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Well-posedness for the fourth-order Moore–Gibson–Thompson equation in the class of Banach-space-valued Hölder-continuous functions

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In this work, we provide a full characterization of well-posedness in vector-valued Hölder continuous function spaces for a fourth-order abstract evolution equation arising from the Moore–Gibson–Thompson equation with memory using operator-valued $\dot{C}^\alpha$-Fourier multipliers. We illustrate our results by providing an example based on the fourth order Moore–Gibson–Thompson equation with Dirichlet boundary conditions.

KEYWORDS
C^α-well posedness, Dirichlet Laplacian, Fourier multipliers, fourth-order Moore–Gibson–Thompson equation

MSC CLASSIFICATION
35A05, 42A45, 35B65

1 | INTRODUCTION

The research about the existence and uniqueness of solutions of higher-order abstract Cauchy problems have been widely considered in the literature.[1–5] Third and higher-order differential equations arise in physics and engineering for describing multi-resonant vibration processes, accelerating and decelerating processes, hyperjerk systems or intermittent motions among others; see for instance previous studies[6–10] and the references therein.

In this paper, we are concerned with the well-posedness analysis in Hölder continuous function spaces for the fourth-order Moore–Gibson–Thompson (MGT) equation introduced by Dell'Oro and Pata[11] in its abstract version:

$$u''''(t) + \alpha u'''(t) + \beta u''(t) - \gamma Au''(t) - \delta Au'(t) - \rho Au(t) = 0, \ \ t \in \mathbb{R}, \tag{1.1}$$

where A is a closed linear operator defined on a Banach space X and $\alpha, \beta, \gamma, \delta, \rho$ are positive real numbers. This model arises from the third-order MGT equation with memory, which has been extensively studied in the literature.[12–14]

In their work,[11] Dell'Oro and Pata analyzed well-posedness of the Equation (1.1) in terms of the existence of the solution semigroup when A is a strictly positive unbounded linear operator whose domain $D(A)$ is densely embedded in a separable real Hilbert space H. They also provide a detailed study on the exponential stability of the solution semigroup in terms of the parameters of the equation. In a previous study,[15] Liu et al. continued this analysis incorporating a delay memory term in the model and stated general decay results. More recently, in another study,[16] Lizama and Murillo-Arcila characterized the well-posedness of (1.1) in the scales of vector-valued Lebesgue, Besov, and Triebel Lizorkin function spaces.

However, there is still work left to be done in connection with Equation (1.1), especially with regard to studying it in general Banach spaces. Particularly, the well-posedness of (1.1) in various spaces of functions such as Hölder spaces still remains unsolved. Our objective in this paper is to solve this open problem. We point out that the corresponding study in Hölder spaces for the third-order MGT equation with and without memory was performed in Cuevas and Lizama[8] and Abadías et al,[17] respectively.

In order to solve our problem, we consider the theory of operator-valued Fourier multipliers. The main convenience of using this technique is avoiding characterization of the solution in terms of families of bounded linear operators such as resolvent families; see Fernández et al.[18] In our work, we will employ a relevant Fourier multiplier theorem given by Arendt et al.[19, Theorem 5.3] This result has been employed extensively in well-posedness research of differential equations in Hölder continuous function spaces as it can be found in other works.[17,20–24]

In this paper, we obtain a full characterization of C^α well-posedness for the abstract evolution Equation (1.1). These results are necessary for the analysis of nonlinear problems,[25] and our characterization does not depend on the geometry of the underlying Banach space X. More precisely, in Theorem 3.4, we prove that if A is a closed linear operator defined on a Banach space X and $\rho(A)$ denotes the resolvent set of A, then the following assertions are equivalent:

(i) Given $f \in C^\alpha(\mathbb{R}, X)$, there exists a unique function $u \in C^{\alpha+2}(\mathbb{R}, D(A)) \cap C^{\alpha+4}(\mathbb{R}, X)$ that solves (1.1) for all $t \in \mathbb{R}$.

(ii) $\left\{ \frac{(is)^4+\alpha(is)^3+\beta(is)^2}{\gamma(is)^2+\delta(is)+\rho} \right\}_{s\in\mathbb{R}} \subseteq \rho(A)$ and

$$\sup_{s\in\mathbb{R}} \left\| (is)^4 \frac{1}{\gamma(is)^2+\delta(is)+\rho} \left(\frac{(is)^4+\alpha(is)^3+\beta(is)^2}{\gamma(is)^2+\delta(is)+\rho} - A \right)^{-1} \right\| < \infty.$$

As a consequence of our result and the closed graph theorem, we obtain the following estimate:

$$||u''''||_{C^\alpha(\mathbb{R},X)} + ||u'''||_{C^\alpha(\mathbb{R},X)} + ||u''||_{C^\alpha(\mathbb{R},X)} + ||Au''||_{C^\alpha(\mathbb{R},X)} + ||Au'||_{C^\alpha(\mathbb{R},X)}$$

$$+||Au||_{C^\alpha(\mathbb{R},X)} \leq C||f||_{C^\alpha(\mathbb{R},X)},$$

where $C > 0$ is constant and independent of $f \in C^\alpha(\mathbb{R}, X)$. We also provide some examples to illustrate our results. More specifically, we prove that under some conditions on the parameters $\alpha, \beta, \gamma, \delta, \rho$, the MGT equation with Dirichlet boundary conditions given by

$$u''''(t) + \alpha u'''(t) + \beta u''(t) - \gamma \Delta u''(t) - \delta \Delta u'(t) - \rho \Delta u(t) = f(t), \ t \in \mathbb{R},$$

is C^α-ill posed in $C^\alpha(\mathbb{R}, C_0(\Omega))$ for $0 < \alpha < 1$.

On the other hand, we also show for a specific choice of the data, that is, $\alpha = \beta = 2, \gamma = \delta = 1$ and $\rho < 1/2$ that model (1.1) is C^α-well-posed when A is a closed operator on $\ell^2(\mathbb{N})$.

2 | PRELIMINARIES

In this section, we introduce the notation, definitions, and results that will be needed throughout the work.

Definition 2.1. Let X be a Banach space and $0 < \alpha < 1$ be fixed. We denote by $\dot{C}^\alpha(\mathbb{R}, X)$ the space

$$\dot{C}^\alpha(\mathbb{R}, X) = \left\{ f : \mathbb{R} \to X : f(0) = 0, \|f||_\alpha := \sup_{t\neq s} \frac{||f(t)-f(s)||}{|t-s|^\alpha} < \infty \right\},$$

and

$$C^\alpha(\mathbb{R}, X) = \{ f : \mathbb{R} \to X : ||f||_{C^\alpha} = ||f||_\alpha + ||f(0)|| < \infty \}.$$

Both $C^\alpha(\mathbb{R}, X)$ and $\dot{C}^\alpha(\mathbb{R}, X)$ are Banach spaces endowed with corresponding norms.

For $k \in \mathbb{N}$, $C^{\alpha+k}(\mathbb{R}, X) = \{ u \in C^k(\mathbb{R}, X) : u^{(k)} \in C^\alpha(\mathbb{R}, X) \}$ is a Banach space when endowed with the norm

$$||u||_{C^{\alpha+k}} = ||u^{(k)}||_{C^\alpha} + ||u(0)||.$$

We now recall the definition of $\dot{C}^{\alpha}$-multipliers due to Arendt et al.[26]

Definition 2.2. Let $M : \mathbb{R}\setminus\{0\} \to \mathcal{B}(X, Y)$ be continuous. M is a $\dot{C}^{\alpha}$-multiplier in $\mathcal{B}(X, Y)$ if there exists a mapping $L : \dot{C}^{\alpha}(\mathbb{R}, X) \to \dot{C}^{\alpha}(\mathbb{R}, Y)$ such that

$$\int_{\mathbb{R}} (Lf)(s)(\mathcal{F}\phi)(s)ds = \int_{\mathbb{R}} (\mathcal{F}(\phi \cdot M))(s)f(s)ds \tag{2.1}$$

for all $f \in \dot{C}^{\alpha}(\mathbb{R}, X)$ and all $\phi \in C_c^{\infty}(\mathbb{R}\setminus\{0\})$, where $C_c^{\infty}(\mathbb{R}\setminus\{0\})$ denotes the space of all C^{∞}-functions in $\mathbb{R}\setminus\{0\}$ having compact support in $\mathbb{R}\setminus\{0\}$ and $\mathcal{F}$ denotes the Fourier transform of a function given by

$$(\mathcal{F}f)(s) := \tilde{f}(s) := \int_{\mathbb{R}} e^{-ist} f(t)dt, \; s \in \mathbb{R}$$

when $f \in L^1(\mathbb{R}, X)$.

Remark 2.3 (Keyantuo et al.[22]). We remark that the space $C_c^1(\Omega)$ of all C^1-functions in Ω having compact support in Ω in Definition 2.2 can be considered instead of the space $C_c^{\infty}(\Omega)$ (see, e.g., Brézis[27, Théorème IV.22]).

Next theorem gives sufficient conditions in terms of norm boundedness to guarantee when $M \in C^2(\mathbb{R}\setminus\{0\}, \mathcal{B}(X, Y))$ is a $\dot{C}^{\alpha}$-multiplier.

Theorem 2.4 (Arendt-Batty-Bu). *Let* $M \in C^2(\mathbb{R}\setminus\{0\}, \mathcal{B}(X, Y))$ *that verifies*

$$\sup_{s\neq 0} ||M(s)|| + \sup_{s\neq 0} ||sM'(s)|| + \sup_{s\neq 0} ||s^2M''(s)|| < \infty. \tag{2.2}$$

Then M is a $\dot{C}^{\alpha}$*-multiplier whenever* $0 < \alpha < 1$.

For more information about weaker conditions to ensure that a certain function is an $\dot{C}^{\alpha}$-multiplier, see Arendt et al.[26, Remark 5.5]

Given a function $u \in L^1_{loc}(\mathbb{R}, X)$, we say that u has subexponential growth if

$$\int_{-\infty}^{\infty} e^{-\varepsilon|t|}\|u(t)\|dt < \infty, \; \text{for each}\, \varepsilon > 0.$$

We denote by $\hat{u}$ the Carleman transform of u

$$\hat{u}(\lambda) = \begin{cases} \int_0^{\infty} e^{-\lambda t}u(t)dt & \mathfrak{Re}\,\lambda > 0, \\ -\int_{-\infty}^0 e^{-\lambda t}u(t)dt & \mathfrak{Re}\,\lambda < 0. \end{cases}$$

For more details about the Carleman transform, see Arendt et al.[19, Chapter 4] or Prüss.[28] A point $s \in \mathbb{R}$ is called a regular point of u if its Carleman transform has a holomorphic extension to a neighborhood of is. The Carleman spectrum of u is defined as $sp_C(u) := \{s \in \mathbb{R} : s \text{ is not regular}\}$. Moreover, $u = 0$ if and only if $sp_C(u) = \emptyset$.

3 | $C\alpha$-WELL-POSEDNESS FOR THE FOURTH-ORDER MGT EQUATION

In this section, we provide a full characterization for well-posedness of the fourth-order MGT equation in Hölder continuous spaces of vector-valued functions. Given a closed linear operator A defined on a Banach space X and $f \in C^{\alpha}(\mathbb{R}, X)$, we analyze the nonhomogeneous MGT equation:

$$u''''(t) + \alpha u'''(t) + \beta u''(t) - \gamma Au''(t) - \delta Au'(t) - \rho Au(t) = f(t), \; t \in \mathbb{R}, \tag{3.1}$$

where $\alpha, \beta, \gamma, \delta, \rho \in \mathbb{R}$ are positive.

We recall that the domain of A, $D(A)$, is a Banach space when endowed with the graph norm. We will now introduce the definition of a C^{α}-well-posed problem as stated in Abadías et al. and Arendt et al.[17,26]

Definition 3.1. Equation (3.1) is said to be C^{α}-well-posed if for each $f \in C^{\alpha}(\mathbb{R}, X)$ there exists a unique function $u \in C^{\alpha+2}(\mathbb{R}, D(A)) \cap C^{\alpha+4}(\mathbb{R}, X)$ that solves (3.1) for all $t \in \mathbb{R}$. Otherwise, Equation (3.1) is said to be C^{α}-ill-posed.

We introduce the following functions that will play an important role in our characterization:

$$g(s) := \frac{1}{\gamma(is)^2 + \delta(is) + \rho}$$

and

$$h(s) := \frac{(is)^4 + \alpha(is)^3 + \beta(is)^2}{\gamma(is)^2 + \delta(is) + \rho}$$

for all $s \in \mathbb{R}$. In what follows, let $M(s) = g(s)(h(s) - A)^{-1}$ whenever it exists.

Remark 3.2. For all $s \in \mathbb{R}$, it follows that $\gamma(is)^2 + \delta(is) + \rho \neq 0$ rendering the functions g and h well-defined for all $s \in \mathbb{R}$.

Theorem 3.3. *Assume that $\alpha, \beta, \gamma, \delta, \rho$ are positive real numbers and $\{h(s)\}_{s\in\mathbb{R}} \subseteq \rho(A)$. If*

$$\sup_{s\in\mathbb{R}} \|s^4 M(s)\| < \infty,$$

then $(is)^4M$, $(is)^3M$, and $(is)^2M$ are $\dot{C}^{\alpha}$-multipliers in $\mathcal{B}(X)$, and $(is)^2M$, isM, and M are $\dot{C}^{\alpha}$-multipliers in $\mathcal{B}(X, D(A))$.

Proof. We point out that $\sup_{s\in\mathbb{R}} \|(is)^j M(s)\| < \infty$ for $j = 0, 1, 2, 3$ is a consequence of having $\sup_{s\in\mathbb{R}} \|s^4 M(s)\| < \infty$ and the continuity of $M(s)$ in $s = 0$. Indeed, given a fixed $j = 0, 1, 2, 3$, it is clear by hypothesis that $\sup_{|s|>\epsilon} \|(is)^j M(s)\| < \infty$ for each $\epsilon > 0$, and the continuity of the function $s \to M(s)$ at $s = 0$ immediately yields to the boundedness of $(is)^j M(s)$ as desired. A simple calculation yields that

$$M'(s) = \frac{g'(s)}{g(s)} M(s) - \frac{h'(s)}{g(s)} M^2(s),$$

and

$$M''(s) = \frac{g''(s)}{g(s)} M(s) - \left(\frac{2g'(s)h'(s)}{g^2(s)} + \frac{h''(s)}{g(s)}\right) M^2(s) + 2\left(\frac{h'(s)}{g(s)}\right)^2 M^3(s),$$

where

$$g'(s) = (-2\gamma s + \delta i)g^2(s), \; g''(s) = -2\gamma g^2(s) + 2g(s)g'(s)(-2\gamma s + \delta i) \tag{3.2}$$

and

$$\begin{aligned} h'(s) &= (4s^3 - 3i\alpha s^2 - 2\beta s)g(s) + ((is)^4 + \alpha(is)^3 + \beta(is)^2)g'(s), \\ h''(s) &= (12s^2 - 6i\alpha s - 2\beta)g(s) + 2(4s^3 - 3i\alpha s^2 - 2\beta s)g'(s) + ((is)^4 + \alpha(is)^3 + \beta(is)^2)g''(s). \end{aligned} \tag{3.3}$$

After some computations and using the identities in (3.2), it is straightforward to see that g, $(is)g$ and $(is)^2g$ are $\dot{C}^{\alpha}$-multipliers in $\mathcal{B}(X)$. In addition, from (3.2) and (3.3), we can state the existence of a constant $C > 0$ such that the following inequalities hold:

$$\left|\frac{sg'(s)}{g(s)}\right| \le C, \; \left|\frac{s^2 g''(s)}{g(s)}\right| \le C, \; \left|\frac{h'(s)}{g(s)}\right| \le Cs^3, \; \left|\frac{h''(s)}{g(s)}\right| \le Cs^2 \tag{3.4}$$

if $|s| > \varepsilon > 0$, for any $\epsilon > 0$.

We define $M_j := (id)^j M$ for $j = 0, 1, 2, 3, 4$. Since $s \mapsto M(s)$ is continuous in $s = 0$, it is sufficient to show that $\sup_{|s|>\varepsilon} \|sM_j'(s)\| < \infty$ and $\sup_{|s|>\varepsilon} \|s^2 M_j''(s)\| < \infty$ for each $j = 0, 1, 2, 3, 4$. We prove the case $j = 4$. The other cases follow analogously. We first show that

$$\begin{aligned}\|sM_4'(s)\| &\le C(\|s^4M(s)\| + \|s^5M'(s)\|)\\ &\le C(\|M_4(s)\| + \|s^4M(s)\| + \|s^8M^2(s)\|)\\ &\le C(\|M_4(s)\| + \|M_4(s)\| + \|M_4^2(s)\|),\end{aligned}$$

where we have used the first and third inequalities in Equation (3.4). We then assert that $\sup_{s\in\mathbb{R}}\|sM_4'(s)\| < \infty$. Furthermore, we also have

$$\|s^2M_4''(s)\| \le C(\|s^4M(s)\| + \|s^5M'(s)\| + \|s^6M''(s)\|).$$

The first two summands are clearly uniformly bounded as it has been previously proved. Using now inequalities from (3.4), we have

$$\begin{aligned}\|s^6M''(s)\| &\le C\left(\left\|\frac{s^6g''(s)}{g(s)}M(s)\right\| + \left\|\frac{s^6h'(s)g'(s)}{g^2(s)}M^2(s)\right\| + \left\|\frac{s^6h''(s)}{g(s)}M^2(s)\right\| + \left\|\frac{s^6h'(s)}{g^2(s)}M^3(s)\right\|\right.\\ &\le C(\|M_4(s)\| + \|M_4^2(s)\| + \|M_4^2(s)\| + \|M_4^3(s)\|)\end{aligned}$$

and then $\sup_{s\in\mathbb{R}}\|s^6M''(s)\| < \infty$. We can then conclude from Theorem 2.4 that M_4 is a $\dot{C}^\alpha$-multiplier in $\mathcal{B}(X)$. At last, the equality

$$AM(s) = h(s)M(s) - g(s)I = g(s)[(is)^4M(s) + \alpha(is)^3M(s) + \beta(is)^2M(s) - I], \tag{3.5}$$

shows that M is a $\dot{C}^\alpha$-multiplier in $\mathcal{B}(X, D(A))$. Finally, $(is)M$ and $(is)^2M$ are $\dot{C}^\alpha$-multipliers in $\mathcal{B}(X, D(A))$, since $(is)g$ and $(is)^2g$ are $\dot{C}^\alpha$-multipliers as remarked before. □

We present now the main result of the paper that characterizes C^α-well-posedness of Equation (3.1) in Hölder continuous spaces in terms of a norm condition for the resolvent operator and the spectrum of A.

Theorem 3.4. *Let A be a closed linear operator defined on a Banach space X and $\alpha, \beta, \gamma, \delta, \rho$ are positive real numbers. Then the following statements are equivalent:*

(i) Equation (3.1) is C^α-well-posed.
(ii) $\{h(s)\}_{s\in\mathbb{R}} \subseteq \rho(A)$ and

$$\sup_{s\in\mathbb{R}} \|(is)^4g(s)(h(s) - A)^{-1}\| < \infty.$$

Proof. We first prove $(i) \Rightarrow (ii)$. Let $s \in \mathbb{R}$ and $x \in D(A)$ so that $Ax = h(s)x$. Then the function $u(t) = e^{ist}x$ solves Equation (3.1) for $f = 0$. By hypothesis, (3.1) is C^α-well-posed. Since the null function is also a solution of (3.1) for $f = 0$, by uniqueness $u = 0$ and thus, $x = 0$. It follows that $(h(s) - A)$ is injective. Now, let $s \in \mathbb{R}$ and $y \in X$. Let $f(t) = e^{ist}y \in C^\alpha(\mathbb{R}, X)$, then if we define $T : C^\alpha(\mathbb{R}, X) \to C^{\alpha+4}(\mathbb{R}, X) \cap C^{\alpha+2}(\mathbb{R}, D(A))$ given by $T(f) = u$ where for a given $f \in C^\alpha(\mathbb{R}, X)$, u is the unique solution of (3.1) associated with f, it implies that T is a well-defined bounded operator. We now fix $\eta \in \mathbb{R}$, then both

$$v_1(t) = u(t + \eta) \ \text{ and } \ v_2(t) = e^{i\eta s}u(t),$$

solve Equation (3.1) with forcing term $e^{i\eta s}f(t)$. Since the solution must be unique, we immediately get $u(t) = e^{ist}u(0)$. Considering $x = u(0) \in D(A)$ and substituing u in (3.1) we get

$$u(t) = e^{ist}g(s)(h(s) - A)^{-1}y,$$

and for $t = 0$

$$u(0) = x = g(s)(h(s) - A)^{-1}y.$$

Therefore, $(h(s)-A)$ is bijective. Taking into account that $\|e^{ist}x\|_\alpha = K_\alpha|s^\alpha\|x\|$, where $K_\alpha = 2\sup_{t>0} t^{-\alpha}\sin(t/2)$, see Arendt et al.,[26, Section 3] we obtain

$$\begin{aligned} K_\alpha|s|^\alpha\|(is)^4 g(s)(h(s)-A)^{-1}y\| = \|u''''\|_\alpha \le \|u\|_{\alpha+4} \le \|T\|\|f\|_\alpha \\ \le \|T\|(\|f\|_\alpha + \|f(0)\|) \le \|T\|\|y\|(K_\alpha|s|^\alpha + 1). \end{aligned}$$

As a consequence, for $\epsilon > 0$, it follows that

$$\sup_{|s|>\epsilon} \|(is)^4 g(s)(h(s)-A)^{-1}\| < \infty.$$

On the other hand, since $g \in C_0(\mathbb{R})$ and $s \to (is)^4 g(s)(h(s)-A)^{-1}$ is continuous at $s=0$, we conclude (ii).

We now show that (ii) implies (i). Indeed, let $f \in C^\alpha(\mathbb{R}, X)$. By Theorem 3.3, we can find $u_0, u_1, u_2 \in C^\alpha(\mathbb{R}, D(A))$ and $u_3, u_4 \in C^\alpha(\mathbb{R}, X)$ such that

$$\int_{\mathbb{R}} u_i(s)(\mathcal{F}\psi_i)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\psi_i \cdot (is)^i \cdot M)(s) f(s) ds, \tag{3.6}$$

for all $\psi_i \in C_c^1(\mathbb{R}\backslash\{0\})$ and $i \in \{0,1,2,3,4\}$ (cf. Remark 2.3). According to Cuevas and Lizama,[8, Theorem 3.5] we can ensure the existence of $y_i \in X$ for all $i \in \{0,1,2,3,4\}$ such that

$$u_0' = u_1 + y_1,\ u_0'' = u_2 + y_2,\ u_0''' = u_3 + y_3, u_0'''' = u_4 + y_4, \tag{3.7}$$

and $u_0 \in C^{\alpha+4}(\mathbb{R}, X)$. From identity (3.5), it follows that

$$(is)^4 M(s) + \alpha(is)^3 M(s) + \beta(is)^2 M(s) - \gamma(is)^2 AM(s) - \delta(is)AM(s) - \rho AM(s) = I,$$

which directly implies

$$\begin{aligned} &\int_{\mathbb{R}} \mathcal{F}(\psi \cdot (is)^4 \cdot M)(s) f(s) ds + \alpha \int_{\mathbb{R}} \mathcal{F}(\psi \cdot (is)^3 \cdot M)(s) f(s) ds + \beta \int_{\mathbb{R}} \mathcal{F}(\psi \cdot (is)^2 \cdot M)(s) f(s) ds \\ &- \gamma \int_{\mathbb{R}} A\mathcal{F}(\psi \cdot (is)^2 \cdot M)(s) f(s) ds - \delta \int_{\mathbb{R}} A\mathcal{F}(\psi \cdot (is) \cdot M)(s) f(s) ds - \rho \int_{\mathbb{R}} A\mathcal{F}(\psi \cdot M)(s) f(s) ds \\ &= \int_{\mathbb{R}} \mathcal{F}(\psi)(s) f(s) ds, \end{aligned} \tag{3.8}$$

for all $\psi \in C_c^1(\mathbb{R}\backslash\{0\})$. Considering identity (3.6), we get that (3.8) is equivalent to

$$\begin{aligned} &\int_{\mathbb{R}} u_0''''(s)(\mathcal{F}\psi)(s)ds + \alpha \int_{\mathbb{R}} u_0'''(s)(\mathcal{F}\psi)(s)ds + \beta \int_{\mathbb{R}} u_0''(s)(\mathcal{F}\psi)(s)ds \\ &- \gamma \int_{\mathbb{R}} Au_0''(s)(\mathcal{F}\psi)(s)ds - \delta \int_{\mathbb{R}} Au_0'(s)(\mathcal{F}\psi)(s)ds - \rho \int_{\mathbb{R}} Au_0(s)(\mathcal{F}\psi)(s)ds = \int_{\mathbb{R}} f(s)\mathcal{F}(\psi)(s)ds, \end{aligned}$$

for all $\psi \in C_c^1(\mathbb{R}\backslash\{0\})$ and then we can find $y \in X$ such that

$$u_0''''(s) + \alpha u_0'''(s) + \beta u_0''(s) - \gamma A u_0''(s) - \delta A u_0'(s) - \rho A u_0(s) = f(s) + y \qquad s \in \mathbb{R}.$$

Since A is invertible by hypothesis, we can define

$$u(t) := u_0(t) + \rho^{-1}A^{-1}y,$$

which can be clearly checked to be a solution of Equation (3.1).

Moreover, since $u \in C^{\alpha+4}(\mathbb{R},X)$, it follows that $u, u', u'' \in C^{\alpha}(\mathbb{R},X)$. By Theorem 3.3, we have that $AM, A(is)M, A(is)^2M$ are $\dot{C}^{\alpha}$-multipliers and then we can find $u_4, u_5, u_6 \in C^{\alpha}(\mathbb{R},X)$ such that

$$\int_{\mathbb{R}} u_{i+4}(s)(\mathcal{F}\psi)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\psi \cdot A(is)^i M)(s) f(s)ds, \text{ for } i = 0,1,2,$$

and for all $\psi \in C_c^1(\mathbb{R}\backslash\{0\})$. Using identity (3.6) for $i = 0, 1, 2$, since A is closed, we have

$$\int_{\mathbb{R}} Au_i(s)(\mathcal{F}\psi)(s)ds = \int_{\mathbb{R}} A\mathcal{F}(\psi \cdot (is)^i \cdot M)(s)f(s)ds = \int_{\mathbb{R}} u_{i+4}(s)(\mathcal{F}\psi)(s)ds \text{ for all } i = 0,1,2.$$

As a consequence, there exist $y_4, y_5, y_6 \in X$ such that $Au_0 = u_4 + y_4, Au_1 = u_5 + y_5, Au_2 = u_6 + y_6$ and therefore $Au, Au', Au'' \in C^{\alpha}(\mathbb{R},X)$.

Let us finally prove the solution is unique. For that purpose, we need to show that given $v, w \in C^{\alpha+4}(\mathbb{R},X) \cap C^{\alpha+2}(\mathbb{R},D(A))$ that both solve Equation (3.1), then it follows that $v = w$. This fact is equivalent to consider a solution $u \in C^{\alpha+4}(\mathbb{R},X) \cap C^{\alpha+2}(\mathbb{R},D(A))$ of the homogeneus Equation (3.1) and prove that $u = 0$. Indeed, computing the Carleman transform $\hat{u}$, we get that $\hat{u}(\lambda) \in D(A)$ and

$$\widehat{u''''}(\lambda) = \lambda^4\hat{u}(\lambda) - \lambda^3 u(0) - \lambda^2 u'(0) - \lambda u''(0) - u'''(0)$$

$$\widehat{u'''}(\lambda) = \lambda^3\hat{u}(\lambda) - \lambda^2 u(0) - \lambda u'(0) - u''(0),\ \widehat{u''}(\lambda) = \lambda^2\hat{u}(\lambda) - \lambda u(0) - u'(0)$$

$$\widehat{Au''}(\lambda) = \lambda^2 A\hat{u}(\lambda) - \lambda Au(0) - Au'(0),\ \widehat{Au'}(\lambda) = \lambda A\hat{u}(\lambda) - u(0) \quad \text{and} \quad \widehat{Au}(\lambda) = A\hat{u}(\lambda)$$

for all $\lambda \in \mathbb{C}\backslash\{i\mathbb{R}\}$. As a consequence, the following identity holds

$$[\lambda^4 + \alpha\lambda^3 + \beta\lambda^2 - \gamma A\lambda^2 - \delta A\lambda - \rho A]\hat{u}(\lambda) = [\lambda^3 + \alpha\lambda^2 + \beta\lambda - \gamma A\lambda - \delta A]u(0)$$

$$+[\lambda^2 + \alpha\lambda + \beta - \gamma A]u'(0) + [\lambda + \alpha]u''(0) + u'''(0)$$

for all $\lambda \in \mathbb{C}\backslash\{i\mathbb{R}\}$. Consequently, the Carleman spectrum of u is empty from hypothesis (*ii*). Therefore, u is the null function by Arendt et al.,[19, Theorem 4.8.2] and the equation is C^{α}-well-posed. □

Remark 3.5.

- Condition $\{h(s)\}_{s\in\mathbb{R}} \subseteq \rho(A)$ in Theorem 3.4 states that invertibility of the operator A is necessary for C^{α}-well-posedness of (3.1).
- If Equation (3.1) is C^{α}-well-posed for some $0 < \alpha < 1$, then it is C^{α}-well-posed for all $0 < \alpha < 1$.
- If condition (*ii*) fails in Theorem 3.4, then Equation (3.1) is said to be C^{α}-ill-posed.

Corollary 3.6. *If Equation (3.1) is C^{α}-well-posed, then there exists $C > 0$ constant and independent of $f \in C^{\alpha}(\mathbb{R},X)$ such that*

$$||u''''||_{C^{\alpha}(\mathbb{R},X)} + ||u'''||_{C^{\alpha}(\mathbb{R},X)} + ||u''||_{C^{\alpha}(\mathbb{R},X)} + ||Au''||_{C^{\alpha}(\mathbb{R},X)} + ||Au'||_{C^{\alpha}(\mathbb{R},X)} + ||Au||_{C^{\alpha}(\mathbb{R},X)} \le C||f||_{C^{\alpha}(\mathbb{R},X)}.$$

Proof. It is a direct consequence of the closed graph theorem. □

4 | APPLICATIONS

In this section, we provide some examples to illustrate our results.

Example 4.1. Let $\alpha, \beta, \gamma, \delta, \rho \in \mathbb{R}^+$ such that $0 < \rho < \alpha\delta - \beta\gamma$ and $(\beta\delta - \alpha\rho)(\alpha\gamma - \delta) > 0$. Let $X = C_0(\Omega) := \{u \in C(\bar{\Omega}) : u|_{\partial\Omega} = 0\}$ with $\Omega \subset \mathbb{R}^d$ Dirichlet regular; see Arendt et al.[19, Theorem 6.1.9] We consider the fourth-order MGT equation given by

$$u''''(t) + \alpha u'''(t) + \beta u''(t) - \gamma\Delta u''(t) - \delta\Delta u'(t) - \rho\Delta u(t) = f(t), \ t \in \mathbb{R}, \tag{4.1}$$

where $A = \Delta$ denotes the Dirichlet Laplacian operator in X. See Dell' Oro and Pata and Liu et al[11,15] for the details about the physical meaning of the parameters involved in the equation. Also, it is well-known that $0 \in \rho(A)$ and $\sigma(A) \subseteq (-\infty, 0)$; see Haase.[29] After some calculations, we get

$$h(s) = \frac{(is)^4 + \alpha(is)^3 + \beta(is)^2}{\gamma(is)^2 + \delta(is) + \rho} = \frac{(s^4 - \alpha is^3 - \beta s^2)(\rho - \gamma s^2 - \delta si)}{|(\rho - \gamma s^2) + \delta si|^2}.$$

Therefore, the identity

$$\mathfrak{Re}\, h(s) = \frac{-\gamma s^6 + (\rho - \alpha\delta + \beta\gamma)s^4 - \beta\rho s^2}{|(\rho - \gamma s^2) + \delta si|^2} \tag{4.2}$$

shows that $\mathfrak{Re}\, h(s) < 0$ for $s \in \mathbb{R}\setminus\{0\}$ since $\rho < \alpha\delta - \beta\gamma$. Furthermore,

$$\mathfrak{Im}\, h(s) = \frac{(\alpha\gamma - \delta)s^5 + s^3(\beta\delta - \alpha\rho)}{|(\rho - \gamma s^2) + \delta si|^2} \neq 0, \tag{4.3}$$

for $s \in \mathbb{R}\setminus\{0\}$ since $(\beta\delta - \alpha\rho)(\alpha\gamma - \delta) > 0$. Let $\{\mu_k\}_{k\in\mathbb{N}}$ be the spectrum of A with $0 > \mu_1 \geq \ldots \geq \mu_k \to \infty$ as $k \to \infty$. Then

$$\begin{aligned} \sup_{s\in\mathbb{R}} ||(is)^4 g(s)(h(s) - \Delta)^{-1}|| &\geq \sup_{\{s\in\mathbb{R} : \mathfrak{Re}\, h(s)=\mu_k, k\in\mathbb{N}\}} ||(is)^4 g(s)(h(s) - \Delta)^{-1}|| \\ \geq \sup_{\{s\in\mathbb{R} : \mathfrak{Re}\, h(s)=\mu_k, k\in\mathbb{N}\}} \frac{|(is)^4 g(s)|}{\text{dist}(h(s), \sigma(\Delta))} &= \sup_{\{s\in\mathbb{R} : \mathfrak{Re}\, h(s)=\mu_k, k\in\mathbb{N}\}} \left| \frac{(is)^4 g(s)}{\mathfrak{Im}\, h(s)} \right| = \infty. \end{aligned}$$

See Engel and Nagel[30, Proposition 1.3, Chapter IV, p.240] for the last equality. We can assert that Equation (3.1) is C^α-ill-posed as a consequence of Theorem 3.4.

Example 4.2. In this example, we show C^α-well-posedness for Equation (3.1) when A is the closed linear operator on the space of sequences $\ell^2(\mathbb{N})$ defined as

$$(Au)_n = nu_n, \qquad D(A) = \{(u_n) \in \ell^2(\mathbb{N}) : (n \cdot u_n) \in l^2(\mathbb{N})\}.$$

Assume $\alpha = \beta = 2$, $\gamma = \delta = 1$, and $\rho < \frac{1}{2}$. It is not difficult to check that

$$h(s) = \frac{-s^6 + \rho s^4 - 2\rho s^2}{|(\rho - s^2) + si|^2} + i\frac{s^5 + s^3(2 - \rho)}{|(\rho - s^2) + si|^2}$$

and then $h(s) \in \rho(A)$ for all $s \in \mathbb{R}$ since $\mathfrak{Im}\, h(s) \neq 0$ for all $s \neq 0$ due to the fact that $\rho < \frac{1}{2}$. Let now $x = (x_n) \in \ell^2(\mathbb{N})$ and $s \neq 0$, then there exists $M > 0$ constant so that:

$$\begin{aligned} ||(is)^4 g(s)(h(s) - A)^{-1}x||^2 &= ||(is)^4((s^4 - 2is^3 - 2s^2) - \frac{1}{g(s)}A)^{-1}x||^2 \\ &= \sum_{n=1}^{\infty} \left| \frac{s^4}{(s^4 - 2is^3 - 2s^2) - \frac{n}{g(s)}} x_n \right|^2 = \sum_{n=1}^{\infty} \frac{s^8}{(s^4 + (n-2)s^2 - \rho n)^2 + (2s^3 + ns)^2} |x_n|^2 \\ &= \sum_{n=1}^{\infty} \frac{s^8}{s^8 + 4s^6 + (n^2 + 2n(1-\rho) + 4)s^4 + (n^2(1-2\rho) + 4n\rho)s^2 + n^2\rho^2} |x_n|^2 < M \end{aligned}$$

using that $\rho < \frac{1}{2}$. Finally, since $s \to (is)^4 g(s)(h(s) - A)^{-1}$ is continuous in $s = 0$, we can then conclude that

$$\sup_{s \in \mathbb{R}} ||(is)^4 g(s)(h(s) - A)^{-1}|| < \infty.$$

Hence, condition (ii) in Theorem 3.4 holds, and C^α-well-posedness of Equation (3.1) follows immediately.

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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