcal{K} \in \Delta \times \mathcal{E}$. Observe that $1_G^\bullet \times \pi_2{}^{\rightarrow}\mathcal{K} \in \Delta \times \mathcal{E}$ and let $\mathcal{M} = (1_G^\bullet \times \pi_2{}^{\rightarrow}\mathcal{K}) \cap \mathcal{K}$. Then $\Gamma_\alpha \in \mathcal{M}$, $\pi_1{}^{\rightarrow}\mathcal{M} = \pi_1{}^{\rightarrow}\mathcal{K} \cap 1_G^\bullet \to 1_G$, $\pi_2{}^{\rightarrow}\mathcal{M} \in \mathcal{E}$ and thus $\mathcal{M} \in \Delta \times \mathcal{E}$. Since $(f \circ \alpha)^{\rightarrow}(1_G^\bullet \times \pi_2{}^{\rightarrow}\mathcal{K}) = f^{\rightarrow}(\pi_2{}^{\rightarrow}\mathcal{K}) \to y$, it follows that $(f \circ \alpha)^{\rightarrow}\mathcal{K} \to y$.

Therefore, $\beta^{\rightarrow}\mathcal{H} = \beta^{\rightarrow}(\mathrm{id}_G \times f^{\rightarrow})\mathcal{K} \cap \pi_2{}^{\rightarrow}\mathcal{H} = (f \circ \alpha)^{\rightarrow}\mathcal{K} \cap \pi_2{}^{\rightarrow}\mathcal{H} \to y$ and thus $\beta^{\rightarrow}\mathcal{H} \in \Sigma$. Hence $\beta : (\Gamma, \Delta \times \Sigma) \to (Y, \Sigma)$ is Cauchy continuous. □

Theorem 4.2. *Assume that $(G, X, \alpha) \in |\boldsymbol{C}|$ and that (Y, f) is a Hausdorff compactification of X in $\boldsymbol{LS}$ and $G \in |\boldsymbol{LS}|$ is also Hausdorff. Following the notation given in Lemma 4.1, $((G, Y, \beta), f)$ is a G-compactification of (G, X, α) whenever $\alpha : (\Gamma_\alpha, \Delta \times \mathcal{E}) \to (X, \mathcal{E})$ is Cauchy continuous.*

Let $(G, X, \alpha) \in |\mathbf{C}|$ and let (X^*, k) be the one-point Hausdorff compactification of X in **LS** defined earlier. Define:

$$\begin{aligned}
\hat{X} &= X^* \\
\hat{X}_g &= k(X_g), g \neq 1_G \quad (\text{recall } X_g^* = k(X_g) \cup \{\omega\}) \\
\hat{X}_{1_G} &= \hat{X} \\
\hat{\alpha}_g(k(x)) &= k(\alpha_g(x)), x \in X_{g^{-1}} \\
\hat{\alpha}_g(\omega) &= \omega \\
\hat{\Gamma}_\alpha &= \{(g, k(x)) \mid (g, x) \in \Gamma_\alpha\} \quad (\text{recall } \Gamma_\alpha^* = \{(g, k(x)) \mid (g, x) \in \Gamma_\alpha\})
\end{aligned}$$

Corollary 4.3. *Suppose that $(G, X, \alpha) \in |\boldsymbol{C}|$, where G is a Hausdorff limit group and $(\hat{X}, k)$ is the one-point Hausdorff compactification of X in $\boldsymbol{LS}$. Then*

(i) If $\alpha : (\Gamma_\alpha, \Delta \times \mathcal{E}) \to (X, \mathcal{E})$ is Cauchy continuous, $((G, \hat{X}, \hat{\alpha}), k)$ is a one-point Hausdorff G-compactification of (G, X, α).

(ii) If (G, X, α) is adherence restrictive and α above is Cauchy continuous, $((G, \hat{X}, \hat{\alpha}), k) \geq ((G, X^*, \alpha^*), k)$.

Proof. Part (i) follows from Theorem 4.2. For part (ii), since (G, X, α) is adherence restrictive, $((G, \hat{X}, \hat{\alpha}), k)$ is a Hausdorff G-compactification of (G, X, α). The ordering above follows from Theorem 5.4 [2]. Observe that $\hat{X}_{g^{-1}} - k(X) = \varnothing$ for each $g \neq 1_G$ and $\hat{X}_{1_G} - k(X) = \{\omega\}$ and $\hat{\alpha}_{1_G}(\{\omega\}) = \{\omega\} = \hat{X}_{1_G} - k(X)$. □

Recall that if (Y, f) and (Z, k) are any two Hausdorff compactifications of X in **LS**, then $(Y, f) \geq (Z, k)$ means that there exists a continuous function $h : Y \to Z$ such that $k = h \circ f$.

Lemma 4.4. *Suppose that $(G, X, \alpha) \in |\boldsymbol{C}|$ and let $(G, Y, \beta) \in |\boldsymbol{C}|$ be as given in Theorem 4.2, where $\alpha : (\Gamma, \Delta \times \mathcal{E}) \to (X, \mathcal{E})$ is Cauchy continuous. Further, assume that $((G, Z, \delta), k)$ is a Hausdorff G-compactification of (G, X, α) in $\boldsymbol{C}$ and $(Y, f) \geq (Z, k)$ in $\boldsymbol{LS}$. Then $(G, Y, \beta) \geq (G, Z, \delta)$.*

Proof. Since $(Y,f) \geq (Z,k)$ in $\mathbf{LS}$, there exists a continuous map $h : Y \to Z$ such that $k = h \circ f$. It remains to show that the following diagram commutes:

$$\begin{array}{ccc} \Gamma_\beta & \xrightarrow{\mathrm{id}_G \times h} & \Gamma_\delta \\ \big\downarrow{\scriptstyle \beta} & & \big\downarrow{\scriptstyle \delta} \\ Y & \xrightarrow{\;\;h\;\;} & Z \end{array}$$

Recall that $\Gamma_\beta = \Gamma_\alpha^* \cup \{(1_G, y) \mid y \in Y\}$, where $\Gamma_\alpha^* = \{(g, f(x)) \mid (g,x) \in \Gamma_\alpha\}$. Since $((G,Z,\delta),k)$ is a Hausdorff G-compactification of (G,X,α), the diagram

$$\begin{array}{ccc} \Gamma_\alpha & \xrightarrow{\mathrm{id}_G \times k} & \Gamma_\delta \\ \big\downarrow{\scriptstyle \alpha} & & \big\downarrow{\scriptstyle \delta} \\ X & \xrightarrow{\;\;k\;\;} & Z \end{array}$$

commutes. Further, Cauchy continuity of α implies that $((G,Y,\beta),f)$ is a Hausdorff G-compactification of (G,X,α). Assume that $(g,f(x)) \in \Gamma_\beta$. Then

$$\begin{aligned} (\delta \circ (\mathrm{id}_G \times h))(g, f(x)) &= \delta(g, (h \circ f)(x)) \\ &= \delta(g, k(x)) \\ &= (\delta \circ (\mathrm{id}_G \times k))(g,x) \\ &= (k \circ \alpha)(g,x) \\ &= (h \circ f \circ \alpha)(g,x) \\ &= h((f \circ \alpha)(g,x)) \\ &= h(\beta \circ (\mathrm{id}_G \times f))(g,x) \\ &= (h \circ \beta)(g, f(x)). \end{aligned}$$

Next, assume that $(1_G, y) \in \Gamma_\beta$ and $y \in Y$. Then $(\delta \circ (\mathrm{id}_G \times h))(1_G, y) = \delta(1_G, h(y)) = h(y) = (h \circ \beta)(1_G, y)$. In either case, $\delta \circ (\mathrm{id}_G \times h) = h \circ \beta$ and $(\mathrm{id}_G, h) : (G,Y,\beta) \to (G,Z,\delta)$ is a morphism in $\mathbf{C}$ and thus $(G,Y,\beta) \geq (G,Z,\delta)$. ☐

A Hausdorff space $X \in |\mathbf{LS}|$ is called *regular* if $\mathrm{cl}\,\mathcal{F} \to x$ in X whenever $\mathcal{F} \to x$ in X. Further, X is said to be *completely regular* if it possesses a regular compactification in $\mathbf{LS}$. Completely regular objects in $\mathbf{LS}$ are characterized in [6]. The next result follows from Theorem 4.2 and Lemma 4.4

Theorem 4.5. *Assume that $(G,X,\alpha) \in |\boldsymbol{C}|$ and X is completely regular. Let (rX, f) denote the largest regular compactification of X in $\boldsymbol{LS}$. Using the notation given in Lemma 4.1, assume that $\alpha : (\Gamma_\alpha, \Delta \times \mathcal{E}) \to (X, \mathcal{E})$ is Cauchy continuous. Then $((G, rX, \beta), f)$ is the largest regular G-compactification of (G,X,α) in $\boldsymbol{C}$.*

Lemma 4.6. *Suppose $(G,X,\alpha) \in |\boldsymbol{C}|$, (Y,f) is a Hausdorff compactification of X in $\boldsymbol{LS}$, and $\alpha : (\Gamma_\alpha, \Delta \times \mathcal{E}) \to (X, \mathcal{E})$ is Cauchy continuous. The Hausdorff G-compactification of (G,X,α) is denoted by $((G,Y,\beta),f)$. Let X^e and Y^e be* *the corresponding envelopes of* X *and* Y*. Define* $h : X^e \to Y^e$ *by* $h(\langle(g,x)\rangle) = \langle(g,f(x))\rangle, g \in G, x \in X$*. Then*

(i) $(g,x) \sim (g_1,x_1)$ *on* $G \times X$ *if and only if* $(g,f(x)) \sim (g_1,f(x_1))$ *on* $G \times Y$*,*
(ii) $(g,y) \sim (g_1,f(x_1))$ *on* $G \times Y$ *implies* $y \in f(X)$*,*
(iii) h *is well-defined,*
(iv) h *is an injection.*

Proof. We prove each part in turn.

(i) Assume that $(g,x) \sim (g_1,x_1)$ on $G \times X$. Then $x \in X_{g^{-1}g_1}$ and $f(x) \in f(X_{g^{-1}g_1})$. If $g^{-1}g_1 \neq 1_G$, then $f(x) \in Y_{g^{-1}g_1}$ and $\beta(g_1^{-1}g, f(x)) = f(\alpha(g_1^{-1}g,x)) = f(x_1)$. Hence $(g,f(x)) \sim (g_1,f(x_1))$. If $g^{-1}g_1 = 1_G$, then $f(x) \in f(X) \subseteq Y = Y_{1_G}$. Also, $x = \alpha(1_G,x) = x_1$ implies that $\beta(1_G,f(x)) = f(\alpha(1_G,x)) = f(x_1)$ and hence $(g,f(x)) \sim (g_1,f(x_1))$. Conversely, suppose that $(g,f(x)) \sim (g_1,f(x_1))$ on $G\times Y$. Then $f(x) \in Y_{g^{-1}g_1}$ and $f(x_1) = \beta(g_1^{-1}g,f(x)) = f(\alpha(g_1^{-1}g,x))$. Since f is an injection $x_1 = \alpha(g_1^{-1}g,x)$. If $g^{-1}g_1 \neq 1_G$, $f(x) \in Y_{g^{-1}g_1} = f(X_{g^{-1}g_1})$ and thus $x \in X_{g^{-1}g_1}$. If $g^{-1}g_1 = 1_G$, then $x \in X_{1_G} = X$ and thus in either case $(g,x) \sim (g_1,x_1)$.

(ii) Suppose that $(g,y) \sim (g_1,f(x_1))$. then $y \in Y_{g^{-1}g_1}$ and $\beta(g_1^{-1}g,y) = f(x_1)$. If $g^{-1}g_1 \neq 1_G$, then $y \in f(X_{g^{-1}g_1})$. However, if $g^{-1}g_1 = 1_G$, $y = \beta(1_G,y) = f(x_1)$ and in either case $y \in f(X)$.

(iii) Assume that $\langle(g,x)\rangle = \langle(g_1,x_1)\rangle$. Then by (i), $\langle(g,f(x))\rangle = \langle(g_1,f(x_1))\rangle$ and thus h is well-defined.

(iv) Finally, suppose that $h(\langle(g,x)\rangle) = h(\langle(g_1,x_1)\rangle)$. Then $(g,f(x)) \sim (g_1,f(x_1))$ on $G \times Y$. According to (i), $(g,x) \sim (g_1,x_1)$ and hence h is an injection.

□

Theorem 4.7. *Under the assumptions listed in Lemma 4.6,* $h : X^e \to Y^e$ *is a homeomorphism onto* $h(X^e)$*.*

Proof. According to Lemma 4.6 (iv), h is an injection. Observe that the diagram below commutes:

$$\begin{array}{ccc} G \times X & \xrightarrow{\theta_X} & X^e \\ \Big\downarrow{\scriptstyle \mathrm{id}_G \times f} & & \Big\downarrow{\scriptstyle h} \\ G \times Y & \xrightarrow{\theta_Y} & Y^e \end{array}$$

where $\theta_X(g,x) = \langle(g,x)\rangle$, $(g,x) \in G \times X$, is a quotient map in **LS**. It follows that h is continuous if and only if $h \circ \theta_X$ is continuous. However, $h \circ \theta_X = \theta_Y \circ (\mathrm{id}_G \times f)$ is continuous and thus h is a continuous injection. Next, suppose that $\mathcal{H} \in F(X^e)$ such that $h^{\to}\mathcal{H} \to h(\langle(g,x)\rangle) = \langle(g,f(x))\rangle$ on Y^e. It remains to verify that $\mathcal{H} \to \langle(g,x)\rangle$ on X^e. There exists $\mathcal{L} \to (g_1,y_1) \sim (g,f(x))$ on $G \times Y$ such that $\theta_Y{}^{\to}\mathcal{L} = h^{\to}\mathcal{H}$. Employing Lemma 4.6 (ii) and (i), $y_1 = f(x_1)$ for some $x_1 \in X$ and $(g_1,x_1) \sim (g,x)$ on $G \times X$. Since $X^e \in \mathcal{H}$, there exists $L \in \mathcal{L}$ such that $\theta_Y(L) \subseteq h(X^e)$. It follows from Lemma 4.6 (ii) that $\pi_2(L) \subseteq f(X)$ and thus $f(X) \in {\pi_2}^{\rightarrow}\mathcal{L}$. Hence $G \times f(X) \in \mathcal{L}$ and $\mathcal{K} = (\mathrm{id}_G \times f)^{\leftarrow}\mathcal{L} \to (g_1, x_1)$ on $G \times X$. Using the commutative diagram above, $h^{\rightarrow}\mathcal{H} = {\theta_Y}^{\rightarrow}\mathcal{L} = (\theta_Y \circ (\mathrm{id}_G \times f))^{\rightarrow}\mathcal{K} = (h \circ \theta_X)^{\rightarrow}\mathcal{K} = h^{\rightarrow}({\theta_X}^{\rightarrow}\mathcal{K})$. Since h is an injection, $\mathcal{H} = {\theta_X}^{\rightarrow}\mathcal{K} \to \langle (g_1, x_1) \rangle = \langle (g, x) \rangle$ on X^e. Hence h is a homeomorphism onto $h(X^e)$. □

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