Applied General Topology

Appl. Gen. Topol. 24, no. 2 (2023), 433-448
doi:10.4995/agt.2023.17743
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On e-spaces and rings of real valued e-continuous functions

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Communicated by S. García-Ferreira

Abstract

Whenever the closure of an open set is also open, it is called e-open and if a space have a base consisting of e-open sets, it is called e-space. In this paper we first introduce and study e-spaces and e-continuous functions (we call a function f from a space X to a space Y an e-continuous at $x \in X$ if for each open set V containing $f(x)$ there is an e-open set containing x with $f(U) \subseteq V$). We observe that the quasicomponent of each point in a space X is determined by e-continuous functions on X and it is characterized as the largest set containing the point on which every e-continuous function on X is constant. Next, we study the rings $C_e(X)$ of all real valued e-continuous functions on a space X. It turns out that $C_e(X)$ coincides with the ring of real valued clopen continuous functions on X which is a $C(Y)$ for a zero-dimensional space Y whose elements are the quasicomponents of X. Using this fact we characterize the real maximal ideals of $C_e(X)$ and also give a natural representation of its maximal ideals. Finally we have shown that $C_e(X)$ determines the topology of X if and only if it is a zero-dimensional space.

2020 MSC: *54C40; 54D35.*

Keywords: *e-space; e-continuous function; real maximal ideal; quasicomponent; zero-dimensional space.*

Received 24 May 2022 – Accepted 10 January 2023

1. Introduction

Throughout this article $C(X)$ denotes the ring of all real-valued continuous functions on a completely regular Hausdorff space X, and $C^*(X)$ is the subring of $C(X)$ consisting of all bounded elements. For each $f \in C(X)$, the set $Z(f) = \{x \in X : f(x) = 0\}$ is called the zero-set of f. If I is an ideal in $C(X)$, we denote $Z[I] = \{Z(f) : f \in I\}$ and $\bigcap Z[I] = \bigcap_{f\in I} Z(f)$. Whenever $\bigcap Z[I]$ is nonempty, I is called fixed; else, free. Maximal ideals of $C(X)$ are precisely of the form $M^p = \{f \in C(X) : p \in \mathrm{cl}_{\beta X} Z(f)\}$ for each $p \in \beta X$, where βX is the Stone-Čech compactification of X. More generally if $A \subseteq \beta X$, we denote $M^A = \{f \in C(X) : A \subseteq \mathrm{cl}_{\beta X} Z(f)\}$. The fixed maximal ideals of $C(X)$ are the sets $M_p = \{f \in C(X) : p \in Z(f)\}$, for $p \in X$. The ideals $O^p = \{f \in C(X) : p \in \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(f)\}$, $p \in \beta X$ in the context of $C(X)$ are lower bounds for prime ideals of $C(X)$ in the sense that for every prime ideal P in $C(X)$, there exists a unique $p \in \beta X$ such that $O^p \subseteq P \subseteq M^p$; see Theorem 7.15 in [4]. More generally, for each $A \subseteq \beta X$ the ideal O^A is defined by the set $\{f \in C(X) : A \subseteq \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(f)\}$. The reader is referred to [3], [4] and [7] for undefined terms and notations concerning $C(X)$ and the concepts of general topology.

Let us give a brief outline of this article. In the next section we first observe that the intersection of two e-open sets is a an e-open set and this follows that the set $C_e(X)$ consisting of real valued e-continuous functions on a space X is a ring under pointwise addition and multiplication of functions. Next we show that every open or dense subspace of an e-space is an e-space but not every subspace (even a closed subspace) is necessarily an e-space. Section 3 is devoted to e-continuous functions. We observe in this section that each e-continuous function on a space X is constant on every quasicomponent of X and every subset of X with this property is in fact a quasicomponent of X. This fact help us to define equivalence classes in βX for characterization of maximal and real maximal ideals of $C_e(X)$. The characterization of spaces X for which $C_e(X)$ or $C_e^*(X)$ (the subring of $C_e(X)$ consisting of bounded ones) coincides with one of the rings $C(X)$ and $C^*(X)$ are characterized. For instance it is shown that $C_e(X) = C(X)$ if and only if X is an e-space (zero-dimensional) and $C_e(X) = C^*(X)$ if and only if X is a pseudocompact e-space (zero-dimensional). We also observe in this section that the rings $C_e(X)$ coincides with the rings of real valued clopen continuous functions on X which are first introduced in [6] under the name of super continuous functions.

Finally in section 4 we characterize the maximal and real maximal ideals of $C_e(X)$. In [1] it is shown that the ring $C_s(X)$ $(=C_e(X))$ of real valued super (clopen) continuous functions on a space X is isomorphic with $C(X_z)$ for a zero-dimensional X_z. In that reference, for characterization of maximal ideals of $C_s(X)$ the authors have given a representation for maximal ideals of $C(X_z)$. In this section we present the characterization of maximal ideals of $C_e(X)$ by some equivalent classes in βX so that if we take X as a zero-dimensional space they coincide with usual ones. Real maximal ideals of $C_e(X)$

are also characterized and we have shown in this section that a space X is zero-dimensional if and only if its topology coincides with the weak topology induced by $C_e(X)$ ($C_e^*(X)$).

2. e-SPACES

A set G in a topological space X is called extremely open (briefly e-open) if G and $\mathrm{cl}_X G$ are open subsets of X. We call a subset of a topological space an e-closed if its complement is an e-open. Equivalently, a set is e-closed if and only if it is closed and its interior is also closed. Clearly every closed-open (clopen) set in a topological space is an e-open set, but not conversely. For example $\mathbb{R} \setminus \{0\}$ is an e-open subset of $\mathbb{R}$ which is not a clopen set. Moreover, for each T_1-space X, the set $X \setminus \{x\}$ is an e-open set for each $x \in X$. In fact, if x is an isolated point of X, then $X \setminus \{x\}$ is clopen, so it is e-open. Otherwise $X \setminus \{x\}$ is open, since X is T_1 and $\mathrm{cl}_X(X \setminus \{x\}) = X$ is open. Thus every dense open subset of a space is an e-open set. In particular an open subset G of $\mathbb{R}$ is e-open in $\mathbb{R}$ if and only if $\mathbb{R} \setminus G$ has an empty interior.

Using the following lemma, we show that the intersection of every two e-open sets is an e-open set.

Lemma 2.1. *Suppose that V is an e-open and U is an open subset of a space X, then $cl_X(V \cap U) = cl_X V \cap cl_X U$.*

Proof. Let $p \in \mathrm{cl}_X V \cap \mathrm{cl}_X U$ and W be an arbitrary neighborhood of p. Since V is e-open, $\mathrm{cl}_X V$ is clopen and therefore it is a neighborhood of p and hence $(W \cap \mathrm{cl}_X V) \cap U \neq \varnothing$. Thus $W \cap (V \cap U) \neq \varnothing$, which implies $p \in \mathrm{cl}_X(V \cap U)$. □

Now the proof of the following result is evident.

Corollary 2.2. *In any space, the intersection of every two e-open sets is an e-open set.*

Corollary 2.2 is not true for an arbitrary intersection (union) of e-open sets. In fact an arbitrary intersection of e-open sets, even clopen sets, need not be even an open set. If X is a space and $x \in X$ is the lone non-isolated point of X, then for each $x \neq y \in X$, the set $X \setminus \{y\}$ is e-open (clopen). Now $G = \bigcap_{x \neq y \in X}(X \setminus \{y\}) = \{x\}$ which is not even open. Also an arbitrary union of e-open sets need not be an e-open set. For example if we take $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ as a subspace of $\mathbb{R}$, then $G_n = \{\frac{1}{2n}\}$, for each $n \in \mathbb{N}$, is clopen and hence e-open but $\bigcup_{n \in \mathbb{N}} G_n$ is not e-open because its closure is $\{\frac{1}{2n} : n \in \mathbb{N}\} \cup \{0\}$ which is not open in X.

Since the intersection of two e-open sets in a topological space (X, τ) is an e-open set by Corollary 2.2, the set of all e-open subsets of X form a base for a topology τ_e on X. Whenever τ_e coincide with τ (i.e., $\tau = \tau_e$), we call the space X an e-space.

Example 2.3. Whenever every open subset of a space has an open closure, i.e., if a space is extremally disconnected, then clearly it is an e-space. In particular, every discrete space is an e-space. For a non-extremally disconnected e-space

we may consider the one-point compactification space $X = \{1, \frac{1}{2}, \cdots, \frac{1}{n}, \cdots\} \cup \{0\}$ as a subspace of $\mathbb{R}$. The set $\mathfrak{B} = \{\{\frac{1}{n}\} : n \in \mathbb{N}\} \cup \{G \subseteq X : X \setminus G$ is finite and $0 \in G\}$ consisting of e-open (clopen) sets is a base for X. Since the closure of the open set $\{1, \frac{1}{3}, \frac{1}{5}, \cdots\}$ is not open, X is an e-space which is not extremally disconnected.

Proposition 2.4. *The trace of any e-open set on an open subspace and also on a dense subspace is e-open.*

Proof. First, suppose that X is an open subspace of Y and let $V \subseteq Y$ be e-open. Let $V_0 = V \cap X$. Then using the previous lemma we have

$$\mathrm{cl}_X V_0 = \mathrm{cl}_Y V_0 \cap X = \mathrm{cl}_Y V \cap \mathrm{cl}_Y X \cap X = \mathrm{cl}_Y V \cap X.$$

Since V is an e-open subset of Y, $\mathrm{cl}_Y V$ is clopen in Y and hence $\mathrm{cl}_X V_0$ is clopen in X. Now, suppose that X is a dense subspace of Y and let $V \subseteq Y$ be e-open. If we put again $V_0 = V \cap X$, it is clear that $\mathrm{cl}_Y V_0 = \mathrm{cl}_Y(V \cap X) = \mathrm{cl}_Y V$ because X is dense in Y. Now $\mathrm{cl}_X V_0 = X \cap \mathrm{cl}_Y V$ implies that $\mathrm{cl}_X V_0$ is open. □

Corollary 2.5. *Every open or dense subspace of an e-space is an e-space.*

Every subspace (even every e-closed subspace) of an e-space need not be an e-space; see the following example.

Example 2.6. Let

$$\mathfrak{B} = \{U \subseteq \mathbb{R} : U \text{ is open in } \mathbb{R} \text{ with usual topology and } [0, \infty) \setminus U \text{ is finite}\}.$$

Clearly $\mathfrak{B}$ may be a base for a topology on $\mathbb{R}$, say τ. Since each member of $\mathfrak{B}$ is an e-open set with respect to the topology τ, $(\mathbb{R}, \tau)$ is an e-space. In fact for each $U \in \mathfrak{B}$, we have $\mathrm{cl}U = \mathbb{R}$ and hence $\mathrm{cl}U$ is open. Now consider the subspace $(-\infty, 0]$ of $(\mathbb{R}, \tau)$ which is e-closed. The collection of all subsets of the form $U \cap (-\infty, 0]$ forms a base for $(-\infty, 0]$, where U is an open subset of $\mathbb{R}$ with usual topology. This implies that the space $(-\infty, 0]$ as a subspace of $(\mathbb{R}, \tau)$ has the usual topology which is not an e-space.

From [3], recall that a T_1-space X is zero-dimensional if each point of X has a neighborhood base consisting of clopen sets. Equivalently, a T_1-space X is zero-dimensional if and only if for each $x \in X$ and each closed set A not containing x, there exists a clopen set containing x which does not meet A. So every zero-dimensional space is a completely regular Hausdorff e-space. The converse is also true by the following proposition.

Proposition 2.7. *A space is a T_3-e-space if and only if it is zero-dimensional.*

Proof. Let X be a T_3-e-space. Let G be an open set in X and $x \in G$. Using the regularity of the e-space X, there exists an open set H such that $x \in H \subseteq \mathrm{cl}_X H \subseteq G$. Now, since X is an e-space, there is an e-open set K in X such that $x \in K \subseteq H$ and hence $x \in K \subseteq \mathrm{cl}_X K \subseteq \mathrm{cl}_X H \subseteq G$, where $\mathrm{cl}_X K$ is clopen, so X is zero-dimensional. □

Example 2.8. Using Proposition 2.7, each T_1-(even T_2-)-e-space need not be zero-dimensional. Whenever X is an infinite set with cofinite topology, then clearly X is a T_1-e-space which is not even T_2, so it is not zero-dimensional. Moreover every T_2-e-space is not necessarily a zero-dimensional space. In fact as in Example 14.2 in [7], we let X be the real line with neighborhoods of any nonzero point being as in the usual topology, while neighborhoods of 0 will have the form $U \setminus A$, where U is a neighborhood of 0 in the usual topology and $A = \{\frac{1}{n} : n = 1, 2, \dots\}$. Now let $Y = X \cap \mathbb{Q}$ as a subspace of X. Clearly Y is a T_2-space which is not T_3 and $\mathfrak{B} = \{((\alpha, \beta) \setminus A) \cap \mathbb{Q} : \alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}\}$ is a base for Y consisting of e-open sets. So Y is an e-space and since Y is not T_3, it is not zero-dimensional.

A topological space X is said to be an extremely T_1-space (briefly a T_1^e-space) if whenever x and y are distinct points in X, there is an e-open set containing each not the other. We call a space X an e-Hausdorff (or a T_2^e-space), if every two different points of X can be separated by two disjoint e-open sets and from [5] a space X is called ultra Hausdorff if every two different points of X can be separated by two disjoint clopen sets. It is easy to see that T_1-spaces and T_1^e-spaces coincide and the following result states that T_2^e-spaces also coincide with ultra Hausdorff spaces.

Proposition 2.9. *A topological space is e-Hausdorff if and only if it is ultra Hausdorff.*

Proof. First we note that whenever $U$and V are two disjoint e-open subsets of a topological space X, then $\mathrm{cl}_X U \cap \mathrm{cl}_X V = \varnothing$ by Lemma 2.1. Next if a space X is an e-Hausdorff, for each two different points x and y in X there exist two disjoint e-open subsets U and V of X containing x and y respectively. But by Lemma 2.1, $\mathrm{cl}_X U$ and $\mathrm{cl}_X V$ are two disjoint clopen sets containing x and y respectively, hence X is an ultra Hausdorff. Since every clopen set is an e-open set, the proof of the converse is evident. □

Corollary 2.10. *Let X be an e-space. Then X is Hausdorff if and only if it is ultra Hausdorff.*

Proposition 2.11. *Every homeomorphic image of an e-space is an e-space.*

Proof. Let X and Y be two homeomorphic spaces, X be an e-space and $\varphi : X \to Y$ be an onto homeomorphism. Let V be an open subset of Y and $y \in V$. Then there is $x \in X$ such that $y = \varphi(x)$. Since φ is continuous $\varphi^{-1}(V)$ is an open subset of X containing x and hence there exists an e-open subset U of X such that $x \in U \subseteq \varphi^{-1}(V)$. But φ is an open function, so $\varphi(U)$ is open and $y \in \varphi(U) \subseteq V$. Now it is enough to show that $\mathrm{cl}_Y \varphi(U)$ is open, i.e., $\varphi(U)$ is an e-open set. Using Theorem 7.9 in [7], we have $\varphi(\mathrm{cl}_X U) = \mathrm{cl}_Y \varphi(U)$ and since φ is open, $\varphi(\mathrm{cl}_X U)$ is an open subset of Y because $\mathrm{cl}_X U$ is an open subset of X. This shows that $\mathrm{cl}_Y \varphi(U)$ is open and we are through. □

Similar to definitions preceding the Proposition 2.9, we may define the e-compactness of the spaces: A space is called e-compact if every e-open cover

of the space has a finite subcover. Clearly every compact space is e-compact and using the following proposition, every e-compact is countably compact.

Proposition 2.12. *If a space X is compact, then it is e-compact and whenever X is an e-compact T_1-space, then it is countably compact.*

Proof. The first part is evident. For the second part, suppose on the contrary that X is not countably compact. Then there is an infinite subset $A = \{x_1, x_2, \dots, x_n, \dots\}$ of X without any cluster point. Now for each $n \in \mathbb{N}$, the set $G_n = X \setminus A_n$, where $A_n = \{x_n, x_{n+1}, \dots\}$ is an e-open set because A_n is closed and $\mathrm{cl}_X G_n = X \setminus B_n$, where $B_n \subseteq A_n$ will be open since B_n is also closed. Clearly $\cup_{n=1}^{\infty} G_n = X$ and no finite number of G_n's cover X, i.e., X is not e-compact.. □

The converse of the second part of the above proposition is not true. To see this consider the space of ordinals $[1, \omega_1)$, where ω_1 is the first uncountable ordinal number. It is known that the space $[1, \omega_1)$ is countably compact. But $\{[1, \alpha) : \alpha \in [2, \omega_1)\}$ is a clopen cover for $[1, \omega_1)$ that can have no finite subcover. We could not prove or disprove the converse of the first part of the proposition, so we cite it here as a question for interested readers.

Question: Is every e-compact space a compact space?

Let A be a subset of a topological space X. An element $x \in X$ is called an e-cluster point of A if each e-open subset of X containing x meets A. The set of all e-cluster points of A is called the e-closure of A and we denote it by $e\text{-cl}_X A$. Clearly for each subset A of a space X, we have $\mathrm{cl}_X A \subseteq e\text{-cl}_X A$ and the inclusion may be proper. For instance if we consider the open interval $(0, 1)$ in $\mathbb{R}$, then $\mathrm{cl}_{\mathbb{R}}(0, 1) = [0, 1]$, but $e\text{-cl}_{\mathbb{R}}(0, 1) = \mathbb{R}$.

As a closure of a set, the e-closure of a set A in a space X is the intersection of all e-closed subsets of X containing A. Whenever

$$S = \{H \subseteq X : \ H \text{ is an } e\text{-closed set and } A \subseteq H\},$$

then $e\text{-cl}_X A = \bigcap_{H \in S} H$. In fact if $x \in \bigcap_{H \in S} H$ and $x \notin e\text{-cl}_X A$, then there is an e-open set G containing x such that $G \cap A = \varnothing$. now $X \setminus G$ is an e-closed set containing A which does not contain x, a contradiction. The reverse inclusion is also routine. The e-interior is defined similarly and the e-interior of a set A is denoted by $e\text{-int}_X A$.

In contrast to the closure of a set which is closed, the e-closure of a set need not be e-closed. To this end, we let $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\}$ be a subspace of $\mathbb{R}$ with usual topology and $A = \{\frac{1}{2}, \frac{1}{4}, \dots\}$. Then $e\text{-cl}A = A \cup \{0\}$ which is not e-closed.

Proposition 2.13. *Let A be an e-compact subset of a space X and X be e-Hausdorff. Then $A = e\text{-cl}_X A$.*

Proof. Let $x \in e\text{-cl}_X A \setminus A$. Since X is e-Hausdorff, for each $a \in A$ there exists a clopen set U_a not containing x by Proposition 2.9. Now $\mathcal{C} = \{A \cap U_a : a \in A\}$ is an e-open cover of the e-compact subspace A and hence it has a finite subcover,

say $\{A\cap U_{a_1},\cdots,A\cap U_{a_n}\}$, so $A=\bigcup_{i=1}^{n}A\cap U_{a_i}$. But $U=\bigcup_{i=1}^{n}U_{a_i}$ is clopen not containing x and hence $X\setminus U$ is e-open containing x which does not meet A, a contradiction. Therefore $A=e\text{-cl}_X A$. □

3. e-CONTINUOUS FUNCTIONS

Definition 3.1. Let X and Y be topological spaces and $f:X\to Y$ be a function. We say that f is e-continuous at a point $x\in X$ if for each open set V in Y containing $f(x)$ there exists an e-open set U in X containing x such that $f(U)\subseteq V$. A function $f:X\to Y$ is called e-continuous if it is e-continuous at each point of X.

Clearly every e-continuous function is continuous, but the converse is not necessarily true. For instance, the identity function $i:\mathbb{R}\to\mathbb{R}$ is continuous but not e-continuous, because the only nonempty e-open subsets of $\mathbb{R}$ are dense open subsets of $\mathbb{R}$. Whenever X is an e-space, then every continuous function on X is e-continuous. In fact if $f:X\to Y$ is continuous, $x\in X$ and V is an open subset of Y containing $f(x)$, then there exists an open set G containing x such that $f(G)\subseteq V$. But X is an e-space, so there is an e-open subset U of X with $x\in U\subseteq G$. Thus $f(U)\subseteq V$ implies f is indeed e-continuous.

As the continuity, the e-continuity of a function may be stated via the inverse images of the open sets under the function and other similar standard conditions. The proof of the following proposition is analogous to that of Theorem 7.2 in [7].

Proposition 3.2. *Let X and Y be topological spaces and $f:X\to Y$ be a function. Then the following statements are equivalent.*

(1) *f is e-continuous.*
(2) *$f^{-1}(V)$ is a union of e-open subsets of X for each open subset V of Y.*
(3) *$f^{-1}(H)$ is an intersection of e-closed subsets of X for each closed subset H of Y.*
(4) *$f(e\text{-}cl_X A)\subseteq cl_Y f(A)$.*

Proposition 3.3. *An e-continuous image of an e-compact space is compact.*

Proof. Let $f:X\to Y$ be e-continuous from X onto Y and X be e-compact. Let $\mathcal{C}=\{V_\alpha:\alpha\in S\}$ be an open cover of Y. For each $x\in X$, there is $\alpha\in S$ such that $f(x)\in V_\alpha$. Since f is e-continuous, there exists an e-open set G_x in X containing x with $f(G_x)\subseteq V_\alpha$. Clearly $X=\bigcup_{x\in X}G_x$ and e-compactness of X implies that $X=\bigcup_{i=1}^{n}G_{x_i}$ for some $x_1,\cdots,x_n\in X$. Since $f(G_{x_i})\subseteq V_{\alpha_i}$, we have $Y=\bigcup_{i=1}^{n}V_{\alpha_i}$, i.e., Y is compact. □

Whenever X is a topological space, we recall that for each $x\in X$, the largest connected subset C_x of X containing x is the component of x. In fact C_x is the union of all connected subsets of X containing x. The quasicomponent Q_x of x in X is the intersection of all clopen subsets of X which contain x. It is well-known that $C_x\subseteq Q_x$ for each x and the inclusion may be proper, see Exercise 26B in [7].

If X and Y are topological spaces, then a function $f : X \to Y$ is called clopen continuous at $x \in X$ if for each neighborhood V of $f(x)$, there is a clopen subset of X containing x such that $f(U) \subseteq V$. The clopen continuous functions and also the rings of real valued clopen continuous functions are studied in [6] and [1] respectively. Clearly every clopen continuous function $f : X \to Y$ is e-continuous and whenever Y is a T_3-space, the converse is also true.

Theorem 3.4. *Let X and Y be topological spaces and $f : X \to Y$ be an e-continuous function. Then the following statements hold.*

(1) *If Y is a T_2-space, then f is constant on each quasi-component Q_x, $x \in X$.*
(2) *If Y is a T_3-space, then f is clopen continuous.*

Proof. (1) Since Y is Hausdorff, for each $y \in Y$,where $y \neq y_0 = f(x_0)$, there exists an open set V in Y such that $y_0 \in Y \setminus \mathrm{cl}_Y V$ and $y \in V$. By the e-continuity of f there is an e-open set G_y in X containing x_0 such that $f(G_y) \subseteq Y \setminus \mathrm{cl}_Y V$. But f is also continuous, so $f(\mathrm{cl}_X G_y) \subseteq \mathrm{cl}_Y f(G_y) \subseteq \mathrm{cl}_Y(Y \setminus V) = Y \setminus V$. Since $Q_{x_0} \subseteq \bigcap_{y_0 \neq y \in Y} \mathrm{cl}_X G_y$ and $y \notin f(\mathrm{cl}_X G_y)$ (for otherwise, $y \in Y \setminus V$ which is impossible), $y \notin f(Q_{x_0})$ for each $y \in Y$ with $y \neq y_0$. This implies that $f(Q_0) = y_0$.

(2) Let $x \in X$ and V be a neighborhood of $f(x)$ in Y. Then there exists an open subset W of Y containing $f(x)$ with $W \subseteq \mathrm{cl}_Y W \subseteq V$ by regularity of Y. Since f is e-continuous, there exists an e-open subset U of X containing x such that $f(U) \subseteq W$. Now $f(\mathrm{cl}_X U) \subseteq \mathrm{cl}_Y f(U) \subseteq \mathrm{cl}_Y W \subseteq V$, because f is continuous. But U is e-open, so $\mathrm{cl}_X U$ is clopen and we are through. □

The converse of part (2) of the above lemma is also true in the sense that Q_x for each x in a space X is in fact the largest subset of X containing x on which every e-continuous function on X is constant.

Proposition 3.5. *Let X be a space and Y be a Hausdorff space containing at least two points. Then for each $x \in X$,*

$$Q_x = \{y \in X : f(x) = f(y), \text{ for each } e\text{-continuous function } f : X \to Y\}.$$

Proof. Whenever $y \in Q_x$, then $f(x) = f(y)$ for each e-continuous function $f : X \to Y$ by Proposition 3.4. Conversely suppose that $y \notin Q_x$. Hence there exists a clopen set U containing x but not y. Now define $f : X \to Y$ with $f(U) = y_1$ and $f(X \setminus U) = y_2$, where y_1 and y_2 are two different points of Y. Clearly f is e-continuous, $f(x) = y_1 \neq y_2 = f(y)$ and we are done. □

If X is a topological space and $f : X \to \mathbb{R}$ is an e-continuous function, then using Theorem 3.4, f is a clopen continuous, because $\mathbb{R}$ is a T_3-space. Therefore the ring of all real valued e-continuous functions $C_e(X)$ on a topological space X coincides with the ring $C_s(X)$ consisting of real valued clopen continuous functions. It is easy to see that $C_e(X)(= C_s(X))$ is an ordered ring which is a subring of $C(X)$ and the following lemma follows that $C_e(X)$ is in fact a lattice ordered ring. Using Theorem 3.4, whenever X is connected, then $C_e(X) = \mathbb{R}$.

We denote by $C_e^*(X)$ the set of all bounded members of $C_e(X)$ and we call X an e-pseudocompact space if $C_e(X) = C_e^*(X)$. Clearly every e-compact space is e-pseudocompact by Proposition 3.3, but the converse is not true. For example $\mathbb{R}$ with usual topology is e-pseudocompact, since it is connected and each member of $C_e(X)$ is constant by Theorem 3.4. But the e-open cover $\{\mathbb{R} \setminus \{n, n+1, \dots\}\}_{n\in\mathbb{N}}$ of $\mathbb{R}$ can have no finite subcover. It is also clear that every pseudocompact space is e-pseudocompact, however the aforementioned example shows that the converse is not necessarily true.

Whenever X is an e-space, then clearly $C(X)$ coincides with $C_e(X)$. By the following proposition, the converse is also true if the space X is a completely regular Hausdorff space. First we need the following lemma.

Lemma 3.6. *Let X, Y and Z be topological spaces and $f : X \to Y$, $g : Y \to Z$ be functions. Then the following statements hold.*

(1) *If f is e-continuous and g is continuous, then $g \circ f : X \to Z$ is e-continuous.*
(2) *If $g\circ f$ is e-continuous and g is an open one-to-one continuous function, then f is e-continuous.*

Proof. It is evident. □

Our proof of the following proposition shows that the parts (1) and (2) are equivalent for any space X. In the other words, for any space X the equality $C^*(X) = C_e^*(X)$ implies $C(X) = C_e(X)$.

Proposition 3.7. *For a completely regular Hausdorff space X, the following statements are equivalent.*

(1) $C^*(X) = C_e^*(X)$.
(2) $C(X) = C_e(X)$.
(3) *The space X is an e-space.*
(4) *The space X is a zero-dimensional space.*

Proof. Clearly (2) implies (1) and if (1) holds we let $f \in C(X)$ and take a homeomorphism $\phi : \mathbb{R} \to (-1, 1)$. Then $\phi \circ f$ is bounded e-continuous by part (1). Now by Lemma 3.6 f will be e-continuous because ϕ is homeomorphism. Hence $C(X) = C_e(X)$, so (1) also implies (2). Using Proposition 2.7, (3) and (4) are equivalent and clearly (3) implies (2). Thus it remains to show that (2) implies (3). We note that the set $\{\mathrm{coz}f : f \in C(X)\}$ is a base for open subsets of X by Theorem 3.2 in [4]. Now if $x \in \mathrm{coz}f$ for some $f \in C(X) = C_e(X)$, then $f(x) \neq 0$, hence for an open set V in $\mathbb{R}$ containing $f(x)$ but not 0, there exists an e-open set U containing x such that $U \subseteq f^{-1}(V) \subseteq \mathrm{coz}f$. This implies that the space X have a base consisting of e-open sets and we are through. □

By Proposition 3.7, whenever X is an e-space, then $C(X) = C_e(X)$. Moreover if X is also e-pseudocompact, then $C_e(X) = C_e^*(X)$ implies that $C(X) = C_e^*(X) \subseteq C^*(X)$ and hence X will be pseudocompact. Therefore in the following proposition, we may replace "pseudocompact" with "e-pseudocompact".

Proposition 3.8. *The following statements for a completely regular Hausdorff space X are equivalent.*

(1) $C(X) = C_e^*(X)$.
(2) $C_e(X) = C^*(X)$.
(3) *The space X is a pseudocompact e-space.*
(4) *The space X is a pseudocompact zero-dimensional space.*

Proof. (1) $\Rightarrow$ (2). First $C_e(X) \subseteq C(X) = C_e^*(X) \subseteq C^*(X)$. Next $C^*(X) \subseteq C(X) = C_e^*(X) \subseteq C_e(X)$. Hence $C_e(X) = C^*(X)$.

(2) $\Rightarrow$ (3). The equality $C_e(X) = C^*(X)$ implies that $C_e^*(X) = C_e(X) \cap C^*(X) = C^*(X)$. Hence by Proposition 3.7 X is an e-space. On the other hand, by the same proposition we have $C_e(X) = C(X)$. Now using part (2), $C(X) = C_e(X) = C^*(X)$, so X is pseudocompact.

(3) $\Rightarrow$ (4) $\Rightarrow$ (1). By Propositions 2.7 and 3.7, the proof is evident. □

4. Characterization of maximal and real maximal ideals of rings of real valued e-continuous functions

From now on we need some notations and details of the proof of Theorem 3.1 in [1] for later use. By Theorem 3.1 in [1], $C_s(X)$, the ring of real valued clopen continuous functions on a space X, is a $C(Y)$ for a zero-dimensional space Y. As we have already observed $C_e(X)$ coincides with $C_s(X)$, so we use $C_e(X)$ instead of $C_s(X)$ in the statement of our Theorem 4.1 which is the same theorem in [1]. On the other hand, in order to familiar with some notations in the proof of Theorem 3.1 in [1] and their applications, we have to give the sketch of the proof.

Theorem 4.1. *For every topological space X, there exists a zero-dimensional space X_z such that $C_e(X) \cong C(X_z)$.*

Proof. Let X_z be the decomposition $\{Q_x : x \in X\}$ on X, where Q_x is the quasicomponent of x and take the collection τ consisting of subsets G of X_z such that $\bigcup_{Q_x \in G} Q_x$ is a union of clopen subset of the space X. It is not hard to see that τ is a topology on X_z and X_z with this topology is Hausdorff. To see that X_z is zero-dimensional, let H be an open set in X_z and $Q_y \in H$ for some $y \in X$. Then by definition $T = \bigcup_{Q_x \in H} Q_x$ is a union of clopen subsets of X and $y \in T$. Therefore there is a clopen subset U of X such that $y \in U \subseteq T$. Now take $G = \{Q_z : z \in U\}$. Since $\bigcup_{Q_z \in G} Q_z = U$ and U is clopen in X, the set G is clopen in X_z and $Q_y \in G \subseteq H$ (to see that $G \subseteq H$ let $Q_x \in G$, then $x \in U \subseteq T$, so there is $Q_z \in H$ with $x \in Q_z \subseteq H$. Therefore $Q_x = Q_z \in H$). This shows that X_z is indeed a zero-dimensional space.

Finally we define $\varphi : C_e(X) \to C(X_z)$ with $\varphi(f) = f_z$ for each $f \in C_e(X)$, where $f_z(Q_x) = f(x)$ for each $x \in X$. By a routine proof we observe that $f_z \in C(X_z)$ for each $f \in C(X)$ and it is easy to see that φ is a one-to-one homomorphism. To complete the proof it remains to show that φ is onto. To this end, let $g \in C(Xz)$. The function $f : X \to \mathbb{R}$ defined by $f(x) = g(Qx)$, for all $x \in X$ is e-continuous. In fact, if $x \in X$, $f(x) = g(Qx) = c$ and

$\varepsilon > 0$ is given, then there is an open set H in X_z containing Q_x such that $g(H) \subseteq (c-\varepsilon, c+\varepsilon)$. Now it is enough to take the e-open subset $G = \bigcup_{Q_z \in H} Q_z$ of X. Clearly $x \in G$ and $f(G) \subseteq (c-\varepsilon, c+\varepsilon)$ which implies that $f \in C_e(X)$. By definitions of f and φ it is clear that $\varphi(f) = g$ and we have thus shown that φ is onto. □

By the above proof, all members of $C(X_z)$ will be of the form f_z for some $f \in C(X)$. Using Corollaries 27.10 and 27.11 in [7], for a locally connected space X we have $C_e(X) \cong C(Y)$, where Y is a discrete space and in particular for a compact locally connected space, we have $C_e(X) \cong \mathbb{R}^n$ for some $n \in \mathbb{N}$.

Now using the definition of the isomorphism φ, the following result is evident.

Lemma 4.2 ([1, Lemma 4.1]). *An ideal I in $C_e(X)$ is fixed if and only if $\varphi(I)$ is a fixed ideal in $C(X_z)$. In particular, φ takes fixed maximal ideals to fixed maximal ideals.*

By this lemma as in Theorems 4.2 and 4.4 in [1], fixed and free maximal ideals of $C_e(X)$ will be characterized as follows.

Proposition 4.3. *([1], Theorem 4.2) For a topological space X, the fixed maximal ideals of $C_e(X)$ are precisely of the form*

$$M_{Q_x} = \{f \in C_e(X) : Q_x \subseteq Z(f)\} = (\bigcap_{y \in Q_x} M_y) \cap C_e(X) \qquad x \in X.$$

The ideals M_{Q_x} are distinct for distinct Q_x and for each $x \in X$, $C_e(X)/M_{Q_x}$ is isomorphic with the real field $\mathbb{R}$.

Proposition 4.4 ([1, Theorem 4.4]). *For every space X, the maximal ideals of $C_e(X)$ are precisely of the form*

$$M^p = \{f \in C_e(X) : p \in cl_{\beta X_z} Z(f_z)\} \quad , \quad p \in \beta X_z$$

As we observe the free maximal ideals of $C_e(X)$ are characterized via the zero-sets of X_z. These maximal ideals M^p so defined are in fact the maximal ideals of $C(X_z)$ which are not necessarily distinct for distinct p. For instance whenever $p \in X$, then $M^p = M^q$, for each $q \in Q_p$. Here we are going to introduce more natural representation of the maximal ideals of $C_e(X)$ by some equivalence classes in βX which do not depend upon X_z. First we construct equivalence classes in βX similar to Q_x's in X. Lemma 3.5 will show us the right way to define such classes. For every $p, q \in \beta X$ define $p \equiv q$ if and only if $f^\beta(p) = f^\beta(q)$, for each $f \in C^*_e(X)$. Clearly, this defines an equivalence relation on βX. Let Q^p be the equivalence class containing p, for every $p \in \beta X$. In case X is a completely regular Hausdorff space, we also note that the mapping $\sigma : X \to X_z$ with $\sigma(x) = Q_x$ for each $x \in X$ has the Stone extension $\bar{\sigma} : \beta X \to \beta X_z$, by Theorem 6.5 in [4]. This extension map is onto as σ is. to characterize the maximal ideals of $C_e(X)$, we first need the following lemmas.

Lemma 4.5. *Let $p, q \in \beta X$. Then $\bar{\sigma}(p) = \bar{\sigma}(q)$ if and only if $q \in Q^p$.*

Proof. For each $f \in C_e(X)$ we have $f_z(Q_x) = f(x)$ for each $x \in X$ and this means that $f \in C_e^*(X)$ if and only if $f_z \in C^*(X_z)$. On the other hand $f = f_z \circ \sigma$ on X for each $f \in C_e(X)$ implies that $f^\beta = f_z^\beta \circ \bar{\sigma}$ for each $f \in C_e^*(X)$. Now suppose that $q \notin Q^p$, then there exists $f \in C_e^*(X)$ such that $f^\beta(p) \neq f^\beta(q)$. This implies that $f_z^\beta \circ \bar{\sigma}(p) \neq f_z^\beta \circ \bar{\sigma}(q)$ and hence $\bar{\sigma}(p) \neq \bar{\sigma}(q)$. Conversely, let $q \in Q^p$ but $\bar{\sigma}(p) \neq \bar{\sigma}(q)$. Then there exists $g \in C^*(X_z)$ with $g^\beta(\bar{\sigma}(p)) \neq g^\beta(\bar{\sigma}(q)$. Since every member of $C^*(X_z)$ is of the form f_z for some $f \in C_e^*(X)$, we may take $g = f_z$ for some $f \in C_e^*(X)$. Now $f_z^\beta \circ \bar{\sigma}(p) \neq f_z^\beta \circ \bar{\sigma}(q)$ implies that $f^\beta(p) \neq f^\beta(q)$, so p is not equivalent to q , i.e., $q \notin Q^p$, a contradiction. □

Note that, σ takes zero-sets of members of $C_e(X)$ to zero-sets in X_z and also the images of any two disjoint zero-sets of members of $C_e(X)$, are disjoint in X_z, since $\sigma(Z(f)) = Z(f_z)$ for each $f \in C_e(X)$. Also using Lemma 3.6, $C_e(X)$ is a lattice ring. In fact if $f \in C_e(X)$ and we take $g : \mathbb{R} \to \mathbb{R}$ with $g(x) = |x|$ for each $x \in \mathbb{R}$, then by Lemma 3.6 we infer that $|f| \in C_e(X)$.

Lemma 4.6. *Let $f \in C_e(X)$. Then $\bar{\sigma}(p) \in cl_{\beta X_z} Z(f_z)$ if and only if $Q^p \cap cl_{\beta X} Z(f) \neq \varnothing$.*

Proof. Let $Q^p \cap \mathrm{cl}_{\beta X} Z(f) = \varnothing$. Then by the relation defined preceding Lemma 4.5, for every $a \in \mathrm{cl}_{\beta X} Z(f)$, there exists $g_a \in C_e^*(X)$ such that $g_a^\beta(p) \neq g_a^\beta(a)$. If we take $g_a^\beta(p) = \delta$ and $g_a^\beta(a) = \alpha$ and define $k_a = |\frac{g_a - \alpha}{\delta - \alpha}| \wedge 1$, then clearly $k_a \in C_e^*(X)$ by the argument preceding the lemma, $0 \leq k_a \leq 1$, $k_a(a) = 0$ and $k_a(p) = 1$. Hence for every $a \in \mathrm{cl}_{\beta X} Z(f)$ we may assume that $0 \leq g_a \leq 1$, $g_a^\beta(p) = 1$ and $g_a^\beta(a) = 0$. Again, letting $h_a = 2(g_a \vee \frac{1}{2} - \frac{1}{2})$, we have $h_a \in C_e^*(X)$, $0 \leq h_a \leq 1$, $a \in \mathrm{int}_{\beta X} Z(h_a^\beta)$ and $h_a^\beta(p) = 1$. Since $\mathrm{cl}_{\beta X} Z(f)$ is compact, there exists a finite subset $\{a_1, a_2, ..., a_n\}$ of $\mathrm{cl}_{\beta X} Z(f)$, such that $\mathrm{cl}_{\beta X} Z(f) \subseteq \bigcup_{i=1}^n Z(h_{a_i}^\beta)$. Then $Z(f) \subseteq Z(h)$, where $h = \prod_{k=1}^n h_{a_k}$ and evidently $h \in C_e^*(X)$, $0 \leq h \leq 1$ and $h^\beta(p) = 1$. Hence $Z(f)$ and $h^{-1}[\frac{2}{3}, 1] = Z(g)$, where $g = h \wedge \frac{2}{3} - \frac{2}{3} \in C_e(X)$ are disjoint zero-sets of $C_e(X)$. Therefore by the argument preceding the lemma, $\sigma(Z(f)) = Z(f_z)$ and $\sigma(Z(g)) = Z(g_z)$ are disjoint zero-sets in X_z. Moreover $p \in \mathrm{cl}_{\beta X} Z(g)$ implies that $\bar{\sigma}(p) \in \bar{\sigma}(\mathrm{cl}_{\beta X} Z(g)) \subseteq \mathrm{cl}_{\beta X_z} \bar{\sigma}(Z(g)) = \mathrm{cl}_{\beta X_z} \sigma(Z(g)) = \mathrm{cl}_{\beta X_z} Z(g_z)$ and so $\bar{\sigma}(p) \notin \mathrm{cl}_{\beta X_z} Z(f_z)$.

Conversely, suppose that $q \in Q^p \cap \mathrm{cl}_{\beta X} Z(f) \neq \varnothing$. Using the previous lemma $\bar{\sigma}(p) = \bar{\sigma}(q)$, and since $q \in \mathrm{cl}_{\beta X} Z(f)$, we have $\bar{\sigma}(q) \in \bar{\sigma}(\mathrm{cl}_{\beta X} Z(f)) \subseteq \mathrm{cl}_{\beta X_z} \bar{\sigma}(Z(f)) = \mathrm{cl}_{\beta X_z} \sigma(Z(f)) = \mathrm{cl}_{\beta X_z} Z(f_z)$. Therefore $\bar{\sigma}(p) \in \mathrm{cl}_{\beta X_z} Z(f_z)$. □

Theorem 4.7. *For every completely regular Hausdorff space X, the maximal ideals of $C_e(X)$ are precisely of the form*

$$M^{Q^p} = \{f \in C_e(X) : Q^p \cap cl_{\beta X} Z(f) \neq \varnothing\} = \left(\bigcup_{q \in Q^p} M^q \right) \cap C_e(X) \quad , \quad p \in \beta X,$$

and they are distinct for distinct Q^p.

Proof. Using the previous lemma and Theorem 4.4, the proof of the first part is evident. For the second part, let $Q^p \neq Q^q$. Then $q \notin Q^p$ and hence $\bar{\sigma}(p) \neq \bar{\sigma}(q)$ by Lemma 4.5. Now there exists $f_z \in C^*(X_z)$ such that $\bar{\sigma}(p) \in \mathrm{cl}_{\beta X_z} Z(f_z)$ and $\bar{\sigma}(q) \notin \mathrm{cl}_{\beta X_z} Z(f_z)$. This means that $Q^p \cap \mathrm{cl}_{\beta X} Z(f) \neq \varnothing$ but $Q^q \cap \mathrm{cl}_{\beta X} Z(f) = \varnothing$ by Lemma 4.6. Therefore $f \in M^{Q^p} \setminus M^{Q^q}$, i.e., $M^{Q^p} \neq M^{Q^q}$. □

We may directly obtain the fixed maximal ideals of $C_e(X)$ from Theorem 4.7 which are characterized in Proposition 4.3.

Corollary 4.8. *Let X be a completely regular Hausdorff space and $p \in \beta X$. Then the maximal ideal M^{Q^p} is fixed if and only if $M^{Q^p} = M_{Q_x}$ for some $x \in X$.*

Proof. Whenever $M^{Q^p} = M_{Q_x}$, clearly M^{Q^p} is fixed. Conversely, let M^{Q^p} be fixed. Then there is $x \in X$ such that $x \in Z(f)$ for each $f \in M^{Q^p}$. Using Proposition 3.4, $f(Q_x) = 0$, $\forall f \in M^{Q^p}$. This means that $M^{Q^p} \subseteq M_{Q_x}$ and hence $M^{Q^p} = M_{Q_x}$ because M^{Q^p} is maximal. □

Remark 4.9. As we observed in Theorem 4.7, $M^{Q^p} = (\bigcup_{q \in Q^p} M^q) \cap C_e(X)$ for each $p \in \beta X$ while for fixed maximal ideals M_{Q_x}, we have $M_{Q_x} = (\bigcap_{y \in Q_x} M_y) \cap C_e(X)$. The reason is that the equivalence classes Q^p's are constructed by a bit different relation which is defined preceding Lemma 4.5 and in contrast to Q_x's, they are not necessarily the quasi components of βX. In fact Q^p for each $p \in \beta X$ is contained in the quasicomponent of p in βX. To see this let $q \in Q^p$ but not in the quasicomponent of p in βX. Then there is a clopen set G in βX such that $p \in G$ but $q \notin G$. Now define $g \in C(\beta X)$ with $g(G) = 0$ and $g(\beta X \setminus G) = 1$. Clearly $g|_X \in C_e^*(X)$ and $g|_X^\beta = g$ separates p and q, i.e., $q \notin Q^p$.

Moreover, as the representation of M^{Q^p}'s, we may wright $M_{Q_x} = (\bigcup_{y \in Q_x} M_x) \cap C_e(X)$, in fact $M_{Q_x} = M_y \cap C_e(X)$ for each $y \in Q_x$. To this end, whenever $f \in C_e(X)$, then for $y \in Q_x$, we have $f(y) = 0$ if and only if $f(Q_x) = f(Q_y) = 0$ by proposition 3.4.

Since every element of βX_z is of the form $\bar{\sigma}(p), p \in \beta X$, the ideal $O^{\bar{\sigma}(p)}$ is defined as usual by the set $\{f_z \in C(X_z) : \bar{\sigma}(p) \in \mathrm{int}_{\beta X_z} \mathrm{cl}_{\beta X_z} Z(f_z)\}$. The related ideals in $C_e(X)$ are $\varphi^{-1}(O^{\bar{\sigma}(p)})$, $p \in \beta X$ and we are to characterize them by the following lemma and corollary.

Lemma 4.10. *Let $f \in C_e(X)$ and $p \in \beta X$. Then $\bar{\sigma}(p) \in int_{\beta X_z} cl_{\beta X_z} Z(f_z)$ if and only if $Q^p \subseteq int_{\beta X} cl_{\beta X} Z(f)$*

Proof. Let $\bar{\sigma}(p) \in \mathrm{int}_{\beta X_z} \mathrm{cl}_{\beta X_z} Z(f_z)$. Then there exists $g_z \in C^*(X_z)$ ($g \in C_e^*(X)$) such that $g_z^\beta(\bar{\sigma}(p)) = 1$ and $\mathrm{coz} g_z^\beta \subseteq \mathrm{int}_{\beta X_z} \mathrm{cl}_{\beta X_z} Z(f_z)$. Therefore $\mathrm{coz} g_z \subseteq Z(f_z)$ which implies that $f_z g_z = 0$ whence $fg = 0$. On the other hand, since $g = g_z \circ \sigma$ on X, we have $g^\beta = g_z^\beta \circ \bar{\sigma}$ and hence $g^\beta(p) = 1$. Now $fg = 0$ implies that $\mathrm{cl}_{\beta X} Z(f) \cup \mathrm{cl}_{\beta X} Z(g) = \beta X$ or equivalently, $\beta X \setminus \mathrm{cl}_{\beta X} Z(g) \subseteq \mathrm{cl}_{\beta X} Z(f)$. Using the equivalence relation defined preceding Lemma 4.5, for

every $q \in Q^p$ we have $g^\beta(q) = g^\beta(p) = 1$. Hence

$$Q^p \subseteq \beta X \setminus \mathrm{cl}_{\beta X} Z(g) \subseteq \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(f).$$

For the converse, suppose that $Q^p \subseteq \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(f)$. By a similar argument in the proof of Lemma 4.5, there exists an e-continuous function $h \in C_e^*(X)$ such that $h^\beta(p) = 1$ and $\beta X \setminus \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(f) \subseteq \mathrm{cl}_{\beta X} Z(h)$. Thus $\mathrm{coz}h^\beta \subseteq \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(f)$, whence $\mathrm{coz}h \subseteq Z(f)$ and hence $fh = 0$ which implies that $f_z h_z = 0$. Moreover, since $h^\beta = h_z^\beta \circ \bar{\sigma}$ and $h^\beta(p) = 1$, we have $h_z^\beta(\bar{\sigma}(p)) = 1$ which means that $h_z \notin M^{\bar{\sigma}(p)}$. Therefore $f_z \in O^{\bar{\sigma}(p)}$ by 7.12(b) in [4], so $\bar{\sigma}(p) \in \mathrm{int}_{\beta X_z}\mathrm{cl}_{\beta X_z} Z(f_z)$. □

Now, by the above lemma and the isomorphism $\varphi : C_e(X) \to C(X_z)$, we have the following corollary.

Corollary 4.11. *For each $p \in \beta X$,*

$$O^{Q^p} := \varphi^{-1}(O^{\bar{\sigma}(p)}) = \{f \in C_e(X) : Q^p \subseteq int_{\beta X}\, cl_{\beta X} Z(f)\}.$$

Corollary 4.12. *An ideal I in $C_e(X)$ is contained in a unique maximal ideal M^{Q^p} if and only if $O^{Q^p} \subseteq I$.*

Using Theorem 4.1, to study the algebraic properties of the rings $C_e(X)$, the space X may be considered as a zero-dimensional space. The following proposition also states that $C_e(X)$ ($C_e^*(X)$) determines the topology of a Hausdorff X if and only if it is zero-dimensional.

Proposition 4.13. *Let X be a Hausdorff space. Then X is zero-dimensional if and only if its topology coincides with the weak topology induced by $C_e(X)$ ($C_e^*(X)$).*

Proof. Whenever X is a zero-dimensional space, then $C_e(X) = C(X)$ by Proposition 3.7. On the other hand X is completely regular, so its topology coincides with the weak topology induced by $C(X) = C_e(X)$ ($C^*(X) = C_e^*(X)$) by Theorem 3.6 in [4]. Conversely suppose that the topology on X coincides with the weak topology induced by $C_e(X)$. Then the collection $\mathfrak{B} = \{f^{-1}(a, b) : f \in C_e(X), a, b \in \mathbb{R}\}$ is a base for X. But each $f^{-1}(a, b)$ is a union of e-open subsets of X by Proposition 3.2 and this shows that the space X has a base consisting of e-open sets, so X is an e-space. On the other hand, using Theorem 3.7 in [4] the space X is completely regular and hence X will be zero-dimensional by Proposition 2.7. □

Remark 4.14. If we consider X to be zero-dimensional, then $C_e(X) = C(X)$ and every two different points $p, q \in \beta X$ can be separated by f^β, for some $f \in C^*(X) = C_e^*(X)$. This means that Q^p is singleton for each $p \in \beta X$ and each maximal ideals M^{Q^p} of $C_e(X)$ is exactly M^p.

The rest of this section is devoted to characterization of real maximal ideals of $C_e(X)$. If R is a commutative ring which contains the real field $\mathbb{R}$, then a maximal ideal M of R is said to be real whenever $\frac{R}{M} \simeq \mathbb{R}$. In C(X) an ideal I is a real maximal ideal if and only if for each $f \in C(X)$, $f - r \in I$ for some

$r \in \mathbb{R}$; see Corollary 3.3 in [2]. From 8B.2 in [4], for every $f \in C(X)$, we denote $\upsilon_f X = \{p \in \beta X : f^*(p) \neq \infty\}$, where $f^* : \beta X \to \mathbb{R}^*$ is the Stone extension of f from X to the one-point compactification $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$. The realcompact spaces between X and βX are precisely the spaces $\upsilon_{C'} X = \bigcap_{f \in C'} \upsilon_f X$ for a subset C' of $C(X)$. The space $\upsilon_{C(X)} X$ is called the Hewitt realcompactification of X which is the smallest; see 8B.3 in [4].

Theorem 4.15. *Let X be a completely regular Hausdorff space and $p \in \beta X$. Then the following statements are equivalent.*

(1) *The maximal ideal M^{Q^p} of $C_e(X)$ is real.*
(2) *$Q^p \cap \upsilon_{C_e(X)} X \neq \varnothing$.*
(3) *$p \in \upsilon_{C_e(X)} X$, or equivalently $Q^p \subseteq \upsilon_{C_e(X)} X$.*

Proof. First note that M^{Q^p} is real if and only if $\varphi(M^{Q^p})$ is real. On the other hand, Lemma 4.6, implies that $\varphi(M^{Q^p}) = M^{\bar{\sigma}(p)}$ and therefore, M^{Q^p} is real if and only if $\bar{\sigma}(p) \in \upsilon X_z$.

(1)⇒(2). Suppose that $Q^p \cap \upsilon_{C_e(X)} X = \varnothing$. Let $q \in Q^p$ be arbitrary. Then by the definition of $\upsilon_{C_e(X)} X$, there exists an e-continuous function $f \in C_e(X)$ such that $f^*(q) = \infty$. Using Theorem 7.6 (a) in [4], $|M^q(f)|$ is infinitely large and therefore by Theorem 5.7 in [4] we infer that the zero-sets $Z_n = \{x \in X : |f(x)| \geq n\}$, $n \in \mathbb{N}$ belong to $Z[M^q]$. Thus Lemma 4.6 implies that

$$\sigma(Z_n) = \{y \in X_z : |f_z(y)| \geq n\} \in Z[M^{\bar{\sigma}(p)}],$$

and hence by Theorems 5.7 and 7.6(a) in [4] we conclude that $f_z^*(\bar{\sigma}(q)) = f_z^*(\bar{\sigma}(p)) = \infty$. This is a contradiction by the preceding argument at the beginig of the proof; see also Theorem 8.4 in [4].

(2)⇒(3). Let $q \in Q^p \cap \upsilon_{C_e(X)} X$ but $p \notin \upsilon_{C_e(X)} X$. Then by the definition of $\upsilon_{C_e(X)} X$, there exists an e-continuous function $f \in C_e(X)$ such that $f^*(p) = \infty$. Since $q \in \upsilon_{C_e(X)} X$, so $f^*(q) = r$, for some real number $r \in \mathbb{R}$. Let $g = r \vee f \wedge (|r| + 1)$, then $g \in C_e^*(X)$ and $g^\beta(q) = f^*(q) = r$. But $q \in Q^p$ and by the definition of Q^p we have $g^\beta(p) = g^\beta(q) = r$, which implies that $f^*(p) = r$, a contradiction.

(3)⇒(1). Using the argument at the beginning of the proof, it is enough to show that $\bar{\sigma}(p) \in \upsilon X_z$. To do this, we need to prove that, $f_z^*(\bar{\sigma}(p)) \neq \infty$, for every $f_z \in C(X_z)$; see Theorem 8.4 in [4]. In fact since $f_z^* \circ \bar{\sigma}$ agree with f^* on X , we have $f_z^* \circ \bar{\sigma} = f^*$. Now $(f_z^* \circ \bar{\sigma})(p) = f^*(p)$ implies that $f_z^*(\bar{\sigma}(p)) \neq \infty$ because $f^*(p) \neq \infty$ and we are done. □

Using Theorem 3.1 in [2] and our Theorem 4.15, we have also the following elementwise characterization of real maximal ideals of $C_e(X)$.

Proposition 4.16. *Let I be an ideal of $C_e(X)$. Then the following statements are equivalent.*

(1) *The ideal I is a real maximal ideal of $C_e(X)$.*
(2) *For each $f \in C_e(X)$, there is $r \in \mathbb{R}$ such that $f - r \in I$.*
(3) *There exists $p \in \beta X$ such that for each $f \in C_e(X)$, $Q^p \cap cl_{\beta X} f^{-1}(r) \neq \varnothing$, for some $r \in \mathbb{R}$.*

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