Hausdorff operators on weighted Banach spaces of type H^∞

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Abstract

Some criteria for the continuity of Hausdorff operators on weighted Banach spaces of analytic functions with sup-norms are presented. The operator is defined in a different way on spaces of entire functions and on spaces of analytic functions on the disc. Both cases are analyzed. Our results complement recent work by Stylogiannis and Galanopoulos, and by Mirotin.

1 Introduction.

The aim of this note is to study continuous Hausdorff operators on weighted Banach spaces of holomorphic functions of type H^∞, both in the case of spaces of entire functions, in which the operator is defined as in [23], and in the case of the disc, where it is defined as in [21]. See the definitions below. In fact, these two papers are the source of motivation and inspiration for our results below. Our main results are Theorem 2.3, Corollary 2.4 and Theorem 3.1. Connections with multiplier operators and composition operators are exhibited.

We describe first the spaces where the operators are defined. Let G be the unit disc $\mathbb{D}$ or the whole complex plane $\mathbb{C}$. We set $R = 1$ for the case of holomorphic functions on the unit disc, and $R = +\infty$ for the case of entire functions. A *weight* v is a continuous function $v : [0, R) \to (0, \infty)$, which is non-increasing on $[0, R)$ and satisfies $\lim_{r\to R} r^n v(r) = 0$ for each $n \in \mathbb{N}$. We extend v to $\mathbb{D}$ if $R = 1$ and to $\mathbb{C}$ if $R = +\infty$ by $v(z) := v(|z|)$. For such a weight v, we define the following *weighted Banach spaces of holomorphic functions* on G

$$H_v^\infty(G) := \{f \in H(G);\ ||f||_v := \sup_{z\in G} v(z)|f(z)| < \infty\},$$

$$H_v^0(G) := \{f \in H(G);\ v|f| \text{ vanishes at infinity on } G\},$$

endowed with the norm $\|.\|_v$. A function g vanishes at infinity on G if for every $\varepsilon > 0$ there is a compact subset K of G such that $|g(z)| < \varepsilon$ if $z \notin K$. If G is an open subset of $\mathbb{C}$, we denote by $H(G)$ the Fréchet space of all holomorphic functions on G endowed with the topology τ_{co} of uniform convergence on the compact subsets of G.

For an analytic function $f \in H(\{z \in \mathbb{C}; |z| < R\})$ and $r < R$, we denote $M(f, r) := \max\{|f(z)|\ ;\ |z| = r\}$. Using the notation O and o of Landau, $f \in H_v^\infty(G)$ if and only if $M(f, r) = O(1/v(r)), r \to R$. Polynomials are contained in $H_v^0(G)$ and the closure of the polynomials in $H_v^\infty(G)$ coincides with $H_v^0(G)$, see e.g. [2].

We recall some examples of weights:

For $R = 1$,

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(i) $v(r) = (1-r^2)^\gamma$ with $\gamma > 0$, which are the so-called standard weights on the disc, for which $H_v^\infty(\mathbb{D})$ are Korenblum type growth spaces; see Section 4.3 in [15].
(ii) $v(r) = \exp(-b(1-r)^{-a}), a, b > 0$, which are called exponential weights, and
(iii) $v(z) = (\log \frac{e}{1-r^2})^{-\alpha}$, $\alpha > 0$, which are called logarithmic weights.

For $R = +\infty$,
(i) $v(r) = \exp(-r^\alpha)$ with $\alpha > 0$,
(ii) $v(r) = \exp(-\exp(r))$, and
(iii) $v(r) = \exp\big(-(\log r)^\alpha\big)$, where $\alpha > 1$.

Banach spaces of the type mentioned above appear naturally in the study of growth conditions of analytic functions and have been considered in many papers. We refer to [1, 2, 5] and the references therein. Lusky [19] obtained the isomorphic classification of these spaces. The space $H_v^\infty(\mathbb{C})$ is denoted in [11] as the general weighted Fock space $\mathcal{F}_\infty^\phi$ of order infinity (i.e. with sup-norms) with $v(z) = \exp(-\phi(|z|))$, and $\phi : [0, \infty[\rightarrow]0, \infty[$ is a twice continuously differentiable increasing function. See [25] for Fock spaces. We refer the reader to [15, 24, 25] for unexplained notation. In what follows $\mathbb{N}$ stands for the natural numbers and we set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

2 Hausdorff operators on weighted Banach spaces of entire functions.

Let μ be a positive measure on $(0, \infty)$. Stylogiannis and Galanopoulos [23] consider formally the *Hausdorff operator induced by the measure* μ defined by

$$\mathcal{H}_\mu(f)(z) := \int_0^\infty \frac{1}{t} f\Big(\frac{z}{t}\Big) d\mu(t), \quad z \in \mathbb{C},$$

where $f \in H(\mathbb{C})$ is an entire function. The operator $\mathcal{H}_\mu$ is studied in [23] on Fock spaces $\mathcal{F}_\alpha^p, 1 \le p \le \infty, \alpha > 0$. This operator is more general than the one considered in [16] for Hardy spaces of the upper half plane, and was first studied in [22] for analytic Bergman spaces of the upper half plane. See also [10, 14, 22]. Here we present some results about the behaviour of $\mathcal{H}_\mu$ when it acts on the weighted spaces $H_v^\infty(\mathbb{C})$ and $H_v^0(\mathbb{C})$.

Proposition 2.1 *Let v be a radial weight on $\mathbb{C}$.*

(1) If the operator $\mathcal{H}_\mu : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ is continuous, then

$$\sup_{n \in \mathbb{N}_0} \int_0^\infty \frac{1}{t^{n+1}} d\mu(t) \le \|\mathcal{H}_\mu\| < \infty, \tag{2.1}$$

and the operator $\mathcal{H}_\mu : H_v^0(\mathbb{C}) \to H_v^0(\mathbb{C})$ is also continuous.

(2) If the operator $\mathcal{H}_\mu : H_v^0(\mathbb{C}) \to H_v^0(\mathbb{C})$ is continuous, then (2.1) holds.

(3) If the operator $\mathcal{H}_\mu : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ is compact, then

$$\lim_{k \to \infty} \int_0^\infty \frac{1}{t^k} d\mu(t) = 0.$$

Proof. (1) Set $g_m(z) := z^m, m \in \mathbb{N}_0$. Since $g_m \in H_v^0(\mathbb{C}) \subset H_v^\infty(\mathbb{C})$ and $\mathcal{H}_\mu$ is well defined and continuous on $H_v^\infty(\mathbb{C})$, we conclude that the function

$$\mathcal{H}_\mu(g_m) = \Big(\int_0^\infty \frac{1}{t^{m+1}} d\mu(t)\Big) z^m$$

belongs to $H_v^\infty(\mathbb{C})$. Moreover, for each $m \in \mathbb{N}_0$,

$$\|\mathcal{H}_\mu(g_m)\|_v = \Big(\int_0^\infty \frac{1}{t^{m+1}} d\mu(t)\Big)\|z^m\|_v \le \|\mathcal{H}_\mu\| \; \|z^m\|_v.$$

This implies

$$\sup_{n\in\mathbb{N}_0} \int_0^\infty \frac{1}{t^{n+1}} d\mu(t) \le \|\mathcal{H}_\mu\| < \infty.$$

As the polynomials are dense in H_v^0 and the continuous operator $\mathcal{H}_\mu$ maps polynomials into polynomials, we conclude that $\mathcal{H}_\mu(H_v^0) \subset H_v^0$.

(2) follows with the same argument, since the polynomials are contained in $H_v^0(\mathbb{C})$.

(3) The sequence $h_m := g_m/\|z^m\|_v$, $m \in \mathbb{N}_0$, is clearly bounded and tends to zero uniformly on compact subsets of $\mathbb{C}$. Indeed, fix $R > 0$, we have $\|z^m\|_v = \sup_{r\ge 0} r^m v(r) \ge (2R)^m v(2R)$. Then, for each $z \in \mathbb{C}$, $|z| \le R$, we get $|h_m(z)| \le (1/2)^m/v(2R)$ for each $m \in \mathbb{N}$, which yields $\lim_{m\to\infty} \sup_{|z|\le R} |h_m(z)| = 0$.

If the operator $\mathcal{H}_\mu : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ is compact, then condition (2.1) holds and the image of the unit ball of $H_v^\infty(\mathbb{C})$ is relatively compact in $H_v^\infty(\mathbb{C})$, hence the norm topology and the weaker Hausdorff topology of uniform convergence on compact subsets of $\mathbb{C}$ coincide on this image. Now, the sequence $(\mathcal{H}_\mu(h_m))_m$ tends to zero uniformly on the compact subsets, since, for each $m \in \mathbb{N}_0$, we have

$$\mathcal{H}_\mu(h_m) = \Big(\int_0^\infty \frac{1}{t^{m+1}} d\mu(t)\Big) h_m,$$

and condition (2.1) holds. Therefore $(\mathcal{H}_\mu(h_m))_m$ tends to zero in $H_v^\infty(\mathbb{C})$. Clearly,

$$\|\mathcal{H}_\mu(h_m)\|_v = \int_0^\infty \frac{1}{t^{m+1}} d\mu(t),$$

which completes the proof. □

Lemma 2.2 *(1) Condition (2.1) implies*

$$\sup_{n\in\mathbb{N}_0} \int_0^\infty \frac{1}{t^{\gamma n+1}} d\mu(t) < \infty,$$

for each $\gamma > 0$.

(2) If the sequence $\big(\int_0^\infty \frac{1}{t^k} d\mu(t)\big)_{k\in\mathbb{N}}$ *tends to zero, then*

$$\lim_{\beta\to\infty} \int_0^\infty \frac{1}{t^\beta} d\mu(t) = 0.$$

Proof. (1) Set

$$M(\mu) := \sup_{n\in\mathbb{N}_0} \int_0^\infty \frac{1}{t^{n+1}} d\mu(t) < \infty,$$

and fix $\gamma > 0$. If $n = 0$, then $\int_0^\infty \frac{1}{t^{\gamma n+1}} d\mu(t) = \int_0^\infty \frac{1}{t} d\mu(t) \le M(\mu)$. We consider now the interval $[1,\infty)$. Since $\gamma n + 1 > 1$ for each $n \ge 1$, we have $\frac{1}{t^{\gamma n+1}} \le \frac{1}{t}$ for each $t \in [1,\infty)$ and $\int_1^\infty \frac{1}{t^{\gamma n+1}} d\mu(t) \le M(\mu)$ for each $n \ge 1$. Now in the interval $(0,1)$ we proceed as follows. Given $n \ge 1$, select $k \in \mathbb{N}$ such that $\gamma n + 1 \le k + 1$. We have $\frac{1}{t^{\gamma n+1}} \le \frac{1}{t^{k+1}}$ for each $t \in (0,1)$. This yields

$$\int_0^1 \frac{1}{t^{\gamma n+1}} d\mu(t) \le \int_0^1 \frac{1}{t^{k+1}} d\mu(t) \le M(\mu).$$

Therefore, for each $n \in \mathbb{N}_0$, we get

$$\int_0^\infty \frac{1}{t^{\gamma n+1}} d\mu(t) = \int_0^1 \frac{1}{t^{\gamma n+1}} d\mu(t) + \int_1^\infty \frac{1}{t^{\gamma n+1}} d\mu(t) \le 2M(\mu).$$

(2) The proof is similar to part (1). □

Theorem 2.3 *Let $\alpha > 0$ and $\beta > 0$. Let v be the weight on $\mathbb{C}$ defined by $v(r) = \exp(-\beta r^\alpha)$. The following conditions are equivalent.*

(i) $\mathcal{H}_\mu : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ *is continuous.*

(ii) $\mathcal{H}_\mu : H_v^0(\mathbb{C}) \to H_v^0(\mathbb{C})$ *is also continuous.*

(iii) $\sup_{n\in\mathbb{N}} \int_0^\infty \frac{1}{t^{n+1}} d\mu(t) < \infty$.

In this case, we have

$$\|\mathcal{H}_\mu\| \le \sup_{n\in\mathbb{N}} \int_0^\infty \frac{1}{t^{n\alpha+1}} d\mu(t).$$

Proof. Proposition 2.1 ensures that (i) implies (ii) and (ii) implies (iii).

Let us assume condition (iii) holds. By Remark 2.2,

$$M(\mu) := \sup_{n\in\mathbb{N}} \int_0^\infty \frac{1}{t^{\alpha n+1}} d\mu(t) < \infty.$$

We show that $\mathcal{H}_\mu : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ is well defined and continuous. In particular, we must prove that $\mathcal{H}_\mu(f)$ is an entire function for each $f \in H_v^\infty(\mathbb{C})$. To do this we apply the Theorem in Mattner [20] and set $F(z,t) := 1/tf(z/t)$, $t \in (0,\infty)$, $z \in \mathbb{C}$, for $f \in H_v^\infty(\mathbb{C})$. For each $z \in \mathbb{C}$ the function $F(z,.)$ is continuous, hence μ-measurable, and $F(.,t)$ is an entire function for each $t \in (0,\infty)$. We have to prove that for each $z_0 \in \mathbb{C}$ there is $\delta > 0$ such that

$$\sup_{|z-z_0|<\delta} \int_0^\infty \frac{1}{t}\Big|f\Big(\frac{z}{t}\Big)\Big| d\mu(t) < \infty.$$

This will follow from our estimates below.

Given an arbitrary function $f \in H_v^\infty(\mathbb{C})$ we have

$$|f(\zeta)| \le \exp(\beta|\zeta|^\alpha)\|f\|_v, \qquad \zeta \in \mathbb{C}.$$

Therefore, for each $z \in \mathbb{C}$ and $t \in (0,\infty)$, we get

$$\frac{1}{t}\Big|f\Big(\frac{z}{t}\Big)\Big| \le \frac{1}{t}\exp\Big(\beta\Big(\frac{|z|}{t}\Big)^\alpha\Big)\|f\|_v = \sum_{n=0}^\infty \frac{1}{n!}\frac{\beta^n|z|^{n\alpha}}{t^{n\alpha+1}}\|f\|_v.$$

Then

$$\int_0^\infty \frac{1}{t}\Big|f\Big(\frac{z}{t}\Big)\Big| d\mu(t) \le \|f\|_v \int_0^\infty \sum_{n=0}^\infty \frac{1}{n!}\frac{\beta^n|z|^{n\alpha}}{t^{n\alpha+1}} d\mu(t) =$$

$$\|f\|_v \sum_{n=0}^\infty \frac{1}{n!}\Big(\int_0^\infty \frac{1}{t^{n\alpha+1}} d\mu(t)\Big)\beta^n|z|^{n\alpha} \le M(\mu)\|f\|_v \sum_{n=0}^\infty \frac{1}{n!}(\beta|z|^\alpha)^n = M(\mu)\|f\|_v \exp(\beta|z|^\alpha).$$

This implies that the operator is well defined and

$$\|\mathcal{H}_\mu(f)\| = \sup_{z\in\mathbb{C}} |\mathcal{H}_\mu(f)(z)|\exp(-\beta|z|^\alpha) \le M(\mu)\|f\|_v.$$

□

Theorem 2.3 for $\alpha = 2$ is proved in [23]. Our proof is close to the one given there.

The monomials are not a Schauder basis of the Banach space $H_v^0(\mathbb{C})$ for $v(r) = \exp(-r^\alpha), r > 0$, by Theorem 2.3 in Lusky [18]. This is why to see that $\mathcal{H}_\mu$ acts as a multiplier on $H_v^\infty(\mathbb{C})$ requires a different proof below.

Corollary 2.4 *Let $\alpha > 0$ and $\beta > 0$. Let v be the weight on $\mathbb{C}$ defined by $v(r) = \exp(-\beta r^\alpha)$. If $\sup_{n\in\mathbb{N}} \int_0^\infty \frac{1}{t^{n+1}} d\mu(t) < \infty$, then the continuous operator $\mathcal{H}_\mu : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ acts on the Taylor expansion as a multiplier; that is,*

$$\mathcal{H}_\mu\Big(\sum_{n=0}^\infty a_n z^n\Big) = \sum_{n=0}^\infty a_n\Big(\int_0^\infty \frac{1}{t^{n+1}} d\mu(t)\Big) z^n.$$

Proof. By the Theorem in [20], see consequence (C3), for each $f \in H_v^\infty(\mathbb{C})$ and each $n \in \mathbb{N}_0$, we have

$$\Big(\frac{d^n}{d\zeta^n}\mathcal{H}_\mu(f)\Big)(z) = \int_0^\infty \frac{1}{t}\frac{\partial^n}{\partial\zeta^n}\Big|_{\zeta=z} f\Big(\frac{\zeta}{t}\Big) d\mu(t) = \int_0^\infty \frac{1}{t^{n+1}} f^{(n)}\Big(\frac{z}{t}\Big) d\mu(t).$$

Hence, the entire function $\mathcal{H}_\mu(f)$ satisfies for each $n \in \mathbb{N}_0$

$$\frac{1}{n!}\mathcal{H}_\mu(f)^{(n)}(0) = \Big(\int_0^\infty \frac{1}{t^{n+1}} d\mu(t)\Big)\frac{f^{(n)}(0)}{n!}.$$

Consequently,

$$\mathcal{H}_\mu\Big(\sum_{n=0}^\infty a_n z^n\Big) = \sum_{n=0}^\infty a_n\Big(\int_0^\infty \frac{1}{t^{n+1}} d\mu(t)\Big) z^n$$

for each $f(z) = \sum_{n=0}^\infty a_n z^n \in H_v^\infty(\mathbb{C})$. □

It is important to point out that not all multipliers

$$T_\gamma\Big(\sum_{n=0}^\infty a_n z^n\Big) = \sum_{n=0}^\infty a_n \gamma_n z^n,$$

with $\gamma \in \ell_\infty$ define a continuous operator from $H_v^\infty(\mathbb{C})$ into itself. This question is related to the solid hull and core of the space $H_v^\infty(\mathbb{C})$ which was investigated for spaces of type H_v^∞ in [6, 7, 9].

Lemma 2.5 *Let v be a radial weight on $\mathbb{C}$ or $\mathbb{D}$. If $m = (m_n)_n \in \ell_1$, then the operators $T_m : H_v^\infty \to H_v^\infty$ and $T_m : H_v^0 \to H_v^0$ given by $T_m(\sum_{n=0}^\infty a_n z^n) = \sum_{n=0}^\infty m_n a_n z^n$ are well defined, continuous, their norm satisfies $\|T_m\| \le \sum_n |m_n|$, and they are also compact.*

Proof. It is enough to show that $T_m : H_v^\infty \to H_v^\infty$ is well defined and continuous, since T_m maps polynomials into polynomials and the subspace of polynomials is dense in H_v^0. Clearly $T_m(f)$ is an entire function for each $f \in H_v^\infty$.

Cauchy inequalities imply $|a_n|\|z^n\|_v \le \|f\|_v$ for each $f = \sum_{n=0}^\infty a_n z^n \in H_v^\infty$ and each $n \in \mathbb{N}_0$, see e.g. the proof of Theorem 2.3 in [4]. This implies

$$\|\sum_{n=0}^\infty m_n a_n z^n\|_v \le \Big(\sum_{n=0}^\infty |m_n|\Big)\sup_n |a_n|\|z^n\|_v \le \Big(\sum_{n=0}^\infty |m_n|\Big)\|f\|_v.$$

This proves the continuity and the estimate of the norm.

To show compactness, for each $k \in \mathbb{N}$, define the operator $T_{m,k}$ on H_v^∞ by

$$T_{m,k}(\sum_{n=0}^{\infty} a_n z^n) = \sum_{n=0}^{k} m_n a_n z^n.$$

This operator is continuous and has finite rank. Moreover, for each $f \in H_v^\infty$, we have

$$\|(T_m - T_{m,k})(f)\|_v \leq \Big(\sum_{n=k+1}^{\infty} |m_n| \Big) \|f\|_v,$$

and T_m is the limit in the operator norm of a sequence of finite rank operators, hence it is compact. □

Proposition 2.6 *Let $\alpha > 0$ and $\beta > 0$. Let v be the weight on $\mathbb{C}$ defined by $v(r) = \exp(-\beta r^\alpha)$. If the sequence $\big(\int_0^\infty \frac{1}{t^k} \, d\mu(t) \big)_{k \in \mathbb{N}_0}$ is in ℓ_1, then the operator $\mathcal{H}_\mu$ is compact on $H_v^\infty(\mathbb{C})$ and on $H_v^0(\mathbb{C})$.*

Proof. This is a consequence of Corollary 2.4 and Lemma 2.5. □

Observe that one cannot use the argument of the proof of Theorem 3.1 in [23] to prove compactness of $\mathcal{H}_\mu$, because the monomials are not a Schauder basis of the Banach space $H_v^0(\mathbb{C})$ for $v(r) = \exp(-r^\alpha), r > 0$, by Theorem 2.3 in Lusky [18], as it was mentioned above. On the other hand, Corollary 2.6 in [18] implies that the monomials are indeed a Schauder basis of $H_v^0(\mathbb{C})$ for $v(r) = \exp(-(\log r)^2), r > 0$.

If the monomials are a Schauder basis of $H_v^0(\mathbb{C})$, then Theorem 5.2 in [7] implies that the spaces $H_v^\infty(\mathbb{C})$ and $H_v^0(\mathbb{C})$ are solid, that is, if an entire function $\sum_{n=0}^\infty a_n z^n$ belongs to one of these spaces and $|b_n| \leq |a_n|, \ n \in \mathbb{N}_0$, then $\sum_{n=0}^\infty b_n z^n$ is also in the same space. We have the following results. The first one improves Lemma 2.5 for certain radial weights on $\mathbb{C}$.

Proposition 2.7 *Let v be the radial weight on $\mathbb{C}$ defined by $v(r) = \exp(-(\log r)^2), r > 0$. If $m = (m_n)_n \in \ell_\infty$, then the operators $T_m : H_v^\infty \to H_v^\infty$ and $T_m : H_v^0 \to H_v^0$, given by $T_m(\sum_{n=0}^\infty a_n z^n) = \sum_{n=0}^\infty m_n a_n z^n$, are well defined and continuous. Moreover T_m is compact if and only if $m \in c_0$.*

Proof. The monomials are a Schauder basis of $H_v^0(\mathbb{C})$ by Corollary 2.6 in [18]. We can apply Theorem 5.2 in [7] to conclude that $H_v^\infty(\mathbb{C})$ and $H_v^0(\mathbb{C})$ are solid. Then, if $m \in \ell_\infty$, T_m is continuous on $H_v^\infty(\mathbb{C})$ and $H_v^0(\mathbb{C})$ by the closed graph theorem.

By Theorem 2.5 in [18] there is $d > 0$ such that for all $f(z) = \sum_{n=0}^\infty a_n z^n \in H_v^\infty(\mathbb{C})$ we have

$$\sup_{n \in \mathbb{N}_0} |a_n| \exp(n^2/4) \leq \|f\|_v \leq d \sup_{n \in \mathbb{N}_0} |a_n| \exp(n^2/4).$$

This implies $\|T_m(f)\|_v \leq d \ (\sup_{n \in \mathbb{N}_0} |m_n|) \ \|f\|_v$ for each $f \in H_v^\infty(\mathbb{C})$. Define, for $k \in \mathbb{N}$,

$$T_{m,k}(\sum_{n=0}^{\infty} a_n z^n) = \sum_{n=0}^{k} m_n a_n z^n, \quad \sum_{n=0}^{\infty} a_n z^n \in H_v^\infty(\mathbb{C}).$$

Assume that $m \in c_0$. By the argument given above, we have $\|(T - T_{m,k})(f)\|_v \leq d \ (\sup_{n \geq k} |m_n|) \ \|f\|_v$ for each $f \in H_v^\infty(\mathbb{C})$. Therefore T_m is compact, since it is the limit of a sequence of finite rank operators in the operator norm.

Conversely, if T_m is compact, then the image of the bounded sequence $(z^n/\|z^n\|_v)_n$, which tends to zero uniformly on compact sets, satisfies that

$$(T_m(z^n/\|z^n\|_v))_n = (m_n z^n/\|z^n\|_v)_n$$

tends also to zero uniformly on compact sets, as $m \in \ell_\infty$, hence it tends to zero in norm too. This implies that $m \in c_0$, because $\|T_m(z^n/\|z^n\|_v)\|_v = |m_n|$ for each $n \in \mathbb{N}_0$. □

Proposition 2.8 *Let v be a radial weight on $\mathbb{C}$ such that the monomials are a Schauder basis of $H_v^0(\mathbb{C})$. If $\sup_{n\in\mathbb{N}} \int_0^\infty \frac{1}{t^{n+1}} d\mu(t) < \infty$, then the operator*

$$T\Big(\sum_{n=0}^{\infty} a_n z^n\Big) = \sum_{n=0}^{\infty} \Big(\int_0^\infty \frac{1}{t^{n+1}}\Big) a_n z^n,$$

is continuous on $H_v^\infty(\mathbb{C})$ and $H_v^0(\mathbb{C})$. Moreover, $T(p)(z) = \mathcal{H}_\mu(p)(z)$ for each polynomial $p(z)$, and the operator $\mathcal{H}_\mu$ has a unique continuous linear extension from the polynomials to $H_v^0(\mathbb{C})$.

Proof. The continuity of T follows again from Theorem 5.2 in [7], which ensures that $H_v^\infty(\mathbb{C})$ and $H_v^0(\mathbb{C})$ are solid. □

Example 2.9 (1) Let χ be the characteristic function of $(1, \infty)$. Consider the measure $d\mu(t) = \chi(t)\frac{1}{t}dt$, where dt is the Lebesgue measure. If $f \in H(\mathbb{C})$ is an entire function, the Hausdorff operator in this case satisfies

$$\mathcal{H}_\mu(f)(z) = \int_0^\infty \frac{1}{t} f\Big(\frac{z}{t}\Big) d\mu(t) = \int_1^\infty \frac{1}{t^2} f\Big(\frac{z}{t}\Big) dt = \int_0^1 f(sz)ds = \frac{1}{z}\int_0^z f(\zeta)d\zeta,$$

hence it coincides with the Hardy operator H, which was investigated for spaces of type $H_v^\infty(\mathbb{C})$ in Theorem 3.12 in [1]. It is shown there that H is continuous on $H_v^\infty(\mathbb{C})$ and that $H^2 = H \circ H$ is compact.

(2) Let v be the weight on $\mathbb{C}$ defined by $v(r) = \exp(-\beta r^\alpha)$ with $\alpha \geq 1$ and $\beta > 0$. Let $g \in H(\mathbb{C})$ be an entire function. According to Corollary 3.12 in [8], the Volterra operator

$$V_g(f)(z) := \int_0^z f(\zeta)g'(\zeta)d\zeta, \quad z \in \mathbb{C},$$

is continuous on $H_v^\infty(\mathbb{C})$ if and only if g is a polynomial of degree less or equal the integer part $[\alpha]$ of α. The Volterra operator is also related to the Hausdorff operator. Indeed, if $n \leq [\alpha]$ and $g(z) = z^n$, we have

$$V_g(f)(z) = n\int_0^z f(\zeta)\zeta^{n-1}d\zeta = nz^n \int_1^\infty \frac{1}{t} f\Big(\frac{z}{t}\Big)\frac{dt}{t^n} = nz^n \mathcal{H}_\mu(f)(z),$$

for the measure $d\mu(t) = \chi(t)(1/t^n)dt$.

3 Hausdorff operators on weighted Banach spaces of analytic functions on the unit disc.

Let μ be a positive Radon measure on the unit disc $\mathbb{D}$ and let K be a μ-measurable function on $\mathbb{D}$. For $w \in \mathbb{D}$, we denote by φ_w the automorphism of the disc defined by

$$\varphi_w(z) = \frac{w - z}{1 - \overline{w}z}, \quad z \in \mathbb{D}.$$

Mirotin [21] defines the *Hausdorff operator associated with* μ *and* K on the disc $\mathbb{D}$ by

$$\mathcal{H}_{K,\mu}(f)(z) := \int_{\mathbb{D}} K(w) f(\varphi_w(z)) d\mu(w), \qquad z \in \mathbb{D},$$

for an analytic function $f \in H(\mathbb{D})$ on the unit disc. He obtained conditions to ensure that the operator acts continuously on the Bloch space, Bergman spaces and Hardy spaces. Similarly defined operators for the Möbius invariant area measure were investigated on some spaces of integrable functions by Karapetyants, Samko and Zhu in [17]. Our purpose in this section is to add a few results about the continuity of the operator $\mathcal{H}_{K,\mu}$ when it acts on spaces of type $H_v^\infty(\mathbb{D})$ and $H_v^0(\mathbb{D})$. It turns out that this question is related to the continuity of the composition operators $C_{\varphi_w} : H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ for the automorphism φ_w, $w \in \mathbb{D}$. By Proposition 2.1 in [5] this composition operator is continuous if and only if

$$\sup_{z \in \mathbb{D}} \frac{\tilde{v}(z)}{\tilde{v}(\varphi_w(z))} < \infty.$$

In this case, the norm of the operator satisfies

$$\|C_{\varphi_w}\| = \sup_{z \in \mathbb{D}} \frac{\tilde{v}(z)}{\tilde{v}(\varphi_w(z))} < \infty.$$

Here $\tilde{v}$ is the *associated weight* to v in the sense of [3], which is defined by

$$\tilde{v}(z) := 1/||\delta_z||_{H_v^\infty(G)'}.$$

The associated weight $\tilde{v}$ is radial, continuous, decreasing $H_v^\infty(G) = H_{\tilde{v}}^\infty(G)$, $H_v^0(G) = H_{\tilde{v}}^0(G)$ and $\|f\|_v = \|f\|_{\tilde{v}}$ for each $f \in H_v^\infty(G)$. All the examples of weights on the unit disc mentioned in the introduction satisfy that there is $C > 0$ such that $v(z) \le \tilde{v}(z) \le Cv(z)$ for each $z \in \mathbb{D}$. Observe that if C_{φ_w} is continuous, then $\|C_{\varphi_w}\| \ge \tilde{v}(0)/\tilde{v}(w) \ge 1$, hence $\sup_{w \in \mathbb{D}} \|C_{\varphi_w}\| = \infty$, since $\lim_{r \to 1} \tilde{v}(r) = 0$. In particular, the spaces $H_v^\infty(\mathbb{D})$ and $H_v^0(\mathbb{D})$ are not Möbius invariant in the sense of [13].

As a consequence of Theorem 2.3 in [5] and its proof, the operator C_{φ_w} is continuous on $H_v^\infty(\mathbb{D})$ for all $w \in \mathbb{D}$ if and only if the weight v satisfies the following condition:

$$(*) \qquad \sup_n \frac{\tilde{v}(1 - 2^{-n})}{\tilde{v}(1 - 2^{-n-1})} < \infty.$$

Standard and logarithmic weights satisfy this condition, but exponential weight do not.

Theorem 3.1 *Let* v *be a radial weight on* $\mathbb{D}$ *satisfying condition* $(*)$. *If the function* $w \in \mathbb{D} \to K(w)\|C_{\varphi_w}\|$ *belongs to* $L^1(\mu)$, *then the operators*

$$\mathcal{H}_{K,\mu} : H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D}),$$

and

$$\mathcal{H}_{K,\mu} : H_v^0(\mathbb{D}) \to H_v^0(\mathbb{D}),$$

are continuous. In this case, we have

$$\|\mathcal{H}_{K,\mu}\| \le \int_{\mathbb{D}} |K(w)| \, \|C_{\varphi_w}\| d\mu(w).$$

Proof. First we prove the result for the operator acting on $H_v^\infty(\mathbb{D})$. To do this, we use again Mattner's Theorem in [20] to ensure that $\mathcal{H}_{K,\mu}(f)$ is an analytic function for each $f \in H_v^\infty(\mathbb{D})$. We may assume that $K(w)$ is defined for all $w \in \mathbb{D}$. Put $F(z,w) := K(w)f(\varphi_w(z)),\ z,w \in \mathbb{D}$, for $f \in H_v^\infty(\mathbb{D})$. For each $z \in \mathbb{D}$, the function $F(z,.)$ is μ-measurable, and for each $w \in \mathbb{D}$, the function $F(.,w)$ is analytic on $\mathbb{D}$. We must show that for each $z_0 \in \mathbb{D}$ there is $\delta > 0$ such that

$$\sup_{|z-z_0|<\delta} \int_{\mathbb{D}} |K(w)| f(\varphi_w(z)| d\mu(w) < \infty.$$

Since the function $w \in \mathbb{D} \to K(w)\|C_{\varphi_w}\|$ belongs to $L^1(\mu)$, we can estimate as follows, for each $z \in \mathbb{D}$,

$$\int_{\mathbb{D}} |K(w)| f(\varphi_w(z)| d\mu(w) = \int_{\mathbb{D}} |K(w)| \tilde{v}(\varphi_w(z)) |f(\varphi_w(z))| \frac{1}{\tilde{v}(\varphi_w(z))} d\mu(w) \leq$$

$$\leq \frac{||f||_v}{\tilde{v}(z)} \Big(\int_{\mathbb{D}} |K(w)| \|C_{\varphi_w}\| d\mu(w) \Big).$$

Therefore $\mathcal{H}_{K,\mu}(f)$ is indeed an analytic function and moreover

$$\|\mathcal{H}_{K,\mu}(f)\|_v \leq \Big(\int_{\mathbb{D}} |K(w)| \ \|C_{\varphi_w}\| d\mu(w) \Big) \|f\|_v, \qquad f \in H_v^\infty(\mathbb{D}).$$

This implies the statement for $H_v^\infty(\mathbb{D})$.

To complete the proof it is enough to show that our assumptions imply $\mathcal{H}_{K,\mu}(H_v^0(\mathbb{D})) \subset H_v^0(\mathbb{D})$. Since the polynomials are dense in $H_v^0(\mathbb{D})$, it is enough to show that $\mathcal{H}_{K,\mu}(H^\infty) \subset H^\infty$. This follows adapting the argument above, having in mind that our assumptions imply that $K \in L^1(\mu)$, because K is μ-measurable and $|K(w)| \leq |K(w)| \ \|C_{\varphi_w}\|$ for each $w \in \mathbb{D}$. □

Remark 3.2 The arguments in the proof of Theorem 3.1 show that if the function $K \in L^1(\mu)$, then the operator $\mathcal{H}_{K,\mu} : H^\infty \to H^\infty$ is continuous and $\|\mathcal{H}_{K,\mu}\| \leq \int_{\mathbb{D}} |K(w)| d\mu(w)$. Compare with Theorem 3 in [20] for Hardy spaces $H^p, 1 \leq p < \infty$ and with Lemma 2.2 in [16] which considers Hardy spaces on the upper half plane.

Our next lemma is certainly known, but we were not able to find a reference.

Lemma 3.3 *(1) If $v(r) = (1-r^2)^\gamma$ with $\gamma > 0$, then*

$$\|C_{\varphi_w}\| = \Big(\frac{1+|w|}{1-|w|} \Big)^\gamma, \qquad w \in \mathbb{D}.$$

(2) If $v(z) = (\log \frac{e}{1-r^2})^{-\alpha},\ \alpha > 0$, then

$$\|C_{\varphi_w}\| \leq \Big(1 + \log \frac{1+|w|}{1-|w|} \Big)^\alpha, \qquad w \in \mathbb{D}.$$

Proof. (1) It is easy to see that

$$(1-|w|)^2 = \inf_{z \in \mathbb{D}} |1 - \overline{w} z|^2 \leq \sup_{z \in \mathbb{D}} |1 - \overline{w} z|^2 = (1+|w|)^2.$$

Therefore, by Proposition 4.1 in [24], we get

$$\sup_{z\in\mathbb{D}} \frac{1-|z|^2}{1-|\varphi_w(z)|^2} = \sup_{z\in\mathbb{D}} \frac{|1-\overline{w}z|^2}{1-|w|^2} = \frac{(1+|w|)^2}{1-|w|^2} = \frac{1+|w|}{1-|w|}.$$

This clearly implies the statement.

(2) We estimate $(v(z)/v(\varphi_w(z)))^{1/\alpha}$, $z \in \mathbb{D}$ again using Proposition 4.1 in [24] as follows.

$$\frac{\log(e/(1-|\varphi_w(z)|^2)}{\log(e/(1-|z|^2)} = \frac{\log\left(\frac{e|1-\overline{w}z|^2)}{(1-|w|^2)(1-|z|^2)}\right)}{\log(e/(1-|z|^2)} \leq$$

$$\frac{\log\left(\frac{e(1+|w|)}{(1-|w|)(1-|z|^2)}\right)}{\log(e/(1-|z|^2)} \leq 1 + \log\frac{1+|w|}{1-|w|}.$$

□

Our next corollary should be compared with the results in [21].

Corollary 3.4 *(1) Let $v(r) = (1-r^2)^\gamma$ with $\gamma > 0$. If the function $w \in \mathbb{D} \to K(w)/(1-|w|)^\gamma$ belongs to $L^1(\mu)$, then $\mathcal{H}_{K,\mu} : H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ is continuous.*

(2) Let $v(r) = (\log \frac{e}{1-r^2})^{-\alpha}$, $\alpha > 0$. If the function $w \in \mathbb{D} \to K(w)\log(1-|w|)$ belongs to $L^1(\mu)$, then $\mathcal{H}_{K,\mu} : H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ is continuous.

Proof. This is a direct consequence of Lemma 3.3 and Theorem 3.1. □

Example 3.5 (1) Consider the weight $v(r) = (1-r^2)^\gamma, 0 \leq r < 1$, with $\gamma > 0$ and the measure $d\mu(w) = dA(w)/(1-|w|^2)^\beta$, $\beta > 0$, where dA is the area measure. If there is $C > 0$ such that $(1/C)(1-|w|^2)^\alpha \leq |K(w)| \leq C(1-|w|^2)^\alpha$ for all $z \in \mathbb{D}$, and $\beta+\alpha-\gamma < 1$, then the Hausdorff operator $\mathcal{H}_{K,\mu} : H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ is continuous. This follows from Corollary 3.4 (1) and Lemma 3.9 in [24].

(2) If the measure μ is concentrated on a sequence $(w_n)_n$ in $\mathbb{D}$, we get the discrete Hausdorff operator, for $d = (d_n)_n \in \mathbb{C}$,

$$\mathcal{H}_d(f)(z) := \sum_{n=1}^{\infty} d_n f(\varphi_{w_n}(z)), \qquad z \in \mathbb{D}.$$

If the sequence reduces to one single point, we obtain the weighted composition operator dC_{φ_w}. Theorem 3.1 implies that if $\sum_{n=1}^\infty |d_n| \, \|C_{\varphi_{w_n}}\| < \infty$, then $\mathcal{H}_d$ is continuous on $H_v^\infty(\mathbb{D})$.

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