

# Strong Fréchet properties of spaces constructed from squares and AD families

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Communicated by D. Werner

## ABSTRACT

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We answer questions of Arhangel'skiĭ using spaces defined from combinatorial objects. We first establish further convergence properties of a space constructed from  $\square(\kappa)$  showing it is Fréchet-Urysohn for finite sets and a  $w$ -space that is not a  $W$ -space. We also show that under additional assumptions it may be not bi-sequential, and so providing a consistent example of an absolutely Fréchet  $\alpha_1$  space that is not bi-sequential. In addition, if we do not require the space being  $\alpha_1$ , we can construct a ZFC example of a countable absolutely Fréchet space that is not bi-sequential from an almost disjoint family of subsets of the natural numbers.

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2020 MSC: 54A35; 54G20.

KEYWORDS: Fréchet-Urysohn; bi-sequential;  $\alpha_1$ -space;  $W$ -space;  $w$ -space.

## 1. INTRODUCTION

Recall that a point  $x \in X$  is a *Fréchet* point if whenever  $x \in \overline{A}$ , there is a sequence  $\{x_n : n \in \omega\} \subseteq A$  converging to  $x$ . A space  $X$  is Fréchet if every point  $x \in X$  is a Fréchet point. Fréchetness is a desired property in a topological space since it allows us to handle the topology through their sequences. However, it is very easy to destroy this property by taking products. There are Fréchet

spaces  $X$  and  $Y$  such that their product  $X \times Y$  is not Fréchet [2]. It is also known that  $X$  and  $Y$  can be even compact [20].

In order to study the behaviour of Fréchetness in products, Arhangel'skiĭ introduced the  $\alpha_i$ -properties for  $i \in \{1, 2, 3, 4, 5\}$ . We will only consider the property  $\alpha_1$  in detail, for a definition of the others, see [1]. A point  $x$  is an  $\alpha_1$ -point if whenever we have countable many sequences  $S_n$  converging to  $x$ , there is a single sequence  $S \rightarrow x$  such that  $|S_n \setminus S| < \omega$  for all  $n \in \omega$ . A space  $X$  is  $\alpha_1$  if every point  $x \in X$  is an  $\alpha_1$ -point.

Besides the  $\alpha_1$ -properties, a strong version of Fréchetness was also introduced:

**Definition 1.1** ([1]). A Tychonoff space  $X$  is *absolutely Fréchet* if every point  $x \in X$  is a Fréchet point in  $\beta X$  (equivalently in some compactification).

The concept of bisequentiality introduced by Ernest Michael [17] was also relevant in Arhangel'skiĭ's paper.

Let  $\mathcal{A} \subseteq \mathcal{P}(X)$  and  $x \in X$ . We will say that  $x \in \overline{\mathcal{A}}$  if  $x \in \overline{A}$  for every  $A \in \mathcal{A}$ . We say that  $\mathcal{A}$  generates a filter, if the intersection of any finite subfamily of  $\mathcal{A}$  is infinite (recall that a filter is a family closed under supersets and finite intersections). Also, for  $\{G_n : n \in \omega\} \subseteq \mathcal{P}(X)$ , we will say that  $G_n \rightarrow x$  if for every open neighborhood  $U$  of  $x$  there is  $n \in \omega$  such that  $G_n \subseteq U$ .

**Definition 1.2** ([17]). A space  $X$  is *bisequential* if for every filter  $\mathcal{F}$  such that  $x \in \overline{\mathcal{F}}$ , there is a family  $\{G_n : n \in \omega\}$  such that  $\mathcal{F} \cup \{G_n : n \in \omega\}$  generates a filter and  $G_n \rightarrow x$ . We also say that  $X$  is *bisequential with respect to a filter*  $\mathcal{U}$  if it satisfies the above definition when we fix  $\mathcal{F} = \mathcal{U}$ .

In [1], the relationship between these properties was studied. The  $\alpha_i$  properties, with the exception of  $\alpha_5$ , get progressively weaker, i.e.,  $\alpha_i \Rightarrow \alpha_{i+1}$  for  $i \leq 3$ . Bisequential spaces are absolutely Fréchet and  $\alpha_3$ , while absolutely Fréchet spaces can only be shown to be  $\alpha_4$ , and they are of course Fréchet.

Most of the inclusions between these classes of spaces have been shown to be strict, nevertheless, the following questions remain open:

- (1) Is there a (countable) absolutely Fréchet space which is not bisequential?
- (2) Is there an  $\alpha_1$  and Fréchet space which is not bisequential?

Malyhin has constructed a consistent example for the second question under  $2^{\aleph_0} < 2^{\aleph_1}$  [16]. Here we will construct consistently a space of size  $\aleph_1$  that answers both questions at the same time and in Section 4 we will construct a countable absolutely Fréchet space which is not bisequential in ZFC, solving question (1).

## 2. SPACES DEFINED FROM $\square(\kappa)$

The example presented in this section was first defined to provide counterexamples to questions about cardinal functions on the  $G_\delta$ -modifications of topological spaces [4]. Given a cardinal  $\kappa$ , a club  $C \subseteq \kappa$  is a subset which is

closed (in the order topology) and unbounded. We assume  $\square(\kappa)$ , namely that there exists a square sequence at  $\kappa$ , i.e., there exists a sequence  $\langle C_\alpha : \alpha < \kappa \rangle$  so that:

- $C_\alpha$  is club in  $\alpha$ ,
- $C_\alpha \cap \beta = C_\beta$  for every  $\beta$  which is a limit point of  $C_\alpha$ ,
- there is no club  $C$  in  $\kappa$  so that  $C \cap \alpha = C_\alpha$  for every  $\alpha$  which is a limit point of  $C$  (such a club is called a *thread*).

The second item is known as the coherence of the sequence and the third item expresses some kind of non-triviality. We will be particularly interested in the case where  $\kappa = \omega_1$ . In this case, an easy example of a  $\square(\omega_1)$  sequence is a *ladder system* on  $\omega_1$  which is a sequence  $\langle L_\alpha : \alpha \in \omega_1 \rangle$  such that:

- If  $\alpha = \beta + 1$  then  $L_\alpha = \{\beta\}$  and
- If  $\alpha$  is a limit ordinal then  $L_\alpha$  is an increasing and unbounded subset of  $\alpha$  of order type  $\omega$ .

Square sequences were introduced by Jensen in his study of the constructible universe [14]. There, an entire family of square principles was introduced and they have become an important combinatorial tool with many applications, mainly in topology. We will not define all of them since we will only work with the above square principle. We recommend [15] for those readers who want to learn more about square principles.

We will make use of Todorčević’s method of minimal walks. In the process of “walking” from  $\beta$  to a lower  $\alpha$  along the sequence, starting at  $\beta_0 = \beta$  and at each step taking  $\beta_{i+1} = \min(C_{\beta_i} \setminus \alpha)$ , we will eventually reach  $\beta_n = \alpha$  as the  $\beta_i$  are a decreasing sequence of ordinals. Let  $\text{Tr}(\alpha, \beta) = \{\beta_i : 0 \leq i \leq n\}$ , and  $\rho_2(\alpha, \beta) = n$ , more precisely defined inductively so that  $\rho_2(\alpha, \alpha) = 0$  and  $\rho_2(\alpha, \beta) = \rho_2(\alpha, \min(C_\beta \setminus \alpha)) + 1$ . Hence, the function  $\rho_2(\alpha, \beta)$  counts the “number of steps” from  $\beta$  to  $\alpha$ . See [22] and [24] for more on this topic.

We will use the following basic facts, which are reformulations of the coherence and the non-triviality of the square sequence respectively:

**Fact 2.1** ([25]).

- (1) For  $\alpha < \beta < \kappa$ ,

$$\sup_{\xi < \alpha} |\rho_2(\xi, \alpha) - \rho_2(\xi, \beta)| < \omega.$$

- (2) For every family  $\mathcal{F} \subseteq [\kappa]^{<\omega}$  of pairwise disjoint finite subsets of  $\kappa$  with  $|\mathcal{F}| = \kappa$  and for every integer  $n$  there exist  $a, b$  both in  $\mathcal{F}$  such that

$$\rho_2(\alpha, \beta) > n$$

for all  $\alpha \in a, \beta \in b$ .

Note that (2) implies the alternative fact that for any pair of families  $\mathcal{F}, \mathcal{G} \subseteq [\kappa]^{<\omega}$  of pairwise disjoint finite subsets of  $\kappa$  with  $|\mathcal{F}| = |\mathcal{G}| = \kappa$  and for every integer  $n$  there exist  $a \in \mathcal{F}$ , and  $b \in \mathcal{G}$  above  $a$  such that

$$\rho_2(\alpha, \beta) > n$$

for all  $\alpha \in a, \beta \in b$ .

We use the notation  $(\rho_2)_\beta$  for the function from  $\beta$  to  $\omega$  defined by  $(\rho_2)_\beta(\xi) = \rho_2(\xi, \beta)$ .

The principle  $\square(\kappa)$  is independent of the axioms of ZFC, it holds under  $V = L$ , but it fails if one assume the  $P$ -ideal dichotomy ( $PID$ ). In general, the failure of  $\square(\kappa)$  for a single  $\kappa > \omega_1$  requires the existence of large cardinals. The proof that  $PID$  implies that there are no square sequences for  $\kappa > \omega_1$  uses the ideal  $\mathcal{I} \subseteq [\kappa]^{<\omega}$  consisting of all subsets  $I$  of  $\kappa$  such that each map  $(\rho_2)_\alpha$  is finite to one on  $I$ . See [23] for a detailed proof of this fact. In [4] a topology on  $\kappa \cup \{\kappa\}$  was defined, where  $\kappa$  is discrete and the set of convergent sequences to  $\kappa$  coincide with this ideal  $\mathcal{I}$ :

Let  $\langle C_\alpha : \alpha < \kappa \rangle$  be the  $\square(\kappa)$  sequence. Define a topology on  $X = \kappa + 1$  so that the points of  $\kappa$  are isolated, and the open neighborhoods for  $\kappa$  are given by sets of the form

$$\{\kappa\} \cup \bigcup_{\alpha \in \text{Lim}(\kappa)} \{\xi < \alpha : \rho_2(\xi, \alpha) > n_\alpha\}$$

where  $n_\alpha < \omega$ .

**Fact 2.2.** *Sets of the form  $U_n(\alpha) = \{\xi < \alpha : \rho_2(\xi, \alpha) > n\} \cup [\alpha, \kappa]$  form a local neighborhood base at  $\kappa$ .*

From this, most of the following statements follow:

**Theorem 2.3** ([4]). *Let  $X_C$  be the space defined above using a square sequence  $C$ , then:*

- (1) *Every subspace of  $X_C$  of size  $< \kappa$  is first countable.*
- (2) *Every subset of  $X_C$  of size  $\kappa$  has the point  $\kappa$  in its closure.*
- (3)  *$X_C$  is a Fréchet and  $\alpha_1$  space.*
- (4)  *$X_C$  is absolutely Fréchet.*

A space is called *Fréchet for finite sets* (abbreviated  $\text{FU}_{\text{fin}}$ ) at  $x$  (see [21], [9] and [10]) if for every collection  $A$  of finite subsets that clusters at  $x$  (i.e., every neighborhood of  $x$  contains a member of  $A$  as a subset) denoted  $x \in \widehat{A}$ , there exists  $\{a_n : n < \omega\} \subseteq A$  so that  $\{a_n : n \in \omega\}$  converges to  $x$ , i.e.,

- for every neighborhood  $U$  of  $x$ , there is some  $n_0 < \omega$  so that  $a_n \subseteq U$  for all  $n \geq n_0$ .

A similar notion for sets of a fixed size  $n$  is called  $\text{FU}_n$ . The property  $\text{FU}_n$  on a space  $Y$  is equivalent to  $Y^n$  being Fréchet when the space  $Y$  has only one non-isolated point, so that a space of this kind being  $\text{FU}_{\text{fin}}$  implies that any finite power is Fréchet [7]. We now show that  $X$  satisfies this stronger version of Fréchetness. We will make use of elementary submodels. For an introduction and some applications of elementary submodels in Topology see [6].

**Theorem 2.4.** *If  $\kappa \geq \omega_1$  is regular and  $X_C$  is the space based on a  $\square(\kappa)$  sequence, then  $X_C$  is  $\text{FU}_{\text{fin}}$ .*

*Proof.* Suppose  $A \subseteq [\kappa]^{<\omega}$  and  $\kappa \in \widehat{A}$ .

**Claim 2.5.** *If  $B \subseteq [\kappa]^{<\omega}$  and  $\{\min(a) : a \in B\}$  is unbounded, then there is some  $\gamma < \kappa$  so that for every  $n < \omega$  there is  $a \in B$  so that  $a \subseteq \gamma$  and  $(\rho_2)_\gamma[a] \cap n = \emptyset$  and therefore there is a subset of  $B$  converging to  $\kappa$ .*

*Proof of Claim:* To prove the claim, assume otherwise, so for each  $\gamma < \kappa$  there is  $n_\gamma$  so that  $(\rho_2)_\gamma[a] \cap n_\gamma \neq \emptyset$  for every  $a \in B$  with  $a \subseteq \gamma$ . Take  $D \subseteq \text{Lim}(\kappa)$  unbounded so that there is  $n$  with  $n_\beta = n$  for all  $\beta \in D$ . Using Fact 2.1 (2) (in its alternative form), we can find  $a \in B$  and  $\beta \in D$  so that  $a \subseteq \beta$  and  $(\rho_2)_\beta[a] \cap n = \emptyset$ , contradicting the choice of  $n$ .

Now we see that there is a convergent subsequence. For every  $n \in \omega$ , pick  $a_n \in B$  such that  $(\rho_2)_\gamma[a_n] \gg n$  (i.e.,  $(\rho_2)_\gamma(\eta) > n$  for every  $\eta \in a_n$ ). Now let  $U_m(\alpha)$  be a basic neighborhood of  $\kappa$ . If

$$k = \sup_{\xi < \min\{\alpha, \gamma\}} |\rho(\xi, \alpha) - \rho_2(\xi, \gamma)|,$$

then  $(\rho_2)_\alpha[a_n \cap \alpha] \gg m$  whenever  $n > m + k$ . This together with the fact that  $[\alpha, \kappa] \subseteq U_m(\alpha)$  implies that  $a_n \subseteq U_m(\alpha)$  for every  $n > m$ . □<sub>Claim</sub>

Fix an elementary submodel  $M$  of a suitable  $H(\theta)$  containing everything relevant so that  $M \cap \kappa \in \kappa$ . If  $\kappa$  is a successor cardinal  $\lambda^+$ , then any elementary submodel of size  $\lambda$  suffices. If  $\kappa$  is a regular limit cardinal, then  $M$  can be formed as a suitably chosen countable increasing sequence of models.

If there is  $a \in A$  such that  $a \cap M = \emptyset$ , then, by elementarity, we are under the assumptions of Claim 2.5 and so there is a convergent sequence as required. We will call  $\delta = M \cap \kappa \in \kappa$ .

Otherwise we have  $a \cap M \neq \emptyset$  for all  $a \in A$  and so  $A_M = \{a \cap M : a \in A\}$  also has  $\kappa$  in its closure as every element in  $A_M$  is contained in its naturally associated element of  $A$ . So there is a subset  $\{a_n : n \in \omega\}$  from  $A_M$  that converges to  $\kappa$  (since the subspace  $\delta \cup \{\kappa\}$  is closed and first countable). If  $a_n \in A$  for infinitely many  $n$  then we are done. Otherwise, by going to a cofinite subset, we may assume  $a_n \notin A$  for all  $n \in \omega$ . Therefore each  $a_n$  is a root of a  $\Delta$ -system  $A_n$  from  $A$  of size  $\kappa$  and, for each  $n \in \omega$ ,  $A'_n = \{a \setminus a_n : a \in A_n\}$  is unbounded and thus clusters at  $\kappa$  and satisfies the hypotheses of Claim 2.5. So, each  $A'_n$  contains a convergent sequence  $\{a_k^n : k \in \omega\}$ .

Now it follows that  $B = \{a_n \cup a_k^n : n, k \in \omega\}$  has  $\kappa$  in its closure, because for every open neighborhood  $U$  of  $\kappa$  there exists  $n \in \omega$  such that  $a_i \subseteq U$  for every  $i \geq n$  and there exists  $k \in \omega$  such that  $a_m^n \subseteq U$  for every  $m \geq k$ . In particular,  $a_n \cup a_k^n \subseteq U$ . Moreover, there is  $\alpha < \kappa$  such that  $B \subseteq [\alpha]^{<\omega}$ . And since the subspace  $\alpha \cup \{\kappa\}$  is first countable, there is a subset from  $B$  converging to  $\kappa$  as required. □

**Corollary 2.6.** *All spaces  $X_C$  defined from a  $\square(\kappa)$  sequence for  $\kappa \geq \omega_1$  have all finite powers Fréchet.*

The game  $G_{O,P}(X, x)$  is played by two players, denoted  $O$  and  $P$ , who alternate turns. Player  $O$  plays first and plays open neighborhoods of  $x$  during her turns. Player  $P$  plays points of  $X$  which are contained in all of the open

neighborhoods played so far by  $O$ . Player  $O$  wins iff the sequence of points played by  $P$  converge to  $x$ .

A space  $X$  is called a  $W$ -space if player  $O$  has a winning strategy in  $G_{O,P}(X, x)$  for every  $x \in X$ .  $X$  is called a  $w$ -space if player  $P$  does not have a winning strategy in  $G_{O,P}(X, x)$  for every  $x \in X$  (see [8]).

**Theorem 2.7.**  $X_C$  is a  $w$ -space which is not a  $W$ -space.

*Proof.* Since  $X$  is Fréchet and  $\alpha_1$ , it is clearly a  $w$ -space, due to a result of Sharma characterizing  $w$ -spaces as Fréchet  $\alpha_2$  spaces [19].

Suppose player  $O$  has a winning strategy  $\sigma$ . For every  $\alpha < \kappa$  there exists a  $\gamma(\alpha)$  such that for every finitely many points  $x_0, \dots, x_n \in \alpha$  chosen by  $P$ , the strategy  $\sigma((x_0, \dots, x_n)) = U_n(\beta)$  for some  $\beta < \gamma(\alpha)$ . Let  $E \subseteq \kappa$  be a club consisting of ordinals  $\alpha$  so that if player  $P$  plays finitely many points below  $\alpha$ , then player  $O$ 's response according to  $\sigma$  is  $U_n(\beta)$  for some  $\beta < \alpha^{+E}$ , where  $\alpha^{+E}$  is the least  $\alpha' \in E \setminus (\alpha + 1)$ .

Let  $\{\alpha_\eta : \eta < \kappa\}$  be the increasing enumeration of  $E$  and consider  $C_{\alpha_\omega}$ . Pick a sequence

$$\alpha_{k_0} < \alpha_{k_0+1} < x_0 < \alpha_{k_1} < \alpha_{k_1+1} < x_1 < \alpha_{k_2} < \alpha_{k_2+1} < x_2 < \dots$$

such that every  $\alpha_{k_i} \in E$  and every  $x_i \in C_{\alpha_\omega}$ . Then  $\kappa \notin \overline{\{x_n : n \in \omega\}}$  as witnessed by  $U_1(\alpha_\omega)$ . Moreover,  $P$  can play the sequence of  $x_n$ 's, since for every  $n \in \omega$

$$x_n \in [\alpha_{i+1}, \kappa) \subseteq [\beta, \kappa) \subseteq U_m(\beta)$$

for every  $i < n$  and every  $\beta < \alpha_{i+1}$ , in particular for  $\beta$  such that  $\sigma(x_0, \dots, x_i) = U_m(\beta)$ , i.e.,  $x_n$  is a legal movement for  $P$ . Therefore  $\sigma$  is not a winning strategy.  $\square$

### 3. BISEQUENTIALITY OF LADDER SYSTEM SPACES

We now turn to the particular case when  $\kappa = \omega_1$ . Since ladder systems do exist in ZFC, we have examples of absolutely Fréchet  $\alpha_1$  spaces  $X_C$ . It remains to show that we can (consistently) find non-bisequential examples.

**Definition 3.1.** We say that a ladder system  $C = \langle C_\alpha : \alpha < \omega_1 \rangle$  is  $\omega$ -crossing if for every countable family  $\{S_n : n \in \omega\}$  of stationary subsets of  $\omega_1$ , there is  $\alpha \in \omega_1$  such that  $C_\alpha \cap S_n \neq \emptyset$  for every  $n \in \omega$ .

**Proposition 3.2.** If  $C = \langle C_\alpha : \alpha < \omega_1 \rangle$  is an  $\omega$ -crossing ladder system, then  $X_C$  is not bisequential.

*Proof.* We have that each  $C_\alpha$  is a closed subset of  $X$  and so no countable filterbase of stationary sets can converge to  $\omega_1$ . Therefore  $X$  is not bisequential with respect to the club filter.  $\square$

**Theorem 3.3.** The generic ladder system added with countable approximations is  $\omega$ -crossing.

*Proof.* Let  $\mathbb{P}$  be the forcing for adding a ladder system with countable conditions (i.e.,  $p \in \mathbb{P}$  iff  $p = \langle L_\alpha : \alpha \in \text{lim}(\eta) \rangle$  is a family of ladders for some  $\eta \in \omega_1$  and ordered by inclusion). Notice that we only have to take care of limit ordinals when defining the ladders. Let  $G$  be a  $\mathbb{P}$ -generic filter over  $V$  and  $\{\dot{S}_n : n \in \omega\}$  a sequence of  $\mathbb{P}$ -names for stationary sets in  $V[G]$ . Take  $M$  a countable elementary submodel of  $H(\theta)$  for  $\theta$  large enough such that  $\mathbb{P}, p, \{\dot{S}_n : n \in \omega\} \in M$ . For  $q \in \mathbb{P}$ , we will say that  $l(q) = \alpha$  if  $q = \langle L_\eta : \eta \in \text{lim}(\alpha) \rangle$ . Let  $\delta = M \cap \omega_1$  and  $\alpha = l(p) \in M$ . Define recursively  $\{q_\eta : \eta \in \omega_1^M\}$  as follows:

- $q_0 = p$ ,
- $q_\eta = \bigcup_{\beta < \eta} q_\beta$  if  $\eta$  is a limit ordinal and
- $q_{\eta+1} \leq q_\eta$  is such that  $q_{\eta+1}$  decides  $\dot{S}_n \cap l(q_\eta)$  for every  $n \in \omega$ .

In  $V$  define  $q = \bigcup_{\eta < \delta} q_\eta = \langle L_\alpha : \alpha \in \text{lim}(\delta) \rangle \in \mathbb{P}$ . Notice that

$$q \Vdash \text{“}\forall n \in \omega (\dot{S}_n \text{ is unbounded in } \delta)\text{”}.$$

Then we can define a ladder  $L_\delta = \{\delta_n : n \in \omega\} \subseteq \delta$  such that  $q \Vdash \text{“}\delta_n \in \dot{S}_n\text{”}$  for every  $n \in \omega$ . Define  $q' = \langle L_\alpha : \alpha \in \text{lim}(\delta + 1) \rangle \in \mathbb{P}$ . Then  $q' \Vdash \text{“}\forall n \in \omega (\dot{L} \cap \dot{S}_n) \neq \emptyset\text{”}$  where  $\dot{L}$  is a name for the generic ladder and hence by density  $L$  is  $\omega$ -crossing.  $\square$

The generic ladder system constructed above is a very weak form of  $\clubsuit$ . This kind of ladders exist under  $\diamond$  and so  $V = L$  also implies there is an absolutely Fréchet,  $\alpha_1$  space of size  $\omega_1$  that is not bi-sequential. Indeed, we can actually find, in both cases, a ladder system that satisfies the slightly stronger statement that for any countable family  $\{A_n : n \in \omega\}$  of uncountable subsets of  $\omega_1$ , there is an  $\alpha$  such that  $C_\alpha \cap A_n \neq \emptyset$  for every  $n$  and so the space is not bi-sequential with respect to *any* filter that clusters at  $\omega_1$ .

**Corollary 3.4.** *It is consistent that there is an absolutely Fréchet  $\alpha_1$  space  $X_C$  of size  $\omega_1$  that fails to be bi-sequential.*

Whether there is a  $\square$ -sequence for which the corresponding space is bi-sequential is not known. We have some weak partial positive results asserting that for all ladder systems, the space is bi-sequential with respect to filters generated by a small family of sets.

Let  $\mathbb{Q}$  be the set of pairs  $\langle f, a \rangle$ , where  $f$  is a finite partial function from  $\omega_1$  to  $\omega$ ,  $a$  is a finite partial function from  $\omega_1 \times \omega$  to  $\omega$  and  $f(\alpha) \leq a(\beta, n)$  if  $(\beta, n) \in \text{dom}(a)$ ,  $\alpha \in \text{dom}(f)$  and  $\rho_2(\alpha, \beta) \leq n$ .  $\mathbb{P}$  is a poset ordered by reverse inclusion.

**Proposition 3.5.**  *$\mathbb{Q}$  is ccc.*

*Proof.* Suppose that  $\{p_\xi = \langle f_\xi, a_\xi \rangle : \xi < \omega_1\}$  is an antichain. We may further assume that:

- (1) the set  $\{\text{dom}(f_\xi) : \xi < \omega_1\}$  forms a  $\Delta$ -system with root  $r_f$  and the functions  $f_\xi$  all agree on  $r_f$ ,

- (2) letting  $A_\xi = \{\alpha : \exists m ((\alpha, m) \in \text{dom}(a_\xi))\}$ , the  $A_\xi$  form a  $\Delta$ -system with root  $r_a$  and the functions  $a_\xi$  all agree on  $\{(\alpha, m) : \alpha \in r_a\}$ ,
- (3)  $\max((\text{dom}(f_\xi) \setminus r_f) \cup (A_\xi \setminus r_a)) < \min((\text{dom}(f_\zeta) \setminus r_f) \cup (A_\zeta \setminus r_a))$  whenever  $\xi < \zeta$ .
- (4) there is some  $n_0 < \omega$  so that  $\{m : \exists \alpha ((\alpha, m) \in \text{dom}(a_\xi))\} \subseteq n_0$  for all  $\xi < \omega_1$ .

Then we can find  $\xi < \zeta < \omega_1$  so that  $\rho_2(\alpha, \beta) > n_0$  whenever  $\alpha \in \text{dom}(f_\xi) \setminus r_f$  and  $\beta \in A_\zeta \setminus r_a$ . We check that  $p_\xi, p_\zeta$  are compatible, with common extension given by  $(f_\xi \cup f_\zeta, a_\xi \cup a_\zeta)$ . Because of the  $\Delta$ -systems, the only remaining possibility for incompatibility occurs when  $\alpha \in \text{dom}(f_\xi) \setminus r_f$  and  $\beta \in A_\zeta \setminus r_a$ , and we must check that  $f_\xi(\alpha) \leq a_\zeta(\beta, n)$  whenever  $(\beta, n) \in \text{dom}(a_\zeta)$  and  $\rho_2(\alpha, \beta) \leq n$ . But this condition is vacuous since we picked  $\xi$  and  $\zeta$  so that any such  $\alpha, \beta$  have  $\rho_2(\alpha, \beta) > n_0$ , and  $n_0$  was greater than any  $n$  which appears in the domain of  $a_\zeta$  by item (4).  $\square$

**Proposition 3.6.** *The poset  $\mathbb{Q}$  forces that  $X_C$  is bisquential for any filter which is generated by ground model sets. In particular, as  $\mathbb{Q}$  is ccc,  $X_C$  is bisquential for the club filter (in the extension).*

*Proof.* Given a generic  $G \subseteq \mathbb{Q}$ , let  $F : \omega_1 \rightarrow \omega$  be the function defined by the first coordinates of the elements of  $G$  and  $A : \omega_1 \times \omega \rightarrow \omega$  the second coordinate generic function.

For each  $k \in \omega$  let  $S_k = \{\alpha : F(\alpha) > k\}$ . We first claim that  $\{S_k : k \in \omega\}$  converges to  $\kappa$ . To see this, fix a basic neighborhood

$$U_n(\alpha) = \{\xi < \alpha : \rho_2(\xi, \alpha) > n\} \cup [\alpha, \kappa]$$

and consider  $k \geq A(\alpha, n)$ . By definition of the partial order, if  $\xi \in S_k$  then  $s(\xi) > k \geq A(\alpha, n) = a(\alpha, n)$  for some  $(s, a) \in G$  and so  $\rho_2(\xi, \alpha) > n$ , i.e.,  $\xi \in U_n(\alpha)$  and so  $S_k \subseteq U_n(\alpha)$  as required.

Now assume  $\mathcal{F}$  is a filter in the extension that clusters at  $\kappa$ , generated by a ground model family  $\mathcal{H}$ . Then  $\mathcal{H}$  also clusters at  $\kappa$  and that each  $S_k$  meets with every  $H \in \mathcal{H}$  is a routine density argument. Thus  $\mathcal{F} \cup \{S_k : k \in \omega\}$  generates a filter and  $X_C$  is bisquential with respect to  $\mathcal{F}$ .  $\square$

In the previous result, it is worth noting that we proved a little bit stronger result, namely, that there is a single sequence in the extension that witnesses that  $X_C$  is bisquential with respect to any filter  $\mathcal{F}$  that clusters at  $\kappa$  and is generated by ground model sets.

**Corollary 3.7.** *MA implies that  $X_C$  is bisquential for any filter which is  $< c$ -generated.*

#### 4. AN ABSOLUTELY FRÉCHET SPACE WHICH IS NOT BISEQUENTIAL IN ZFC

A family  $\mathcal{A} \subseteq [\omega]^\omega$  is *almost disjoint (ad)* if  $A \cap B$  is finite for every distinct  $A, B \in \mathcal{A}$ .  $\mathcal{A}$  is a mad family if it is an ad family and it is maximal with respect to this property. The ideal  $\mathcal{I}(\mathcal{A})$  generated by  $\mathcal{A}$  is the family of all subsets of  $\omega$  that can be covered by finitely many elements of  $\mathcal{A}$ . For a family  $\mathcal{W} \subseteq \mathcal{P}(\omega)$ ,



$\mathcal{W}^\perp = \{X \subseteq \omega : \forall W \in \mathcal{W} |X \cap W| < \omega\}$  and for an ideal  $\mathcal{I}$ ,  $\mathcal{I}^+ = \mathcal{P}(\omega) \setminus \mathcal{I}$ . In the case of the ideal generated by an ad family we will write  $\mathcal{I}^+(\mathcal{A})$  instead of  $\mathcal{I}(\mathcal{A})^+$ . An ad family  $\mathcal{A}$  is completely separable if for every  $X \in \mathcal{I}^\oplus(\mathcal{A})$ , where

$$\mathcal{I}^\oplus(\mathcal{A}) = \{Y \subseteq \omega : |\{A \in \mathcal{A} : |Y \cap A| = \omega\}| \geq \omega\},$$

there is  $A \in \mathcal{A}$  such that  $A \subseteq X$ . It is usual to define a completely separable mad family to be a family  $\mathcal{A}$  such that for every  $X \in \mathcal{I}^+(\mathcal{A})$  there is  $A \in \mathcal{A}$  such that  $A \subseteq X$ . For a mad family both properties are equivalent but the existence of a completely separable mad family in ZFC remains open, while the existence of a completely separable ad family (not maximal!) is a theorem of ZFC:

**Fact 4.1** ([3]). *There is a completely separable ad family in ZFC.*

Given an ad family  $\mathcal{A}$  we can define the  $\psi$ -space of  $\mathcal{A}$  as the space  $\omega \cup \mathcal{A}$  where  $\omega$  is discrete and the set  $\{n \in \omega : n \in A\}$  converges to  $A$  for every  $A \in \mathcal{A}$  [18]. We denote by  $\psi(\mathcal{A})$  this space and since it is locally compact,  $\psi(\mathcal{A})^* = \psi(\mathcal{A}) \cup \{\infty\}$  will denote its one-point compactification. Following [7], we will call the subspace  $\omega \cup \{\infty\}$  of  $\psi(\mathcal{A})^*$  the *ad space generated by  $\mathcal{A}$* . We will say that  $\mathcal{A}$  satisfies a topological property  $\mathcal{P}$  if the ad space generated by  $\mathcal{A}$  does. For a great survey on almost disjoint families and its  $\psi$ -space we refer the reader to [12] and [11].

In [5], an almost disjoint family which is  $\alpha_1$ , absolutely Fréchet and fails to be bisquential is constructed using CH. This hypothesis is used in order to carry on the recursive construction. Completely separable (m)ad families can be used to get rid of this extra axiom due to the next result:

**Fact 4.2.** *If  $\mathcal{A}$  is a completely separable ad family and  $X \in \mathcal{I}^\oplus(\mathcal{A})$ , then  $|\{A \in \mathcal{A} : A \subseteq X\}| = \mathfrak{c}$ .*

Even though it is not sufficient to get the same properties using a completely separable ad family instead of CH, it is good enough to get an absolutely Fréchet space that is not bisquential.

**Theorem 4.3.** *There exists an absolutely Fréchet ad family  $\mathcal{A}$  which is not bisquential.*

*Proof.* Let  $\mathcal{E} = \{a_\alpha : \alpha < \mathfrak{c}\} \subseteq [\omega]^\omega$  be a completely separable ad family. We can assume that  $\{a_n : n \in \omega\}$  forms a partition of  $\omega$  into infinite sets by replacing  $a_n$  with  $a'_n = (a_n \cup \{n\}) \setminus \bigcup_{i < n} a'_i$  if necessary. Enumerate  $[\omega]^\omega = \{X_\alpha : \omega \leq \alpha < \mathfrak{c}\}$ . We will construct recursively two families  $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$  and  $\mathcal{B} = \{B_\alpha : \omega \leq \alpha < \mathfrak{c}\}$  such that

- (1)  $\mathcal{A} \subseteq \mathcal{E}$  (hence, it is almost disjoint).
- (2) For every  $B \in \mathcal{B}$ , either  $B \in \mathcal{E} \setminus \mathcal{A}$  or  $B \in \mathcal{E}^\perp$ .
- (3) If  $X_\alpha \in \mathcal{I}^+(\mathcal{A})$ , then  $B_\alpha \subseteq X_\alpha$ .
- (4) If  $|\{n \in \omega : |X_\alpha \cap A_n| = \omega\}| = \omega$ , then  $A_\alpha \subseteq X_\alpha$ .

For  $n < \omega$  define  $A_n = a_n$ . Assume we have constructed  $\mathcal{A}_\delta = \{A_\alpha : \alpha < \delta\}$  and  $\mathcal{B}_\delta = \{B_\alpha : \alpha < \delta\}$  with the desired properties for an infinite ordinal  $\delta < \mathfrak{c}$ .

If  $X_\delta \notin \mathcal{I}^+(\mathcal{A}_\delta)$  define  $B_\delta \in \mathcal{E} \setminus \mathcal{A}$  arbitrarily. Suppose  $X_\delta \in \mathcal{I}^+(\mathcal{A}_\delta)$ . If  $X_\delta \in \mathcal{I}^\oplus(\mathcal{A}_\delta)$ , there is  $a \in \mathcal{E} \setminus \mathcal{A}_\delta$  such that  $a \subseteq X_\delta$  by fact 4.2. Define  $B_\delta = a$ . On the other hand, if  $X_\delta \in \mathcal{I}^+(\mathcal{A}_\delta) \setminus \mathcal{I}^\oplus(\mathcal{A}_\delta)$ , there exists  $X' \in [X_\delta]^\omega$  such that  $X' \in \mathcal{E}^\perp$ . In this case define  $B_\delta = X'$ .

Now assume that  $X_\delta$  is as in 4. Then using fact 4.2 again, there is  $a \in \mathcal{E} \setminus (\mathcal{A}_\delta \cup B_\delta)$  such that  $a \subseteq X_\delta$ . Define  $A_\delta = a$ . Otherwise define  $A_\delta \in \mathcal{E} \setminus (\mathcal{A}_\delta \cup B_\delta)$  arbitrarily. This finishes the construction.

We shall prove that  $\mathcal{A}$  is absolutely Fréchet but not bisequential. Notice that in the ad space generated by  $\mathcal{A}$ , a subset  $X \subseteq \omega$  converges to  $\infty$  iff  $X \in \mathcal{A}^\perp$  and  $\infty \in \overline{X}$  iff  $X \in \mathcal{I}^+(\mathcal{A})$ . Since  $\infty$  is the only non-isolated point in the ad space generated by  $\mathcal{A}$ , we will prove that  $\infty$  is a Fréchet point.

Suppose  $\infty \in \overline{X}$ . There is  $\alpha \in [\omega, \mathfrak{c})$  such that  $X = X_\alpha$ . Hence, since  $\mathcal{I}^+(\mathcal{A}) \subseteq \mathcal{I}^+(\mathcal{A}_\alpha)$ , it follows that  $B_\alpha \subseteq X$  and considering that either,  $B_\alpha \in \mathcal{E} \setminus \mathcal{A}$  or  $B_\alpha \in \mathcal{E}^\perp$ , we conclude that  $B_\alpha \in \mathcal{A}^\perp$ . In view of every infinite subset of  $\mathcal{A}$  converges to  $\infty$  and  $\omega$  is discrete in  $\psi(\mathcal{A})^*$ , it follows that  $\psi(\mathcal{A})^*$  witnesses that  $\mathcal{A}$  is absolutely Fréchet.

We now turn our attention to bisequentiality. Let  $\mathcal{J}$  be the following ideal:

$$\mathcal{J} = \{X \subseteq \omega : \exists n \in \omega \forall m > n (|X \cap A_m| < \omega)\}.$$

Define  $\mathcal{F}$  as the dual filter (i.e.,  $\mathcal{F} = \{X \subseteq \omega : \omega \setminus X \in \mathcal{J}\}$ ). Since every element  $A \in \mathcal{A}$  is disjoint from one member of  $\mathcal{F}$ , it follows that  $\infty \in \overline{\mathcal{F}}$ . Notice that  $\mathcal{F} \cup \{G\}$  generates a filter iff  $G \in \mathcal{F}^+ := \mathcal{J}^+$  iff  $|\{n \in \omega : |G \cap A_n| = \omega\}| = \omega$ . Let  $\{G_n : n \in \omega\} \subseteq \mathcal{F}^+$  and assume without loss of generality that it is a decreasing sequence of sets. Note that we will finish if we find  $A \in \mathcal{A}$  such that  $\forall n \in \omega (A \cap G_n \neq \emptyset)$  because  $\{\infty\} \cup \omega \setminus A$  is a neighborhood of  $\infty$  in the ad space generated by  $\mathcal{A}$  and none of the  $G_n$ 's is contained there. Find an increasing sequence  $\{k_n : n \in \omega\}$  such that  $|G_n \cap A_{k_n}| = \omega$  for every  $n \in \omega$  and define  $X = \bigcup_{n \in \omega} (G_n \cap A_{k_n})$ . There is  $\alpha \in [\omega, \mathfrak{c})$  such that  $X = X_\alpha$ . Notice that  $X$  satisfies point 4, then  $A_\alpha \subseteq X_\alpha$  and since  $A_\alpha$  is almost disjoint from every  $A_{k_n}$  and  $\{G_n : n \in \omega\}$  is decreasing,  $A_\alpha \cap G_n \neq \emptyset$  for every  $n \in \omega$ .  $\square$

## 5. QUESTIONS

We do not know if the spaces constructed from  $\square$ -sequences can ever be bi-sequential.

**Question 5.1.** *For regular  $\kappa \geq \omega_1$  can there be a  $\square(\kappa)$  sequence for which the space described in Section 2 is bi-sequential?*

**Question 5.2.** *For  $\kappa = \omega_1$  is there a ZFC example of a ladder system that gives a non-bisequential example? Or a ZFC example that is bi-sequential?*

**ACKNOWLEDGEMENTS.** *The authors would like to thank the anonymous referee for their review on the paper. The second author also wishes to thank Michael Hrušák for several discussions on constructions of ladder systems and almost disjoint families with related combinatorial properties.*

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