Lipschitz structure of metric and Banach spaces

PhD. Dissertation

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Abstract

Since the inception of Banach Space Theory, the study of complemented and uncomplemented subspaces of Banach spaces has been one of the main themes of the area. Specifically, in non-separable Banach spaces, there have been many efforts in constructing a theoretical framework to describe the linear complementation structure of Banach spaces. Classical concepts such as the Separable Complementation Property, Projectional Resolutions of the Identity, and the Plichko Property have been and continue to be studied in this area.

Similarly, Lipschitz maps between Banach spaces have also played a main role in the development of the theory. Questions such as the Lipschitz classification of Banach spaces, differentiability of Lipschitz maps, or the existence of Lipschitz retractions onto subsets and subspaces of Banach spaces, have been and continue to be active topics of research with a wealth of results and applications.

In this thesis we analyse the Lipschitz retractional structure of non-separable metric and Banach spaces, as an analogous theory to the linear complementation one in Banach spaces. We also discuss the connection of this topic with the ongoing program to study the structure of Lipschitz-free Banach spaces, and to the problem of finding bounded linear extension operators for Lipschitz functions.

First, we generalize some classical tools of the linear theory to the non-linear setting: We define the concept of Lipschitz retractional skeletons as a generalization of Projectional skeletons. As applications of these concepts, we show that the Lipschitz-free space of a Plichko Banach space is again Plichko. We also use Lipschitz retractional skeletons to characterize metric spaces whose Lipschitz-free spaces enjoy the Plichko property witnessed by Dirac measures, and we show that the Lipschitz-free space of any $\mathbb{R}$-tree is 1-Plichko witnessed by molecules.

Next, we pass on to defining the (α, β) Lipschitz Retraction Property (Lipschitz $\mathrm{RP}(\alpha, \beta)$ for short) for a pair of infinite cardinals $\alpha \leq \beta$. These are the non-linear analogues to the classical Complementation Properties. We observe that $C(K)$ spaces enjoy the Lipschitz $\mathrm{RP}(\aleph_0, \aleph_0)$, which in turn implies that their associated Lipschitz-free space satisfy the Separable Complementation Property.

As a continuation of the previous study, we construct, for every infinite cardinal Λ, a complete metric space which fails the Lipschitz $\mathrm{RP}(\Lambda, \Lambda)$. In the countable case, we are able to produce a complete metric space, called the skein space, with a stronger property than the negation of the Lipschitz $\mathrm{RP}(\aleph_0, \aleph_0)$: Every separable subset of the skein space with at least two points fails to be a Lipschitz retract.

Finally, we generalize a result of Heinrich and Mankiewicz to the non-linear setting, by showing that for any metric space M, every subset is contained in another subset of the same density character which admits a bounded linear extension operator for the space of Lipschitz functions.

Keywords: Non-linear Functional Analysis, Lipschitz maps, Lipschitz retractions, Non-separable Banach spaces, Non-separable metric spaces, Lipschitz-free spaces, Local Complementation, Linear Extensions of Lipschitz maps.

Resumen

Desde el comienzo de la Teoría de Espacios de Banach, el estudio de los subespacios complementados y no complementados ha sido uno de los principales temas del área. Específicamente, en espacios de Banach no separables, han habido grandes esfuerzos en construir un marco teórico para describir la estructura de subespacios linealmente complementados en espacios de Banach. Concepctos clásicos como la Propiedad del Complemento Separable, Resoluciones Proyectivas de la Identidad, y la Propiedad de Plichko han sido y continúan siendo estudiadas en esta disciplina.

En igual medida, las aplicaciones de Lipschitz en espacios de Banach también han jugado un papel importante en el desarrollo de la teoría. Cuestiones como la clasificación de Lipschitz de los espacios de Banach, la diferenciabilidad de las funciones de Lipschitz, o la existencia de retracciones de Lipschitz a subconjuntos y subespacios de espacios de Banach, son líneas de investigación activas con abundantes resultados y aplicaciones.

En esta tesis analizamos la estructura de retractos de Lipschitz en espacios métricos y espacios de Banach no separables, de forma análoga a la teoría de complementación lineal en espacios de Banach. También discutimos la conexión de este tema con el progreso actual en el estudio de la estructura de los espacios de Lipschitz-free, y con el problema de la existencia de operadores de extensión lineales para funciones de Lipschitz.

En primer lugar, generalizamos algunas herramientas clásicas de la teoría lineal al marco no lineal: Definimos el concepto de esqueletos retractivos de Lipschitz como una generalización a los esqueletos proyectivos. Como aplicación de estas nociones, demostramos que el espacio de Lipschitz-free asociado a un espacio de Banach con la propiedad de Plichko tiene a su vez la propiedad de Plichko. Utilizamos también los esqueletos retractivos de Lipschitz para caracterizar aquellos espacios métricos cuyo espacio de Lipschitz-free tiene la propiedad de Plichko con medidas de Dirac, y mostramos que el espacio de Lipschitz-free asociado a cualquier $\mathbb{R}$-árbol es 1-Plichko con moléculas elementales.

A continuación, pasamos a definir la Propiedad del Retracto de Lipschitz (α, β) (o la Lipschitz $\mathrm{RP}(\alpha, \beta)$) para un par de cardinales infinitos $\alpha \leq \beta$. Esta es la propiedad no lineal análoga a la clásica Propiedad del Complemento. Observamos que los espacios $C(K)$ tiene la Lipschitz $\mathrm{RP}(\aleph_0, \aleph_0)$, lo cual implica que sus espacios de Lipschitz-free asociados poseen la Propiedad del Complemento Separable.

Siguiendo con el estudio previo, construimos, para cada cardinal infinito Λ, un espacio métrico completo sin la Lipschitz $\mathrm{RP}(\Lambda, \Lambda)$. En el caso numerable, podemos mejorar este resultado produciendo un espacio métrico completo que satisface una propiedad más fuerte que la negación de la Lipschitz $\mathrm{RP}(\aleph_0, \aleph_0)$: Todo subconjunto separable con almenos dos puntos no es un retracto de Lipschitz.

Finalmente, generalizamos un resultado de Heinrich y Mankiewicz al marco no lineal al mostrar que en cada espacio métrico M, todo subconjunto está contenido en otro subconjunto con el mismo carácter de densidad que además admite un operador lineal de extensión de funciones Lipschitz.

Palabras clave: Análisis Funcional No Lineal, Aplicaciones de Lipschitz, Retractos de Lipschitz, Espacios de Banach No Separables, Espacios métricos No Separables, Espacios de Lipschitz-free, Complementación Local, Extensiones lineales de Aplicaciones de Lipschitz.

Resum

Des del principi de la Teoria d'Espais de Banach, l'estudi dels subespais complementats i no complementats ha estat un dels principals temes de l'àrea. Específicament, en espais de Banach no separables, hi ha hagut un gran esforç de construir un marc teòric per descriure l'estructura de subespais linealment complementats en espais de Banach. Conceptes clàssics com la Propietat del Complement Separable, Resolucions Projectives de la Identitat, i la Propietat de Plichko han estat i continuen sent estudiades en aquesta disciplina.

En igual mesura, les aplicacions de Lipschitz en espais de Banach també han jugat un paper important en el desenvolupament de la teoria. Qüestions com la classificació de Lipschitz dels espais de Banach, la diferenciabilitat de les funcions de Lipschitz, o l'existència de retraccions de Lipschitz a subconjunts i subespais d'espais de Banach, són línies d'investigació actives amb abundants resultats i aplicacions.

En aquesta tesi analitzem l'estructura de retractes de Lipschitz en espais mètrics i espais de Banach no separables, de manera anàloga a la teoria de complementació lineal en espais de Banach. També discutim la connexió d'aquest tema amb el progrés actual en l'estudi de l'estructura dels espais de Lipschitz-free, i amb el problema de l'existència d'operadors d'extensió lineals per a funcions de Lipschitz.

En primer lloc, generalitzem algunes eines clàssiques de la teoria lineal al marc no lineal: Definim el concepte d'esquelets retractius de Lipschitz com una generalització dels esquelets projectius. Com aplicació d'aquestes nocions, demostrem que l'espai de Lipschitz-free associat a un espai de Banach amb la propietat de Plichko té la propietat de Plichko. Utilitzem també els esquelets retractius de Lipschitz per a caracteritzar aquells espais mètrics que generen espais de Lipschitz-free amb la propietat de Plichko amb mesures de Dirac, i mostrem que l'espai de Lipschitz-free associat a qualsevol $\mathbb{R}$-arbre és 1-Plichko amb molècules elementals.

A continuació, passem a definir la Propietat del Retracte de Lipschitz (α, β) (o la Lipschitz RP(α, β)) per a un parell de cardinals infinits $\alpha \leq \beta$. Aquesta és la propietat no lineal anàloga a la clàssica Propietat del Complement. Observem que els espais $C(K)$ tenen la Lipschitz RP$(\aleph_0, \aleph_0)$, la qual cosa implica que els espais de Lipschitz-free associats posseeixen la Propietat del Complement Separable.

Seguint amb l'estudi previ, construïm, per a cada cardinal infinit Λ, un espai mètric complet sense la Lipschitz RP(Λ, Λ). En el cas numerable, podem millorar aquest resultat produint un espai mètric complet que satisfà una propietat més

forta que la negació de la Lipschitz $\mathrm{RP}(\aleph_0, \aleph_0)$: Tot subconjunt separable amb almenys dos punts no és un retracte de Lipschitz.

Finalment, generalitzem un resultat de Heinrich i Mankiewicz al marc no lineal al demostrar que en cada espai mètric M, tot subconjunt està contingut en altre subconjut amb el mateix caràcter de densitat que a més admet un operador lineal d'extensió de funcions Lipschitz.

Contents

CHAPTER 1

Introduction

The main topic of this thesis is the study of Lipschitz maps in non-separable metric and Banach spaces. This is a subfield of Geometric Non-linear Functional Analysis, which encompasses the study of uniformly continuous and, in particular, Lipschitz functions between metric and Banach spaces. This subject has been deeply studied in the last century, and has many branches and applications, mainly related to Linear Functional Analysis, and also to other topics such as Probability and Measure Theory. This is illustrated by one classical result, due to Mazur and Ulam [41], which says that every isometric bijection between Banach spaces is necessarily linear. As put by Benyamini and Lindenstrauss in their fundamental book [5], the Mazur−Ulam Theorem implies that the linear structure of Banach spaces is entirely determined by their structure as metric spaces.

We are particularly interested in the concept of *Lipschitz retraction*, that is, a Lipschitz function from a metric space onto one of its subsets which fixes all points in the image (called a *Lipschitz retract*). Clearly, singletons are trivially Lipschitz retracts in every metric space, whereas in every connected metric space, finite sets with more than one point fail to be Lipschitz (or even continuous) retracts. Lipschitz retractions can be seen as the non-linear generalization of linear projections in Banach spaces. It follows from the Hahn Banach theorem that subspaces with finite dimension or finite codimension are always linearly complemented. These are usually called the trivial linear complements.

In the linear theory, and specifically in the study of non-separable Banach spaces, there is a strong incentive to determine how rich is the structure of non-trivial complemented subspaces of a given density character in a Banach space, since a good description often provides enough information to deduce additional structural properties and to derive geometrical properties. As a relevant example for our later discussion, the Plichko property can be described through the presence of commutative projectional skeletons, and it implies the existence of a Locally Uniformly Rotund renorming and a strong Markushevich basis. On the other hand, it is known that there exist Banach spaces in which every infinite-dimensional subspace fails to have any non-trivial linearly complemented subspace. These spaces are called *hereditarily indecomposable* Banach spaces, and despite their highly pathological properties, they are quite ubiquitous, according to the celebrated Gowers dichotomy ([20]).

Similarly, in metric spaces, where Lipschitz maps are the canonical morphisms, it is very natural to search for non-trivial Lipschitz retracts in order to understand the original metric structure. Additionally, Lipschitz retracts often inherit good properties from their superspace, which further motivates the study of these maps. Since Lipschitz retractions are a weaker notion than linear projections, it is natural to wonder if there also exist pathological metric and Banach spaces which lack non-trivial Lipschitz retracts; or if, on the contrary, non-trivial Lipschitz retracts can always be found in every metric or Banach space.

An important motivation and tool for the study of Lipschitz retractions in metric and Banach spaces is the class of Lipschitz-free Banach spaces: Given a complete metric space M with an arbitrary distinguished point $0 \in M$, the vector space $\mathrm{Lip}_0(M)$ formed by all real-valued Lipschitz functions which vanish at the distinguished point 0 is a Banach space when endowed with the norm given by the best Lipschitz constant. Additionally, $\mathrm{Lip}_0(M)$ is a dual space, and the canonical predual is the *Lipschitz-free space of* M, which is denoted by $\mathcal{F}(M)$. The term “Lipschitz-free space" was coined by Godefroy and Kalton in their influential article [17]. Their publication motivated the study of this topic in Geometric Functional Analysis, which remains a highly active area two decades later. However, Lipschitz-free spaces had been studied before, either implicitly or under different names: The earliest appearance of such a construction was given by Arens and Eells in [3]. The first version of the monograph [49] by Weaver (see [50] for the second version) appeared four years before the publication of Godefroy and Kalton, and is still an essential reference in this topic.

Many efforts in the recent study of Lipschitz-free spaces have been put into understanding their structure as Banach spaces (see e.g.: [1, 10, 11, 18, 32, 30, 34] for a small sample). In this context, Lipschitz retractions play a crucial role, since a Lipschitz retraction from a metric space M onto a subset S induces a linear projection from $\mathcal{F}(M)$ onto $\mathcal{F}(S)$.

Although the converse of the previous statement does not hold, a linear projection from $\mathcal{F}(M)$ onto a subspace of the form $\mathcal{F}(S)$ does provide a bounded linear extension operator $E\colon \mathrm{Lip}_0(S) \to \mathrm{Lip}_0(M)$, which, on bounded sets, is also continuous for the topology of pointwise convergence of Lipschitz functions. This suggests more natural structural questions in metric spaces: given a metric space M and a subset S of M, when does there exist a bounded linear extension operator $E\colon \mathrm{Lip}_0(S) \to \mathrm{Lip}_0(M)$? This perspective to extension of Lipschitz functions has been studied by A. Brudnyi and Y. Brudnyi (see e.g. [8, 6] or the monograph [7]), and by other authors such as Godefroy and Ozawa [18, 16] in relation to the Approximation Properties in Lipschitz-free spaces.

In our context, we say that S is *locally complemented* in M if there exists a bounded linear extension operator $E\colon \mathrm{Lip}_0(S) \to \mathrm{Lip}_0(M)$. Although the notion of local complementability was introduced by Kalton [31] in a different context, as shown by Fakhoury in [13], the two notions coincide in Banach spaces. It follows from the landmark article of Lindenstrauss and Tzafriri [40] that non-Hilbert Banach spaces always contain closed subspaces which are not locally complemented. However, using Model Theory, Heinrich and Mankiewicz showed in [26] that (non-separable) Banach spaces have a relatively rich structure of locally complemented Banach spaces, in the sense that every closed linear subspace is contained in a locally complemented linear subspace of the same density character.

Outline of the thesis

To summarize the previous discussion, in Banach spaces we can consider the following three structural notions for a given subspace, in descending order of strength: linear complement, Lipschitz retract, and local complement. Since local complementability can be expressed in terms of linear extension operators of Lipschitz maps, both Lipschitz retracts and local complements are also natural concepts in the more general setting of metric spaces. In this thesis we study these three concepts in the context of non-separable complete metric and Banach spaces. The main goals of this analysis are the following:

(A) Studying the linear concepts of projectional skeletons and the Plichko property in the class of Lipschitz-free Banach spaces. Additionally, we seek to define analogous non-linear notions in the underlying metric space, and to describe the relationship between the linear and non-linear variants.
(B) Constructing pathological metric spaces which lack good Lipschitz retractional structures of a given density character.
(C) Extending the Heinrich and Mankiewicz result to the class of complete metric spaces, proving that, in metric spaces, the local complementation structure of a given density character is as well behaved as the corresponding linear one in Banach spaces.

We discuss now how this thesis is organised. Including the present introduction and the conclusion, the main body of the text contains 6 chapters. Chapter 2 is dedicated to providing the necessary background for the rest of the thesis. First, we present classical concepts regarding linear projections in Banach spaces, and recall the definitions of projectional skeletons and Plichko spaces, as well as discuss the relationship between both notions. In this part we also discuss the so-called Complementation Properties, with special emphasis on the Separable Complementation Property. Next, we introduce some basic concepts and results regarding Lipschitz maps and Lipschitz retractions in metric spaces. Finally, we take a look at Lipschitz-free spaces, briefly discussing their construction, their basic properties, and some auxiliary results that we will use in later chapters.

Chapter 3 is related to goal (A) above. Recall that a Banach space X is said to be *Plichko* if there exists a linearly dense set Δ in X and a norming subspace N in X^* such that the set $\Delta_f = \{x \in \Delta \colon \langle f, x\rangle \neq 0\}$ is countable for all $f \in N$.

If N can be taken to be X^*, then X is *Weakly Lindelöf Determined*. All non-separable Lipschitz-free spaces contain an isomorphic copy of $\ell_1(\Gamma)$, where Γ is an uncountable set (see [23]). For this reason, non-separable Lipschitz-free spaces are not Weakly Lindelöf Determined. Nevertheless, there are no known examples of Lipschitz-free space which fail to be Plichko. In fact, in most cases, it is not straightforward to show that the Plichko property holds either.

In this discussion, the concept of projectional skeleton introduced by Kubiś in [36] is particularly relevant, as Kubiś showed that a Banach space is Plichko if and only if it admits a commutative projectional skeleton. We generalize this concept, introducing *Lipschitz retractional skeletons*, and use them to show that the Lipschitz-free space associated to a Plichko Banach space is again Plichko. Additionally, we characterize which metric spaces yield Lipschitz-free spaces with the Plichko property witnessed by Dirac measures, and show that the Lipschitz-free space of any $\mathbb{R}$-tree is Plichko witnessed by molecules.

We finish Chapter 3 by extending the linear Complementation Properties to the non-linear setting: Given a metric space M, and a pair or cardinal numbers $\alpha \leq \beta$, we say that M has the (α, β) *Lipschitz Retraction Property (Lipschitz RP(α, β) for short)* if every subset of density character α is contained in a Lipschitz retract of density character β. We observe that $C(K)$ spaces always enjoy the Lipschitz RP$(\aleph_0, \aleph_0)$.

In Chapter 4, working towards goal (B), we construct counterexamples to the Lipschitz RP(Λ, Λ) for every infinite cardinal Λ. We first tackle the countable case, seeking a stronger property than the negation of the Lipschitz RP$(\aleph_0, \aleph_0)$: We produce a complete metric space in which every non-singleton separable subset is not a Lipschitz retract. This complete metric space, which we call the *skein space*, is constructed in three steps:

(1) First, we define the elementary pieces of the space. These are compact metric spaces isometric to subsets of the planar circumference endowed with the arc-length distance. We call these pieces *threads*. Each thread has two distinguished endpoints.
(2) Next, we produce the building blocks of the skein space. Each building block, called a *threading space*, is constructed by gluing a certain family of uncountably many totally disconnected threads by their endpoints, which are now shared by all threads in a given threading space. The threading spaces we construct satisfy that every separable space containing the two endpoints is not a Lipschitz retract.
(3) Finally, we construct the skein space by transfinite induction. Informally, the goal of this last step is to keep attaching threading spaces inductively until every pair of close enough points acts as the two endpoints of a threading space. This way, using the properties of the threading spaces constructed in the second step, every separable space with at least two points will fail to be a Lipschitz retract.

In the second part of Chapter 4, we deal with the Lipschitz $\mathrm{RP}(\Lambda, \Lambda)$ for any infinite cardinal Λ. This construction, while being significantly simpler than the skein space, is also heavily reliant on the disconectedness of the resulting metric space.

In Chapter 5 we discuss the local complementation property in metric spaces. The main result, as hinted at in goal (C), is the generalization of the Heinrich and Mankiewicz theorem for metric spaces. Specifically, we prove that in every complete metric space M, every subset N of M is contained in another subset S of the same density character such that there exists a linear extension operator $E\colon \mathrm{Lip}_0(S) \to \mathrm{Lip}_0(M)$ with $\|E\| = 1$. The proof of our result relies on a different proof of the Heinrich and Mankiewicz theorem given by Sims and Yost in [48].

Chapter 6 is a brief conclusion overviewing the contributions made in the thesis.

In chapters 3, 4 and 5, some open questions are discussed throughout the exposition. These questions and some additional ones, are collected in the final section of each of these chapters, together with a brief discussion of each of them.

The research collected in this thesis has been carried out with the thesis supervisors Antonio José Guirao, Petr Hájek and Vicente Montesinos, at Universitat Politècnica de València and the Czech Technical University:

- The content of Chapter 3 appears in the published article [25] and in the preprint [21], which has been accepted for publication in the Mediterranean Journal of Mathematics.
- The construction of the skein space of Chapter 4 appears in [24]. The second part comes from [25].
- All new results from Chapter 5 are published in [25].

Notation

It is assumed that the reader is familiar with the fundamental notions of Functional Analysis, and particularly of Banach Space Theory. The first four chapters of the monograph [12] cover all needed background in this area. Although we will introduce Lipschitz functions from basic concepts, some prior understanding of the topic shall be helpful, for which we recommend the first three chapters of [50]. Basic knowledge in General Topology will also be required, as well as some familiarity with transfinite induction.

We briefly discuss now the notation that will be used in this document.

In metric spaces, we will use M to refer to a metric space in place of (M, d) whenever there is no ambiguity regarding the distance M is endowed with. Given a point $p \in M$ and a positive number $r > 0$, we write $B(p, r)$ to denote the open ball centered at p of radius r. Given a subset A of a metric space M, we will denote by $\overline{A}$ the closure of A in M. Given a point p and a subset A in M, the distance from p to A will be written as $d(p, A) = \inf\{d(p, q) \colon q \in A\}$. The distance between two subsets A and B in M, will be written as $d(A, B) = \inf\{d(p, q) \colon p \in A, q \in B\}$

All Banach spaces in this document are real. We will also use X to refer to a Banach space in place of $(X, \|\cdot\|)$ whenever there is no ambiguity regarding the norm X is endowed with. The unit ball of a Banach space X will be denoted by B_X, and its unit sphere by S_X. The topological dual of a Banach space X will be denoted by X^*, and given a point $x \in X$ and a functional $x^* \in X^*$, and we use $\langle x^*, x\rangle$ to denote the dual action of x^* on x. Given a subset A of a Banach space X, $\overline{span}(A)$ denotes the closed linear span of A, and $\overline{conv}(A)$ denotes the closed convex span of A.

CHAPTER 2

Background

In this chapter we go over the basic notions and results in Linear and Non-Linear Functional Analysis which are necessary to put the main results of this thesis in the proper context. Most of the material we present is well known, but we also introduce some concepts and results which are more specific to our purposes. We include short proofs of some non-trivial statements, although we omit long and involved ones in order to keep the text more focused. In cases where proofs are not presented, appropriate references are given.

2.1. Linear notions in the structure of Banach spaces

Let X be a Banach space. Given any linear subspace Y of X, there exists another linear subspace Z such that $Y \cap Z = \{0\}$ and $X = Y + Z$. This is usually written as $X = Y \oplus Z$, and Z is called an *algebraic complement of* Y. If $X = Y \oplus Z$, then every vector $x \in X$ can be written uniquely as a sum $x = y_x + z_x$ for some $y_x \in Y$ and $z_x \in Z$. This defines a linear map $P_Y \colon X \to Y$, which satisfies, by uniqueness of the decomposition, that $P_Y y = y$ for all $y \in Y$. This map is a linear projection of X onto Y.

More generally, if X is a Banach space and Y is a linear subspace, a *linear projection of* X *onto* Y is any linear map $P \colon X \to Y$ such that $Py = y$ for all $y \in Y$. Notice that up until this point we have not used the topology of the Banach space X, since we do not require the subspaces to be closed, or the projections to be bounded.

If Y is a closed subspace of X, and there exists a closed subspace Z such that $X = Y \oplus Z$, then Z is a *topological complement of* Y. It can be shown that in this case the associated projection $P_Y \colon \ \to Y$ is bounded. In fact, a subspace Y of a Banach space X admits a topological complement if and only if there exists a linear and bounded projection $P \colon X \to Y$ onto Y. This leads us to the following classical definition:

Definition 2.1. Let X be a Banach space. A closed subspace Y of X is *linearly complemented in* X if there exists a linear and bounded projection $P \colon X \to Y$ onto Y.

Every subspace of finite dimension and of finite codimension is always complemented. However, in general, not every closed subspace is linearly complemented in a given Banach space. Indeed, as Lindenstrauss and Tzafriri famously showed in [40], the only Banach spaces in which every closed subspace is linearly complemented are Hilbert spaces. On the other end of the spectrum we find the *indecomposable* spaces, which are Banach space in which the only complemented subspaces are those with finite dimension or codimension. If X is an indecomposable space with the property that every infinite-dimensional subspace is again indecomposable, then X is said to be *hereditarily indecomposable*. Although we will not study these Banach spaces in this thesis, it is worth mentioning that this latter class of Banach spaces are a well studied object, with a wealth of remarkable and powerful results.

In non-separable Banach spaces, a natural line of research concerns the study of separable complemented subspaces, since often a rich separable complementation structure allows obtaining good information on the whole Banach space. In the next subsections, we briefly introduce some concepts that are often used to describe the aforementioned separable complementation structure of Banach spaces. It is worth mentioning now that along this thesis, we will be presenting non-linear analogues to the following concepts, as well as studying these linear notions in the context of Lipschitz-free spaces.

2.1.1. projectional skeletons and the Plichko property. The next definition was introduced by Kubiś in [36]. Recall that a partially ordered set Γ is *directed* if for every pair $s_1, s_2 \in \Gamma$ there exists an element $t \in \Gamma$ such that $s_1, s_2 \leq t$. It is said to be *σ-complete* if every sequence $\{s_n\}_{n\in\mathbb{N}}$ such that $s_n \leq s_{n+1}$ for all $n \in \mathbb{N}$, has the supremum in Γ. A subset Γ' of Γ is *cofinal* if for every $s \in \Gamma$ there exists $t \in \Gamma'$ such that $s < t$.

Definition 2.2. Let X be a Banach space. A *projectional skeleton* on X is a family $\{P_s\}_{s\in\Gamma}$ of bounded linear projections on X indexed by a directed and σ-complete partially ordered set Γ, such that the following conditions hold:

(i) P_sX is separable for all $s \in \Gamma$.
(ii) $P_sP_t = P_tP_s = P_s$ whenever $s, t \in \Gamma$ and $s \leq t$.
(iii) If $(s_n)_n$ is an increasing sequence of indices in Γ, then $P_sX = \overline{\bigcup_{n\in\mathbb{N}} P_{s_n}X}$, where $s = \sup_{n\in\mathbb{N}} s_n$.
(iv) $X = \bigcup_{s\in\Gamma} P_sX$.

If $r \geq 1$, we say that an *r-projectional skeleton* is a projectional skeleton where the norm of every projection is less than or equal to r. We say that a projectional skeleton is *commutative* if $P_sP_t = P_tP_s$ for all $s, t \in \Gamma$, regardless of whether they are comparable or not.

It was observed in [36] that in every projectional skeleton $\{P_s\}_{s\in\Gamma}$ indexed by a directed and σ-complete partially ordered set Γ, we can find a cofinal subset Γ' and a real number $r \geq 1$ such that $\|P_s\| \leq r$ for all $s \in \Gamma'$. For this reason, one may always assume that the operator norms of the projections of a projectional skeleton are uniformly bounded.

On the other hand, we have the concept of Plichko spaces. Recall that given a Banach space X and a real number $r \geq 1$, a closed subspace N of X^* is *r-norming* if $\sup\{\langle f, x\rangle \colon f \in N,\ \|f\| \leq 1\} \geq \frac{1}{r}\|x\|$.

DEFINITION 2.3. Let X be a Banach space, and let $r \geq 1$. We say that X is *r-Plichko* if there exists a pair (Δ, N), where $\Delta \subset X$ is a linearly dense subset of X and N is an r-norming subspace of X^* such that for every functional $f \in N$, the set

$$S_\Delta(f) = \{x \in \Delta \colon \langle f, x\rangle \neq 0\}$$

is countable. We say that the pair (Δ, N) is a *witness* of the Plichko property in X. Since N is essentially determined by the linearly dense set Δ, we sometimes say, equivalently, that X is Plichko witnessed by the set Δ.

This class of Banach spaces was studied by Plichko in several articles ([46, 45, 44, 43]) under a different name. In the survey [28], Kalenda named these spaces Plichko spaces. We refer to this survey and to the monograph [22] for a detailed study of this notion. It is known, for instance, that all Plichko Banach spaces admit a Locally Uniformly Rotund norm, and that the Plichko property is equivalent to the existence of a countably norming Markushevich basis. Some classical examples of non-separable Banach spaces with the Plichko property are $c_0(\Gamma)$ and $\ell_1(\Gamma)$ for an uncountable set Γ, as well as all reflexive Banach spaces. On the other hand, as will be easily deduced in the next subsection, the space ℓ_∞ is not Plichko.

It is clear from the definition that a non-separable Banach space admitting a projectional skeleton is “full" of separable linearly complemented subspaces. Less obvious is the fact that Plichko Banach spaces also have a very rich separable complementation structure. In fact, Kubiś showed the following result:

THEOREM 2.4 (Kubiś [36]). *Let $r \geq 1$. A Banach space X is r-Plichko if and only if it admits a commutative r-projectional skeleton.*

2.1.2. Complementation properties. Projectional skeletons provide information on the existence of separable complemented subspaces, with additional structural properties. A more direct approach is given by the so-called Complementation Properties, which can be defined for any two given cardinal numbers:

DEFINITION 2.5. Given α, β two cardinal numbers with $\alpha \leq \beta$, we say that a Banach space X has the *(α, β) Complementation Property (CP(α, β)* for short), if for every closed subspace $Y \subset X$ with $\operatorname{dens}(Y) = \alpha$ there exists another subspace Z that contains Y, such that $\operatorname{dens}(Z) \leq \beta$ and Z is linearly complemented in X. We say that X has the *Separable Complementation Property (SCP)* if it has the CP$(\aleph_0, \aleph_0)$.

The SCP is the most studied of the Complementation Properties. The following well known result confirms the intuitive idea that projectional skeletons are a stronger concept than the SCP.

Proposition 2.6. *If a Banach space admits a projectional skeleton, then it has the SCP. In particular, if X is Plichko, then it has the SCP.*

Proof. Let X be a Banach space, let $\{P_s\}_{s\in\Gamma}$ be a projectional skeleton in X, and let Y be a separable subspace of X. Consider a sequence $\{y_n\}_{n\in\mathbb{N}}$ dense in Y. Since $X = \bigcup_{s\in\Gamma} P_s(X)$, for every $n \in \mathbb{N}$ there exists $s_n \in \Gamma$ such that $y_n \in P_{s_n}(X)$. By directedness of Γ, we may assume that $s_n \leq s_m$ for all $n \leq m \in \mathbb{N}$. Then, if $s = \sup_{n\in\mathbb{N}} s_n$, we obtain that $y_n \in P_s(X)$ for all $n \in \mathbb{N}$, and thus Y is contained in the separable and linearly complemented subspace $P_s(X)$. □

It is worth noting that the fact that Plichko spaces enjoy the SCP was known well before the study of projectional skeletons. Kubiś constructed in [37] a Banach space which has the SCP but is not Plichko.

On the other hand, the space ℓ_∞ does not have the SCP, since every infinite-dimensional linearly complemented subspace of ℓ_∞ is isomorphic to ℓ_∞ itself, and thus non-separable. It is also clear that nonseparable indecomposable Banach spaces do not have the $\mathrm{CP}(\alpha, \beta)$ for any two cardinals $\alpha \leq \beta$ strictly smaller than their density character. A remarkable result of Koszmider, Shelah, and Świętek ([35]) shows that, under the Generalized Continuum Hypothesis, there exist indecomposable Banach spaces with arbitrarily large density character.

2.2. Basics in Lipschitz maps on metric spaces

We start with the definition of Lipschitz map between two metric spaces:

Definition 2.7. Let (M, d_M) and (N, d_N) be two metric spaces. A map $F\colon M \to N$ is said to be *Lipschitz* if the supremum

$$\|F\|_{\mathrm{Lip}} = \left\{ \frac{d_N\left(F(x), F(y)\right)}{d_M(x,y)} : x \neq y \in M \right\}$$

is finite. The value $\|F\|_{\mathrm{Lip}}$ is the *Lipschitz constant of F*.

Given a non-negative real number K, we say that a map $F\colon M \to N$ is K-Lipschitz if it is Lipschitz and $\|F\|_{\mathrm{Lip}} \leq K$.

It should be noted that every linear and bounded operator between Banach spaces is in particular a Lipschitz map, with Lipschitz constant equal to the operator norm.

As is the case with linear and bounded operators, we have the following result regarding compositions of Lipschitz maps:

Proposition 2.8. *Let M, N and S be metric spaces, and let $F\colon M \to N$ and $G\colon N \to S$ be Lipschitz maps. Then $G \circ F\colon M \to S$ is a Lipschitz map with $\|G \circ F\|_{Lip} \leq \|G\|_{Lip} \cdot \|F\|_{Lip}$.*

PROOF. Let $p, q \in M$. Since F is $\|F\|_{\text{Lip}}$-Lipschitz, we have that $d_N(F(p), F(q)) \leq \|F\|_{\text{Lip}} d_M(p, q)$. Applying now that G is $\|G\|_{\text{Lip}}$-Lipschitz, we obtain that

$$d_S(G(F(p)), G(F(q))) \leq \|G\|_{\text{Lip}} d_N(F(p), F(q)) \leq \|G\|_{\text{Lip}} \cdot \|F\|_{\text{Lip}} d_M(p, q),$$

which finishes the proof. □

We will mainly focus on two specific types of Lipschitz maps: Lipschitz retractions and real-valued Lipschitz functions. We start by formally defining Lipschitz retractions:

DEFINITION 2.9. Let K be a non-negative real number, let M be a metric space and let S be a subset of M. A map $R\colon M \to S$ is a *K-Lipschitz retraction* if it is a K-Lipschitz map such that $R(p) = p$ for all $p \in S$. In this case, the set S is called a *K-Lipschitz retract of M*.

Again, it is clear that a linear and bounded projection in a Banach space is in particular a Lipschitz retraction. On the other hand, it is often the case that a linear subspace of a Banach space is not linearly complemented, but it is a Lipschitz retract. For instance, while c_0 is famously not linearly complemented in ℓ_∞, it is straightforward to check that the map $R\colon \ell_\infty \to c_0$ given by

$$(R(\{x_n\}_{n\in\mathbb{N}}))_k = \begin{cases} 0, & \text{if } |x_k| < \limsup_n |x_n| \\ \operatorname{sign}(x_k)(|x_k| - \limsup_n |x_n|) & \text{if } |x_k| \geq \limsup_n |x_n| \end{cases}$$

is a 2-Lipschitz retraction onto c_0. This is an example of a Banach space being a Lipschitz retract of its bidual. It was asked by Lindenstrauss in [38] whether every Banach space is a Lipschitz retract of its bidual. A non-separable Banach space which fails this condition was given by Kalton in [30], while the separable case remains an open question. We will discuss this conjecture several times during this thesis.

Given a real number K, if a metric space M is a K-Lipschitz retract in every metric space that contains it, then we say that M is an *absolute K-Lipschitz retract*. It can be shown that if a metric space S is a K_1-Lipschitz retract of an absolute K_2-Lipschitz retract, then S is an absolute K_1K_2-absolute Lipschitz retract. The space ℓ_∞ is an absolute 1-Lipschitz retract, and thus it follows that c_0 is an absolute 2-Lipschitz retract.

Next, given a metric space M, we will denote by $\text{Lip}(M)$ the set of all real-valued Lipschitz functions defined on M. Note that the Lipschitz constant of a map $f \in \text{Lip}(M)$ is computed as

$$\|f\|_{\text{Lip}} = \sup_{p\neq q\in M} \frac{|f(p) - f(q)|}{d(p, q)}.$$

The ordered field structure of the real numbers allows us to multiply Lipschitz functions by scalars and to perform addition of Lipschitz functions.

PROPOSITION 2.10. *Let M be a metric space, let $f, g \in Lip(M)$, and let $\lambda \in \mathbb{R}$. Then:*

(1) $\|\lambda f\|_{Lip} = |\lambda| \|f\|_{Lip}$.
(2) $\|f + g\|_{Lip} \leq \|f\|_{Lip} + \|g\|_{Lip}$.

PROOF. Fix two points $p, q \in M$ for the rest of the proof.

To show (1), notice that $|\lambda f(p) - \lambda f(q)| = |\lambda||f(p) - f(q)|$. Therefore:

$$\sup_{p \neq q \in M} \frac{|\lambda f(p) - \lambda f(q)|}{d(p,q)} = |\lambda| \frac{|f(p) - f(q)|}{d(p,q)} = |\lambda| \|f\|_{\mathrm{Lip}}.$$

For (2), the triangle inequality shows through direct computation that $|(f+g)(p) - (f+g)(q)| \leq |f(p) - f(q)| + |g(p) - g(q)| \leq (\|f\|_{\mathrm{Lip}} + \|g\|_{\mathrm{Lip}}) d(p,q)$. $\square$

Notice that we have that, in particular, the set $\mathrm{Lip}(M)$ is a vector space.

We also have the following property about pointwise limits of Lipschitz functions:

PROPOSITION 2.11. *Let M be a metric space, and let $\{f_i\}_{i \in \mathbb{N}}$ be a net in $Lip(M)$ with uniformly bounded Lipschitz constant, which converges pointwise to a function $f \colon M \to \mathbb{R}$. Then $f \in Lip(M)$ with $\|f\|_{Lip} \leq \liminf_{i \in I} \|f_i\|_{Lip}$.*

PROOF. A simple computation shows that for every $p, q \in M$ we have:

$$|f(p) - f(q)| = \lim_{i \in I} |f_i(p) - f_i(q)| \leq \liminf_{i \in I} \|f_i\|_{\mathrm{Lip}} d(p,q).$$

$\square$

With this, we can show that pointwise suprema and infima of Lipschitz functions behaves well with respect to the Lipschitz constant. We use the notation $f \vee g = \max\{f, g\}$ and $f \wedge g = \min\{f, g\}$. We have the following easy properties:

PROPOSITION 2.12. *Let $\{f_i\}_{i \in I}$ be a subset of $Lip(M)$ with uniformly bounded Lipschitz constant. Then*

(1) $\left\|\bigvee_{i \in I} f_i\right\|_{Lip} \leq \sup_{i \in I} \|f_i\|_{Lip}$.
(2) $\left\|\bigwedge_{i \in I} f_i\right\|_{Lip} \leq \sup_{i \in I} \|f_i\|_{Lip}$.

PROOF. We show first that given two Lipschitz functions $f, g \in \mathrm{Lip}(M)$, then $\|f \vee q\|_{\mathrm{Lip}} \leq \max\{\|f\|_{\mathrm{Lip}}, \|g\|_{\mathrm{Lip}}\}$ and $\|f \wedge q\|_{\mathrm{Lip}} \leq \max\{\|f\|_{\mathrm{Lip}}, \|g\|_{\mathrm{Lip}}\}$. Fix $p, q \in M$, and suppose without loss of generality that $(f \vee g)(p) \geq (f \vee g)(q)$. Assume first that $f(p) \geq g(q)$. Then:

$$|(f \vee g)(p) - (f \vee g)(q)| = f(p) - (f \vee g)(q) \leq f(p) - f(q) \leq \|f\|_{\mathrm{Lip}} d(p,q).$$

Otherwise, if $g(p) \geq f(p)$ we obtain the inequality $|(f \vee g)(p) - (f \vee g)(q)| \leq \|g\|_{\mathrm{Lip}} d(p,q)$. It follows that $\|f \vee q\|_{\mathrm{Lip}} \leq \max\{\|f\|_{\mathrm{Lip}}, \|g\|_{\mathrm{Lip}}\}$. The second part can be shown from Proposition 2.10 (1) and the fact that $f \wedge g = -((-f) \vee (-g))$.

Now, given a net $\{f_i\}_{i\in I}$ in $\mathrm{Lip}(M)$, the functions $\bigvee_{i\in I} f_i$ and $\bigwedge_{i\in I} f_i$ are, respectively, the pointwise suprema and infima of the net given by the suprema and infima of all finite subfamilies of I. Therefore, the result follows by applying Proposition 2.11. □

We can now show a fundamental theorem in the study of Lipschitz functions is the following extension theorem, due to McShane [42].

THEOREM 2.13 (McShane's Extension Theorem). *Let M be a metric space, and let S be a subset of M. For every function $f \in Lip(S)$ there exists a Lipschitz function $g \in Lip(M)$ such that g extends f and $\|g\|_{Lip} = \|f\|_{Lip}$.*

PROOF. For every $p \in M$, define

$$g(p) = \bigwedge_{x\in S} \left(f(x) + \|f\|_{\mathrm{Lip}} d(p,x)\right).$$

First we show that $g : M \to \mathbb{R}$ extends f. Indeed, if we fix $y \in S$, for every $x \in S$ we have that $f(y) - f(x) \le +\|f\|_{\mathrm{Lip}} d(y,x)$, with equality achieved when $x = y$. This implies that $g(y) = f(y)$.

Now, to show that $\|g\|_{\mathrm{Lip}} \le \|f\|_{\mathrm{Lip}}$, by Proposition 2.12 it is enough to show that the function $h_x \colon M \to \mathbb{R}$ given by $h_x(p) = f(x) + \|f\|_{\mathrm{Lip}} d(p,x)$ is $\|f\|_{\mathrm{Lip}}$-Lipschitz for all $x \in S$. Indeed, we have that for $p, q \in M$:

$$|h_x(p) - h_x(q)| = \|f\|_{\mathrm{Lip}} |d(p,x) - d(q,x)| \le \|f\|_{\mathrm{Lip}} d(p,q).$$

□

2.3. Lipschitz-free spaces

As can be deduced from Proposition 2.10, the function $\|\cdot\|_{\mathrm{Lip}} \colon \mathrm{Lip}(M) \to \mathbb{R}$ defines a seminorm in the vector space $\mathrm{Lip}(M)$. It is not a norm, since all constant functions in a metric space have Lipschitz constant 0, and thus we have non-zero vectors in $\mathrm{Lip}(M)$ whose norm vanishes. However, if we fix a distinguished point $0 \in M$, we have that in the subspace $\mathrm{Lip}_0(M) = \{f \in \mathrm{Lip}(M) \colon f(0) = 0\}$, the Lipschitz constant is a well defined norm. It can be shown that the normed space $(\mathrm{Lip}_0(M), \|\cdot\|_{\mathrm{Lip}})$ is complete, and thus it is a Banach space. Moreover, the choice of the distinguished point in M we have made to define $\mathrm{Lip}_0(M)$ is not relevant, in the sense that choosing a different distinguished point yields an isometrically isomorphic space. Additionally, if M is a metric space and $\widehat{M}$ is its completion, the Banach spaces $\mathrm{Lip}_0(M)$ and $\mathrm{Lip}_0(\widehat{M})$ are isometrically isomorphic as well.

Given a metric space M with a distinguished point $0 \in M$, and given a point $p \in M$, the Dirac measure (or Dirac function) $\delta(p) \colon \mathrm{Lip}_0(M) \to \mathbb{R}$, given by $\delta(p)(f) = f(p)$ for all $f \in \mathrm{Lip}_0(M)$, defines a linear and bounded functional, and it is therefore a point in the dual space $\mathrm{Lip}_0(M)^*$. Moreover, the dual of the closed linear subspace spanned by $\delta(M) \subset \mathrm{Lip}_0(M)^*$ is isometrically isomorphic to $\mathrm{Lip}_0(M)$. This canonical predual, that is, the subspace $\overline{\mathrm{span}}(\delta(p) \colon p \in M) \subset$

$\mathrm{Lip}_0(M)^*$ equipped with the inherited norm, is denoted by $\mathcal{F}(M)$, and is called the *Lipschitz-free space of M*. Notice that it follows that Lipschitz-free spaces have the same density character as the underlying metric space. Again, completeness and the choice of the distinguished point yield isometrically isomorphic Lipschitz-free spaces.

It is important to note that the map $\delta\colon M \to \mathcal{F}(M)$ is a (non-linear) isometric embedding. Therefore, we may regard every complete metric space as a closed subset of its Lipschitz-free space.

The subject of Lipschitz-free spaces is a very active area of research, and many results have been obtained in this domain. We refer to the monograph [50], to the seminal article [17] and to the survey [15] for an extensive study of these spaces. We will discuss now those properties which are more relevant to our discussion, and which we will use in the main body of the thesis.

Apart from Dirac measures, we may distinguish another type of points in Lipschitz-free spaces of great relevance:

Definition 2.14. Let M be a metric space, and let $p \neq q \in M$. The element $m_{p,q} = \frac{\delta(p)-\delta(q)}{d(p,q)} \in B_{\mathcal{F}(M)}$ is called an *elementary molecule.*

Note that given a function $f \in \mathrm{Lip}_0(M)$ and two points $p \neq q \in M$, we have that $\langle m_{p,q}, f\rangle = \frac{f(p)-f(q)}{d(p,q)}$. This implies that $\|f\|_{\mathrm{Lip}} = \sup\{\langle m_{p,q}, f\rangle \colon p \neq q \in M\}$. As an application of Hahn-Banach's Theorem, we obtain that

$$B_{\mathcal{F}(M)} = \overline{\mathrm{conv}}\left\{m_{p,q}\colon p \neq q \in M\right\}.$$

One of the most important properties of Lipschitz-free spaces is the following result:

Theorem 2.15. *Let M and S be two metric spaces. For every Lipschitz map $F\colon M \to S$ such that $F(0) = 0$ there exists a linear operator $\widehat{F}\colon \mathcal{F}(M) \to \mathcal{F}(S)$ with $\|\widehat{F}\| = \|F\|_{Lip}$ and such that $\widehat{F} \circ \delta_M = \delta_S \circ F$.*

Proof. Consider the linear operator $T\colon \mathrm{Lip}_0(N) \to \mathrm{Lip}_(M)$ given by $Tg = F\colon g$ for all $g \in \mathrm{Lip}_0(N)$. It follows from Proposition 2.8 that $\|T\| \leq \|F\|_{\mathrm{Lip}}$. In addition, T is weak* to weak* continuous (note that the weak* topology in bounded subsets of $\mathrm{Lip}_0(M)$ coincides with the topology of pointwise convergence). It follows that T is the adjoint of an operator $\widehat{F}\colon \mathcal{F}(M) \to \mathcal{F}(N)$, which satisfies $\|\widehat{F}\| = \|F\|_{\mathrm{Lip}}$ and $\widehat{F} \circ \delta_M = \delta_S \circ F$. $\square$

A special case of the previous theorem is when $\iota\colon S \to M$ is an isometric embedding of S into M. Then $\widehat{i}\colon \mathcal{F}(S) \to \mathcal{F}(M)$ is a linearly isometric embedding of $\mathcal{F}(S)$ into $\mathcal{F}(M)$. This means, in particular, that when S is a subset of M containing 0, we can consider $\mathcal{F}(S)$ as a closed linear subspace of $\mathcal{F}(M)$. We will use this identification throughout the thesis. With this in mind, we have the following result with a direct proof:

Proposition 2.16. *Let M be a metric space and let S be a closed subset of M containing 0. If $F\colon M \to S$ is a Lipschitz retraction onto N, then $\widehat{F}$ is a linear projection from $\mathcal{F}(M)$ onto $\mathcal{F}(S)$.*

Proof. Since the Set $\delta(S)$ is linearly dense in $\mathcal{F}(S)$, it suffices to show that $\widehat{F}$ fixes $\delta(p)$ for all $p \in S$. A direct computation shows that

$$\widehat{F}\delta(p) = \delta(F(p)) = \delta(p).$$

□

Given a Banach space X, there exists a linear left inverse to the map $\delta\colon X \to \mathcal{F}(X)$. This map is called the *barycenter map*, and is usually denoted by $\beta_X\colon \mathcal{F}(X) \to X$ (see e.g.: [17]). Using this linear inverse and Theorem 2.15 we obtain the following universal property of Lipschitz-free spaces:

Theorem 2.17. *Let M be a metric space and let X be a Banach space. If $F\colon M \to X$ is a Lipschitz map with $F(0) = 0$, there exists a linear map $\widehat{F}\colon \mathcal{F}(M) \to \mathcal{F}(X)$ such that $\widehat{F} \circ \delta = F$ and $\|\widehat{F}\| = \|F\|_{Lip}$.*

Additionally, thanks to the next result, we have that when a Lipschitz-free space $\mathcal{F}(M)$ admits a projectional skeleton, we may assume without loss of generality that the complemented subspaces generated by the skeleton are the Lipschitz-free spaces of separable subsets of M. In the proof, we use the concept of *support of an element $\mu \in \mathcal{F}(M)$ (denoted supp(μ))*, defined as the intersection of all closed subsets S in M such that $\mu \in \mathcal{F}(S)$. This notion was introduced in [2], where, among other properties and applications, it is shown that for every $\mu \in \mathcal{F}(M)$, the support of μ is a separable set such that $\mu \in \mathcal{F}(\mathrm{supp}(\mu))$.

Proposition 2.18 ([25]). *Let M be a complete metric space (resp. Banach space), and let $\{P_s\}_{s\in\Gamma}$ be a projectional skeleton on $\mathcal{F}(M)$. Then there exists a σ-closed cofinal subset of Γ and a family $\{A_s\}_{s\in\Gamma'}$ of separable subsets (resp. separable linear subspaces) of M such that $P_s(\mathcal{F}(M)) = \mathcal{F}(A_s)$.*

In particular, $\mathcal{F}(M)$ admits a projectional skeleton $\{P_s\}_{s\in\Gamma}$ such that $P_s(\mathcal{F}(M))$ is $\mathcal{F}(A_s)$ where A_s is a separable subset (resp. separable linear subspace) of M for all $s \in \Gamma$.

Proof. Let $s = s_0 \in \Gamma$ be an arbitrary index. The set $P_{s_0}(\mathcal{F}(M))$ is a separable subset of $\mathcal{F}(M)$. Since the support of a point in $\mathcal{F}(M)$ is a closed separable subset of M, we obtain that the set

$$A_{s_0} = \overline{\bigcup_{\mu\in P_{s_0}(\mathcal{F}(M))} \mathrm{supp}(\mu)}$$

is a closed separable subset of M as well. Moreover, since $\mathrm{supp}(\mu) \subset A_{s_0}$ for all $\mu \in P_{s_0}(\mathcal{F}(M))$, we have that $P_{s_0}(\mathcal{F}(M)) \subset \mathcal{F}(A_{s_0})$, which is a separable subset of $\mathcal{F}(M)$. By the properties of projectional skeletons, we can find $s_1 \in \Gamma$, with $s_0 \leq s_1$ such that $\mathcal{F}(A_{s_0}) \subset P_{s_1}(\mathcal{F}(M))$.

By induction, we construct $(s_n)_n \subset \Gamma$ such that $s_n \leq s_{n+1}$ and $\{A_{s_n}\}_n$ are closed separable subsets of M satisfying

$$P_{s_n}\big(\mathcal{F}(M)\big) \subset \mathcal{F}(A_{s_n}) \subset P_{s_{n+1}}\big(\mathcal{F}(M)\big).$$

This implies that $\bigcup_{n\in\mathbb{N}} P_{s_n}\big(\mathcal{F}(M)\big) = \bigcup_{n\in\mathbb{N}} \mathcal{F}(A_n)$. Consider now $t_s = \sup(s_n) \in \Gamma$. We have then that

$$P_{t_s}\big(\mathcal{F}(M)\big) = \overline{\bigcup_{n\in\mathbb{N}} P_{s_n}\big(\mathcal{F}(M)\big)} = \overline{\bigcup_{n\in\mathbb{N}} \mathcal{F}(A_n)}.$$

Note as well that $\overline{\bigcup_{n\in\mathbb{N}} \mathcal{F}(A_n)} = \mathcal{F}\left(\overline{\bigcup_{n\in\mathbb{N}} A_n}\right)$. Indeed, it is clear that $\overline{\bigcup_{n\in\mathbb{N}} \mathcal{F}(A_n)} \subset \mathcal{F}\left(\overline{\bigcup_{n\in\mathbb{N}} A_n}\right)$. Conversely, note that $\overline{\bigcup_{n\in\mathbb{N}} \mathcal{F}(A_n)}$ is a closed linear subspace which contains $\delta(p)$ for every $p \in \overline{\bigcup_{n\in\mathbb{N}} A_n}$, so it follows that $\mathcal{F}\left(\overline{\bigcup_{n\in\mathbb{N}} A_n}\right) \subset \overline{\bigcup_{n\in\mathbb{N}} \mathcal{F}(A_n)}$.

Set $A_{t_s} = \overline{\bigcup_{n\in\mathbb{N}} A_n}$ and $\Gamma' = \{t_s\}_{s\in\Gamma}$. Clearly Γ' is σ-complete and cofinal, and the result follows.

It is easy to modify this argument to see that if X is a Banach space and $\mathcal{F}(X)$ admits a projectional skeleton $\{P_s\}_{s\in\Gamma}$, we can assume that $P_s\big(\mathcal{F}(X)\big) = \mathcal{F}(Y_s)$ where $\{Y_s\}$ is a family of separable closed linear subspaces. $\square$

Finally, we will also need a very useful result, due to Kalton, that characterizes the norming subspaces of $\mathrm{Lip}_0(M)$.

THEOREM 2.19 (Proposition 3.3 in [32]). *Let M be a metric space, let $r \geq 1$, and let N be a closed subspace of $Lip_0(M)$. Then N is r-norming if and only if for every finite set A of M containing 0, every $\varepsilon > 0$, and every function $g \in Lip_0(A)$, there exists a function $f \in N$ such that f extends g and $\|f\|_{Lip} \leq (r+\varepsilon)\|g\|_{Lip}$.*

CHAPTER 3

Lipschitz retractional structure of metric and Banach spaces

In this chapter we start by studying the non-linear versions of the classical complementation properties we introduced in Chapter 2. It is divided into three sections: In the first section we define Lipschitz retractional skeletons and their relationship with projectional skeletons, and we use them to show that the Lipschitz-free space of a Plichko Banach space is again Plichko. In the second section we study the Plichko property and its witnesses in Lipschitz-free spaces of several classes of metric spaces. We characterize metric spaces whose Lipschitz-free space is Plichko witnessed by Dirac measures, and we show that the Lipschitz-free space of any $\mathbb{R}$-tree is Plichko witnessed by elementary molecules. Finally, in the third section we look at Lipschitz analogues to the Complementation Properties, proving some positive results and motivating the constructions of the next chapter.

3.1. Lipschitz retractional skeletons and the Plichko property

We start by defining the concept analogous to projectional skeletons in the metric setting, by replacing linear projections with Lipschitz retractions. It is worth noting that the related concept of retractional skeletons in compact topological spaces, using continuous retractions, has been studied in the literature. We refer to the monograph [27] for more details in this topic.

Definition 3.1. Let M be a metric space. A *Lipschitz retractional skeleton* on M is a family $\{R_s\}_{s\in\Gamma}$ of Lipschitz retractions on M indexed by a directed and σ-complete partially ordered set Γ, such that the following conditions hold:

(i) $R_s(M)$ is a separable subset of M for every $s \in \Gamma$.
(ii) If $s, t \in \Gamma$ such that $s \leq t$, then $R_s \circ R_t = R_t \circ R_s = R_s$.
(iii) If $(s_n)_{n\in\mathbb{N}}$ is a totally ordered sequence in Γ, then $R_s(M) = \overline{\bigcup_{n\in\mathbb{N}} R_{s_n}(M)}$ where $s = \sup_{n\in\mathbb{N}} s_n$.
(iv) $M = \bigcup_{s\in\Gamma} R_s(M)$.

Given $r \geq 1$, a r-Lipschitz retractional skeleton is a Lipschitz retractional skeleton with Lipschitz constant uniformly bounded by r. A Lipschitz retractional skeleton is *commutative* if $R_s \circ R_t = R_t \circ R_t$ for every $s, t \in \Gamma$, regardless of whether they are comparable or not.

Since every linear and continuous projection is in particular a Lipschitz retractions, it is clear that the previous concept is weaker than the notion of projectional skeletons, when considering Banach spaces as metric spaces. However, thanks to the linearization property of Lipschitz free spaces (Theorem 2.15), we can show that the existence of a Lipschitz retractional skeleton in a metric space implies that the associated Lipschitz-free space admits a projectional skeleton. As we will show in Chapter 4, the converse statement of this next result fails in a strong sense for general metric spaces.

PROPOSITION 3.2 ([25]). *Let M be a complete metric space and $r \geq 1$. Suppose that M admits a (commutative) r-Lipschitz retractional skeleton on M. Then $\mathcal{F}(M)$ admits a (commutative) r-projectional skeleton.*

PROOF. Let $\{R_s\}_{s\in\Gamma}$ be an r-Lipschitz retractional skeleton in M. Using Theorem 2.15, let $P_s\colon = \widehat{R_s}\colon \mathcal{F}(M) \to \mathcal{F}(M)$ be the linear maps such that $\|P_s\| = \|R_s\|_{\text{Lip}}$ and $P_s(\delta(x)) = \delta(R_s(x))$ for all $x \in M$. Let us check that this family is a projectional skeleton on $\mathcal{F}(M)$.

In the first place, since $R_s(M)$ is separable for all $s \in \Gamma$, and $P_s(\mathcal{F}(M))$ is equal to $\mathcal{F}(R_s(M))$, we obtain that $P_s(\mathcal{F}(M))$ is separable for all $s \in \Gamma$. Next, suppose $s, t \in \Gamma$ with $s \leq t$ and take $x \in M$. We have then that

$$P_sP_t(\delta(x)) = \delta(R_sR_t(x)) = \delta(R_s(x)) = P_s(\delta(x)),$$

and similarly for $P_tP_s(\delta(x))$. Since P_sP_t, P_tP_s and P_s are bounded linear maps and $\delta(M)$ is a linearly dense subset of $\mathcal{F}(M)$, we obtain that $P_sP_t = P_tP_s = P_s$ as desired.

Next, suppose that $(s_n)_n$ is an increasing sequence of indices in Γ, and let $s = \sup_{n\in\mathbb{N}} s_n$. Consider $x \in M$ and $\varepsilon > 0$. By hypothesis there exists $n_0 \in \mathbb{N}$ and $y \in M$ such that $d\big(R_s(x), R_{s_{n_0}}(y)\big) < \varepsilon$. Hence, since the δ map is an isometry, we have that $\|\delta\big(R_s(x)\big) - \delta\big(R_{n_0}(y)\big)\| < \varepsilon$. This implies that $\|P_s(\delta(x)) - P_{n_0}(\delta(y))\| < \varepsilon$. Hence, $P_s(\delta(x)) \in \overline{\bigcup_{n\in\mathbb{N}} P_{s_n}(\mathcal{F}(M))}$.

Now, since (s_n) is increasing, by the remark we made about condition (ii) of Definition 2.2, the family $\{P_{s_n}(\mathcal{F}(M))\}_n$ is increasing as well. This implies that $\overline{\bigcup_{n\in\mathbb{N}} P_{s_n}(\mathcal{F}(M))}$ is a linear subspace of $\mathcal{F}(M)$. Then, by the linearity of P_s and the fact that $\delta(M)$ is linearly dense in $\mathcal{F}(M)$, we obtain that

$$P_s(\mathcal{F}(M)) = \overline{\bigcup_{n\in\mathbb{N}} P_{s_n}(\mathcal{F}(M))},$$

as desired.

Finally, to prove that $\mathcal{F}(M) = \bigcup_{s\in\Gamma} P_s(\mathcal{F}(M))$, we use the concept of support of an element of $\mathcal{F}(M)$. For all $\mu \in \mathcal{F}(M)$ the set $\text{supp}(\mu) \subset M$ is a closed separable subset such that $\mu \in \mathcal{F}(\text{supp}(\mu))$. Hence, if for every $\mu \in \mathcal{F}(M)$ we find $s \in \Gamma$ such that $\text{supp}(\mu) \subset R_s(M)$, we will obtain that $\mu \in \mathcal{F}(R_s(M)) = P_s(\mathcal{F}(M))$, completing the proof.

To this end, consider $\mu \in \mathcal{F}(M)$, and let $(x_n)_n \subset \text{supp}(\mu)$ be a dense sequence. By hypothesis, for x_1 we can find $s_1 \in \Gamma$ such that $x_1 \in R_{s_1}(M)$. Suppose that we have constructed $(s_i)_{i=1}^n$ in Γ such that $s_i \leq s_{i+1}$ for $1 \leq i \leq n-1$ and such that $x_i \in R_{s_i}(M)$. By hypothesis there exists $s^* \in \Gamma$ such that $x_{n+1} \in R_{s^*}(M)$. Since Γ is directed, we can find $s_{n+1} \in \Gamma$ such that $s_i \leq s_{n+1}$ for $1 \leq i \leq n$ and $s^* \leq s_{n+1}$.

This way we inductively construct an increasing sequence $(s_n)_n$ such that $x_n \in R_{s_n}(M)$. By item (iii) in the hypothesis, there exists $s \in \Gamma$ such that $R_s(M) = \overline{\bigcup_{n\in\mathbb{N}} R_{s_n}(M)}$. Since the dense sequence $(x_n)_n$ is contained in $\bigcup_{n\in\mathbb{N}} R_{s_n}(M)$, it follows that $\operatorname{supp}(\mu) \subset R_s(M)$. We conclude that $\{P_s\}_{s\in\Gamma}$ is a projective skeleton. The last two statements follow immediately. $\square$

This, together with Kubiś' equivalence of the Plichko property in terms of projectional skeletons, yields the following result about Lipschitz free spaces of r-Plichko Banach spaces.

Corollary 3.3 ([25]). *Let X be a Banach space and let $r \geq 1$. If X is r-Plichko, then $\mathcal{F}(X)$ is r-Plichko.*

Proof. By Theorem 2.4, if X is r-Plichko, then it admits a commutative r-projectional skeleton. Since linear projections are in particular Lipschitz retractions, by Proposition 3.2 the space $\mathcal{F}(X)$ also admits a commutative r-projectional skeleton. By the converse implication of Theorem 2.4, $\mathcal{F}(X)$ is r-Plichko. $\square$

Recall that a Banach space X is *Weakly Lindelöf Determined (WLD for short)* if there exists a linearly dense subset $\Delta \in X$ such that the set $\{x \in \Delta\colon \langle f, x\rangle \neq 0\}$ is countable for all $f \in X^*$. Clearly, every WLD Banach space is Plichko. Moreover, the WLD property is inherited by linear subspaces, and the space $\ell_1(\Gamma)$ is not WLD if Γ is uncountable. Since, as shown in [23], the space $\ell_1(\Gamma)$ embeds linearly into $\mathcal{F}(M)$, where Γ is the density character of M, we have that non-separable Lipschitz-free spaces fail to be WLD. The previous corollary shows that the weaker Plichko property can occur in Lipschitz-free spaces. In fact, there are currently no examples of metric spaces whose Lipschitz-free space is not Plichko. In relation to this question, it is not known if the converse of the Corollary 3.3 is true.

3.2. Witnesses of the Plichko property in Lipschitz-free spaces

When trying to show directly that the Lipschitz-free space of a metric space M has the Plichko property, some natural candidates for the linearly dense set witnessing such property are subsets of molecules. We start by studying the situation for witnesses formed by Dirac measures.

3.2.1. The Plichko property witnessed by Dirac measures. If D is a subset of M such that $\Delta_D = \{\delta(p)\colon p \in D\}$ is linearly dense in $\mathcal{F}(M)$, it is straightforward to see that D must be dense in M. Therefore, if Δ_D witnesses the Plichko property in $\mathcal{F}(M)$, it follows that the associated norming subspace $N_D = \{f \in \mathrm{Lip}_0(M)\colon f \text{ is countably supported in } D\}$ must be contained in the set of separately supported functions in M. This hints at the fact that the Plichko property in Lipschitz-free spaces witnessed by a subset of Dirac measures is a rather strong condition in the underlying metric space. Indeed, we show that it can be characterized by a strong form of local separability in a metric space. Namely, among other equivalent properties, we will obtain that $\mathcal{F}(M)$ is λ-Plichko witnessed by set of Dirac measures if and only if the open ball $B\left(p, rd(p,0)\right)$ is separable for all $p \in M$ and all $r < \frac{1}{\lambda}$.

Before stating the full characterization, let us prove that this property implies that the space of separately supported functions is norming. Since we will use this subspace repeatedly during this section, we fix the notation

$$S_0 = \{f \in \mathrm{Lip}_0(M) \colon \mathrm{supp}(f) \text{ is separable } \} \subset \mathrm{Lip}_0(M)$$

for every metric space M and for the rest of the chapter. Here, $\mathrm{supp}(f)$ denotes the support of the Lipschitz function f in the usual sense; that is, the smallest closed subset of M such that $f(x) = 0$ for all $x \notin \mathrm{supp}(f)$. This set is clearly a closed linear subspace of $\mathrm{Lip}_0(M)$.

Proposition 3.4 ([25]). *Let M be a metric space and $\lambda \geq 1$. If for every $p \in M$, and every $0 < r < \frac{1}{\lambda}$, the set $B(p, r \cdot d(p,0))$ is separable, then S_0 is λ-norming.*

Proof. By Kalton's Lemma 2.19, it is enough to show that for every finite set $F \subset M$ with $0 \in F$, every $\varepsilon > 0$ and every Lipschitz function $f \in \mathrm{Lip}_0(F)$ with $\|f\|_{\mathrm{Lip}} = 1$, there exists a function $g \in S_0$ such that $g_{|F} = f$ and $\|g\|_{\mathrm{Lip}} \leq \lambda(1+\varepsilon)$. Using McShane's extension theorem, this is equivalent to proving that for every finite set $F \subset M$ with $0 \in F$, every $\varepsilon > 0$ and every function $f \in \mathrm{Lip}_0(M)$ with $\|f\|_{\mathrm{Lip}} = 1$, there exists a function $g \in S_0$ such that $g_{|F} = f_{|F}$ and $\|g\|_{\mathrm{Lip}} \leq \lambda(1+\varepsilon)$.

Fix $f \in \mathrm{Lip}_0(M)$ with $\|f\|_{\mathrm{Lip}} = 1$. Define the subsets $P = \{p \in M : f(p) > 0\}$, $N = \{p \in M : f(p) < 0\}$ and $Z = \{p \in M : f(p) = 0\}$.

Fix $x_0 \in P$ and $\varepsilon > 0$, and define $\tau_{x_0}(p) = \max\{f(x_0) - \lambda(1+\varepsilon)d(p,x_0), 0\}$. Put $D_{x_0} = \{p \in M,\ \tau_{x_0}(p) > 0\}$ (D_{x_0} is the topological interior of the support of τ_{x_0}). We claim that $D_{x_0} \subset P$. Indeed, let $p \in D_{x_0}$. Then $\tau_{x_0}(p) = f(x_0) - \lambda(1+\varepsilon)d(p,x_0) > 0$. Equivalently, $d(p,x_0) < \big(\lambda(1+\varepsilon)\big)^{-1} f(x_0)$.

Also, since $\|f\|_{\mathrm{Lip}} = 1$, we have that $|f(x_0) - f(p)| < \big(\lambda(1+\varepsilon)\big)^{-1} f(x_0)$. Thus,

$$f(p) \geq f(x_0) - (\lambda(1+\varepsilon))^{-1} f(x_0) = f(x_0)\Big(1 - (\lambda(1+\varepsilon))^{-1}\Big) > 0,$$

as we claimed. It is also clear that $x_0 \in D_{x_0}$. It follows that $P = \bigcup_{x\in P} D_x$.

Similarly, for $x_0 \in N$ and $\varepsilon > 0$, we define $\tau_{x_0}(p_0) = \min\{f(x_0) + \lambda(1+\varepsilon)d(p,x_0), 0\}$ and $D_{x_0} = \{p \in M,\ \tau_{x_0}(p) < 0\}$. Following the same reasoning as before, we get $N = \bigcup_{x\in N} D_x$. In particular, if $x \in P$ and $y \in N$, we get $D_x \cap D_y = \emptyset$.

Now let $F \subset M$ be a finite set with $0 \in F$. Put $FP = F \cap P$, $FN = F \cap N$ and $FZ = F \cap Z$. Define a function $g\colon M \to M$ in the following way:

$$g(p) = \begin{cases} \bigvee_{x \in FP} \tau_x(p), & \text{if } p \in P \\ \bigwedge_{x \in FN} \tau_x(p), & \text{if } p \in N \\ 0, & \text{if } p \in Z \end{cases}$$

This function has the desired properties, that is:

(i) $g(p) = f(p)$ for all $p \in F$,
(ii) $g(0) = 0$,
(iii) $g \in S_0$, and
(iv) $\|g\|_{\mathrm{Lip}} \leq \lambda(1+\varepsilon)$.

Let us check this. Let $p \in F$. Suppose that $p \in FP$. Then $g(p) \geq \tau_p(p) = f(p)$ by definition. Let x be an arbitrary point in FP. Then, since $\|f\|_{\mathrm{Lip}} = 1$ and $\lambda \geq 1$:

$$\begin{aligned} \tau_x(p) &= f(x) - \lambda(1+\varepsilon)d(p,x) = f(x) - \lambda d(p,x) - \lambda\varepsilon d(p,x) \\ &\leq f(x) - d(p,x) \leq f(x) - (f(x) - f(p)) = f(p) \end{aligned}$$

Hence $g(p) = f(p)$. By a similar argument we see that if $q \in FN$, then $g(q) = f(q)$, and clearly if $z \in FZ$, by definition $g(z) = f(z) = 0$. We have proven (i) and (ii) since $0 \in F$.

To see (iii), we need to prove that g has a separable support. Note that $\operatorname{supp}(g) = \bigcup_{x \in F} \operatorname{supp}(\tau_x)$. Since F is finite, it suffices to show that $\operatorname{supp}(\tau_x)$ is separable for every $x \in F$. Suppose $x_0 \in FP$ and let $p \in M$ with $d(p, x_0) > (\lambda(1+\varepsilon))^{-1} d(x_0, 0)$. Then $\lambda(1+\varepsilon)d(p,x_0) > d(x_0, 0)$, so

$$f(x_0) - \lambda(1+\varepsilon)d(p,x_0) < f(x_0) - d(x_0,0) \leq f(0) = 0,$$

which implies that $\tau_{x_0}(p) = 0$. Thus, $\operatorname{supp}(\tau_{x_0}) \subset B(x_0, (\lambda(1+\varepsilon))^{-1} d(x_0, 0))$, which is separable by hypothesis. The same reasoning applies if $x_0 \in FN$, so we conclude that g has separable support and thus condition (iii) is satisfied.

Property (iv) follows from the definition of τ_x for every $x \in M$, and Proposition 2.12. □

Once we have that our geometric property implies that $S_0(M)$ is norming, in order to show that it also implies the Plichko property witnessed by Dirac measures, it is enough to find a dense set $D \subset M$ such that the associated linear subspace $N_D = \{f \in \mathrm{Lip}_0(M) \colon f \text{ is countably supported in } D\}$ is $S_0(M)$. This will hold if and only if the intersection of D with every separable subset of M is countable. The existence of such a dense set D is implied by the weaker property of local separability. The following proof is an elementary application of Zorn's Lemma, but we include it for completeness.

LEMMA 3.5. *Let M be a metric space such that every point in M has a separable neighbourhood (M is locally separable). Then there exists a dense set D in M such that for every separable subset S of M, the intersection $D \cap S$ is countable.*

PROOF. Consider the following set:

$$T = \big\{\{A_i\}_{i\in I} \subset \mathcal{P}(M)\colon A_i \text{ is non-empty, open and separable for all } i \in I,\\ A_i \cap A_j = \emptyset \text{ for all } i \neq j \in I\big\},$$

which is non-empty since M is locally separable. The set T can be ordered by inclusion, and it is straightforward to check that every chain in T has an upper bound given by the union of every family in the chain. Hence, by Zorn's Lemma we can consider $F_0 = \{A_i\}_{i\in I}$ a maximal family in T. Then, since F_0 is maximal in T and M is locally separable, we have that $M = \overline{\bigcup_{i\in I} A_i}$.

Choose for every $i \in I$ a countable set D_i dense in A_i, and set $D = \bigcup_{i\in I} D_i$. Let us check that D satisfies the thesis of the Lemma: Let S be a separable subset of M. Then S has the countable chain condition, so there exists a countable subset $F_0' = \{A_{i_n}\}_{n\in\mathbb{N}}$ of F_0 such that

$$S \cap \bigcup_{i\in I} A_i = S \cap \bigcup_{n\in\mathbb{N}} A_{i_n}.$$

Therefore, since $D_i \subset A_i$ for all $i \in I$, we obtain that $S \cap D = \bigcup_{n\in\mathbb{N}} S \cap D_{i_n}$, which is countable since it is the countable union of countable sets. $\square$

We now proceed to state the full characterization as the main result of this subsection.

THEOREM 3.6 ([21]). *Let M be a metric space with distinguished point $0 \in M$, and let $\lambda \geq 1$. The following statements are equivalent:*

(i) For all $p \in M$ and for all $r < \frac{1}{\lambda}$, the ball $B(p, r \cdot d(p, 0))$ is separable.
(ii) $\mathcal{F}(M)$ is λ-Plichko witnessed by a subset of $\delta(M)$.
(iii) $\mathcal{F}(M)$ admits a commutative λ-projectional skeleton $\{P_s\}_{s\in\Gamma}$ such that $P_s(\delta(p)) \in \{0, \delta(p)\}$ for all $p \in M$.
(iv) M admits a commutative λ-Lipschitz retractional skeleton $\{R_s\}_{s\in\Gamma}$ such that $R_s(p) = \{0, p\}$ for all $p \in M$.
(v) The closed subspace $\{f \in Lip_0(M)\colon supp(f) \text{ is separable}\}$ is a λ-norming subspace of $Lip_0(M)$.

PROOF. Let us start by proving that (i) and (v) are equivalent. Indeed, Proposition 3.4 is precisely the statement that (i) implies (v). To show the converse, suppose by contradiction that $S_0(M)$ is a λ-norming subspace of $\mathrm{Lip}_0(M)$ and that there exists a point $p_0 \in M$ different from 0 and $0 < r_0 < \frac{1}{\lambda}$ such that the open ball $B(p_0, r_0 \cdot d(p_0, 0))$ is non-separable.

Consider the function $f_0 \in \mathrm{Lip}_0(\{0, p_0\})$ defined by $f_0(0) = 0$ and $f_0(p_0) = d(p_0, 0)$. This function is clearly 1-Lipschitz, and thus by Kalton's Lemma 3.3 in

[32], choosing $\varepsilon_0 > 0$ such that $r_0 \cdot (\lambda + \varepsilon_0) < 1$ there exists a $(\lambda+\varepsilon_0)$-Lipschitz function g_0 in $S_0(M) \subset \mathrm{Lip}_0(M)$ such that $g_0(0) = 0$ and $g_0(p_0) = d(p_0, 0)$. Since we are assuming that $B(p_0, r_0 \cdot d(p_0, 0))$ is non-separable, there must exist a point $x_0 \in M$ with $d(p_0, x_0) < r_0 \cdot d(p_0, 0)$ such that $g_0(x_0) = 0$. However, this implies that

$$\begin{aligned} |g_0(p_0)| = |g_0(p_0) - g_0(x_0)| &\leq (\lambda + \varepsilon_0) d(p_0, x_0) \\ &< r_0 \cdot (\lambda + \varepsilon_0) d(p_0, 0) < d(p_0, 0), \end{aligned}$$

which contradicts the choice of g_0.

We continue with (i) implies (ii): If a complete metric space M satisfies property (i), Proposition 3.4 shows that $S_0(M)$ is a λ-norming subspace of $\mathrm{Lip}_0(M)$. By hypothesis, the set $M \setminus \{0\}$ is locally separable, so by Lemma 3.5, we can find $D' \subset M \setminus \{0\}$ dense such that D' intersects every separable subset of $M \setminus \{0\}$ in a countable set. Clearly, the set $D = D' \cup \{0\}$ also satisfies that it is dense in M and for every separable subset S of M, the intersection $D \cap S$ is countable. Put $\Delta = \{\delta(x) \colon x \in D\}$. Then Δ is linearly dense in $\mathcal{F}(M)$, and for every $f \in S_0(M)$ we have that

$$\{x \in D \colon \langle f, \delta(x) \rangle \neq 0\} \subset \mathrm{supp}(f) \cap D$$

is countable. We conclude that $\mathcal{F}(M)$ is λ-Plichko.

(ii) implies (iii): Write $\Delta \subset \delta(M)$ to denote the set witnessing the λ-Plichko property in $\mathcal{F}(M)$. Then $S_0(M)$ is countably supported in Δ, so by Proposition 21 in [36], the Lipschitz-free space $\mathcal{F}(M)$ admits a commutative λ-projectional skeleton $\{P_s\}_{s\in\Gamma}$ which generates $S_0(M)$. Additionally, since Δ is a subset of $\delta(M)$ which is linearly dense in $\mathcal{F}(M)$, there exists a dense subset D of M such that $\Delta = \delta(D)$. Now, by Corollary 20 in [9] we may assume that $P_s(\delta(p)) \in \{0, \delta(p)\}$ for all $p \in D$ and for all $s \in \Gamma$. By density of D in M and continuity of the projections in the projectional skeleton, we conclude that $P_s(\delta(p)) \in \{0, \delta(p)\}$ for all $p \in M$ and all $s \in \Gamma$.

(iii) implies (iv): Suppose that $\mathcal{F}(M)$ admits a commutative λ-projectional skeleton $\{P_s\}_{s\in\Gamma}$ such that $P_s(\delta(p)) \in \{0, \delta(p)\}$ for all $p \in M$. For every $s \in \Gamma$, the image of the subset $\delta(M) \subset \mathcal{F}(M)$ is contained in $\delta(M)$. Hence, when restricting P_s to the subset $\delta(M)$, we obtain a map $P_{s|\delta(M)} \colon \delta(M) \to \delta(M)$ which is λ-Lipschitz. Since the map $\delta \colon M \to \delta(M)$ is an isometry, this restriction induces a λ-Lipschitz map

$$R_s \colon M \to M$$

for every $s \in \Gamma$, defined by $R_s(p) = \delta^{-1}(P_s(\delta(p)))$. For every $s \in \Gamma$, the map R_s is a retraction onto $\delta^{-1}(P(\delta(M)))$, which is a closed separable subset of M. It is direct to check that $\{R_s\}_{s\in\Gamma}$ is a commutative λ-Lipschitz retractional skeleton in M, and it clearly satisfies that $R_s(p) \in \{0, p\}$ for all $p \in M$.

(iv) implies (v): Suppose that M admits a commutative λ-Lipschitz retractional skeleton $\{R_s\}_{s\in\Gamma}$ such that $R_s(p) = \{0, p\}$ for all $p \in M$. We show that $S_0(M)$ is a λ-norming subspace of M once more by using Kalton's Lemma 3.3 in

[32]. Suppose that $A \subset M$ is a finite set and that $F \in \mathrm{Lip}_0(A)$ is a 1-Lipschitz function defined in this set. The set A is finite and thus separable, so there exists $s_0 \in \Gamma$ such that $S_{s_0} = R_{s_0}(M)$ is a separable set containing A. By McShane's extension theorem, we may find another 1-Lipschitz function $\overline{F} \in \mathrm{Lip}_0(S_{s_0})$ such that $\overline{F}_{|A} = F$.

Define now $f_{s_0} \colon M \to \mathbb{R}$ by $f_{s_0}(p) = \overline{F}(R_{s_0}(p))$ for all $p \in M$. The map f_{s_0} is λ-Lipschitz because R_{s_0} is a λ-Lipschitz retraction. Since $R_{s_0}(p) = 0$ if $p \notin S_{s_0}$ and $\overline{F}(0) = 0$, we obtain that the support of f_{s_0} is contained in the separable subset S_{s_0}, which implies that f_{s_0} belongs to $S_0(M)$. We conclude that $S_0(M)$ is λ-norming by Lemma 3.3 in [32].

Since the equivalence between (i) and (v) has already been discussed, we finish the proof of the theorem. □

The geometric condition in (i) of the previous theorem is simple enough to allow us to construct many non-separable metric spaces whose Lipschitz-free spaces have the Plichko property witnessed by Dirac measures. The following example shows that such metric spaces can be found in the Banach spaces $\ell_p(\Gamma)$ for every $1 \leq p \leq \infty$ and every uncountable cardinal Γ. For $1 \leq p \leq \infty$ and $\gamma \in \Gamma$, we write e_γ to denote the vector in $\ell_p(\Gamma)$ and $c_0(\Gamma)$ such that $e_\gamma(\gamma) = 1$ and $e_\gamma(\nu) = 0$ for $\nu \in \Gamma \setminus \{\gamma\}$.

Example 3.7. Let $p \in [1, \infty]$ and let Γ be uncountable. The Banach space $\ell_p(\Gamma)$ contains a complete metric space M_p of density character Γ such that $\mathcal{F}(M_p)$ is 1-Plichko witnessed by Dirac measures.

To show this, fix $p \in [1, \infty]$. For each $\gamma \in \Gamma$, denote $E_\gamma = [0, e_\gamma] = \{(x_\nu)_{\nu \in \Gamma} \in \ell_p(\Gamma) \colon x_\gamma \in [0,1], \text{ and } x_\nu = 0 \text{ for } \nu \neq \gamma\}$. Put $M_p = \bigcup_{\gamma \in \Gamma} E_\gamma$. Then M_p is a complete metric space of density character Γ, and it is easy to check that for every $x \in M_p$ different from 0, the open ball $B(x, d(x, 0))$ is contained in the separable segment E_γ such that $x \in E_\gamma$. Hence, M_p satisfies property (i) for $\lambda = 1$ in Theorem 3.6 and we conclude that $\mathcal{F}(M_p)$ is 1-Plichko witnessed by Dirac measures.

In the previous example, the constructed metric space M_p has a stronger property than the geometric condition we demand: Every point x in M_p is contained in a separable subset E_x such that for every point $y \in E_x$ the open ball $B(y, d(y, 0))$ is contained in E_x, and for every point $z \notin E_x$ the open ball $B(z, d(z, 0))$ is contained in $M \setminus E_x$. Informally, these metric spaces can be seen as the union of (uncountably many) open and closed separable components "glued" at the distinguished point 0 in such a way that the distance between points in two different components is always greater than the distance of each individual point to 0. This fact may suggest that the geometric condition in (i) of Theorem 3.6 is only satisfied by metric spaces that are very close to being separable in the previous sense. However, we show in the next example that this is not the case:

EXAMPLE 3.8. There exists a non-separable metric space N_2 isometric to a subset of $\ell_2(\Gamma)$ for uncountable Γ containing e_γ for all $\gamma \in \Gamma$, whose Lipschitz-free space is 1-Plichko witnessed by Dirac measures with the following property:

For every $\gamma \in \Gamma$ and every separable subset S_γ in N_2 containing e_γ, there exists a point $x \in N_2 \setminus S_\gamma$ such that e_γ belongs to the open ball $B(x, d(x, 0))$.

To construct such a space, consider $M_2 \subset \ell_2(\Gamma)$ of Example 3.7. Define $N_2 = M_2 \cup \{e_\gamma + e_\nu \colon \gamma \neq \nu \in \Gamma\}$. As before, for every $x \in M_2$, the open ball $B(x, d(x, 0))$ in N_2 is contained in the separable set E_γ such that $x \in E_\gamma$. Additionally, for every pair of indices $\gamma \neq \nu \in \Gamma$, we have that $d(e_\gamma + e_\nu, 0) = \sqrt{2}$, and thus the open ball $B(e_\gamma + e_\nu, d(e_\gamma + e_\nu, 0))$ in N_2 is contained in the union of $E_\gamma \cup E_\nu \cup \{e_\gamma + e_\nu\}$, which is separable as well. The metric space N_2 satisfies condition (i) with $\lambda = 1$ in Theorem 3.6 and we have that $\mathcal{F}(N_2)$ is 1-Plichko witnessed by Dirac measures.

Now, for every $\gamma \in \Gamma$, given any separable subset S_γ in N_2 containing e_γ, there exists $\nu \in \Gamma$ such that $e_\gamma + e_\nu$ does not belong to S_γ (since that would contradict the separability of S_γ). It is simple to verify that the point e_γ belongs to the open ball $B(e_\gamma + e_\nu, d(e_\gamma + e_\nu, 0))$.

In this last example, we may also intuitively identify separable components whose union forms the whole complete metric space (each set of the form E_γ or $E_{\gamma,\nu}$ for every $\gamma, \nu \in \Gamma$). However, as we have proven, given a point p in one component, there may be points in different components closer to p than the value $d(p, 0)$. Let us formalize this intuition by characterizing metric spaces with the Plichko property witnessed by Dirac measures in terms of their metric structure:

DEFINITION 3.9. Let M be a complete metric space, and let $0 < r \leq 1$. We say that a collection $\mathcal{S}$ of subsets of M is a *separable r-slab decomposition* if

(1) For all $N \in \mathcal{S}$, the set $N \setminus \{0\}$ is an open separable set,
(2) For all $N_1, N_2 \in \mathcal{S}$ with $N_1 \neq N_2$, it holds that $N_1 \cap N_2 \subset \{0\}$.
(3) For all $p \in M$ such that the ball $B(p, r \cdot d(p, 0))$ is contained in $\overline{\bigcup_{N \in \mathcal{S}} N}$, there exists a countable subfamily $\mathcal{S}_p \subset \mathcal{S}$ such that $B(p, r \cdot d(p, 0))$ is contained in $\overline{\bigcup_{N \in \mathcal{S}_p} N}$.

We say that a separable r-slab decomposition is *total* if $M = \overline{\bigcup_{N \in \mathcal{S}} N}$.

Note that, for a total separable r-slab decomposition, the condition that the ball $B(p, r \cdot d(p, 0))$ is contained in $\bigcup_{N \in \mathcal{S}} N$ is satisfied automatically for all $p \in M$.

An important remark is that the geometric condition (i) of Theorem 3.6 trivializes condition (3) of the definition of separable r-slab decomposition, in the sense that if for some $0 < r \leq 1$ it holds that $B(p, r \cdot d(p, 0))$ is separable for all $p \in M$, then any collection satisfying condition (1) satisfies automatically condition (3). Indeed, suppose $\mathcal{S}$ is a family of separable sets in such a metric space, and fix $p \in M$ such that $B(p, r \cdot d(p, 0)) \subset \overline{\bigcup_{N \in \mathcal{S}} N}$. Since $B(p, r \cdot d(p, 0))$ is separable, there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $B(p, r \cdot d(p, 0))$ such that $B(p, r \cdot d(p, 0)) \subset \overline{\{x_n \colon n \in \mathbb{N}\}}$. Consider for each $n \in \mathbb{N}$ a sequence

$\{y_k^n\}_{k\in\mathbb{N}}$ in $\bigcup_{N\in\mathcal{D}_0} N$ converging to x_n, and choose $N_k^n \in \mathcal{S}$ such that $y_k^n \in N_k^n$ for all $n, k \in \mathbb{N}$. It follows that $B(p, r\cdot d(p,0)) \subset \overline{\bigcup_{n,k\in\mathbb{N}} N_k^n}$.

PROPOSITION 3.10 ([21]). *Let M be a complete metric space and let $\lambda \geq 1$. The following statements are equivalent:*

(i) $\mathcal{F}(M)$ is λ-Plichko witnessed by a subset of $\delta(M)$.

(ii) For every $0 < r < \frac{1}{\lambda}$, given a separable r-slab decomposition $\mathcal{S}$ of M there exists a total separable r-slab decomposition $\mathcal{S}'$ such that $\mathcal{S} \subset \mathcal{S}'$.

(iii) For every $0 < r < \frac{1}{\lambda}$, M admits a total separable r-slab decomposition.

PROOF. We start by showing that (i) implies (ii). Assume first that $\mathcal{F}(M)$ is λ-Plichko witnessed by a set $\Delta \subset \delta(M)$. Fix $0 < r < \frac{1}{\lambda}$, and fix a separable r-slab decomposition $\mathcal{S}$. Consider the following family:

$$\Omega := \big\{\mathcal{D} = \{N_i\}_{i\in I} \colon \mathcal{S} \subset \mathcal{D},\ \mathcal{D} \text{ is a separable } r\text{-slab decomposition}\big\},$$

which can be partially ordered by inclusion. We will use Zorn's Lemma in the family Ω. Consider $\{\mathcal{D}_i\}_{i\in I}$ a totally ordered subset of Ω, and define $\mathcal{D}_0 = \bigcup_{i\in I} \mathcal{D}_i$. We will show that $\mathcal{D}_0$ belongs to Ω. It is clear that $\mathcal{D}_0$ contains $\mathcal{S}$, so we only have to check that $\mathcal{D}_0$ is a separable r-slab decomposition. Condition (1) of the definition of separable r-slab decomposition is direct, while condition (2) follows from the fact that I is totally ordered. To check condition (3), fix $p \in M$ such that $B(p, r\cdot d(p,0)) \subset \overline{\bigcup_{N\in\mathcal{D}_0} N}$. Using (i) of Theorem 3.6, we have that $B(p, r\cdot d(p,0))$ is separable, so there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $B(p, r\cdot d(p,0))$ such that $B(p, r\cdot d(p,0)) \subset \overline{\{x_n \colon n \in \mathbb{N}\}}$.

Consider for each $n \in \mathbb{N}$ a sequence $\{y_k^n\}_{k\in\mathbb{N}}$ in $\bigcup_{N\in\mathcal{D}_0} N$ converging to x_n, and choose $N_k^n \in \mathcal{D}_0$ such that $y_k^n \in N_k^n$ for all $n, k \in \mathbb{N}$. It follows that

$$B(p, r\cdot d(p,0)) \subset \overline{\bigcup_{n,k\in\mathbb{N}} N_k^n},$$

so $\mathcal{D}_0$ is a separable r-slab decomposition. We can apply Zorn's Lemma to find a maximal element $\mathcal{S}' \in \Omega$. Let us prove that $\mathcal{S}'$ is total.

Indeed, suppose by contradiction that there exists a point $p \in M \setminus \{0\}$ such that p lays outside the closure of the set $\bigcup_{N\in\mathcal{S}'} N$. Then, applying condition (i) of Theorem 3.6 we can find an open and separable set A_p containing p such that $A_p \cap N = \emptyset$ for all $N \in \mathcal{S}'$. Consider $\mathcal{S}_p = \mathcal{S}' \cup \{A_p \cup \{0\}\}$. We will show that $\mathcal{S}_p$ is a separable r-slab decomposition, contradicting the maximality of $\mathcal{S}'$. Conditions (1) and (2) of the definition of separable r-slab decomposition are clearly satisfied by the choice of A_p and the fact that $\mathcal{S}'$ is a separable r-slab decomposition. To check condition (3), consider any $x \in M$. By (i) of Theorem 3.6 again, we get that the ball $B(x, r\cdot d(x,0))$ is separable. Since $N \setminus \{0\}$ is open for all $N \in \mathcal{S}'$, we obtain that for every $x \in A_p$, $B(x, r\cdot d(x,0))$ intersects only countably many sets of $\mathcal{S}'$ in a nonempty set. It follows that is a separable r-slab decomposition which contains $\mathcal{S}$, contradicting the maximality of $\mathcal{S}'$ in Ω.

Since $\mathcal{S} = \{0\}$ is trivially a separable r-slab decomposition, it is clear that (ii) implies (iii).

Finally, to show that (iii) implies (i), suppose there exists a total separable r-slab decomposition $\mathcal{S}$ in M for every $0 < r < \frac{1}{\lambda}$. We will prove that condition (i) in Theorem 3.6 holds for the dense subset $\bigcup_{N\in\mathcal{S}} N$, which is enough to prove that $\mathcal{F}(M)$ is λ-Plichko witnessed by Dirac measures.

Fix $0 < r < \frac{1}{\lambda}$, consider a separable set $N \in \mathcal{S}$, and fix $p \in N \setminus \{0\}$. First, since $\mathcal{S}$ is total, it follows that $B(p, r \cdot d(p, 0))$ is contained in $\overline{\bigcup_{N\in\mathcal{S}} N}$. Next, since $\mathcal{S}$ is a separable r-slab decomposition, by condition (3) there exists a countable subfamily $\mathcal{S}_0 \subset \mathcal{S}$ such that $B(p, r \cdot d(p, 0))$ is contained in $\overline{\bigcup_{N\in\mathcal{S}_0} N}$. This shows that $B(p, r \cdot d(p, 0))$ is contained in a countable union of separable sets, and it is thus separable itself. $\square$

3.2.2. The Plichko property witnessed by molecules. The case of $\mathbb{R}$-trees. As we have shown in the previous subsection, restricting the witness of the Plichko property in Lipschitz-free spaces to a subset of Dirac measures imposes strong structural and geometric conditions in the underlying metric space. By (i) in Theorem 3.6, it forces, in particular, that every point except for possibly one must have a separable neighbourhood. Since non-separable Banach spaces do not satisfy this condition, we have by Theorem 3.3 that Lipschitz-free spaces of non-separable Plichko Banach spaces have the Plichko property with witnesses not contained in the set of Dirac measures.

In this subsection we study the subset of the Lipschitz-free space formed by molecules, which contains multiples of the Dirac measures. In particular, we show that the class of $\mathbb{R}$-trees yields Lipschitz-free spaces with the 1-Plichko property witnessed by a subset of molecules. Let us start by formally defining the concept of $\mathbb{R}$-tree.

In a metric space M, an arc between two points $p, q \in M$ is a continuous map $F \colon [a, b] \to M$ with $a < b$, such that $F(a) = p$ and $F(b) = q$. We also call the set $[p, q]_M = F([a, b])$ an arc in M if such a map F exists. A metric space (T, d) is an $\mathbb{R}$-*tree* if for every pair of points, $x \neq y \in T$ there exists a unique arc $[x, y]_T \subset T$ which is isometric to the real line segment $[0, d(x, y)]$. Fixing a distinguished point 0 in a $\mathbb{R}$-tree T, called the *root of* T, we can define the following partial order in T: Given $p, q \in T$, we say that $p \leq q$ if and only if $[0, p]_T \subset [0, q]_T$. Notice that in this order, every arc of the form $[0, p]_T$ for $p \in T$ is totally ordered. We will use the notation $[p, q)_T$ to denote the set $[p, q]_T \setminus \{q\}$. The sets $(p, q]_T$ and $(p, q)_T$ are defined similarly.

In [14], Godard showed that Lipschitz-free spaces over $\mathbb{R}$-trees are isometrically isomorphic to a space of the form $L_1(\mu)$. Since $L_1(\mu)$ is a 1-Plichko space for every measure μ, it follows that $\mathcal{F}(T)$ is 1-Plichko for any $\mathbb{R}$-tree. However, it is straightforward to see that $\mathbb{R}$-trees do not satisfy in general the geometric condition

(i) of Theorem 3.6, and thus the 1-Plichko property of the associated Lipschitz-free space is not witnessed by Dirac measures. We show in this subsection that molecules are enough for this purpose.

We will need to describe subsets of $\mathbb{R}$-trees in some level of detail. A subset of T will be called an $\mathbb{R}$-*subtree* if it is an $\mathbb{R}$-tree which contains the root $0 \in T$. We start with the following observation regarding $\mathbb{R}$-subtrees:

Given an $\mathbb{R}$-subtree A of T, we get that if $p \in \overline{A}$, then $[0,p)_T$ is contained in A. Indeed, consider a sequence $\{x_n\}_{n\in\mathbb{N}}$ in A converging to p. Consider for every $n \in \mathbb{N}$ the point

$$y_n = \max\,[0,x_n]_T \cap [0,p]_T.$$

Since $y_n \leq x_n$, every point in $[x_n,y_n)_T$ is bigger than y_n. Then, we have by choice of y_n that $[x_n,y_n)_T$ does not intersect the path $[y_n,p]_T$, so $[x_n,y_n)_T \cup [y_n,p]_T$ is the unique path in T joining x_n and p (note that this holds trivially as well if $x_n = y_n$). This implies that $y_n \in [x_n,p]_T$, and that, in particular, the sequence $\{y_n\}_{n\in\mathbb{N}}$ converges to p. Since $y_n \in [0,p]_T$, it also holds that $y_n \leq p$ for all $n \in \mathbb{N}$. Combining both previous facts we obtain that $[0,p)_T \subset \bigcup_{n\in\mathbb{N}}[0,y_n]_T$. Finally, since $y_n \in [0,x_n]_T$ and A is an $\mathbb{R}$-subtree of T, we get that $y_n \in A$ and $[0,y_n]_T \subset A$. We conclude that $[0,p)_T \subset A$. As an immediate consequence to this observation, we obtain that if A and B are $\mathbb{R}$-subtrees of an $\mathbb{R}$-tree T, then $\overline{A \cap B} = \overline{A} \cap \overline{B}$.

We have the following standard result, which we briefly prove for completeness:

LEMMA 3.11. *Let T be a $\mathbb{R}$-tree. For any closed $\mathbb{R}$-subtree S of T the map $P_S\colon T \to S$ given by $P_S(t) = \max S \cap [0,t]_T$ for all $t \in T$, defines a 1-Lipschitz retraction such that $d(S,t) = d(P_S(t),t)$ for all $t \in T$.*

PROOF. Let S be an $\mathbb{R}$-subtree of T. Note first that P_S is well defined, since the arc $[0,t]_T$ is totally ordered and the set $S \cap [0,t]_T$ is closed. Moreover, since S is an $\mathbb{R}$-subtree containing the root, we have that $[0,t]_T$ is contained in S if $t \in S$, so it follows that $P_S(t) = t$ for all $t \in S$.

We will show next that $d(S,t) = d(P_S(t),t)$ for all $t \in T$. If $t \in S$ then the equality follows trivially. If $t \notin S$, then, for any $s \in S$ the set $[t,P_S(t)]_T \cup [P_S(t),s]_T$ is the unique arc from t to s in T. By definition of $\mathbb{R}$-tree, we have that $d(t,s) = d(t,P_S(t)) + d(P_S(t),s)$, from which it follows that $d(t,s) \geq d(t,P_S(t))$ for every $s \in S$. Since $P_S(t) \in S$, we obtain that $d(S,t) = d(P_S(t),t)$ as desired.

Finally, we show that P_S is 1-Lipschitz. Let $p,q \in T$ with $p \neq q$. By the previous property of P_S, it is clear that $d(P_S(p),P_s(q)) \leq d(p,q)$ if at least one of p or q belongs to S, so we may assume that neither p nor q belong to S. It is also trivial that $d(P_S(p),P_s(q)) \leq d(p,q)$ if $P_S(p) = P_s(q)$, so we only have to check this inequality when $P_S(p) \neq P_s(q)$. In this case, we have that $[p,P_S(p)]_T \cup [P_S(p),P_S(q)]_T \cup [P_S(q),q]_T$ is the unique arc between p and q in T, which implies that

$$d(p,q) = d(p,P_S(p)) + d(P_S(p),P_S(q)) + d(P_S(q),q),$$

from which it follows that $d(P_S(p),P_s(q)) \leq d(p,q)$. $\square$

In the proof of the main theorem of this subsection we will construct another tree structure in the partially ordered sense: We say that a partially ordered set $(\Omega, \leq)$ with a least element $0 \in \Omega$ is a *partially ordered tree* if for any $\alpha \in \Omega$, the set $[0,\alpha]_\Omega = \{\beta \in \Omega \colon \beta \leq \alpha\}$ is well ordered. We say that a partially ordered tree Ω is *complete* if every chain has a supremum in Ω. Note that in a complete partially ordered tree Ω, given two elements $\alpha, \beta \in \Omega$, the element $\alpha \wedge \beta = \sup [0,\alpha]_\Omega \cap [0,\beta]_\Omega$ is the infimum of the pair $\{\alpha, \beta\}$.

We have the following useful remark about partially ordered trees:

Lemma 3.12. *Let $(\Omega, \leq)$ be a partially ordered tree. Let $\alpha_1 \leq \alpha_2 \in \Omega$. If $\beta \in \Omega$ satisfies $\alpha_1 \not\leq \beta$, then $\alpha_1 \wedge \beta = \alpha_2 \wedge \beta$.*

Proof. Since $\alpha_1 \wedge \beta \leq \alpha_2 \wedge \beta$ and $\alpha_2 \wedge \beta \leq \beta$, it is enough to show that $\alpha_2 \wedge \beta \leq \alpha_1$. Suppose otherwise for the sake of contradiction. Since the set $[0,\alpha_2]_\Omega$ is totally ordered and contains both points α_1 and $\alpha_2 \wedge \beta$, it follows that $\alpha_1 \leq \alpha_2 \wedge \beta$. This implies that $\alpha_1 \leq \beta$, a contradiction. $\square$

We will construct a partially ordered tree in the set of closed $\mathbb{R}$-subtrees of a given complete $\mathbb{R}$-tree, partially ordered by the inclusion relation. However, we will first build partially ordered trees consisting of non-necessarily closed $\mathbb{R}$-trees. In order to obtain trees of closed $\mathbb{R}$-trees preserving some desired qualities, we will make use of the following technical lemma:

Lemma 3.13. *Let T be a complete $\mathbb{R}$-tree, and let $(\Omega^\circ, \subset)$ be a partially ordered tree consisting of $\mathbb{R}$-subtrees of T. Suppose that Ω° is complete and that for any $S_1 \neq S_2 \in \Omega^\circ$, the set $S_1 \cap S_2$ is a closed $\mathbb{R}$-subtree which belongs to Ω°. Then the partially ordered tree $(\Omega, \subset)$ defined by*

$$\Omega = \{\overline{S} \colon S \in \Omega^\circ\}$$

is a complete partially ordered tree such that for any $S_1 \neq S_2 \in \Omega^\circ$, the set $\overline{S_1} \cap \overline{S_2}$ is $S_1 \cap S_2$ and belongs to Ω.

Proof. In order to show that $(\Omega, \subset)$ is a partially ordered tree, it suffices to show that for any $S \in \Omega^\circ$, we have $[0, \overline{S}]_\Omega = \{H \colon H \in [0, S)_{\Omega^\circ}\} \cup \{\overline{S}\}$. Hence, let $H \in \Omega^\circ$ such that $\overline{H} \subset \overline{S}$, with $H \neq S$. We have that $H \cap S$ is closed, and, using that H and S are $\mathbb{R}$-subtrees, we obtain:

$$H \subset \overline{H} = \overline{H} \cap \overline{S} = \overline{H \cap S} = H \cap S.$$

Therefore, $H = \overline{H}$ and $H \in [0, S)_{\Omega^\circ}$.

Next, consider an increasing net $\{S_i\}_{i \in I}$ in Ω°. It holds that $\overline{\bigcup_{i \in I} \overline{S_i}} = \overline{\bigcup_{i \in I} S_i}$, so it follows that Ω is complete. Finally, for any $S_1, S_2 \in \Omega^\circ$ with $\overline{S_1} \neq \overline{S_2}$ we have that $S_1 \neq S_2$, so $S_1 \cap S_2$ is closed and belongs to Ω°. It follows that $\overline{S_1} \cap \overline{S_2} = \overline{S_1 \cap S_2} = S_1 \cap S_2$ is in Ω. $\square$

A point l in a rooted $\mathbb{R}$-tree T is called a *leave* if it is maximal with respect to the order induced by the tree T. The set of all leaves of an $\mathbb{R}$-tree T will be denoted by $\mathcal{L}(T)$. It is straightforward to show that $T = \bigcup_{l\in\mathcal{L}(T)}[0,l]_T$. We introduce one final notation before the main result of the subsection: Given a family $\mathcal{S}$ of closed $\mathbb{R}$-subtrees of an $\mathbb{R}$-tree T, and a point $p \in T$, we define:

$$R(p,\mathcal{S}) = \sup_{S\in\mathcal{S}} (S \cap [0,p]_T) \in [0,p]_T.$$

Note that if $p < q \in T$ and $p \notin \bigcup_{S\in\mathcal{S}} S$, then $R(p,\mathcal{S}) = R(q,\mathcal{S})$.

THEOREM 3.14 ([21]). *Let T be an $\mathbb{R}$-tree. Then $\mathcal{F}(T)$ is 1-Plichko witnessed by a pair (Δ, N) where Δ is a linearly dense set of molecules.*

PROOF. We divide the proof into five steps. Let us briefly and informally comment on the strategy of the proof: We will construct a transfinite and increasing sequence $\{\Omega_\alpha\}_{\alpha<\omega_1}$ of partially ordered trees consisting of closed separable $\mathbb{R}$-subtrees of T, in such a way that the whole $\mathbb{R}$-tree T is covered by the end of the construction. This partially ordered tree assigns an ordinal height to each point in the $\mathbb{R}$-tree T: namely, the minimum ordinal α such that the point appears in a $\mathbb{R}$-subtree of Ω_α. We will then consider certain elementary molecules of the form $m_{p,q} \in \mathcal{F}(M)$ such that the height of q is the successor of the height of p, and p,q range over a sufficiently small dense subset of T. The $\mathbb{R}$-tree structure of T will allow us to extend any Lipschitz function defined on a separable subset to a function which is constant in all but countably many $\mathbb{R}$-subtrees included in any Ω_α. This will imply that only countably many of the chosen elementary molecules do not vanish at the constructed extension.

1.- Setup of the inductive construction

Let ω_1 be the first uncountable ordinal. We will construct by transfinite induction an increasing family of partially ordered trees $\{\Omega_\alpha\}_{\alpha<\omega_1}$ consisting of separable closed $\mathbb{R}$-trees. Simultaneously, we will construct for each $\alpha < \omega_1$ a family of countable sets $\mathcal{D}_\alpha = \{D_S\}_{S\in\Omega_\alpha}$. Let us state precisely the properties of the sets Ω_α and $\mathcal{D}_\alpha$ for a given ordinal $\alpha < \omega_1$:

(O1) The pair $(\Omega_\alpha, \subset)$ is a partially ordered tree consisting of separable, closed $\mathbb{R}$-trees of T.
(O2) Ω_α is complete as a partially ordered tree.
(O3) Given $S \in \Omega_\alpha$ and $\beta < \alpha$, if there exists $H \in \Omega_\beta$ such that $S \subset H$, then $S \in \Omega_\beta$.
(O4) Given $S \neq H \in \Omega_\alpha$, there exists $\beta < \alpha$ such that $S \cap H$ belongs to Ω_β.
(O5) Maximality of successor generations: If α is the successor of an ordinal β, for any leaf $l \in \mathcal{L}(T) \setminus \left\{\bigcup_{S\in\Omega_\alpha} S\right\}$ there exists an element $S \in \Omega_\alpha$ such that $S \cap [0,l]_T$ is not contained in any element of Ω_β.
(D1) For every $S \in \Omega_\alpha$, the set D_S is a countable and dense subset of S.
(D2) Given $S_1 \subset S_2 \in \Omega_\alpha$, we have that $S_1 \cap D_{S_2} = D_{S_1}$.

To start the inductive construction, simply put $\Omega_0 = \{\{0\}\}$ and $\mathcal{D}_0 = \{\{0\}\}$.

2.- The limit ordinal case

Let $\alpha < \omega_1$ be a limit ordinal, and suppose that we have constructed Ω_β and $\mathcal{D}_\beta$ for all ordinals $\beta < \alpha$. Put

$$\Omega_\alpha^\circ = \left\{ \bigcup_{\beta<\alpha} S_\beta \colon S_\beta \in \Omega_\beta,\ S_\beta \subset S_{\beta+1} \text{ for all } \beta < \alpha \right\}$$

and

$$\Omega_\alpha = \left\{ \overline{S} \colon S \in \Omega_\alpha^\circ \right\}.$$

Since α is a countable ordinal, all $\mathbb{R}$-subtrees in Ω_α° and Ω_α are separable. We will show that Ω_α° is a partially ordered tree, and that conditions (O2) and (O4) hold for Ω_α° (that is, they hold considering Ω_α° in place of Ω_α). Note that condition (O4) for Ω_α° and inductive hypothesis imply in particular that the intersection of two different elements in Ω_α° is closed. Then, applying Lemma 3.13 we will have that Ω_α is itself a partially ordered tree satisfying (O1), (O2) and (O4). We will then check (O3) for Ω_α. Since we are at the limit ordinal case, we do not need to show (O5).

We start by showing (O4) for Ω_α°. Let $S \neq H \in \Omega_\alpha^\circ$. We can write $S = \bigcup_{\beta<\alpha} S_\beta$ and $H = \bigcup_{\beta<\alpha} H_\beta$, where $S_\beta, H_\beta \in \Omega_\beta$, and $S_\beta \subset S_{\beta+1}$ and $H_\beta \subset H_{\beta+1}$ for all $\beta < \alpha$. We may assume without loss of generality that $S \not\subseteq H$. Suppose first that there exists $\gamma_0 < \alpha$ such that $H = H_{\gamma_0} \in \Omega_{\gamma_0}$. Since $S \not\subseteq H$, there exists β_0 such that S_{β_0} is not contained in H. Then, using inductive hypothesis and Lemma 3.12 we have that

$$S \cap H = \bigcup_{\beta_0 \leq \beta < \alpha} S_\beta \cap H = S_{\beta_0} \cap H.$$

We conclude, using inductive hypothesis, that $S \cap H$ belongs to $\Omega_{\max\{\beta_0,\gamma_0\}}$, which proves (O4) in this case.

Suppose now that H does not belong to Ω_β for any $\beta < \alpha$. Then, we can find a sequence of ordinals $\{\beta_n\}_{n\in\mathbb{N}}$ with $\beta_n < \beta_{n+1} < \alpha$ for all $n \in \mathbb{N}$ such that $H_{\beta_{n+1}}$ does not belong to Ω_{β_n}, and such that $\alpha = \sup_{n\in\mathbb{N}} \beta_n$. Since S is not contained in H, by passing to a subsequence, we can assume as well that $S_{\beta_n} \not\subseteq H_{\beta_n}$ for all $n \in \mathbb{N}$. This implies, by Lemma 3.12 again, that

$$S_\beta \cap H_{\beta_n} = S_{\beta_n} \cap H_{\beta_n}$$

for all $\beta \geq \beta_n$, which in turn shows that $S \cap H_{\beta_n} = S_{\beta_n} \cap H_{\beta_n}$ for all $n \in \mathbb{N}$.

Suppose that $H_{\beta_n} \subset S_{\beta_n}$ for all $n \in \mathbb{N}$. Then, since $[0, S_{\beta_{n+1}}]_{\Omega_{n+1}}$ is totally ordered and $S_{\beta_n}, H_{\beta_{n+1}}$ belong to this initial segment, we have that either $S_{\beta_n} \subset H_{\beta_{n+1}}$ or $H_{\beta_{n+1}} \subset S_{\beta_n}$. The second possibility implies by (O3) in the inductive hypothesis that $H_{\beta_{n+1}} \in \Omega_{\beta_n}$, which contradicts the choice of $H_{\beta_{n+1}}$. Therefore, we conclude that in this case, $S_{\beta_n} \subset H_{\beta_{n+1}}$ for all $n \in \mathbb{N}$. However, this also leads to a contradiction, since it implies that $S \subset H$.

Hence, there exists $n_0 \in \mathbb{N}$ such that $H_{\beta_{n_0}} \not\subseteq S_{\beta_{n_0}}$. By Lemma 3.12, we get that $S_{\beta_{n_0}} \cap H_\beta = S_{\beta_{n_0}} \cap H_{\beta_{n_0}}$ for $\beta_{n_0} \leq \beta < \alpha$. Fix now $\beta_{n_0} \leq \beta < \alpha$. Then, both sets $S_\beta \cap H_\beta$ and $S_{\beta_{n_0}}$ belong to the initial segment $[0, S_\beta]_{\Omega_\beta}$, which is totally ordered. Suppose for the sake of contradiction that $S_{\beta_{n_0}} \subset S_\beta \cap H_\beta$. Then $S_{\beta_{n_0}} \subset H_\beta$, which implies that $S_{\beta_{n_0}}$ is the set $S_{\beta_{n_0}} \cap H_\beta = H_{\beta_{n_0}} \cap S_{\beta_{n_0}}$. Then $S_{\beta_{n_0}}$ is contained in $H_{\beta_{n_0}}$, contradicting the initial choice of the sequence $\{\beta_n\}_{n\in\mathbb{N}}$. Hence, we have that $S_\beta \cap H_\beta$ is contained in $S_{\beta_{n_0}}$, from which we obtain that

$$S_\beta \cap H_\beta = S_{\beta_{n_0}} \cap H_\beta = S_{\beta_{n_0}} \cap H_{\beta_{n_0}}.$$

Since this holds for $\beta_{n_0} \leq \beta < \alpha$, we conclude that $S \cap H = S_{\beta_{n_0}} \cap H_{\beta_{n_0}} \in \Omega_{\beta_{n_0}}$, which concludes the proof of (O4) for Ω_α°.

Finally, we prove that Ω_α° is a partially ordered tree satisfying also (O2). We start by showing that for $S = \bigcup_{\beta<\alpha} S_\beta \in \Omega_\alpha^\circ$, the set $[0, S]_{\Omega_\alpha^\circ}$ is well ordered. Indeed, note that given $H \in [0, S]_{\Omega_\alpha^\circ}$ with $H \neq S$, we have by (O4) that there exists an ordinal $\beta_0 < \alpha$ such that $H \in \Omega_{\beta_0}$ and $S_{\beta_0} \not\subseteq H$. By Lemma 3.12, we have that $S_\beta \cap H = S_{\beta_0} \cap H$ for $\beta_0 \leq \beta < \alpha$. This implies that

$$H = S \cap H = \bigcup_{\beta \geq \beta_0} S_\beta \cap H = S_{\beta_0} \cap H.$$

Therefore, H is a subset of S_{β_0}. Now, with this reasoning, given two sets $H_1, H_2 \in [0, S]_{\Omega_\alpha^\circ}$ with $H_1, H_2 \neq S$, we can find an ordinal β_0 such that $H_1, H_2 \in \Omega_{\beta_0}$ and $H_1, H_2 \subset S_{\beta_0}$. Since Ω_{β_0} is a partially ordered tree, we conclude that H_1 and H_2 are comparable, and thus $[0, S]_{\Omega_\alpha^\circ}$ is totally ordered. It is also well ordered, since given a nonempty subset $A \subset [0, S]_{\Omega_\alpha^\circ}$, we have two possibilities: either $A = \{S\}$, or A contains an element H strictly contained in S, In the first case A trivially has a least element, while in the second case we have, using (O4), that H belongs to Ω_{β_0} for some $\beta_0 < \alpha$. Moreover, it is straightforward to see that a least element of $A \cap [0, H]_{\Omega_{\beta_0}}$ in Ω_{β_0} is also a least element of A in Ω_α°. The conclusion now follows by inductive hypothesis.

It only remains to show that Ω_α° is complete. Let $\{S_i\}_{i\in I}$ be a completely ordered subset of Ω_α°. For every $i \in I$, define $\beta_i \leq \alpha$ as the least ordinal such that $S_i \in \Omega_{\beta_i}$. Suppose first that there exists $i_0 \in I$ such that $\beta_{i_0} = \alpha$, i.e.: S_{i_0} does not belong to Ω_β for any $\beta < \alpha$. Then, since Ω_α° satisfies (O4), we obtain that for any $i \in I$ such that $S_{i_0} \subset S_i$, we necessarily have that $S_i = S_{i_0}$. Hence, $\bigcup_{i\in I} S_i = S_{i_0}$ which belongs to Ω_α°. Otherwise, suppose that $\beta_i < \alpha$ for all $i \in I$. If $\sup_{i\in I} \beta_i < \alpha$, the result follows from inductive hypothesis. Otherwise, consider a sequence $(i_n)_{n\in\mathbb{N}}$ such that $\beta_{i_n} < \beta_{i_{n+1}}$ and $\sup_{n\in\mathbb{N}} \beta_{i_n} = \alpha$. Then, $S_{i_n} \subset S_{i_{n+1}}$, since otherwise we would have that $S_{i_{n+1}} \subset S_{i_n}$, which by (O3) implies that $S_{i_{n+1}} \in \Omega_{\beta_{i_n}}$, contradicting the minimality of $\beta_{i_{n+1}}$. Similarly, given $i \in I$, there exists $n \in \mathbb{N}$ such that $\beta_i < \beta_{i_n}$, which implies by the same argument that $S_i \subset S_{i_n}$. We conclude that

$$\bigcup_{i\in I} S_i = \bigcup_{n\in\mathbb{N}} S_{i_n},$$

which is clearly an element of Ω_α°.

We have now shown that Ω_α° is a partially ordered tree satisfying (O2) and (O4), and thus it follows easily from Lemma 3.13 that Ω_α satisfies (O1), (O2) and (O4). We show now that Ω_α also satisfies (O3). Let $S = \bigcup_{\beta<\alpha} S_\beta \in \Omega_\alpha^\circ$, and suppose there exists $\beta_0 < \alpha$ and $H \in \Omega_{\beta_0}$ such that $\overline{S} \subset H$. Then $S_\beta \subset H$ for all $\beta < \alpha$, and by inductive hypothesis on Ω_β we get that $S_\beta \in \Omega_{\beta_0}$ for all $\beta < \alpha$. Since Ω_{β_0} is complete, we obtain that $\overline{S} \in \Omega_{\beta_0}$ as well.

To finish the limit ordinal case, it only remains to define $\mathcal{D}_\alpha$. We do so by defining, for every $S = \bigcup_{\beta<\alpha} S_\beta \in \Omega_\alpha^\circ$ the set $D_{\overline{S}} = \bigcup_{\beta<\alpha} D_{S_\beta}$. Since α is countable, it holds that $D_{\overline{S}}$ is countable for every $S \in \Omega_\alpha^\circ$. Since D_{S_β} is dense in S_β, it also holds that $D_{\overline{S}}$ is dense in $\overline{S}$. Finally, given $S = \bigcup_{\beta<\alpha} S_\beta \in \Omega_\alpha^\circ$, if H is another element of Ω_α° with $\overline{H} \subset \overline{S}$ and $\overline{H} \neq \overline{S}$, we have by Lemma 3.13 that $\overline{S} \cap \overline{H} = S \cap H = \overline{S \cap H} = \overline{H}$. In particular, this implies that H is closed and that H is a subset of S. Arguing as before, we obtain that there exists $\beta_0 < \alpha$ such that $H \in \Omega_{\beta_0}$ and $H \subset S_{\beta_0}$. The inductive hypothesis now implies that $H \cap D_{S_\beta} = D_H$ for all $\beta \geq \beta_0$ and thus $H \cap D_S = D_H$. We have then that conditions (D1) and (D2) are satisfied, and we conclude the limit ordinal case of the inductive process.

3.- The successor ordinal case

Suppose now that Ω_α has been defined for a countable ordinal α. If $T = \bigcup_{S\in\Omega_\alpha} S$ we stop the inductive process. Otherwise, there exists at least one leaf of T which is not contained in $\bigcup_{S\in\Omega_\alpha}$. Before defining $\Omega_{\alpha+1}$ we need some previous definitions and observations.

For every $p \in \bigcup_{S\in\Omega_\alpha} S$ we may define the ordinal height(p) as the least ordinal $\beta \leq \alpha$ such that $p \in H$ for some $H \in \Omega_\beta$. Using minimality, such a set is unique by condition (O4). Hence, we may define $H(p) \in \Omega_{\text{height}(p)}$ as the (unique) smallest $\mathbb{R}$-subtree in Ω_α containing p. Indeed, observe that given a point $p \in \bigcup_{S\in\Omega_\alpha}$, for any element $S \in \Omega_\alpha$ containing p, it holds that $H(p) \subset S$. Otherwise, the set $H(p) \cap S$ is strictly contained in $H(p)$, and thus by (O3) and (O4) there exists an ordinal $\beta_0 <$ height(p) such that $H(p) \cap S \in \Omega_{\beta_0}$. Since $p \in H(P) \cap S$, this contradicts the minimality of the ordinal height(p). In particular, this implies that given $p \leq q \in \bigcup_{S\in\Omega_\alpha} S$, it holds that $H(p) \subset H(q)$.

Next, consider any leave $l \in \mathcal{L}(T)$ such that $l \notin \bigcup_{S\in\Omega_\alpha} S$. Consider $R(l, \Omega_\alpha) = \sup_{S\in\Omega_\alpha} S \cap [0, l]_T$. We will show that $R(l, \Omega_\alpha) \in \bigcup_{S\in\Omega_\alpha} S$. To show the existence of a set in Ω_α containing $R(l, \Omega_\alpha)$, we may assume that $R(l, \Omega_\alpha) \neq 0$ and consider a strictly increasing sequence $\{p_n\}_{n\in\mathbb{N}}$ in $[0, R(l, \Omega_\alpha))_T$ converging to $R(l, \Omega_\alpha)$. By definition of $R(l, \Omega_\alpha)$, height(p_n) is at most α and the set $H(p_n)$ belongs to Ω_α for every $n \in \mathbb{N}$. Since $H(p_n)$ is contained in $H(p_{n+1})$ for all n, using (O2), there exists a closed separable $\mathbb{R}$-subtree S_l in Ω_α containing $\bigcup_{n\in\mathbb{N}} H(p_n)$. Such a closed set must contain the limit point $R(l,\Omega_\alpha)$.

We can now define $\Omega_{\alpha+1}$. Using Zorn's Lemma, consider a maximal set of leaves $\mathcal{L}_{\alpha+1} \subset \mathcal{L}(T) \setminus \left\{\bigcup_{S\in\Omega_\alpha} S\right\}$ such that for any two different $l_1, l_2 \in \mathcal{L}_{\alpha+1}$, there exists $S \in \Omega_\alpha$ such that $[0,l_1]_T \cap [0,l_2]_T \subset S$. Define:

$$\Omega_{\alpha+1} = \Omega_\alpha \cup \{H(R(l,\Omega_\alpha)) \cup [0,l]_T \colon l \in \mathcal{L}_{\alpha+1}\}.$$

Let us check that $\Omega_{\alpha+1}$ satisfies (O1)-(O5). Clearly every set in $\Omega_{\alpha+1}$ is a closed and separable $\mathbb{R}$-subtree of T. Moreover, given any $l \in \mathcal{L}_{\alpha+1}$, it holds that

$$[0, H(R(l,\Omega_\alpha)) \cup [0,l]_T]_{\Omega_{\alpha+1}} = [0, H(R(l,\Omega_\alpha))]_{\Omega_\alpha} \bigcup (H(R(l,\Omega_\alpha)) \cup [0,l]_T).$$

Indeed, if $S \in \Omega_{\alpha+1}$ is a proper subset of $H(R(l,\Omega_\alpha)) \cup [0,l]_T$, then necessarily S belongs to Ω_α, since for any $l' \in \mathcal{L}_{\alpha+1}$ different from l, the segments $[0,l]_T$ and $[0,l']_T$ are not comparable. Then, since neither S nor $H(R(l,\Omega_\alpha))$ contain any point strictly bigger than $R(l,\Omega_\alpha)$, and $H(R(l,\Omega_\alpha))$ contains the segment $[0,R(l,\Omega_\alpha)]_T$, we necessarily have that S is contained in $H(R(l,\Omega_\alpha))$. We conclude that every initial segment of $\Omega_{\alpha+1}$ is well ordered, and thus (O1) holds.

To show (O2), consider a totally ordered family $\{S_i\}_{i\in I}$ in $\Omega_{\alpha+1}$. If $S_i \in \Omega_\alpha$ for all $i \in I$, the supremum of $\{S_i\}_{i\in I}$ belongs to Ω_α by inductive hypothesis. Otherwise, there exists $i_0 \in I$ and $l_0 \in \mathcal{L}_{\alpha+1}$ such that $S_{i_0} = H(R(l_0,\Omega_\alpha)) \cup [0,l_0]_T$. As before, since $[0,l_0]_T$ and $[0,l]_T$ are not comparable for any other leaf $l \in \mathcal{L}(T)$, we must have that $S_i = S_{i_0}$ for all S_i containing S_{i_0}. It follows that S_{i_0} is the supremum (it is indeed the maximum) of the family $\{S_i\}_{i\in I}$ and (O2) is proven.

Condition (O3) follows easily from inductive hypothesis and the fact that no leaf in $\mathcal{L}_{\alpha+1}$ is contained in any set of Ω_α. To show property (O4), consider two leaves $l_1 \neq l_2 \in \mathcal{L}_{\alpha+1}$. On the one hand, we have that

$$[0,l_1]_T \cap [0,l_2]_T = [0, \min\{R(l_1,\Omega_\alpha), R(l_2,\Omega_\alpha)\}]_T \subset H(R(l_1,\Omega_\alpha)) \cap H(R(l_2,\Omega_\alpha)).$$

Indeed, the first equality follows from the fact that there exists $S \in \Omega_\alpha$ such that $[0,l_1]_T \cap [0,l_2] \cap S$, while the second is an immediate consequence of the fact that $H(R(l_1,\Omega_\alpha)) \cap H(R(l_2,\Omega_\alpha))$ is an $\mathbb{R}$-subtree of T.

On the other hand, by definition of $R(l_1,\Omega_\alpha)$ and $R(l_2,\Omega_\alpha)$ we also have that

$$H(R(l_1,\Omega_\alpha)) \cap [0,l_2]_T \subset H(R(l_1,\Omega_\alpha)) \cap H(R(l_2,\Omega_\alpha))$$
$$H(R(l_2,\Omega_\alpha)) \cap [0,l_1]_T \subset H(R(l_1,\Omega_\alpha)) \cap H(R(l_2,\Omega_\alpha))$$

Therefore, we obtain that

$$(H(R(l_1,\Omega_\alpha)) \cup [0,l_1]_T) \cap (H(R(l_2,\Omega_\alpha)) \cup [0,l_2]_T) = H(R(l_1,\Omega_\alpha)) \cap H(R(l_2,\Omega_\alpha)),$$

which belongs to Ω_α by inductive hypothesis.

Finally, we show (O5). By contradiction, suppose there exists a leaf $l_0 \in \mathcal{L}_{\alpha+1} \setminus \left\{\bigcup_{S\in\Omega_{\alpha+1}} S\right\}$ such that for every element $H \in \Omega_{\alpha+1}$ there exists $S \in \Omega_\alpha$

such that $[0, l_0]_T \cap H$ is contained in S. However, this directly contradicts the maximality of the family $\mathcal{L}_{\alpha+1}$.

Once (O1)-(O5) have been shown, consider for every $l \in \mathcal{L}_{\alpha+1}$ a countable dense set $D'_l \subset (R(l, \Omega_\alpha), l]_T$, and put

$$D_{H(R(l,\Omega_\alpha)) \cup [0,l]_T} = D_{H(R(l,\Omega_\alpha))} \cup D'_l.$$

It is straightforward to check that (D1) and (D2) are satisfied. The induction process is finished.

4.- T is covered by $\{\Omega_\alpha\}_{\alpha<\omega_1}$

Once Ω_α is constructed for every $\alpha < \omega_1$, the first thing we check is that for every point $p \in T$ there exists $\alpha < \omega_1$ and $S \in \Omega_\alpha$ such that $p \in S$. In order to do this, it is enough to prove it for any leaf $l \in \mathcal{L}(T)$. Suppose for the sake of contradiction that there exists $l_0 \in \mathcal{L}(T)$ such that $l_0 \notin \bigcup_{S \in \Omega_\alpha} S$ for every $\alpha < \omega_1$. We will construct by induction a transfinite sequence $\{p_\alpha\}_{\alpha<\omega_1}$ in $[0, l_0]_T$ such that for every $\alpha < \omega_1$

(P1) $p_\alpha \in \bigcup_{S \in \Omega_\alpha} S$
(P2) For all $\beta < \alpha$ we have that $p_\beta < p_\alpha$.

Put $p_0 = 0$. Suppose p_β has been defined for all β smaller than a limit ordinal α. Define $p_\alpha = \sup\{p_\beta\}$. Since p_β is in $[0, l_0]_T$ for all $\beta < \alpha$, we have that $p_\alpha \in [0, l_0]_T$. Condition (P1) is satisfied since Ω_α is complete as a partially ordered tree. Finally, using inductive hypothesis, condition (P2) holds since $p_\alpha \geq p_{\beta+1} > p_\beta$ for all $\beta < \alpha$.

Suppose now that p_α has been defined for a countable ordinal α. Since $l_0 \notin \bigcup_{S \in \Omega_{\alpha+1}} S$, condition (O5) ensures that there exists an element $S \in \Omega_{\alpha+1}$ such that $[0, l_0]_T \cap S$ is not contained in any element of Ω_α. Since $H(p_\alpha) \in \Omega_\alpha$ and contains the segment $[0, p_\alpha]_T$, this implies that there exists an element $p_{\alpha+1} \in [0, l_0]_T \cap S$ such that $p_{\alpha+1} > p_\alpha$. Conditions (P1) and (P2) are easily seen to be satisfied in this case, and the inductive process is finished.

Now, the set $\{p_\alpha\}_{\alpha<\omega_1}$ is an strictly increasing uncountable chain in the separable set $[0, l_0]_T$, a contradiction. Indeed, since the metric topology in $[0, l_0]_T$ is the order topology, with an strictly increasing uncountable chain we would be able to define uncountable family of pairwise disjoint nonempty sets in $[0, l_0]_T$.

Therefore, we have that $T = \bigcup_{\alpha<\omega_1} \bigcup_{S \in \Omega_\alpha} S$. This implies that the set $D = \bigcup_{\alpha<\omega_1} \bigcup_{S \in \Omega_\alpha} D_S$ is dense in T. Moreover, for every $S, H \in \bigcup_{\alpha<\omega_1} \Omega_\alpha$, we have by (D2) that $(H \cap S) \cap D_S = D_{H \cap S}$, so we obtain that $D_{H \cap S}$ is contained in D_S. It follows that for every $S \in \bigcup_{\alpha<\omega_1} \Omega_\alpha$ we have $D \cap S = D_S$, which is a countable dense set in S.

Note as well that since $T = \bigcup_{\alpha<\omega_1} \bigcup_{S \in \Omega_\alpha} S$, we can define $\text{height}(p)$ and $H(p) \in \Omega_{\text{height}(p)}$ for every point $p \in T$.

5.- The witnessing set of molecules.

We can now define the subset of the molecules of $\mathcal{F}(T)$ that witnesses the 1-Plichko property. Define:

$$\Delta = \left\{m_{R(p,\Omega_\alpha),p} \in \mathcal{F}(T) \colon p \in D \text{ and } \operatorname{height}(p) = \alpha + 1\right\}$$

We will show that Δ is linearly dense and that the set of countably supported functions on Δ is 1-norming.

In order to show that Δ is linearly dense in $\mathcal{F}(M)$, it is enough to prove that $\delta(p)$ belongs to the closed linear span of Δ for every $p \in T$. We prove it by induction on the height of $p \in T$.

If $\operatorname{height}(p) = 0$, then $\delta(p) = 0$, which trivially belongs to the closed linear span of Δ. Suppose first that the closed linear span of Δ contains $\delta(q)$ for all $q \in T$ with $\operatorname{height}(q) < \alpha$, for some limit ordinal $\alpha < \omega_1$. Then, by definition of Ω_α, there exists for every $\beta < \alpha$ a set $S_\beta \in \Omega_\beta$ such that $H(p) = \overline{\bigcup_{\beta<\alpha} S_\beta}$. Then $\delta(p)$ is the limit of a sequence $\{\delta(q_\beta)\}_{\beta<\alpha}$ such that $q_\beta \in S_\beta$ for all $\beta < \alpha$. It follows by inductive hypothesis that $\delta(p)$ belongs to the closed linear span of Δ.

Suppose now that $\operatorname{height}(p) = \alpha + 1$ and that $\delta(q) \in \overline{\operatorname{span}}\Delta$ for all $q \in T$ with $\operatorname{height}(q) \leq \alpha$. We may assume that $p \in D$. Note that $R(p, \Omega_\alpha)$ is a point of height less or equal than α, and thus $\delta\left(R(p, \Omega_\alpha)\right.$. Since $m_{R(p,\Omega_\alpha),p}$ belongs to Δ, it follows that:

$$\delta(p) = \delta\left(R(p, \Omega_\alpha) - d\left(R(p, \Omega_\alpha), p\right) m_{R(p,\Omega_\alpha),p} \in \operatorname{span}\Delta.\right.$$

We conclude that $\mathcal{F}(M) = \overline{\operatorname{span}}\Delta$.

To finish the proof, we show that $N = \{f \in \operatorname{Lip}_0(M) \colon f \text{ is countably supported in } \Delta\}$ is 1-norming. In order to prove this, we will show that any 1-Lipschitz function defined on a separable subset of T and which vanishes at 0 can be extended to a 1-Lipschitz function defined in the whole $\mathbb{R}$-tree T and belonging to N. Using Kalton's Lemma 2.19, this will yield that N is 1-norming.

Therefore, fix a separable subset A of T containing 0, and fix a 1-Lipschitz function $g \in \operatorname{Lip}_0(A)$. Since A is separable, there exists a countable ordinal α and a family $\mathcal{S} \subset \Omega_\alpha$ of closed separable $\mathbb{R}$-subtrees such that $A \subset \overline{\bigcup_{S\in\mathcal{S}} S}$.

Using McShane's Theorem 2.13, we can extend g to a 1-Lipschitz function $\hat{g} \in \operatorname{Lip}_0\left(\overline{\bigcup_{S\in\mathcal{S}} S}\right)$. Finally, we define the function $f \in \operatorname{Lip}_0(T)$ by

$$f(p) = \begin{cases} \hat{g}(p), & \text{if } p \in \overline{\bigcup_{S\in\mathcal{S}} S} \\ \hat{g}\left(R\left(p, \mathcal{S}\right)\right), & \text{otherwise.} \end{cases}$$

Note that since $\overline{\bigcup_{S\in\mathcal{S}} S}$ is a closed $\mathbb{R}$-subtree of T, we can write f as $\hat{g} \circ R_{\mathcal{S}}$, where $R_{\mathcal{S}}$ is the 1-Lipschitz retraction associated to $\overline{\bigcup_{S\in\mathcal{S}} S}$ described in Lemma 3.11. This yields immediately that f is 1-Lipschitz.

Now, consider a point $p \in D$ with $\operatorname{height}(p) = \beta + 1$ for some countable ordinal β. We will show that if $\langle m_{R\left(p,\Omega_\beta\right),p}, f\rangle \neq 0$, then $p \in \overline{\bigcup_{S\in\mathcal{S}} S}$. Since $D \cap \bigcup_{S\in\mathcal{S}} S$ is countable, this will show that $f \in N$, which finishes the proof.

Suppose then that $p \notin \overline{\bigcup_{S\in\mathcal{S}} S}$. If $R(p,\Omega_\beta)$ also does not belong to $\overline{\bigcup_{S\in\mathcal{S}} S}$, then, since $R(p,,\Omega_\beta) \leq p$, we obtain that $R(R(p,\mathcal{S}),\Omega_\beta) = R(p,\mathcal{S})$, and it follows that $\langle m_{R(p,\Omega_\beta),p}, f\rangle = 0$ by definition of f.

Otherwise, suppose that $R(p,\Omega_\beta)$ is contained in $\overline{\bigcup_{S\in\mathcal{S}} S}$. We will show that $R(p,\Omega_\beta) = R(p,\mathcal{S})$. Since $R(p,\Omega_\beta) < p$ and it belongs to $\overline{\bigcup_{S\in\mathcal{S}} S}$, it holds that $R(p,\mathcal{S}) \geq R(p,\Omega_\beta)$. Suppose for the sake of contradiction that $R(p,\mathcal{S}) > R(p,\Omega_\beta)$. By definition of $R(p,\mathcal{S})$, and since $p \notin \overline{\bigcup_{S\in\mathcal{S}} S}$, there exists $S_0 \in \mathcal{S} \subset \Omega_\alpha$ such that $R(p,\Omega_\beta) < \max\{S_0 \cap [0,p]_T\} < p$. Consider the $\mathbb{R}$-subtree $H(p) \cap S_0$, which contains the point $\max\{S_0 \cap [0,p]_T\}$. Since the height of p is $\beta+1$, we have that $H(p) \in \Omega_{\beta+1}$. Now, using (O3) and (O4), and the fact that $H(p) \cap S_0$ is strictly contained in $H(p)$, we deduce that $H(p) \cap S_0$ is in Ω_β. However, this implies that $R(p,\Omega_\beta) \geq \max\{S_0 \cap [0,p]_T\}$, which leads to a contradiction. We conclude that $f \in N$ and the proof is finished.

□

3.3. Lipschitz analogues to Complementation Properties

We have now studied Lipschitz retractional skeletons as the natural extension of the concept of projectional skeletons to the metric space setting. In this section, we continue by analyzing the situation that arises when considering the non-linear analogues to the classical Complementation Properties we discussed in Chapter 2.

Definition 3.15. Given α, β two cardinal numbers with $\alpha \leq \beta$, we say that a metric space M has the *(α,β) Lipschitz Retraction Property (Lipschitz RP(α,β)* for short), if for every closed subset $N \subset M$ with $\operatorname{dens}(N) = \alpha$ there exists another subset S that contains N, such that $\operatorname{dens}(S) \leq \beta$ and S is a Lipschitz retract of M. We say that M has the *Separable Lipschitz Retraction Property (Lipschitz SRP)* if it has the Lipschitz RP$(\aleph_0,\aleph_0)$.

Analogously to the linear setting, if a metric space M admits a Lipschitz retractional skeleton, then M has the Lipschitz SRP (a similar proof to Proposition 2.6 works in the non-linear case). Hence, the Lipschitz SRP is a useful tool to produce counterexamples of metric spaces which do not admit Lipschitz retractional skeletons. We will expand on this approach in Chapter 4, where we construct metric spaces failing the Lipschitz RP(Λ,Λ) for every infinite cardinal Λ.

Additionally, Proposition 2.16 readily implies that if M has the Lipschitz RP(α,β), then $\mathcal{F}(M)$ has the CP(α,β) for any two cardinal numbers $\alpha \leq \beta$. It is an open question whether every Lipschitz-free has the SCP.

For now, let us give results which describe the situation in the class of $C(K)$ spaces for a compact Hausdorff topological space K. Recall from Chapter 2 that, as shown in [35], under the Generalized Continuum Hypothesis, for any cardinality Λ there exists a compact Hausdorff space K of such that $C(K)$ is an indecomposable space with density character Λ. However, in the non-linear setting, we have that these spaces have a much richer Lipschitz retractional structure:

THEOREM 3.16 ([25]). *The Banach space $C(K)$ of real continuous functions has the Lipschitz SRP for any compact Hausdorff space K. Therefore, the Lipschitz-free space of any $C(K)$ space has the SCP.*

PROOF. Let Y be a separable linear subspace of $C(K)$. Then, there exists a separable linear subspace of $C(K)$ that contains Y and is isometric to a $C(K')$ space for some compact metric space K' (see Exercise 5.88 in [12]). By Theorem 3.5 in [29], Y is an absolute 2-Lipschitz retract, so in particular it is 2-Lipschitz retract of $C(K)$, which concludes the proof. $\square$

In the previous proof, we used that $C(K)$ space are absolute 2-Lipschitz retracts, where the constant 2 is optimal for general $C(K)$ spaces. This way we obtained a slightly stronger result: every separable subspace of a $C(K)$ space is contained in a separable 2-Lipschitz retract of $C(K)$. We briefly show that the constant 2 is also optimal for the Lipschitz SRP in ℓ_∞. The proof of the following result is based on the classical proof that shows that c_0 is not better than a 2-Lipschitz retract of ℓ_∞.

THEOREM 3.17 ([21]). *Let S be a separable subset of ℓ_∞ containing c_0, and suppose there exists a Lipschitz retraction $R\colon \ell_\infty \to S$. Then $\|R\|_{Lip} \geq 2$.*

PROOF. Let S be a separable subset of ℓ_∞ containing c_0. We can write $S = \{x_n = (x_n(k))_{k\in\mathbb{N}} \in \ell_\infty \colon n \in \mathbb{N}\}$. Consider the sequence $z \in \ell_\infty$ given by

$$z(k) = \begin{cases} -\operatorname{sign}(x_k(k)) & \text{for } k \in \mathbb{N} \text{ with } x_k(k) \neq 0 \\ 1 \text{ for } k \in \mathbb{N} \text{ with } x_k(k) = 0. \end{cases}$$

For every $k \in \mathbb{N}$, let $P_k \colon \ell_\infty \to \ell_\infty$ be the linear projection given by $(P_k x)(i) = x(i)$ if $i \leq k$ and $(P_k x)(i) = 0$ if $i > k$ for every $x \in \ell_\infty$. With this notation, define for every $k \in \mathbb{N}$ the sequence $y_k = 2P_k(z)$, which belongs to S since S contains c_0. It is straightforward to check that $\|z - y_k\|_\infty = 1$ for all $k \in \mathbb{N}$.

For every $n \in \mathbb{N}$ and every $k \geq n$, it holds that $\|x_n - y_k\|_\infty \geq 2$, since $|y_k(n)| = 2$ and the n-th coordinate of x_n is 0 or has the opposite sign. It follows that $\limsup_{k\to+\infty} \|x - y_k\|_\infty \geq 2$ for every $x \in S$. Therefore:

$$2 \leq \limsup_{k\to+\infty} \|R(z) - y_k\|_\infty \leq \limsup_{k\to+\infty} \|R\|_{\mathrm{Lip}} \|z - y_k\|_\infty = \|R\|_{\mathrm{Lip}},$$

and the result is proven. $\square$

Using the previous result and the universal property of Lipschitz-free spaces, we obtain the following corollary:

COROLLARY 3.18 ([21]). *Let $1 \leq r < 2$. Then $\mathcal{F}(\ell_\infty)$ is not r-Plichko.*

Proof. Suppose $\mathcal{F}(\ell_\infty)$ is r-Plichko for some $r \geq 1$. By Theorem 2.4 and Proposition 2.18, there exists an r-projectional skeleton $\{P_s\}_{s\in\Gamma}$ and a family of separable subspaces $\{Y_s\}_{s\in\Gamma}$ of ℓ_∞ such that $P_s(\mathcal{F}(\ell_\infty)) = \mathcal{F}(Y_s)$ for all $s \in \Gamma$. Using properties (iii) and (iv) of the definition of projectional skeletons, we have that there exists $s_0 \in \Gamma$ such that $c_0 \subset Y_{s_0}$. Write $\beta_{Y_{s_0}} : \mathcal{F}(Y_{s_0}) \to Y_{s_0}$ to denote the extension of the identity map in Y_{s_0} to the Lipschitz-free space, which exists by Theorem 2.17. The map $\beta_{Y_{s_0}} \circ P_{s_0} \circ \delta \colon \ell_\infty \to Y_{s_0}$ is a Lipschitz retraction onto Y_{s_0} with Lipschitz constant less than or equal to r. By Theorem 3.17 we obtain that $r \geq 2$. □

3.4. Open problems

In the next chapter we will construct examples of complete metric spaces failing the Lipschitz $\mathrm{RP}(\Lambda,\Lambda)$ for every infinite cardinal Λ. However, as we mentioned throughout the previous discussion, there are currently no examples of Banach spaces known to fail any of these properties.

Problem 3.19. Given an infinite cardinal Λ, does there exist a Banach space failing the Lipschitz $\mathrm{RP}(\Lambda,\Lambda)$? In particular, does there exist a Banach space without the Lipschitz SRP?

The separable case is especially interesting, since the existence of a Banach space without the Lipschitz SRP would imply that there exists a separable Banach space which is not a Lipschitz retract of its bidual, giving a complete answer to the Lindenstrauss conjecture discussed in Chapter 2. In Chapter 5 we introduce the tools to derive the connection between these two questions. In particular, we explicitly obtain this link in the discussion after Problem 5.9.

Regarding the linear Complementation Properties in Lipschitz-free spaces, no counterexample is known at all:

Problem 3.20. Given an infinite cardinal Λ, does there exist a complete metric space whose Lipschitz-free space fails the $\mathrm{CP}(\Lambda,\Lambda)$? In particular, does there exist a complete metric space whose Lipschitz-free space fails the SCP?

For the separable case in both Problems 3.19 and 3.20, a slightly easier question is to find a Banach space without a Lipschitz retractional skeleton, and a complete metric space whose Lipschitz-free space does not have a projectional skeleton, respectively. However, it seems that the most natural strategy to find such counterexamples is through the Lipschitz SRP and the SCP.

We have characterized the Plichko property witnessed by Dirac measures in Lipschitz-free spaces. It would be interesting to strengthen this characterization:

Problem 3.21. Characterize the Plichko property witnessed by molecules in Lipschitz-free spaces.

A particularly relevant case is that of the Lipschitz-free space of Plichko Banach space:

Problem 3.22. Let X be a Banach space with the Plichko property. Is the Plichko property in $\mathcal{F}(X)$ witnessed by molecules?

CHAPTER 4

Pathological metric spaces

In this chapter we present the construction of two classes of complete metric spaces which lack a rich structure of Lipschitz retracts of a prescribed density character; namely, we construct for every infinite cardinal Λ a complete metric space failing the Lipschitz $\mathrm{RP}(\Lambda, \Lambda)$. For the Lipschitz SRP (that is, when $\Lambda = \aleph_0$), it is actually possible to obtain a complete metric space in which every separable set with at least two points fails to be a Lipschitz retract.

4.1. Failing the Lipschitz SRP: The skein space.

As mentioned above, the goal of this section is to prove the following result:

THEOREM 4.1 ([24]). *There exists a complete metric space M of cardinality continuum such that every separable subset of M with at least two points is not a Lipschitz retract of M. In particular, M fails the Lipschitz SRP.*

In order to obtain a metric space with this property, we will construct for each ordinal α a complete metric space $\mathrm{Sk}(\alpha)$, called the *skein space of order α*. We will show that if α is an ordinal with uncountable cofinality, the space $\mathrm{Sk}(\alpha)$ satisfies Theorem 4.1.

The construction of the skein spaces is self-contained though technical. It is divided into three subsections, going from subsection 4.1.1 to 4.1.3. In 4.1.1 we define the basic pieces of the construction, called *threads*. These threads are isometric to subsets of one-dimensional circles with the distance given by the arc-length. We will define uncountable families of totally disconnected threads which satisfy certain metric properties related to Lipschitz functions between these threads (see Theorem 4.9).

In 4.1.2 we will use these uncountable families of threads to define the building blocks of the final metric space. These building blocks are called *threading spaces*, and each block is built from one of the uncountable families defined in 4.1.1. All threads that form each one of these threading spaces are attached to two anchor points $\{0, 1\}$ in the threading space, and every one of these threading spaces satisfies the weaker property that every separable subset containing both anchor points is not a Lipschitz retraction of the whole threading space.

In subsection 4.1.3 we finish by using these threading spaces to construct the final metric space via a transfinite inductive process of length ω_1. We call the resulting complete metric space the *skein space*. Very informally, the skein space satisfies that any pair of points behaves as the pair of anchor points of one of the threading spaces constructed in section 4.1.2. This way we have that any separable space with more than one point contains two anchor points of a threading space, and hence it is not a Lipschitz retract of the whole skein.

4.1.1. Construction of the fundamental pieces: Threads with infinitely many gaps.

4.1.1.1. *Threads.* Let $l, a > 0$ with $a \leq l$. We say that a metric space $(T, d_{l,a})$ is an $\mathbb{R}$-*thread of length* l *and width* a if T is a closed subset of the real segment $[0, l]$ containing 0 and l, and the metric $d_{l,a}$ is defined by

$$d_{l,a}(x, y) = \min\{|x - y|, x + (l - y) + a, y + (l - x) + a\}$$

for every $x, y \in T$. Our main example will be constructed inductively by repeated adjoining of metric spaces, isometric to a thread described above, to the previous space. In this sense, the adjoined new pieces are certainly meant to be distinct sets. However, keeping in mind this feature, there is no danger of confusion if we simply call any metric space T a *thread of length* l *and width* a if T is isometric to an $\mathbb{R}$-thread of length l and width a as defined above (and work with it using the above description).

Let us mention some basic facts about threads. First, notice that every thread is a compact metric space. Also, we may define in every thread $(T, d_{l,a})$ the natural order and the Lebesgue measure since the set T is a subset of the real line. Then, for every $x, y \in T$ with $x \leq y$ we define the set $[x, y]_T \subset T$ as $[x, y] \cap T$, where $[x, y]$ is the usual real segment. The set $[x, y]_T$ with the inherited metric is again a thread.

If T is a thread of length l, we say that a closed subset I of T is an *extended interval* of T if I is of the form $[p, q]_T = [p, q] \cap T$ or $[0, p]_T \cup [q, l]_T$ for a pair of points $p, q \in T$ with $p < q$. In either case, the points p and q are called the *extreme points* of I. See Figure 1 for a representation of a thread and the two kinds of extended intervals it contains.

Notice as well that in a thread of length l and width a, the distance between the extreme points 0 and l is exactly the width a. We can also realize that every thread is locally isometric to T with the usual metric inherited from the real line; indeed, if the distance between two points of T is less than the width of the thread, then this distance coincides with the usual metric. As a consequence, we have that if the length and the width of a thread coincide, then the thread is isometric to a subset of the real segment $[0, l]$.

The way we compute the distance in threads implies that Lipschitz functions from threads into other metric spaces are similar to Lipschitz functions from intervals. Specifically we have the following result:

Proposition 4.2. *Let T be a thread of length l_T and width a_T, let M be a metric space, and let $K \geq 0$. A function $F \colon T \to M$ is K-Lipschitz if and only if $d(F(0), F(l_T)) \leq K a_T$, and for every $x, y \in T$ we have $d\big(F(x), F(y)\big) \leq K|y - x|$.*

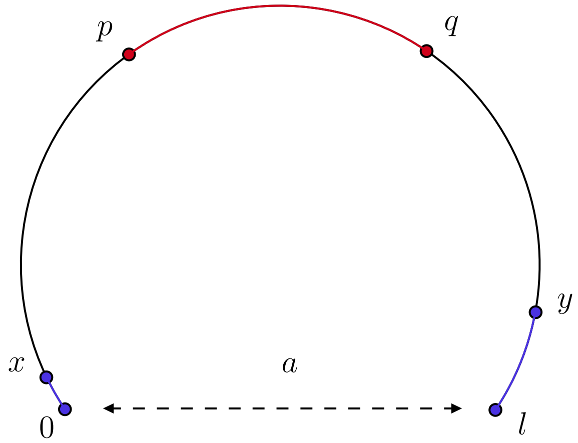

FIGURE 1. Thread of length l and width a. The red and blue lines correspond to the two kinds of extended intervals possible in a thread.

PROOF. Evidently, if F is K-Lipschitz, we directly obtain that $d(F(0), F(l_T)) \leq Ka_T$ and the inequality:

$$d\big(F(x), F(y)\big) \leq Kd(x, y) \leq K|y - x|.$$

Suppose now that the inequality is true for every pair of points in T, and take $x \leq y \in T$. If $d(x, y) = y - x$, then we obtain directly that $d\big(F(x), F(y)\big) \leq Kd(x, y)$. Otherwise, we have that $d(x, y) = x + a_T + (l_T - y)$. Therefore,

$$\begin{aligned} d\big(F(x), F(y)\big) &\leq d\big(F(x), F(0)\big) + d\big(F(0), F(l_T)\big) + d\big(F(l_T), F(y)\big) \\ &\leq K(|x| + a_T + |l_T - y|) = Kd(x, y). \end{aligned}$$

Hence, F is K-Lipschitz. □

4.1.1.2. *Lipschitz functions between threads with gaps.* In a thread T, we say that a non-trivial open interval $(x, y) \subset \mathbb{R}$ is a *gap* of T if $x, y \in T$ and $(x, y) \cap T = \emptyset$. The points x, y of a gap $C = (x, y)$ in a thread T are called the *endpoints* of C, and the value $d(x, y)$ is the *length* of the gap. It is readily seen that a closed subset of $\mathbb{R}$ can have at most countably many distinct gaps. Hence, given any complete thread $T \subset \mathbb{R}$, we may consider the sequence $\{C_k^T\}_{k\in\mathbb{N}}$ of gaps in T. Moreover, since every thread T is bounded, its sequence of gaps can be ordered so that $\text{length}(C_{k+1}^T) \leq \text{length}(C_k^T)$ for all $k \in \mathbb{N}$.

We are going to study in detail the behavior of Lipschitz maps between threads with infinitely many gaps. We have the following property.

LEMMA 4.3. *Let T and S be two threads of length l_T, l_S and width a_T, a_S respectively. Let $K \geq 1$, and suppose that there is no gap in T with length greater* *than or equal to a_S/K. Then for every K-Lipschitz function $F\colon T \to S$ we have that*

$$|F(q) - F(p)| \leq K|q - p|, \qquad \text{for all } p, q \in T.$$

PROOF. For any pair of points $p, q \in T$ with $p \leq q$ there exists an increasing finite sequence $(x_k)_{k=1}^n \subset T$ with $x_1 = p$ and $x_n = q$ such that $d(x_k, x_{k+1}) < a_S/K$ for all $1 \leq k \leq n-1$. This implies that $d\big(F(x_{k+1}), F(x_k)\big) = \big|F(x_{k+1}) - F(x_k)\big|$. Hence, we have

$$\begin{aligned} |F(q) - F(p)| \leq \sum_{k=1}^{n} \big|F(x_{k+1}) - F(x_k)\big| &= \sum_{k=1}^{n} d\big(F(x_{k+1}), F(x_k)\big) \\ \leq K \sum_{k=1}^{n} d(x_{k+1}, x_k) &\leq K \sum_{k=1}^{n} (x_{k+1} - x_k) = K(q - p). \end{aligned}$$

The result is proven.

□

Next, we are going to prove an elementary proposition which will allow us to assume without loss of generality that the Lipschitz maps we consider are non-decreasing.

PROPOSITION 4.4. *Let $K \geq 1$. Let T, S be two threads of length l_T, l_S and width a_T, a_S respectively, and let $F\colon T \to S$ be a K-Lipschitz function such that $F(0) = 0$ and $F(l_T) = l_S$. Then there exists a non-decreasing Lipschitz function $\widehat{F}\colon T \to S$ with $\|\widehat{F}\|_{Lip} \leq \|F\|_{Lip}$ such that $\widehat{F}(0) = 0$ and $\widehat{F}(l_T) = l_S$.*

PROOF. Put $K = \|F\|_{\mathrm{Lip}}$. Notice that if T has a gap (p, q) of length greater than or equal to a_S/K, then the result follows directly putting $\widehat{F}(x) = 0$ if $x \leq p$, and $\widehat{F}(x) = l_S$ if $x \geq q$. Suppose then that there are no gaps in T with length greater than or equal to a_S/K.

Now define $\widehat{F}\colon T \to S$ by

$$\widehat{F}(x) = \max_{y \leq x} F(y).$$

Clearly, $\widehat{F}$ is non-decreasing with $F \leq \widehat{F}$, $\widehat{F}(0) = 0$ and $\widehat{F}(l_T) = l_S$. It only remains to see that $\|\widehat{F}\|_{\mathrm{Lip}} \leq K$. Using Proposition 4.2, we only need to prove that given $p, q \in T$ with $p \leq q$, we have

$$d\big(\widehat{F}(q), \widehat{F}(p)\big) \leq K(q - p). \tag{4.1}$$

Observe that $\widehat{F}(q) = F(z)$ for some $z \leq q$. If $z \leq p$ we necessarily have that $\widehat{F}(q) = \widehat{F}(p)$ and the equation is trivially satisfied. Otherwise, using Lemma 4.3

with $p \leq z$ yields

$$\begin{aligned} d\big(\widehat{F}(q), \widehat{F}(p)\big) &\leq \widehat{F}(q) - \widehat{F}(p) = F(z) - \widehat{F}(p) \\ &\leq F(z) - F(p) \leq K(z-p) \leq K(q-p), \end{aligned}$$

and equation (4.1) is proven.

We conclude that $\|\widehat{F}\|_{\mathrm{Lip}} \leq K$ and the result is proven. $\square$

We can also use Lemma 4.3 to prove a similar result to the one above.

PROPOSITION 4.5. *Let T and S be two threads with length l_T and l_S, and width a_T and a_S respectively. Let $K \geq 1$. Suppose there exists a K-Lipschitz function $F\colon T \to S$ such that $F(0) = A$ and $F(l_T) = B$, for two points $A, B \in S$ with $A < B$. If T does not have any gap of length greater than or equal to a_S/K, then the function $\widehat{F}\colon T \to [A, B]_S$ defined by*

$$\widehat{F}(x) = \begin{cases} A, & \text{if } F(x) \leq A \\ F(x), & \text{if } F(x) \in [A, B]_S \\ B, & \text{if } F(x) \geq B \end{cases}$$

is K-Lipschitz as well.

PROOF. As before, by Proposition 4.2 we only need to check that for every $p, q \in T$ with $p \leq q$, we have:

$$d\big(\widehat{F}(q), \widehat{F}(p)\big) \leq K(q-p).$$

We will only prove the case when $F(p) \leq A$ and $F(q) \in [A, B]_S$, since the remaining possibilities are shown similarly. By Lemma 4.3, we have in this case that

$$\begin{aligned} d\big(\widehat{F}(p), \widehat{F}(q)\big) &= d\big(A, F(q)\big) \leq F(q) - A \\ &\leq F(q) - F(p) \leq K(q-p). \end{aligned}$$

We conclude that $\|\widehat{F}\|_{\mathrm{Lip}} \leq K$.

$\square$

Let us now give some definitions and prove some technical results which will be heavily used in the proof of the main theorem of the section. Let T and S be two threads, and suppose there is a Lipschitz function $F\colon T \to S$ which is non-decreasing. We say that a gap (p_T, q_T) in T *jumps over a gap* (p_S, q_S) *in S with respect to F* if $F(p_T) \leq p_S$ and $F(q_T) \geq q_S$ (see Figure 2).

The first lemma we prove says intuitively that if we have a non-decreasing Lipschitz function F between two threads T and S that fixes the extreme points of the threads, then every gap in S must be jumped by a gap in T with respect to F. Although this result is fairly intuitive, we include the (simple) proof for completeness.

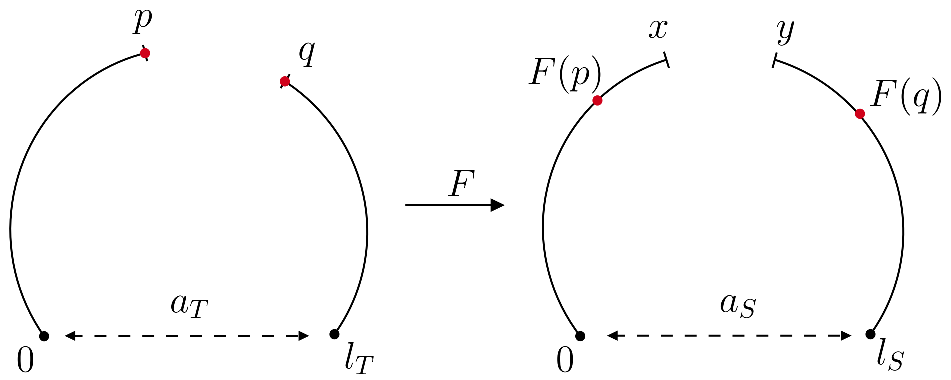

FIGURE 2. The gap (p, q) jumps over (x, y) with respect to F.

LEMMA 4.6. *Let T and S be two threads of length l_T and l_S respectively. Suppose that there is a non-decreasing Lipschitz function $F\colon T \to S$ such that $F(0) = 0$ and $F(l_T) = l_S$. Let C^S be a gap in S. Then there exists a gap in T that jumps over C^S with respect to F.*

PROOF. Define $p_S, q_S \in S$ such that $C^S = (p_S, q_S)$. Consider the points:

$$e_- = \max\{F(x) \in S\colon\ x \in T,\ F(x) \leq p_S\},$$
$$e_+ = \min\{F(y) \in S\colon\ y \in T,\ F(y) \geq q_S\}.$$

These minimum and maximum values always exist since we have that $F(0) = 0$ and $F(l_T) = l_S$, and T is compact. Hence, we can find

$$x_- = \max\{x \in T\colon\ F(x) = e_-\},$$
$$y_+ = \min\{y \in T\colon\ F(y) = e_+\}.$$

Since F is non-decreasing and $e_- < e_+$, we have that $x_- < y_+$ and $(x_-, y_+)_T = \emptyset$. Moreover, both x_- and y_+ belong to T again by compactness, so (x_-, y_+) is a gap in T. The gap (x_-, y_+) jumps over (p_S, q_S) with respect to F. $\square$

The second lemma we prove can also be easily deduced and is intuitively clear. It shows that if a small enough gap (in a sense that is made explicit in the statement of the lemma) C^T in a thread T jumps over several gaps in a thread S simultaneously with respect to a Lipschitz function F, then the length of C^T must be bigger than the length of the smallest subinterval of $[0, 1]$ that contains all the gaps C^T jumps over, divided by the Lipschitz constant of F.

LEMMA 4.7. *Let $K > 1$, and let T and S be two threads of length l_T and l_S respectively. Denote by a_S the width of S. Suppose that there is a non-decreasing Lipschitz function $F\colon T \to S$ with $\|F\|_{Lip} = K$ such that $F(0) = 0$ and $F(l_T) = l_S$. Let C^T be a gap in T such that length$(C^T) < a_S/K$, and let $\big((x_j, y_j)\big)_{j=1}^k$ be a finite* *collection of different gaps in S. If C^T jumps over (x_j, y_j) with respect to F for all $1 \leq j \leq k$, then*

$$K \cdot \mathit{length}(C^T) \geq \max_{j \neq j'} |y_j - x_{j'}|.$$

Proof. Put $C^T = (p, q)$ with $p, q \in T$. Since C^T jumps over (x_j, y_j) for all $1 \leq j \leq k$, we have that $F(p) \leq x_j$ and $F(q) \geq y_j$. Hence, we have that

$$F(q) - F(p) \geq \max_{j \neq j'} |y_j - x_{j'}|.$$

Since $d(p, q) < a_S/K$, when computing the distance between $F(p)$ and $F(q)$ in the thread S, we necessarily have that $d\big(F(p), F(q)\big) = F(q) - F(p)$. Therefore, applying that F is K-Lipschitz we obtain:

$$\max_{j \neq j'} |y_j - x_{j'}| \leq d(F(q), F(p)) \leq K \cdot \text{length}(C^T),$$

and the result is proven. □

We are going to define also a particular kind of intervals which will be useful in the proof of Theorem 4.9. Let $(a, b) \subset [0, 1]$ be a nontrivial open interval, and let $r > 0$. We define the *sweeping of* $[a, b]$ *by* r as the interval

$$\mathcal{D}_r(a, b) = (b - r, a + r).$$

Notice that if $r \leq (b - a)/2$, then $\mathcal{D}_r(a, b) = \emptyset$. We can prove two simple properties about this concept.

Proposition 4.8. *The following properties are satisfied:*

1. *Let $(a, b) \subset [0, 1]$ be a nontrivial open interval, and let $r > 0$. Then the Lebesgue measure of the sweeping $\mathcal{D}_r(a, b)$ is less than $2r$.*
2. *Let $r > 0$, and let T and S be threads of length l_T and l_S respectively. Let $F\colon T \to S$ be a non-decreasing K-Lipschitz map such that $F(0) = 0$ and $F(l_T) = l_S$. Suppose that there is a gap C^T in T such that $\mathit{length}(C^T) < a_S/K$, and such that C^T jumps over two gaps C_1^S, C_2^S in S with respect to F. Moreover, suppose that $C_2^S \not\subseteq \mathcal{D}_r(C_1^S)$. Then $K \cdot \mathit{length}(C^T) > r$.*

Proof. Statement (1) is easy to see. For statement (2), put $C_1^S = (x_1, y_1)$, $C_2^S = (x_2, y_2)$ with $x_1, x_2, y_1, y_2 \in S$. Notice that if $C_2^S \not\subseteq \mathcal{D}_r(C_1^S)$, this means that either $y_1 - r - x_2 > 0$, or $y_2 - x_1 - r > 0$. In any case, we obtain that

$$\max\{|y_1 - x_2|, |y_2 - x_1|\} > r,$$

Now, since C^T jumps over C_1^S and C_2^S simultaneously and $\text{length}(C^T) < a_S/K$, the result follows from Lemma 4.7. □

In figure 3 we have a representation of the situation in (2) of the previous result.

Finally, we are able to prove the main result of the first part of the process:

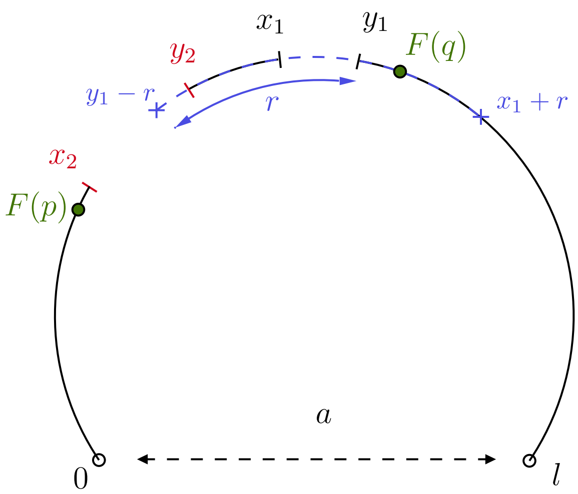

FIGURE 3. The gap $C^T = (p, q)$ jumps over (x_1, y_1) and (x_2, y_2), and $(x_2, y_2) \nsubseteq \mathcal{D}_r\big((x_1, y_1)\big) = (y_1 - r, x_1 + r)$. Hence, $K \cdot \text{length}(C^T) \geq r$.

THEOREM 4.9. *Let $K \geq 1$, let $\{S_n\}_{n\in\mathbb{N}}$ be a countable family of threads of length l_n and width a_n respectively for each $n \in \mathbb{N}$, and let $\varepsilon > 0$ be such that for every $n \in \mathbb{N}$:*

- *The Lebesgue measure of S_n is bigger than ε.*
- *If I is an open interval such that $I \cap S_n$ is nonempty, then there exist infinitely many gaps of S_n contained in I.*

Then, there exists a decreasing sequence $\gamma^* = (\gamma^*_k)_{k\in\mathbb{N}}$ of positive real numbers with the following property:

Let $1 \leq K' \leq K$, let T be a thread of length l_T, and let $\{C^T_k\}_{k\in\mathbb{N}}$ be the sequence of gaps of T ordered decreasingly according to their length. If length$(C^T_k) \leq \gamma^*_k$ for all $k \in \mathbb{N}$, then for every $n \in \mathbb{N}$ such that length$(C^T_1) < a_n/K'$ there does not exist any K'-Lipschitz function $T \to S_n$ such that $F(0) = 0$ and $F(l_T) = l_n$.

PROOF. For each $n \in \mathbb{N}$, let $\{G^n_k\}_{k\in\mathbb{N}}$ be the sequence of gaps of S_n ordered decreasingly according to their length, and put $\alpha^n_k = \text{length}(G^n_k)$. Then $\alpha^n = (\alpha^n_k)_{k\in\mathbb{N}}$ is the decreasing sequence of lengths of the gaps in the thread S_n for each $n \in \mathbb{N}$.

We are going to construct inductively by a diagonal method the sequence $\gamma^* = (\gamma^*_k)_{k\in\mathbb{N}}$ with the following properties:

(1) $\gamma^*_k < 2^{-(k+1)}K^{-1}\varepsilon$, for all $k \in \mathbb{N}$.
(2) Let $1 \leq K' \leq K$, let T be a thread, and let $\{C^T_i\}_{i\in\mathbb{N}}$ be the sequence of gaps of T ordered decreasingly according to their length. If length$(C^T_i) \leq \gamma^*_i$ for all $i \leq k$, then for every $i \in \mathbb{N}$ with $i \leq k$ such that length$(C^T_1) <$

a_i/K there is no Lipschitz map $F\colon T \to S_i$ with $\|F\|_{\text{Lip}} \le K'$ such that $F(0) = 0$ and $F(l_T) = l_i$.

For $k = 1$, define γ_1^* such that $0 < \gamma_1^* < \min\{2^{-2}K^{-1}\varepsilon, K^{-1}\alpha_1^1\}$. Property (1) above is satisfied. Let $1 \le K' \le K$, let T be a thread, let $\{C_i^T\}_{i\in\mathbb{N}}$ be the sequence of its gaps in decreasing length order, and suppose that $\text{length}(C_1^T) \le \gamma_1^*$. Suppose by contradiction that $\text{length}(C_1^T) < a_1/K'$, and that there exists a Lipschitz map $F\colon T \to S_1$ with $\|F\|_{\text{Lip}} \le K'$ and $F(0) = 0$ and $F(l_T) = l_1$. We assume F to be non-decreasing by Proposition 4.4.

The thread S_1 has the gap G_1^1 of length α_1^1. By Lemma 4.6, there exists $i \in \mathbb{N}$ such that the gap C_i^T in T jumps over G_1^1. Then, by Lemma 4.7, since $\text{length}(C_i^T) \le \text{length}(C_1^T) < a_1/K'$, we have that $K' \cdot \text{length}(C_i^T) > \alpha_1^1$. However, we know that

$$K' \cdot \text{length}(C_i^T) \le K \cdot \text{length}(C_1^T) \le K\gamma_1^* < \alpha_1^1,$$

a contradiction. The first step of the induction is done.

Suppose we have selected $\{\gamma_i^*\}_{i=1}^k$ satisfying the desired properties for $k \in \mathbb{N}$. Before continuing with the proof, let us informally give some intuition of the technical argument that follows. We want to define the next element in the sequence (that is, γ_{k+1}^*) small enough so that any thread T with gaps smaller than the first $k+1$ elements of γ^*, and smaller than a_{k+1}, cannot be mapped with a K-Lipschitz function that preserves the extremes into the thread S_{k+1}. However, since the first k elements of γ^* are already set and do not depend on the next thread S_{k+1}, the first k gaps in T could jump over many gaps in S_{k+1}, and in many different ways. Nevertheless, as we are going to see, the fact that S_{k+1} is of large enough measure and contains infinitely many gaps in each intersecting open interval, and the way in which we chose the first k elements in γ^*, ensures that there will always be infinitely many gaps in S_{k+1} that cannot be jumped over in any way with the first k gaps of T with any suitable K-Lipschitz function. Hence, we will be able to define γ^{k+1} small enough so that the biggest of these “unjumped" gaps in S_{k+1} cannot be jumped over either by the remaining gaps of T. We just need to account for all the possibilities in which the first k gaps of T might behave under a K-Lipschitz function.

Let $\sigma = \{j_i\}_{i=1}^k$ be an ordering of the sequence $\{1, \dots, k\}$. For convenience of the notation, put $j_0 = 0$, and $n_{j_0} = 1$, and define

$$D_{(j_0,j_1)} = \mathcal{D}_{K\gamma_{j_1}^*}(G_{n_{j_0}}^{k+1}).$$

$D_{(j_0,j_1)}$ is the sweeping of $G_{n_{j_0}}^{k+1}$, the biggest gap of S_{k+1}, by $K\gamma_{j_1}^*$. The measure of $D_{(j_0,j_1)}$ is at most $2K\gamma_{j_1}^* < 2^{-j_1}\varepsilon < \varepsilon$. Hence, since S_{k+1} has measure greater than ε, the set $S_{k+1} \setminus D_{(j_0,j_1)} = S_{k+1} \cap ([0, l_{k+1}] \setminus D_{(j_0,j_1)})$ is nonempty. Since $[0, l_{k+1}] \setminus D_{(j_0,j_1)}$ is a finite union of open intervals, by hypothesis there must exist infinitely many gaps in $S_{k+1} \setminus D_{(j_0,j_1)}$. We can then consider

$$n_{(j_0,j_1)} = \min\{n > n_{j_0} \colon\ G_n^{k+1} \not\subseteq D_{(j_0,j_1)}\}.$$

Intuitively, $G^{k+1}_{n_{(j_0,j_1)}}$ is the biggest gap of S_{k+1} smaller than $G^{k+1}_{n_{j_0}}$ which is not contained in the sweeping $D_{(j_0,j_1)}$. We continue the process defining

$$D_{(j_0,j_1,j_2)} = D_{(j_0,j_1)} \cup \mathcal{D}_{K\gamma^*_{j_2}}(G^{k+1}_{n_{(j_0,j_1)}}).$$

The measure of $D_{(j_0,j_1,j_2)}$ is at most $(2^{-j_1} + 2^{-j_2})\varepsilon < \varepsilon$, and its complement in $[0, l_{k+1}]$ is still a finite union of open intervals, so by the properties of S_{k+1} we can make the same argument as before to find

$$n_{(j_0,j_1,j_2)} = \min\{n > n_{(j_0,j_1)} \colon\ G^{k+1}_n \nsubseteq D_{(j_0,j_1,j_2)}\},$$

which will be the biggest gap of S_{k+1} smaller than $G^{k+1}_{n_{(j_0,j_1)}}$ not contained in $D_{(j_0,j_1,j_2)}$. Hence, it is not contained in neither the sweeping $\mathcal{D}_{K\gamma^*_{j_1}}(G^{k+1}_{n_{j_0}})$ nor $\mathcal{D}_{K\gamma^*_{j_2}}(G^{k+1}_{n_{(j_0,j_1)}})$.

Repeating this process k times, we can define $n_\sigma = n_{(j_0,\ldots,j_{k-1})} \in \mathbb{N}$ such that $G^{k+1}_{n_\sigma}$ is the biggest gap of S_{k+1} not contained in $\mathcal{D}_{K\gamma^*_{j_k}}(G^{k+1}_{n_{(j_0,\ldots,j_{i-1})}})$ for any $1 \leq i \leq k$, and smaller than $G^{k+1}_{n_{(j_0,\ldots,j_{i-1})}}$ for every $1 \leq i \leq k-1$. Notice that this last condition can be written as:

$$\alpha^{k+1}_{n_\sigma} < \min_{1\leq i\leq k} \alpha^{k+1}_{n_{(j_0,\ldots,j_{i-1})}}. \tag{4.2}$$

Now, let $\Omega_k = \{\sigma = \{j_i\}_{i=1}^k \colon \sigma \text{ is an ordering of } \{1,\ldots,k\}\}$. Clearly Ω_k is a finite set, so we can define $n_{\Omega_k} = \max\{n_\sigma \colon \sigma \in \Omega_k\}$. The corresponding gap $G^{k+1}_{n_{\Omega_k}}$ is smaller than or equal to each $G^{k+1}_{n_\sigma}$. Equivalently, we have that

$$\alpha^{k+1}_{n_{\Omega_k}} \leq \min_{\sigma\in\Omega_k} \alpha^{k+1}_{n_\sigma}. \tag{4.3}$$

Finally, choose γ^*_{k+1} so that $0 < \gamma^*_{k+1} < \min\{2^{-(k+2)}K^{-1}\varepsilon, K^{-1}\alpha^{k+1}_{n_{\Omega_k}}\}$. Again, property (1) of the induction is satisfied. Let $1 \leq K' \leq K$, let T be a thread such that its sequence of gaps $\{C^T_i\}_{i\in\mathbb{N}}$ ordered decreasingly in length, satisfy that $\text{length}(C^T_i) \leq \gamma^*_i$ for all $i \leq k+1$. Applying inductive hypothesis, since the result is assumed to be true for k, we only need to prove that if $\text{length}(C^T_1) < a_{k+1}/K'$, there is no Lipschitz map $F\colon T \to S_{k+1}$ with $\|F\|_{\text{Lip}} \leq K'$ and $F(0) = 0$ and $F(l_T) = l_{k+1}$. Suppose by contradiction that such a map F exists. Again, we may assume F to be non-decreasing.

Put again $j_0 = 0$ and $n_{j_0} = 1$, and consider the gap $G^{k+1}_{n_{j_0}}$ in S_{k+1} (that is, the biggest gap of S_{k+1}). By Lemma 4.6, there exists $j_1 \in \mathbb{N}$ such that the gap $C^T_{j_1}$ in S jumps over $G^{k+1}_{n_{j_0}}$, which has length $\alpha^{k+1}_{n_{j_0}}$. Since $K' \cdot \text{length}(C^T_{k+1}) < \alpha^{k+1}_{n_{\Omega_k}} < \alpha^{k+1}_{n_{j_0}}$, by Lemma 4.7 we have that $j_1 \leq k$. Therefore, we can define $n_{(j_0,j_1)} = \min\{n > n_{j_0} \colon\ G^{k+1}_n \nsubseteq D_{(j_0,j_1)}\}$ as before.

Consider now the gap $G^{k+1}_{n_{(j_0,j_1)}}$ in S_{k+1}, and take j_2 such that $C^T_{j_2}$ jumps over $G^{k+1}_{n_{(j_0,j_1)}}$. Again, since $K' \cdot \text{length}(C^T_{k+1}) < \alpha^{k+1}_{n_{\Omega_k}} < \alpha^{k+1}_{n_{(j_0,j_1)}}$, we obtain that $j_2 \leq k$. Moreover, j_2 is different from j_1. Indeed, if $j_2 = j_1$, then $C^T_{j_1}$ jumps over both $G^{k+1}_{n_{j_0}}$

and $G^{k+1}_{n_{(j_0,j_1)}}$. By the choice of $n_{(j_0,j_1)}$, $G^{k+1}_{n_{(j_0,j_1)}} \nsubseteq \mathcal{D}_{K\gamma^*_{j_1}}(G^{k+1}_{n_{j_0}})$, so by Proposition 4.8, we have that $K' \cdot \mathrm{length}(C^T_{j_1}) > K\gamma^*_{j_1}$, a contradiction.

We can repeat this process k-times until we obtain a sequence $\sigma = \{j_i\}_{i=1}^k$ of k different numbers in $\{1, \ldots, k\}$ (so $\sigma \in \Omega_k$) such that $C^T_{j_i}$ jumps over the gap $G^{k+1}_{n_{(j_0,\ldots,j_{i-1})}}$ in S_{k+1} for all $1 \leq i \leq k$. To finish the proof, consider the gap $G^{k+1}_{n_\sigma}$ in S_{k+1}, where n_σ is defined as above for $\sigma \in \Omega_k$, and take $\tilde{i}$ such that $C^T_{\tilde{i}}$ jumps over $G^{k+1}_{n_\sigma}$. Reasoning as before, by the choice of γ^*_{k+1} and using equation (4.3) we have that $\tilde{i} \leq k$. Then $\tilde{i} = j_{i_0}$ for some $1 \leq i_0 \leq k$. We have chosen i_0 such that $C^T_{j_{i_0}}$ jumps over $G^{k+1}_{n_{(j_0,\ldots,j_{i_0-1})}}$ as well. Moreover, $G^{k+1}_{n_\sigma}$ is not contained in $\mathcal{D}_{K\gamma^*_{j_{i_0}}}\big(G^{k+1}_{n_{(j_0,\ldots,j_{i_0-1})}}\big)$. Therefore, by Proposition 4.8, we have that $K' \cdot \mathrm{length}(C^T_{j_{k_0}}) > K\gamma^*_{j_{k_0}}$, a contradiction.

The induction is finished and the result follows.

□

Compact subsets of the real line with positive measure that contain no nontrivial intervals have been considered many times before: the well known *fat Cantor sets* are examples of this kind of objects. For our purposes, we need to find threads with these properties and whose gaps are smaller than any given decreasing sequence of positive real numbers. For completeness, we include the construction of such threads in the following subsection.

Let us first finish this subsection by making a simple remark:

Remark 4.10. *Let M be a complete metric space, let $K \geq 1$, and let S_1, S_2 be two closed subsets of M such that $d(S_1, S_2) = \varepsilon > 0$. Then, if T is a thread such that its sequence of gaps $\{C^T_k\}_{k\in\mathbb{N}}$ satisfies length$(C^T_k) < \varepsilon/K$ for all $k \in \mathbb{N}$, then there is no K-Lipschitz map $F\colon T \to S_1 \cup S_2$ such that $F(0) \in S_1$ and $F(p) \in S_2$ for some $p \in T$.*

Proof. Consider the point

$$P = \min\{x \in T \colon F(x) \in S_2\}.$$

The point P is not 0 since $F(0) \in S_1$, and for all $x \in [0, P)_T$ we have that $F(x) \in S_1$. However, since every gap in T is smaller than ε/K, there exists $x \in [0, P)_T$ such that $d(x, P) < \varepsilon/K$. Then $d(F(x), F(P)) < \varepsilon$, which contradicts the fact that $d(S_1, S_2) = \varepsilon$. □

4.1.1.3. *Construction of threads with infinitely many gaps.* Our objective now is to define a collection of closed subsets of the real segment $[0, 1]$ containing 0 and 1 which, when given a thread metric, will satisfy the hypothesis of Theorem 4.9 for $\varepsilon = 1/2$. For the rest of this section, fix $\mathbb{Q} \cap (0, 1) = (q_n)_{n=1}^\infty$, an ordering of the rational numbers in the interval $(0, 1)$. Consider a decreasing sequence of real numbers $\gamma = (\gamma_i)_{i=1}^\infty$ such that

(i) $\gamma_i > 0$ for all $i \in \mathbb{N}$,

(ii) $\gamma_i < 2^{-(i+1)}$ for all $i \in \mathbb{N}$.
(iii) $q_1 + \gamma_1 < 1$.

Put $\Delta = \{\gamma = (\gamma_i)_i \colon \gamma$ is decreasing and satisfies (i), (ii) and (iii)$\}$ for the rest of the section.

For any given $\gamma \in \Delta$, we define $\{G_i\}_{i\in\mathbb{N}}$ inductively as the following open subintervals of $(0,1)$:

$$G_1^\gamma = (q_1, q_1 + \gamma_1),$$

and for $i \geq 2$:

$$G_i^\gamma = (q_{n_i}, q_{n_i} + \gamma_i), \text{ where } n_i = \min\left\{ n \in \mathbb{N} \colon (q_n, q_n + \gamma_i) \subset (0,1) \setminus \left(\bigcup_{j<i} G_j^\gamma \right) \right\}.$$

Note that property (ii) of γ guarantees that n_i exists for all $i \in \mathbb{N}$.

Using this, we define the closed subset $T_\gamma \subset [0,1]$ as

$$T_\gamma = [0,1] \setminus \left(\bigcup_{i=1}^{\infty} G_i^\gamma \right)$$

for any $\gamma \in \Delta$. The definition of $\{G_i^\gamma\}_{i\in\mathbb{N}}$ and T_γ for every $\gamma \in \Delta$ is fixed for the rest of the section.

PROPOSITION 4.11. *Let $\gamma = (\gamma_i)_{i=1}^\infty \in \Delta$, and let $\{G_i^\gamma\}_{i\in\mathbb{N}}$ and T_γ be defined as above. Then T_γ is a compact subset of $[0,1]$ that satisfies:*

(1) The Lebesgue measure of T_γ is greater than or equal to $1/2$.
(2) The points 0 and 1 belong to T_γ.
(3) The sequence of gaps of T_γ is the sequence $\{G_i^\gamma\}_{i\in\mathbb{N}}$, and length$(G_i^\gamma) = \gamma_i$ for all $i \in \mathbb{N}$.
(4) The set T_γ does not contain any nontrivial interval. Consequently, if $(x, x+\delta) \cap T_\gamma \neq \emptyset$ for some $x \in [0,1]$ and $\delta > 0$, then $(x, x+\delta)$ contains infinitely many gaps of T_γ.

PROOF. Notice that the Lebesgue measure of T_γ is greater than or equal to $1 - \sum_{i=1}^\infty 2^{-(i+1)} = 1/2$ for all possible γ by property (ii), and the points 0 and 1 are not in G_i^γ for any $i \in \mathbb{N}$ by construction and property (iii), so (1) and (2) are clear.

To see (3), we need to prove that G_i^γ is a gap in T^γ for all $i \in \mathbb{N}$, and that every gap of T^γ is one of G_i^γ for some $i \in \mathbb{N}$. Consider an interval $G_i^\gamma = (q_{n_i}, q_{n_i} + \gamma_i)$. We have directly by construction that $G_i^\gamma \cap T = \emptyset$, so we only need to see that the endpoints of G_i^γ are in T to prove that it is a gap of T. If one of the endpoints p_i of G_i^γ is not in T, there must exist $j \in \mathbb{N}$ such that $p_i \in G_j^\gamma$. Since G_j^γ is open and p_i is in the closure of G_i^γ, we have that $G_i^\gamma \cap G_j^\gamma \neq \emptyset$, a contradiction with the choice of n_i and n_j. We conclude that G_i^γ is a gap of T_γ.

Next, let $x, y \in T_\gamma$ with $x < y$ and $(x,y)_{T_\gamma} = \emptyset$. For a point $p \in (x,y)$, since $p \notin T_\gamma$, there must exist $i \in \mathbb{N}$ such that $p \in G_i^\gamma$. The interval G_i^γ is contained in (x, y) because both x and y belong to T_γ. Moreover, since the endpoints of G_i^γ belong to T_γ, we necessarily have that $G_i^\gamma = (x, y)$, and we are done with (3).

Finally, the set T_γ is nowhere dense, since it contains no intervals. Indeed, suppose there is an interval $(x, x+\delta) \subset T_\gamma$ for some $x \in [0,1]$ and $\delta > 0$ with $x+\delta < 1$. The subinterval $(x, x+\delta/2)$ contains a rational number q_{n_0}. Since $(\gamma_i)_{i=1}^\infty$ is decreasing and converging to 0, there must exist i_0 such that $\gamma_i < \delta/2$ for all $i \geq i_0$. Then, for all $i \geq i_0$, the natural number n_0 satisfies that $(q_{n_0}, q_{n_0}+\gamma_i) \subset T_\gamma$, and in particular

$$(q_{n_0}, q_{n_0} + \gamma_i) \subset (0,1) \setminus \left(\bigcup_{j<i} G_j^\gamma \right).$$

Therefore, there must exist $i_1 \geq i_0$ such that $n_0 = \min \Bigg\{ n \in \mathbb{N} \colon (q_n, q_n + \gamma_{i_1}) \subset (0, l) \setminus \left(\bigcup_{j<i_1} G_j^\gamma \right) \Bigg\}$, which implies that $G_{i_1}^\gamma = (q_{n_0}, q_{n_0} + \gamma_{i_1})$; a contradiction with the assumption that $(q_{n_0}, q_{n_0} + \delta/2) \subset T_\gamma$. $\square$

Now, given any $\gamma \in \Delta$ and any $0 < a \leq 1$, we may assign the metric $d_{1,a}$ as defined at the beginning of this section to the set T_γ, such that $(T_\gamma, d_{1,a})$ is a thread. We will denote by $T_\gamma(1, a)$ the thread of length 1 and width a formed by endowing the subset T_γ as defined above for $\gamma \in \Delta$ with the metric $d_{1,a}$.

With Proposition 4.11 we have that any countable family of these threads satisfies the hypothesis of Theorem 4.9. In fact, any countable family of subthreads with measure uniformly bounded from below also satisfies the hypothesis of Theorem 4.9. Moreover, given such a countable family of threads $\{T_n\}_{n\in\mathbb{N}}$ and $K \geq 1$, for the sequence $\gamma^* = \{\gamma_k^*\}_{k\in\mathbb{N}}$ obtained by Theorem 4.9 we can always find another sequence $\gamma^0 = \{\gamma_k^0\}_{k\in\mathbb{N}} \in \Delta$ such that $\gamma_k^0 \leq \gamma_k^*$ for all $k \in \mathbb{N}$. Hence, there exists a thread of the form $T_{\gamma^0}(1, a)$ that cannot be mapped with a K-Lipschitz function preserving the extreme points onto any T_n for any $n \in \mathbb{N}$, provided $a < \text{width}(T_n)/K$.

Observe that thanks to the properties of the generic sets T_γ, we can obtain the following fact about the thread $T_\gamma(1, a)$:

PROPOSITION 4.12. *Let $\gamma \in \Delta$, and let $T_\gamma(1, a)$ be the thread of length 1 and width a associated with γ. Then $T_\gamma(1, a)$ is totally separated, i.e.: If p and q are two different points in $T_\gamma(1, a)$, there exist two disjoint open and closed subsets $S_1, S_2 \subset T_\gamma(1, a)$ such that $p \in S_1$, $q \in S_2$, and $T_\gamma(1, a) = S_1 \cup S_2$. As a consequence, for any point $p \in T_\gamma(1, a)$ and any $\varepsilon > 0$, there exists an open and closed subset S of diameter less than ε such that $p \in S$.*

PROOF. Put $T = T_\gamma(1, a)$. Let p and q be two different points in T. Suppose without loss of generality that $p < q$. If the interval $(p, q)_T$ is empty, then the result follows considering $S_1 = [0, p]_T$ and $S_2 = [q, 1]_T$. Otherwise, if $(p, q)_T$ is nonempty, by property (4) in Proposition 4.11 there exists a gap (x, y) in T such that $p < x$ and $y < q$. Put now $S_1 = [0, x]_T$ and $S_2 = [y, 1]_T$ and the result is proven.

The second statement follows immediately from the first and from the linear structure of threads. $\square$

When dealing with Lipschitz functions between threads, the image of the extreme points of a thread plays an important role for some technical arguments (as can be seen in the previous subsection). This is the motivation for introducing the next result.

PROPOSITION 4.13. *Let T and S be two threads of length l_T and l_S, and width a_T and a_S respectively. Let $K \geq 1$. Suppose T is totally separated. Consider S_1 and S_2 open subsets of S, and take D_1 and D_2 dense subsets of S_1 and S_2 respectively.*

Let $F\colon T \to S$ be a K-Lipschitz function such that $F(0) = A$ and $F(l_S) = B$ with $A \in S_1$ and $B \in S_2$. Then for every $\varepsilon > 0$, there exists a pair of points $P, Q \in T$ with $P < Q$, a pair of points $\widehat{A} \in D_1$ and $\widehat{B} \in D_2$, and a $(K + \varepsilon)$-Lipschitz function $\widehat{F}\colon [P, Q]_T \to S$ such that $\widehat{F}(P) = \widehat{A}$ and $\widehat{F}(Q) = \widehat{B}$.

PROOF. Since S_1 is open in S and $F(0) \in S_1$, there exists a positive number $r > 0$ such that for any point $p \in [0, r]_T$ we have $F(p) \in S_1$. The thread T is totally separated, so we can find $P \in [0, r]_T$ and $\delta_P > 0$ such that $d\big([0, P]_T, T \backslash \big([0, P]_T\big)\big) = \delta_P$. Similarly, we may find $Q \in T$ with $P < Q$ and $\delta_Q > 0$ such that $F(Q) \in S_2$ and $d\big([Q, l_T]_T, T \setminus \big([Q, l_T]_T\big)\big) = \delta_Q$.

By density, we can find $\widehat{A} \in S_1$ and $\widehat{B} \in S_2$ such that $d\big(F(P), \widehat{A}\big) < 2^{-1}\varepsilon \cdot \delta_P$ and $d\big(F(Q), \widehat{B}\big) < 2^{-1}\varepsilon \cdot \delta_Q$. Define now $\widehat{F}\colon [P, Q]_T \to S$ so that

$$\widehat{F}(x) = \begin{cases} \widehat{A}, & \text{if } x = P, \\ F(x), & \text{if } x \in (P, Q)_T, \\ \widehat{B}, & \text{if } x = Q. \end{cases}$$

It is now routine to check that $\widehat{F}$ is $(K + \varepsilon)$-Lipschitz. $\square$

4.1.2. Construction of the building blocks: Threading metric spaces. We now want to use the threads $T_\gamma(1, a)$ we defined for $\gamma \in \Delta$ and $0 < a \leq 1$ to construct non-separable complete metric spaces that will act as building blocks of the final metric space. To do this, we first formalize the notion of *attachment* of metric spaces, which will allow us to "glue" metric spaces in a convenient way. This concept has been used in many contexts in the literature, but we choose to include a definition tailored to our necessities.

DEFINITION 4.14. Let (M, d_M) be a complete metric space. Consider $\mathcal{N} = \{(N_\gamma, d_\gamma)\}_{\gamma \in \Gamma}$ be a collection of pairwise disjoint and disjoint with M complete metric spaces, and let $\mathcal{S} = \{S_\gamma\}_{\gamma \in \Gamma}$ be a collection of sets such that: for each $\gamma \in \Gamma$ the set S_γ is a compact subset of N_γ, and there exists an isometry $\Phi_\gamma\colon S_\gamma \to M$

onto a subset of M. The *attachment of* M *with* $\mathcal{N}$ *by* $\mathcal{S}$ is the pair $(M(\mathcal{N},\mathcal{S}), d_{\mathcal{N},\mathcal{S}})$, where

$$M(\mathcal{N},\mathcal{S}) = M \cup \Big(\bigcup_{\gamma\in\Gamma} N_\gamma \setminus S_\gamma \Big),$$

and

$$d_{\mathcal{N},\mathcal{S}} \colon M(\mathcal{N},\mathcal{S}) \times M(\mathcal{N},\mathcal{S}) \longrightarrow \mathbb{R}^+$$

is a map defined by

$$d_{\mathcal{N},\mathcal{S}}(p,q) = \begin{cases} d_M(p,q), & \text{if } p,q \in M, \\ d_\gamma(p,q), & \text{if } p,q \in N_\gamma \setminus S_\gamma \text{ for some } \gamma \in \Gamma, \\ \min_{x\in S_\gamma}\{d_\gamma(p,x) + d_M(\Phi_\gamma(x),q)\}, & \text{if } p \in N_\gamma \setminus S_\gamma \text{ for some } \gamma\in\Gamma, \text{ and } q \in M, \\ H_{\gamma_1,\gamma_2}(p,q), & \text{if } p \in N_{\gamma_1}\setminus S_{\gamma_1},\ q \in N_{\gamma_2} \setminus S_{\gamma_2} \text{ for } \gamma_1 \neq \gamma_2 \in \Gamma, \end{cases}$$

where $H_{\gamma,\eta} \colon N_\gamma \times N_\eta \to \mathbb{R}^+$ is the map defined by

$$H_{\gamma,\eta}(p,q) = \min\{d_\gamma(p,x) + d_M(\Phi_\gamma(x),\Phi_\eta(y)) + d_\eta(y,q) \colon x \in S_\gamma,\ y \in S_\eta\}.$$

Notice that both minima used in the definition of $d_{\mathcal{N},\mathcal{S}}$ are well defined by compactness of the sets S_γ for each $\gamma \in \Gamma$. Moreover, it is straightforward to check that the map $d_{\mathcal{N},\mathcal{S}}$ defines a complete metric in $M(\mathcal{N},\mathcal{S})$.

It is also clear from the definition that the metric space $M(\mathcal{N},\mathcal{S})$ contains M isometrically, as well as an isometric copy of N_γ for each $\gamma\in\Gamma$. We may write $M \subset M(\mathcal{N},\mathcal{S})$ and $N_\gamma \subset M(\mathcal{N},\mathcal{S})$ by virtue of this fact.

With this concept, we can now define the aforementioned building blocks of the main metric space we seek to construct:

Definition 4.15. Consider a metric space $M = \{A,B\}$ formed by two points at a distance $0 < a \leq 1$. Let $\mathcal{N}_a = \{T_\gamma(1,a)\}_{\gamma\in\Delta}$, where Δ is the set of sequences defined in Section 1, and $T_\gamma(1,a)$ is the thread associated with γ of width a. We may consider $T_\gamma(1,a)$ and $T_\eta(1,a)$ to be disjoint for $\gamma\neq\eta$. For each $T_\gamma(1,a)$, put $S_\gamma = \{0_\gamma, 1_\gamma\}$, the set of the two extreme points of $T_\gamma(1,a)$. We let $\mathcal{S}_a = \{S_\gamma\}_{\gamma\in\Delta}$, and $\Phi_\gamma : S_\gamma \to M$ as $\Phi_\gamma(0_\gamma) = A,\ \Phi_\gamma(1_\gamma) = B$.

We define the *threading space* $\mathrm{Th}(A,B)$ to be the attachment of M with $\mathcal{N}_a$ by $\mathcal{S}_a$. We say that $\mathrm{Th}(A,B)$ is *anchored* at A and B, and these two points are called the *anchors* of $\mathrm{Th}(A,B)$. If a threading space $\mathrm{Th}(A,B)$ is fixed and there is no room for ambiguity, we write $T_\gamma(1,a) \subset \mathrm{Th}(A,B)$ to denote the isometric copy of the thread $T_\gamma(1,a)$ contained in the threading space $\mathrm{Th}(A,B)$.

Note that in a threading space $\mathrm{Th}(A,B)$ we have that $T_\gamma(1,a) \cap T_\eta(1,a) = \{A,B\}$ for any two different $\gamma,\eta\in\Delta$. Moreover, if $p,q \in \mathrm{Th}(A,B)$ belong to different threads, the distance $d(p,q)$ is computed according to the definition of attachment, which in this case results in

$$d(p,q) = \min\{d(p,A) + d(A,q), d(p,B) + d(B,q)\}.$$

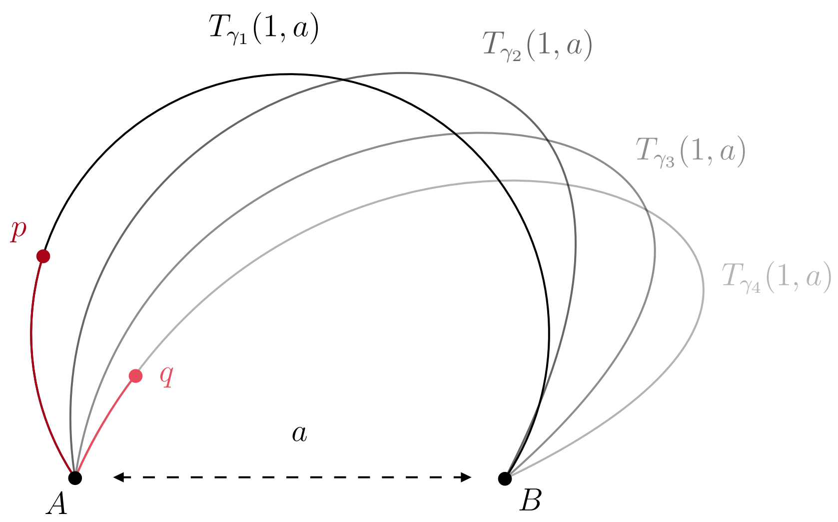

FIGURE 4. Subset of $\mathrm{Th}(A,B)$. The distance from p to q is $d(p,A)+d(A,q)$.

See Figure 4 for a representation of a subset of a threading space.

By definition, in a metric space $M(\mathcal{N},\mathcal{S})$ formed by attachment (and in threading spaces in particular), given a point $p\in N_\gamma$ for some $\gamma\in\Gamma$ and a point $x_1\in M$, there exists a point $s_1\in S_\gamma$ such that $d(p,x_1)=d(p,s_1)+d(s_1,x_1)$. However, it is possible that for a different point $x_2\in M$, the point $s_2\in S_\gamma$ such that the identity $d(p,x_2)=d(p,s_2)+d(s_2,x_2)$ holds, is different from s_1. Points in N_γ that always use the same point in S_γ to compute their distance to the rest of the space are especially relevant to our discussion.

In general, given a metric space M and a closed subset N, we say that a point $p\in N$ is *bound to* $s\in N$ in N if for every $x\in M\setminus N$ we have $d(p,x)=d(p,s)+d(s,x)$.

For example, in a threading space $\mathrm{Th}(A,B)$, it is not hard to check that a point p in a thread $T_\gamma(1,a)$ is bound to A in $T_\gamma(1,a)$ if and only if $d(p,A)<d(p,B)$ and $d(p,A)\le\frac{1-a}{2}$ (the analogous result for B holds as well).

In the final subsection we will deal with Lipschitz functions defined on a single thread and with image in metric spaces formed by attachment. We finish this section by defining two simple concepts and proving a result that will help us simplify this type of maps.

Now, if T is a thread, M is a metric space, N is a closed subset of M, and $F\colon T\to M$ is a Lipschitz function, we say that an extended interval (as defined in the beginning of subsection 4.1.1) I is *maximal* with respect to F and N if

$F(I) \subset N$ and every extended interval $J = [a', b']_T$ in T that contains I and such that $F(J) \subset N$ is equal to I. We have the following straightforward result:

PROPOSITION 4.16. *Let T be a thread of length l and width a. Let M be a metric space, let N be a closed subset of M, and let $F\colon T \to M$ be a Lipschitz function. If an extended interval I in T with extremes $a, b \in T$ is maximal with respect to F and N, and there exists $s \in N$ such that both $F(a)$ and $F(b)$ are bound to s in N, then the function $\widehat{F}\colon T \to M$ defined by:*

$$\widehat{F}(x) = \begin{cases} s, & \text{if } x \in I, \\ F(x), & \text{if } x \in T \setminus I \end{cases}$$

is Lipschitz with $\|\widehat{F}\|_{Lip} = \|F\|_{Lip}$.

PROOF. Put $K = \|F\|_{\mathrm{Lip}}$. We will start by proving that

$$d\big(\widehat{F}(0), \widehat{F}(l)\big) \leq K \cdot a. \tag{4.4}$$

If both 0 and l belong to the extended interval I then it follows trivially. Similarly, if 0 and l belong to $T \setminus I$ then the inequality follows since F is K-Lipschitz. Hence, suppose first that $0 \in I$ and $l \in T \setminus I$. Then, we have necessarily that $a = 0$, and so $F(a)$ is bound to s in N. This implies that

$$\begin{aligned} d\big(\widehat{F}(0), \widehat{F}(l)\big) &= d\big(s, F(l)\big) \leq d\big(F(0), s\big) + d\big(s, F(l)\big) \\ &= d\big(F(0), F(l)\big) \leq K \cdot a \end{aligned}$$

A similar argument shows that if $0 \in T \setminus I$ and $l \in I$ the same inequality holds. Hence we conclude that equation (4.4) is satisfied.

Next, we prove that for every $x, y \in T$ with $x < y$ we have

$$d(\widehat{F}(x), \widehat{F}(y)) \leq K(y - x). \tag{4.5}$$

As before, we may only check this holds for $x, y \in T$ with $x < y$ such that $x \in T \setminus I$ and $y \in I$. By maximality of I, there exists $t \in T$ with $x \leq t < y$ such that $F(t) \notin N$. Additionally, since t is not in I but y does belong to the extended interval, one of the extremes a or b of I belongs to $(t, y]_T$. We may suppose without loss of generality that $x \leq t < a \leq y$. Notice that $d\big(F(t), s\big) \leq d\big(F(t), F(a)\big)$ because a is bound to s in N. Then we have:

$$\begin{aligned} d\big(\widehat{F}(x), \widehat{F}(y)\big) &\leq d\big(F(x), F(t)\big) + d\big(F(t), s\big) \\ &\leq d\big(F(x), F(t)\big) + d\big(F(t), F(a)\big) + d\big(F(a), F(y)\big) \\ &\leq K\big((t - x) + (a - t) + (y - a)\big) = K(y - x). \end{aligned}$$

This proves that equation (4.5) holds as well.

Using both equations (4.4) and (4.5) we can apply Proposition 4.2 to obtain that $\|\widehat{F}\|_{\mathrm{Lip}} \leq K$. ∎

The process used to construct the skein metric space which fails to have any non-trivial separable Lipschitz retracts is to keep attaching threading spaces inductively such that any two distinct points of the skein act as the anchors to a threading space contained in the skein[1]. As we are going to see in the final section, this construction presents its own technical difficulties. However, the main results of the first two sections will be very useful in this regard.

4.1.3. Construction of the skein metric spaces. The final metric space will be constructed using transfinite induction. Let us discuss this process in general for limit ordinal numbers:

Let κ be a limit ordinal number. Suppose that $\{(M_\alpha, d_\alpha)\}_{\alpha<\kappa}$ is a transfinite sequence of metric spaces that are *increasing*, in the sense that $M_\alpha \subset M_\beta$ if $\alpha < \beta$ and the restriction of d_β to M_α results in the metric d_α. Then we may define the metric space (M_κ, d_κ) where $M_\kappa = \bigcup_{\alpha<\kappa} M_\alpha$, and d_κ is defined for any $p, q \in M_\kappa$ as $d_\kappa(p, q) = d_\alpha(p, q)$ where $\alpha < \kappa$ is the least ordinal number such that $p, q \in M_\alpha$. It is straightforward to check that the metric d_κ is well defined and (M_κ, d_κ) is indeed a metric space.

We will call (M_κ, d_κ) the metric space *generated by* $\{(M_\alpha, d_\alpha)\}_{\alpha<\kappa}$, and as usual we may omit the mention of the metric d_κ when referring to it if there is no room for ambiguity. If κ is an ordinal with uncountable cofinality (i.e., the supremum of any countable sequence of ordinals $(\alpha_n)_n$ such that $\alpha_n < \kappa$ for all $n \in \mathbb{N}$ is strictly smaller than κ), then the metric space M_κ generated by $\{M_\alpha\}_{\alpha<\kappa}$ is complete, provided each M_α is complete for every $\alpha < \kappa$. To see this, consider any Cauchy sequence $(p_n)_n$ in M_κ. Each p_n belongs to M_{α_n} for some ordinal $\alpha_n < \kappa$. Since κ has uncountable cofinality, the supremum $\alpha^* = \sup_n(\alpha_n)$ is strictly smaller than κ. Hence, the sequence $(p_n)_n$ belongs to the complete metric space M_{α^*}, and therefore it is convergent in M_{α^*} to a point p^*. The point p^* belongs to M_κ, and clearly $(p_n)_n$ converges to p^* in M_κ as well.

4.1.3.1. *Construction of the skein metric spaces.* We are going to construct by transfinite induction an increasing class of complete metric spaces $\{\mathrm{Sk}(\beta)\}_\beta$ for every ordinal $\beta \leq \omega_1$, called the *β-skein metric spaces*. The complete metric space failing to have any non-trivial separable Lipschitz retract is the ω_1-skein space $\mathrm{Sk}(\omega_1)$.

Consider at the first step the 0-skein metric space $M_0 = \{A, B\}$ formed by two points at distance 1/2, and put $G_0 = \{A, B\}$. Suppose we have defined increasingly the α-skein spaces $\{\mathrm{Sk}(\alpha)\}_{\alpha<\beta}$ up to an ordinal $\beta \leq \omega_1$. If β is a limit ordinal, simply define $\mathrm{Sk}(\beta)$ as the completion of generated metric space $\bigcup_{\alpha<\beta} \mathrm{Sk}(\alpha)$ in the way described above, which contains isometrically the previous skein spaces $\mathrm{Sk}(\alpha)$ for all $\alpha < \beta$. Notice that if β has uncountable cofinality (i.e.: if $\beta = \omega_1$), it is not necessary to take the completion.

[1]"*skein*: a length of yarn or thread collected together into the shape of a loose ring" (Cambridge dictionary. n.d.).

Suppose now that $\beta = \lambda + 1$ for an ordinal $\lambda < \omega_1$. For every p in the skein $\mathrm{Sk}(\lambda)$ and every $q \in G_\lambda = \mathrm{Sk}(\lambda) \setminus \left(\bigcup_{\alpha<\lambda} \mathrm{Sk}(\alpha) \right)$ with $p \neq q$ and $d(p,q) \leq 1/2$, we may consider the threading space $\mathrm{Th}(p,q)$ as defined in subsection 4.1.2. Take now the family of complete metric spaces $\mathcal{N}_\lambda = \{\mathrm{Th}(p,q)\}_{\{p,q\}\in\Gamma_\lambda}$, where

$$\Gamma_\lambda = \big\{ \{p,q\} \subset \mathrm{Sk}(\lambda) \colon p \in \mathrm{Sk}(\lambda),\ q \in G_\lambda,\ 0 < d(p,q) \leq 1/2 \big\},$$

which we may take to be pairwise disjoint and disjoint with $\mathrm{Sk}(\lambda)$. For any $\{p,q\} \in \Gamma_\lambda$, we have by definition of the threading space $\mathrm{Th}(p,q)$ that there is an isometry $\Phi_{\{p,q\}}$ from the set of anchor points $\mathrm{An}_{\{p,q\}}$ of $\mathrm{Th}(p,q)$ onto the set $\{p,q\}$ in $\mathrm{Sk}(\lambda)$. Therefore, considering $\mathcal{S}_\lambda = \{\mathrm{An}_{\{p,q\}}\}_{\{p,q\}\in\Gamma_\lambda}$ we can define $\mathrm{Sk}(\beta)$ as the attachment of $\mathrm{Sk}(\lambda)$ with $\mathcal{N}_\lambda$ by $\mathcal{S}_\lambda$. The resulting metric space $\mathrm{Sk}(\beta)$ is the β-skein, and it is a complete metric space containing isometrically the previous skein space $\mathrm{Sk}(\lambda)$. The induction process is finished, and we have defined the β-skein metric space for every ordinal number $\beta \leq \omega_1$.

Intuitively, we may describe the previous process in the following way: If $\beta \leq \omega_1$ is a limit ordinal, then the β-skein space is the completion (if necessary) of the union of all previous skein spaces. If β is the successor of an ordinal λ, then the β-skein is formed by attaching a threading space at every pair of points closer than 1/2 and such that at least one of them was newly introduced at the previous step λ.

Note that although we may formally continue the inductive process for ordinals greater than ω_1, since no new points are introduced at this step, the process becomes stationary and the skein space $\mathrm{Sk}(\beta)$ is $\mathrm{Sk}(\omega_1)$ for every $\beta \geq \omega_1$.

For a subset S of a skein space $\mathrm{Sk}(\beta)$, we may define its *(skein) order*, written $\mathrm{ord}(S)$, as the least ordinal $\alpha \leq \beta$ such that $S \subset \mathrm{Sk}(\alpha)$. For a point $p \in \mathrm{Sk}(\beta)$, we write $\mathrm{ord}(p) = \mathrm{ord}(\{p\})$. For any ordinal β, the *(skein) generation of order β* is the set $G_\beta = \mathrm{Sk}(\beta) \setminus \left(\bigcup_{\alpha<\beta} \mathrm{Sk}(\alpha) \right)$.

Figure 5 is a conceptual representation of a subset of the skein $\mathrm{Sk}(3)$, which contains 3 different generations (the gaps in the threads have been ignored for the sake of clarity). The distance between the points x and y in the figure are computed by $d(x,y) = d(x,p) + d(p,q) + d(q,y)$.

Crucially, in the skein space $\mathrm{Sk}(\omega_1)$, the corresponding generation G_{ω_1} is empty, and every point in the ω_1-skein $\mathrm{Sk}(\omega_1)$ belongs to a previous generation. This means that, in this space, every pair of points p and q such that $d(p,q) \leq 1/2$ belong to a set Γ_α where α is strictly smaller than ω_1, and thus an isometric copy of the threading space $\mathrm{Th}(p,q)$ is contained in $\mathrm{Sk}(\beta)$. Moreover, in this case, the order of any separable subset of the skein space $\mathrm{Sk}(\omega_1)$ is strictly smaller than ω_1.

Notice that for any two different pairs of different points $(p_1,q_1),(p_2,q_2) \in \mathrm{Sk}(\omega_1)\times\mathrm{Sk}(\omega_1)$ such that $d(p_1,q_1) = d(p_2,q_2) \leq 1/2$, the threading spaces $\mathrm{Th}(p_1,q_1)$ and $\mathrm{Th}(p_2,q_2)$ are contained in $\mathrm{Sk}(\omega_1)$ and are isometric. Moreover, for any $\gamma \in \Delta$,

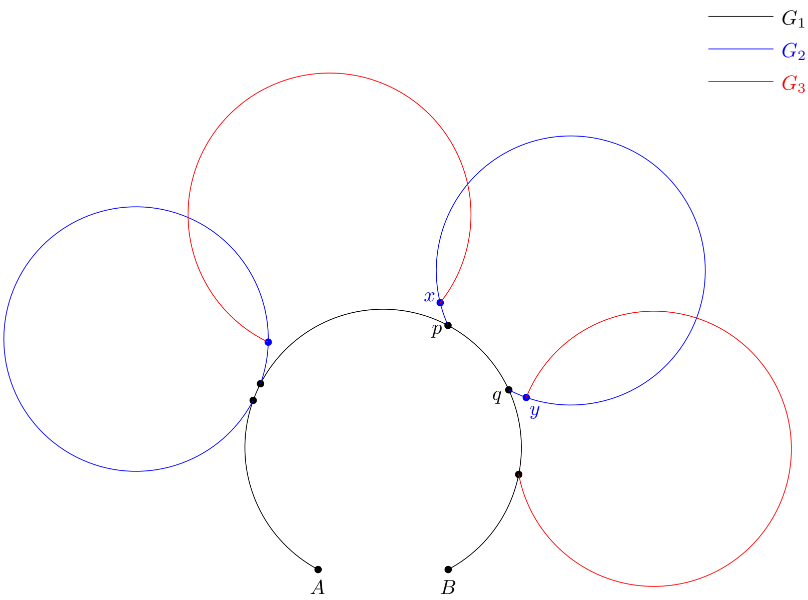

FIGURE 5. Subset of the skein space $\mathrm{Sk}(3)$. The points x and y are bound to p and q respectively.

each of these two threading spaces contains an isometric copy of the thread $T_\gamma(1, a)$, where $a = d(p_1, q_1)$. To differentiate the different copies of the same thread in M that arise due to this fact, we will denote by $T_\gamma(p, q)$ the thread $T_\gamma(1, d(p, q))$ contained in the threading space $\mathrm{Th}(p, q) \subset \mathrm{Sk}(\omega_1)$.

Finally, note also that for a given successor ordinal number $\beta + 1$ and any $(p, q) \in \Gamma_\beta$ and $\gamma \in \Delta$, it holds that $T_\gamma(p, q) \setminus \{p, q\}$ is open in the skein $\mathrm{Sk}(\beta+1)$. Hence, we conclude that any open subset of a thread $T_\gamma(p, q) \subset \mathrm{Sk}(\beta + 1)$ with $(p, q) \in \Gamma_\beta$ and $\gamma \in \Delta$ which does not contain the extreme points $\{p, q\}$ is also open in $\mathrm{Sk}(\beta + 1)$.

The skein space $\mathrm{Sk}(\omega_1)$ contains separable subsets with different structures, all of which fail to be Lipschitz retracts of $\mathrm{Sk}(\omega_1)$. We are going to prove some results that let us reduce the kind of separable subsets we have to consider to a smaller class. In particular, first we are going to show that it is enough to prove that separable subsets without isolated points are not Lipschitz retracts. Secondly, we will introduce some concepts and prove some results to deal with points in limit ordinal generations. We structure these two topics in two different subsections:

4.1.3.2. *First reduction: subspaces with isolated points.* This first reduction is relatively straightforward to see. It is based on two quick observations about the skein space $\mathrm{Sk}(\omega_1)$ and about threads with small gaps. The first observation is a general fact about threads which we have already stated and proven in remark 4.10. The second one we present in the following simple lemma:

LEMMA 4.17. *Let $p, q \in Sk(\omega_1)$ be two different points. There exists a finite sequence $\{x_k\}_{k=0}^n \subset Sk(\omega_1)$ with $x_0 = p$ and $x_n = q$ such that $d(x_k, x_{k+1}) \leq 1/2$ for all $0 \leq k \leq n-1$.*

PROOF. We prove the result by transfinite induction on $\mathrm{ord}(\{p,q\}) < \omega_1$. If $\mathrm{ord}(\{p,q\}) = 0$, then $\{p,q\} = \{A,B\}$ and the result follows directly. Suppose $\mathrm{ord}(\{p,q\}) = \beta < \omega_1$, and suppose the result is true for any set of two points with order $\alpha < \beta$. Consider $\beta_p = \mathrm{ord}(p)$. If β_p is a limit ordinal, then p is the limit of a sequence in $\bigcup_{\alpha<\beta_p} \mathrm{Sk}(\alpha)$, and in particular we can choose x_p with $\mathrm{ord}(x_p) < \beta_p$ such that $d(x_p, p) \leq 1/2$. If $\beta_p = \lambda + 1$ for a countable ordinal λ, then by construction of $\mathrm{Sk}(\omega_1)$ we have that p belongs to the threading space $\mathrm{Th}(x_p, y_p)$ for some $x_p, y_p \in \mathrm{Sk}(\lambda)$. Since p is in a thread of length 1 with extremes x_p, y_p, the distance from p to one of these extremes is less than or equal to $1/2$. Assume without loss of generality that $d(x_p, p) \leq 1/2$. We conclude that in any case there exists x_p with $\mathrm{ord}(x_p) < \mathrm{ord}(p)$ such that $d(x_p, p) \leq 1/2$, and arguing in the same way there exists x_q with $\mathrm{ord}(x_q) < \mathrm{ord}(q)$ such that $d(x_q, q) \leq 1/2$.

The points x_p, x_q satisfy that $\mathrm{ord}(\{x_p, x_q\}) < \beta$, so by inductive hypothesis there exists a sequence $\{x_k\}_{k=}^n \subset M$ with $x_0 = x_p$ and $x_n = x_q$ such that $d(x_k, x_{k+1}) \leq 1/2$ for all $0 \leq k \leq n$. The result follows now adding the points p and q at the beginning and at the end of the sequence respectively. □

Let us mention that this previous lemma can be improved so that the distance between the points in the sequence is less than $1/4$, since this is the biggest possible gap in the threads we are considering. However, we do not consider this improvement to be relevant enough and prefer to prove it with a simpler and shorter argument, since we will only need to use the lemma as it is stated now.

Now we can prove the first reduction result:

PROPOSITION 4.18. *Let S be a closed subset of $Sk(\omega_1)$ with at least two different points. If there exists $p \in S$ such that p is isolated in S, then S is not a Lipschitz retract of $Sk(\omega_1)$.*

PROOF. Put $\varepsilon = d\big(p, S\backslash\{p\}\big)$, which is positive since p is isolated in S. Suppose there exists a Lipschitz retraction $R\colon \mathrm{Sk}(\omega_1) \to S$, and put $K = \|R\|_{\mathrm{Lip}}$.

Consider any point $q \in S$ different from p. By Lemma 4.17 there exists a finite sequence $\{x_k\}_{k\in\mathbb{N}} \subset \mathrm{Sk}(\omega_1)$ such that $x_0 = p$ and $x_n = q$, and such that $d(x_k, x_{k+1}) \leq 1/2$ for all $0 \leq k \leq n-1$. By construction of the skein space $\mathrm{Sk}(\omega_1)$, there exists an isometric copy of the threading space $\mathrm{Th}(x_k, x_{k+1})$ in $\mathrm{Sk}(\omega_1)$, so we may assume that these threading spaces are contained in $\mathrm{Sk}(\omega_1)$. For every $0 \leq k \leq n-1$, the threading space $\mathrm{Th}(x_k, x_{k+1})$ itself contains the threads $T_\gamma(x_k, x_{k+1})$ for every $\gamma \in \Delta$. Choose $\gamma^* = (\gamma_i^*)_{i\in\mathbb{N}} \in \Delta$ such that $\gamma_i^* < \varepsilon/K$ for every $i \in \mathbb{N}$, and write T_k^* to denote the thread $T_{\gamma^*}(x_k, x_{k+1})$ contained in the threading space $\mathrm{Th}(x_k, x_{k+1})$ with extremes x_k and x_{k+1} for every $0 \le k \le n-1$.

Define

$$k_0 = \min\bigl\{k \in \{0, \dots, n-1\} \colon R\bigl(T_k^*\bigr) \not\subseteq \{p\}\bigr\},$$

which exists since $q \in T_{n-1}^*$ and $R(q) = q \in S \setminus \{p\}$. By definition of k_0, there exists a point $y_0 \in T_{k_0}^*$ such that $R(y_0) \in S \setminus \{p\}$. If $k_0 = 0$, the point y_0 cannot be the lower extreme $x_0 = p$ of the thread T_0^*, since $R(p) = p$. If $k_0 \neq 0$, again we have that y_0 cannot be x_{k_0} because x_{k_0} is also in the previous thread $T_{k_0-1}^*$ as its higher extreme point, which would contradict the minimality of k_0. We conclude then that $R(x_{k_0}) = p$. Since the gaps of $T_{k_0}^*$ are given by the sequence γ^*, they are all smaller than ε/K. We can then apply Remark 4.10 to reach a contradiction with the existence of the retraction R. □

4.1.3.3. *Second reduction: points in limit ordinal generations.* In the construction of the skein space $\mathrm{Sk}(\omega_1)$, we have a better understanding of the points belonging to successor ordinal generations than we do of points in limit ordinal generations. Indeed, for a point p of order $\alpha + 1$ we know that there exist two points x and y, with at least one of them in generation α such that p belongs to a thread T_γ for a sequence $\gamma \in \Delta$ with extreme points x and y. However, a point in a limit ordinal generation can initially only be described as a limit of a sequence of points in previous generations, and it does not belong to a thread or to any other defined structure. This subsection is dedicated to finding ways to describe these limit points in order to compensate for the comparatively low *a priori* knowledge we have of them.

For a closed subset S of a metric space M and $r > 0$, define the *open ball around S of radius r*, denoted by $B(S, r)^\circ$ as the set

$$B(S, r)^\circ = \{p \in M \colon d(p, S) < r\}.$$

The main result of this subsection is the following:

PROPOSITION 4.19. *In the skein space $Sk(\omega_1)$, for every ordinal number $\beta < \omega_1$, the β-skein $Sk(\beta)$ is a 1-Lipschitz retraction of the ball $B(Sk(\beta), 1/8)^\circ$.*

In fact, as we are going to see, a stronger result is satisfied, which is helpful in the inductive argument we use to prove it.

Let us introduce some useful concepts: For an ordinal number $\beta < \omega_1$ and a point $p \in \mathrm{Sk}(\omega_1)$, we may consider the set $P_\beta(p) = \bigl\{x \in \mathrm{Sk}(\beta) \colon d(p, x) = d\bigl(p, \mathrm{Sk}(\beta)\bigr)\bigr\}$. Since the β-skein $\mathrm{Sk}(\beta)$ is not compact when $\beta > 0$, we cannot easily ensure that $P_\beta(p)$ is nonempty in every case. In the case where $P_\beta(p)$ is nonempty for a point $p \in \mathrm{Sk}(\omega_1)$ and an ordinal $\beta < \omega_1$, the members of $P_\beta(p)$ will be called the *ancestors of p of order β*.

If a point $p \in \mathrm{Sk}(\omega_1)$ has order $\beta + 1$ for some ordinal $\beta < \omega_1$, then it belongs to a threading space $\mathrm{Th}(x, y)$ for a pair of points (x, y) with $\mathrm{ord}\{x, y\} = \beta$, and it is straightforward to see that the set of ancestors of p of order β is nonempty and is contained in $\{x, y\}$. Since every thread in $\mathrm{Th}(x, y)$ has length 1, if $d(p, \mathrm{Sk}(\beta)) < 1/2$, then $P_\beta(p)$ is unique and is equal to either x or y. The other point will be called the *pseudo-ancestor of p of order β*, and will be denoted by $Q_\beta(p)$. In this way, every point p in a successor ordinal generation $G_{\beta+1}$ such that the distance $d(p, \mathrm{Sk}(\beta))$ is smaller than $1/2$ will belong to the threading space $\mathrm{Th}(P_\beta(p), Q_\beta(p))$. Notice that this concept is only defined for points in successor ordinal generations and with respect to the preceding ordinal.

For each ordinal number $\beta < \omega_1$, we say that a subset S of $\mathrm{Sk}(\omega_1)$ containing $\mathrm{Sk}(\beta)$ is β-stable if for every point $p \in S$ there exists an ancestor $P_\beta(p)$ and it is unique, and moreover, the resulting well defined map $P_\beta \colon S \to M$ is a 1-Lipschitz retraction. Hence, the main result of this subsection will be proven if we show that $B(\mathrm{Sk}(\beta), 1/8)^\circ$ is β-stable for all $\beta < \omega_1$.

We prove the following even stronger result:

PROPOSITION 4.20. *Let $\beta < \omega_1$ be an ordinal number. If two points p and q belong to the ball $B(Sk(\beta), 1/8)^\circ$, then the ancestors $P_\beta(p)$ and $P_\beta(q)$ exist and are unique. Moreover, if $P_\beta(p) \neq P_\beta(q)$, then $d(p, q) = d\big(p, P_\beta(p)\big) + d\big(P_\beta(p), P_\beta(q)\big) + d\big(P_\beta(q), q\big)$.*

In particular, the ball $B(Sk(\beta), 1/8)^\circ$ is β-stable.

PROOF. Put $\alpha = \mathrm{ord}\{p, q\}$. We are going to prove the result by induction on α. If α is smaller than β, then both p and q belong to the skein $\mathrm{Sk}(\beta)$ and the result follows trivially. Hence, we will start the induction assuming $\alpha = \beta + 1$. Let us divide the proof into three parts: the base case, the successor ordinal case, and the limit ordinal case. The base case is in fact the most technical part of the proof:

1.- The base case: $\alpha = \beta + 1$

Suppose that $\alpha = \beta + 1$. Since $\mathrm{ord}\{p, q\} = \beta + 1$, at least one of p and q is in generation $G_{\beta+1}$. Assume without loss of generality that p belongs to generation $G_{\beta+1}$. As we discussed earlier, since the distance from p to $\mathrm{Sk}(\beta)$ is less than $1/8$ and in particular less than $1/2$, we have that the ancestor of order β of p, $P_\beta(p)$, exists and is unique, and p is in the threading space anchored at its ancestor and pseudo-ancestor of order β, denoted by $\mathrm{Th}\big(P_\beta(p), Q_\beta(p)\big)$. Moreover, since $d(p, P_\beta(p)) < 1/4$, we have that p is bound to $P_\beta(p)$ in $\mathrm{Sk}(\omega_1) \setminus \mathrm{Sk}(\beta)$ (we briefly discussed this when introducing the concept of boundness before Proposition 4.16). In other words, we have that the distance from p to any point $x \in \mathrm{Sk}(\beta)$ is computed by

$$d(p, x) = d\big(p, P_\beta(p)\big) + d\big(P_\beta(x), x\big) \qquad \text{for every } x \in \mathrm{Sk}(\beta). \tag{4.6}$$

Now, if the point q is in the skein $\mathrm{Sk}(\beta)$, then q is its own ancestor of order β, and the result follows directly by the previous identity. Suppose then that q is also in generation $G_{\beta+1}$. By the same discussion as above, q belongs to the threading space $\mathrm{Th}\big(P_\beta(q), Q_\beta(q)\big)$, and q satisfies the corresponding identity to

(4.6). There are two possibilities: either both threading spaces $\mathrm{Th}\big(P_\beta(p), Q_\beta(p)\big)$ and $\mathrm{Th}\big(P_\beta(q), Q_\beta(q)\big)$ are the same, or p and q belong to different threading spaces.

If p and q belong to two different threading spaces, then the result follows from equation (4.6) (applied to both p and q) and the construction of the skein $\mathrm{Sk}(\beta{+}1)$. Otherwise, if both threading spaces $\mathrm{Th}\big(P_\beta(p), Q_\beta(p)\big)$ and $\mathrm{Th}\big(P_\beta(q), Q_\beta(q)\big)$ are the same, we may assume that the pseudo-ancestor of p, $Q_\beta(p)$, and the ancestor of q, $P_\beta(q)$, are the same point (otherwise $P_\beta(p) = P_\beta(q)$ and there is nothing left to prove). Now, on the one hand we have that $d\big(p, P_\beta(p)\big) < 1/8$ and $d\big(q, P_\beta(q)\big) < 1/8$ by hypothesis; and on the other hand the width of the threading spaces in the skein spaces we defined is less than $1/2$, so $d\big(P_\beta(p), P_\beta(q)\big) \leq 1/2$. Hence, we necessarily have that

$$d\big(p, P_\beta(p)\big) + d\big(P_\beta(p), P_\beta(q)\big) + d\big(P_\beta(q), q\big) < 3/4 < |q - p|,$$

from which the result follows, whether p and q belong to the same thread in the threading space $\mathrm{Th}\big(P_\beta(p), P_\beta(q)\big)$ or not.

2.- The successor ordinal case: $\alpha = \eta + 1$ for $\eta > \beta$

Suppose now that $\alpha = \eta + 1$ for some countable ordinal $\eta > \beta$, and that the result holds for every pair of points of order strictly less than $\eta + 1$. Let us prove first that both ancestors $P_\beta(p)$ and $P_\beta(q)$ exist and are unique and that the ancestor operation commutes for p and q at order η, that is: $P_\beta\big(P_\eta(p)\big) = P_\beta(p)$ and $P_\beta\big(P_\eta(q)\big) = P_\beta(q)$. Since the argument is exactly the same for both points, we will only prove it for p, and again we may assume without loss of generality that p is in the generation $G_{\eta+1}$. Since the distance from p to $\mathrm{Sk}(\beta)$ is less than $1/8$, we have as well that $d\big(p, \mathrm{Sk}(\eta)\big) < 1/8$ since $\mathrm{Sk}(\beta) \subset \mathrm{Sk}(\eta)$. Therefore, by the first step of the induction process we have that the ancestor of p of order η is unique and

$$d(p, x) = d\big(p, P_\eta(p)\big) + d\big(P_\eta(p), x\big), \qquad \text{for all } x \in \mathrm{Sk}(\eta). \tag{4.7}$$

Moreover, now $P_\eta(p) \in \mathrm{Sk}(\eta)$, and with the previous equation we can deduce that $P_\eta(p)$ belongs to the ball $B(\mathrm{Sk}(\beta), 1/8)^\circ$ as well, so by induction again we have that $P_\beta\big(P_\eta(p)\big)$ is unique, and $d\big(P_\eta(p), x\big) = d\big(P_\eta(p), P_\beta\big(P_\eta(p)\big)\big) + d\big(P_\beta\big(P_\eta(p)\big), x\big)$. These two identities result in the following equation:

$$d(p, x) = d\big(p, P_\eta(p)\big) + d\big(P_\eta(p), P_\beta\big(P_\eta(p)\big)\big) + d\big(P_\beta\big(P_\eta(p)\big), x\big).$$

Applying equation (4.7) for $P_\beta\big(P_\eta(p)\big) \in \mathrm{Sk}(\beta)$ we can put the first two terms of the right-hand side in the previous equation as $d\big(p, P_\eta(p)\big) + d\big(P_\eta(p), P_\beta\big(P_\eta(p)\big)\big) = d\big(p, P_\beta\big(P_\eta(p)\big)\big)$, and finally obtain:

$$d(p, x) = d\big(p, P_\beta\big(P_\eta(p)\big)\big) + d\big(P_\beta\big(P_\eta(p)\big), x\big), \qquad \text{for all } x \in \mathrm{Sk}(\beta). \tag{4.8}$$

Now, from equation (4.8) it is easy to prove that $P_\beta(p)$ is unique and $P_\beta\big(P_\eta(p)\big) = P_\beta(p)$.

To finish with this case, suppose that $P_\beta(p) \neq P_\beta(q)$. Then $P_\eta(p) \neq P_\eta(q)$ by what we have just proven. We can now apply the inductive hypothesis several times and deduce that:

$$\begin{aligned}
d(p,q) = \quad & d\big(p, P_\eta(p)\big) + d\big(P_\eta(p), P_\eta(q)\big) + d\big(P_\eta(q), q\big) \\
= \quad & d\big(p, P_\eta(p)\big) + d\big(P_\eta(p), P_\beta(p)\big) + \\
& d\big(P_\beta(p), P_\beta(q)\big) + d\big(P_\beta(q), P_\eta(q)\big) + d\big(P_\eta(q), q\big) \\
= \quad & d\big(p, P_\beta(p)\big) + d\big(P_\beta(p), P_\beta(q)\big) + d\big(P_\beta(q), q\big).
\end{aligned}$$

This finishes the successor ordinal case.

3.- The limit ordinal case

Suppose finally that α is a limit ordinal. As in the previous case, we start by proving that $P_\beta(p)$ and $P_\beta(q)$ exist and are unique. Similarly, we assume that $\mathrm{ord}(p) = \alpha$, and we only prove it for p. Consider a sequence $\{p_n\}_{n\in\mathbb{N}}$ of points in $\mathrm{Sk}(\alpha)$ convergent to p and such that $\mathrm{ord}(p_n) < \alpha$ for all $n \in \mathbb{N}$. Moreover, since the ball $B(\mathrm{Sk}(\beta), 1/8)^\circ$ is an open set of $\mathrm{Sk}(\omega_1)$ which contains p, we may suppose that $d\big(p_n, \mathrm{Sk}(\beta)\big) < 1/8$ as well for all $n \in \mathbb{N}$. Therefore, by inductive hypothesis, $P_\beta(p_n)$ is unique for all $n \in \mathbb{N}$, and

$$d(p_n, x) = d\big(p_n, P_\beta(p_n)\big) + d\big(P_\beta(p_n), x\big), \quad \text{for all } x \in \mathrm{Sk}(\beta) \text{ and all } n \in \mathbb{N}. \tag{4.9}$$

We are going to prove first that the sequence $\{P_\beta(p_n)\}_{n\in\mathbb{N}}$ is convergent. Indeed, since $\mathrm{ord}(p_n) < \alpha$ for all $x \in \mathbb{N}$, for all $n, m \in \mathbb{N}$ such that $P_\beta(p_n) \neq P_\beta(p_m)$, we have that $d(p_n, p_m) = d\big(p_n, P_\beta(p_n)\big) + d\big(P_\beta(p_n), P_\beta(p_m)\big) + d\big(P_\beta(p_m), p_m\big)$. In particular, $d\big(P_\beta(p_n), P_\beta(p_m)\big) \leq d(p_n, p_m)$ for all $n, m \in \mathbb{N}$, whether $P_\beta(p_n) \neq P_\beta(p_m)$ or not. Since the sequence $\{p_n\}_{n\in\mathbb{N}}$ converges, it is a Cauchy sequence, which implies that the sequence $\{P_\beta(p_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence as well, and thus convergent in the complete metric space $\mathrm{Sk}(\beta)$. Denote the limit of this sequence by P^*, which belongs to the set $\mathrm{Sk}(\beta)$.

Taking the limit when n tends to infinity in equation (4.9), we obtain that

$$d(p, x) = d\big(p, P^*\big) + d\big(P^*, x\big), \qquad \text{for all } x \in \mathrm{Sk}(\beta).$$

Similarly to the successor ordinal case, from this equation it follows that $P_\beta(p) = P^*$ and it is unique.

Now, suppose that $P_\beta(p) \neq P_\beta(q)$, and consider two sequences $\{p_n\}_{n\in\mathbb{N}}$ and $\{q_n\}_{n\in\mathbb{N}}$ in $B(\mathrm{Sk}(\beta), 1/8)^\circ$ converging to p and q respectively, and satisfying that $\mathrm{ord}\{p_n, q_n\} < \alpha$. By the previous argument, we have that:

$$\begin{aligned}
d(p,q) &= \lim_n d(p_n, q_n) = \lim_n \Big(d\big(p_n, P_\beta(p_n)\big) + d\big(P_\beta(p_n), P_\beta(q_n)\big) + d\big(P_\beta(q_n), q_n\big)\Big) \\
&= d\big(p, P_\beta(p)\big) + d\big(P_\beta(p), P_\beta(q)\big) + d\big(P_\beta(q), q\big),
\end{aligned}$$

which concludes the proof. □

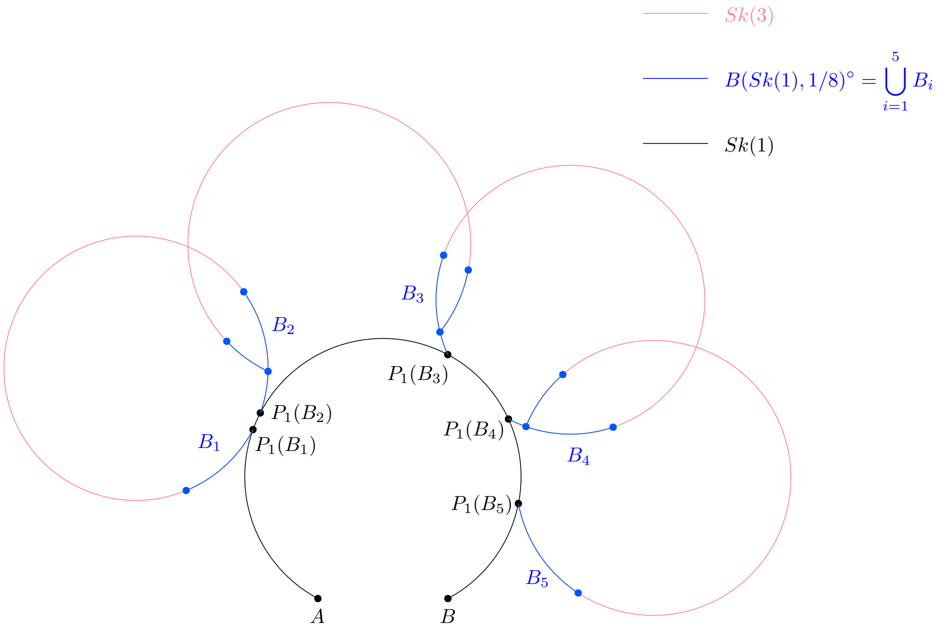

FIGURE 6. The skein Sk(1) is a 1-Lipschitz retract of the ball $B(\mathrm{Sk}(1), 1/8)$ with the ancestor map.

In Figure 6 we observe conceptually Proposition 4.20. In this diagram we portray again a subset of the skein Sk(3), and the ball $B(\mathrm{Sk}(1), 1/8)^\circ$ (colored in blue) is partitioned into 5 subsets $\{B_i\}_{i=1}^5$ such that every point in the same B_i has the same ancestor of order 1. The ancestor map clearly defines in this case a 1-Lipschitz retraction onto Sk(1).

Finally, with this proposition, the second reduction result follows directly:

PROOF OF 4.19. It follows directly from Proposition 4.20. $\square$

In the proof of the main theorem, given a separable subset, we will consider a bigger separable subset that is in some sense closed for the operation of taking ancestors closer than 1/8. Specifically, we have the following Lemma:

LEMMA 4.21. *Given a separable subset S of the metric space $Sk(\omega_1)$, there exists a separable set $\widehat{S} \subset Sk(\omega_1)$ containing S such that: for every point $x \in \widehat{S}$ and every ordinal $\beta < \omega_1$ such that $d(x, Sk(\beta)) < 1/8$, the unique ancestor of order β of x belongs to $\widehat{S}$.*

PROOF. For any point $x \in \mathrm{Sk}(\omega_1)$, put $\beta_0(x) = \mathrm{ord}(x)$, which is a countable ordinal. Trivially we have that the ancestor $P_{\beta_0}(x) = x$ is unique. We might define then $\beta_1(x) = \min\{\beta < \omega_1 \colon d\big(P_{\beta_0}(x), \mathrm{Sk}(\beta)\big) < 1/8\}$, which satisfies that $P_{\beta_1}\big(P_{\beta_0}(x)\big)$ is unique as well.

Then, we can inductively define a decreasing sequence of ordinal numbers

$$\beta_n(x) = \min\{\beta < \omega_1 \colon d\big(\big(P_{\beta_{n-1}(x)} \circ \cdots \circ P_{\beta_0(x)}\big)(x), \mathrm{Sk}(\beta)\big) < 1/8\}$$

for each $n \in \mathbb{N}$. Since $\beta_{n+1}(x) \leq \beta_n(x)$ for every $n \in \mathbb{N}$ and the ordinal numbers are well ordered, there must exist $n_0(x) \in \mathbb{N}$ such that $\beta_n(x) = \beta_{n_0}(x)$ for all $n \geq n_0$.

Now, given a separable subset S of the metric space M, take D a countable and dense subset of S. Consider the set $\widehat{D}$ defined by:

$$\widehat{D} = \bigcup_{x\in D} \bigcup_{n\in\mathbb{N}} \bigcup_{\beta=\beta_n(x)}^{\beta_{n-1}(x)} P_\beta\big(\big(P_{\beta_{n-1}(x)} \circ \cdots \circ P_{\beta_0(x)}\big)(x)\big),$$

which is countable, contains D, and satisfies that for any point $x \in \widehat{D}$ and any ordinal β with $d(x, \mathrm{Sk}(\beta)) < 1/8$, the ancestor $P_\beta(x)$ belongs to $\widehat{D}$ as well.

Finally, put $\widehat{S} = \overline{\widehat{D}}$. The set $\widehat{S}$ is separable and it contains S. For any point $x \in \widehat{S}$ and any ordinal $\beta < \omega_1$ such that x belongs to the ball $B(\mathrm{Sk}(\beta), 1/8)^\circ$, we have that x is the limit of a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points in $\widehat{D}$ which are also in $B(\mathrm{Sk}(\beta), 1/8)^\circ$. As we argued in the proof of Proposition 4.20, we have that the sequence $\{P_\beta(x_n)\}_{n\in\mathbb{N}}$, which is contained in $\widehat{D}$, converges to $P_\beta(x)$. The statement of the lemma now follows directly. $\square$

We will use the previous lemma as well the fact that every separable subset of the skein $\mathrm{Sk}(\omega_1)$ is contained in the closure of the union of countably many threads:

LEMMA 4.22. *Let S be a separable subset of the skein $Sk(\omega_1)$. Then there exists a countable family of pairs $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ in $Sk(\omega_1) \times Sk(\omega_1)$ and a countable family of sequences $\{\gamma^n\}_{\in\mathbb{N}}$ in Δ such that the following property is satisfied:*

For every point $x \in S$ in a successor ordinal generation there exists a natural number n_x such that x belongs to the interior of the thread $T_{\gamma^{n_x}}(x_{n_x}, y_{n_x}) \subset Sk(\omega_1)$.

PROOF. Suppose by contradiction that the result fails. Since S is separable, there are only countably many ordinals $\alpha < \omega_1$ such that the intersection of S with generation G_α is nonempty. Hence, there must exist one successor ordinal $\alpha_0 + 1$ such that for every countable collection of pairs $\big\{\{x_n, y_n\}\big\}_{n\in\mathbb{N}}$ in Γ_{α_0} and any countable family of sequences $\{\gamma^n\}_{\in\mathbb{N}}$ in Δ, there exists a point in $S \cap G_{\alpha_0+1}$ that lies outside the interior of the thread $T_{\gamma^n}(x_n, y_n) \subset \mathrm{Sk}(\alpha_0 + 1)$ for all $n \in \mathbb{N}$.

Since every point in $S \cap G_{\alpha_0+1}$ belongs to the interior of a thread anchored at a pair of points in Γ_{α_0}, by a standard transfinite induction argument we may find an uncountable set of different points $\{p_i\}_{i\in I}$ in $S \cap G_{\alpha_0+1}$ and an uncountable family of different threads $\{T_{\gamma^i}(x_i, y_i)\}_{i\in I}$ with $\{x_i, y_i\} \in \Gamma_{\alpha_0}$ and $\gamma^i \in \Delta$ for all $i \in I$, such that p_i belongs to the interior of the thread $T_{\gamma^i}(x_i, y_i)$ for all $i \in I$. However, this implies that the family $\{T_{\gamma^i}(x_i, y_i)^\circ \cap S\}_{i\in I}$ is an uncountable collection of nonempty and open subsets of S which are pairwise disjoint, which contradicts the separability of S. $\square$

4.1.3.4. *Proving the general case.* We proceed now to prove the main result of the section:

Proof of Theorem 4.1. Consider the complete skein $\mathrm{Sk}(\omega_1)$. We will prove that it does not contain any non-singleton separable Lipschitz retracts.

We will proceed by contradiction. Let S be a separable subset of $\mathrm{Sk}(\omega_1)$ containing at least two points. We may assume that S has no isolated points by Proposition 4.18. Suppose there exists a Lipschitz retraction $R\colon \mathrm{Sk}(\omega_1) \to S$ onto S, and put $K = \|R\|_{\mathrm{Lip}}$. We are going to find a specific thread T^* in $\mathrm{Sk}(\omega_1)$ such that when restricting the map R to T^*, the resulting K-Lipschitz function can be transformed to yield a contradiction. Because of the length of the proof, we divide it in two parts: The first part describes the process to define the problematic thread T^*, while the second part deals with the map $R_{|T^*}$, and how to transform it to arrive at a contradiction. We will also highlight important facts throughout the proof to help in its readability.

1.- Defining the conflicting thread T^*

We start by finding two points to anchor the thread T^*:

Fact 1. There exist two points p and q in S closer than $1/2$, and such that there exists a successor ordinal β_0 with $p, q \in B(\mathrm{Sk}(\beta_0), 1/8)^\circ$ which satisfies that $P_{\beta_0}(p) \neq P_{\beta_0}(q)$.

Moreover, the ancestor $P_{\beta_0}(p)$ is in generation β_0, and is contained in a thread $T_{\gamma^0}\big(A_0, B_0\big)$ with $\big\{A_0, B_0\big\} \in \Gamma_{\beta_0 - 1}$.

Proof of Fact 1. Define α_0 as the least ordinal such that $S \cap \mathrm{Sk}(\alpha_0)$ is nonempty,

We divide the proof in two cases:

Case 1: There exist two different points p and q in $B(\mathrm{Sk}(\alpha_0), 1/8)^\circ \cap S$ closer than $1/2$ such that $P_{\alpha_0}(p) \neq P_{\alpha_0}(q)$.

In this case, if the ordinal α_0 is a successor ordinal, putting $\beta_0 = \alpha_0$ we are done, since by definition of α_0 both $P_{\alpha_0}(p)$ and $P_{\alpha_0}(q)$ belong to generation G_{α_0}.

Suppose then that α_0 is a limit ordinal. Since p and q belong to the ball $B(\mathrm{Sk}(\alpha_0), 1/8)^\circ$, the ordinal

$$\alpha_1 = \min\{\beta < \omega_1 \colon \{p, q\} \subset B(\mathrm{Sk}(\beta), 1/8)^\circ\}$$

is less than α_0. Both $P_{\alpha_1}(p)$ or $P_{\alpha_1}(q)$ are well defined and unique. Moreover, by minimality, α_1 must be a successor ordinal, and at least one of $P_{\alpha_1}(p)$ or $P_{\alpha_1}(q)$ must belong to generation G_{α_1}. Therefore, we can put $\beta_0 = \alpha_1$ and Fact 1 is proven for Case 1.

Case 2: There exists a point $A \in S \cap \mathrm{Sk}(\alpha_0)$ such that for all $x \in B(A, 1/8)^\circ \cap S$ we have that $P_{\alpha_0}(x) = A$.

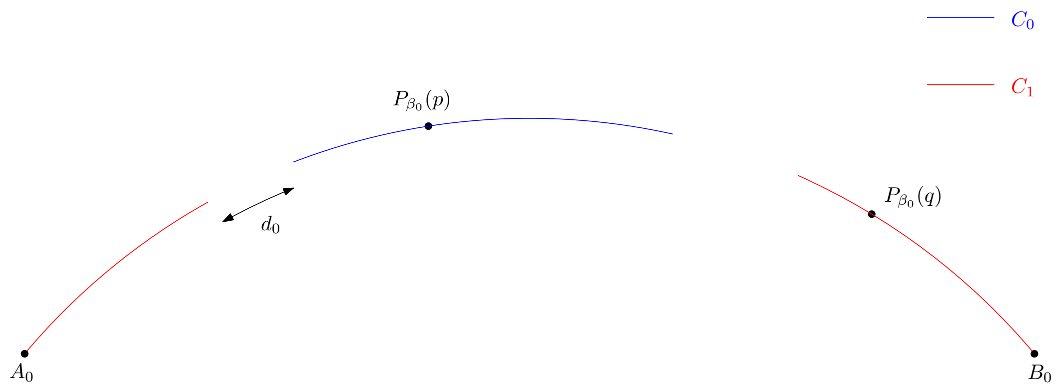

FIGURE 7. Thread of the skein $\mathrm{Sk}(\beta_0)$ showing one possible arrangement of the ancestors of p and q and choice of C_0 and C_1.

Notice that the above statement follows from negating the assumption of Case 1. In this case define the ordinal

$$\eta_0 = \min\{\eta < \omega_1 \colon P_\eta(x) \neq A \text{ for some } x \in B(A, 1/8)^\circ \cap S\},$$

Such an ordinal number must exist since A is not isolated in S by assumption. Moreover, η_0 must be a successor ordinal since every point in a limit ordinal generation is the limit of the succession given by its previous (existing) ancestors. Take now any point $p \in B(A, 1/8)^\circ \cap S$ such that $P_{\eta_0}(p) \neq A$, and set $q = A$. Putting $\beta_0 = \eta_0$, we have that both p and q belong to $B(\mathrm{Sk}(\beta_0), 1/8)^\circ$ and $P_{\beta_0}(p) \neq P_{\beta_0}(q) = q$. Moreover, the ancestor $P_{\beta_0}(p)$ belongs to generation G_{β_0} by minimality. $\square$

With this in mind, we can apply Proposition 4.12 to the thread $T_{\gamma^0}(A_0, B_0)$ and the point $P_{\beta_0}(p)$, to find a compact subset $C_0 \subset T_{\gamma^0}(A_0, B_0)$ with diameter less than $d\big(P_{\beta_0}(p), P_{\beta_0}(q)\big)$ such that $P_{\beta_0}(p) \in C_0$ and C_0 is open and closed in $\mathrm{Sk}(\beta_0)$. Put $C_1 = \mathrm{Sk}(\beta_0) \setminus C_0$. Then the point $P_{\beta_0}(q)$ belongs to C_1, and since C_0 is compact and disjoint from the closed set C_1, the distance $d\big(C_0, C_1\big)$ is strictly positive. Put $d_0 = d\big(C_0, C_1\big) > 0$.

Figure 7 summarizes one possible layout of the elements we have defined so far in the skein $\mathrm{Sk}(\beta_0)$.

Now, the separation between the sets C_0 and C_1 allows us to use Remark 4.10 to obtain the following fact:

FACT 2. If $T = [0, l]$ is a thread whose gaps are all smaller than $d_0/2K$, there cannot be any $2K$-Lipschitz map $F \colon T \to \mathrm{Sk}(\beta_0)$ such that $F(0) = P_{\beta_0}(p) \in C_0$ and $F(l) = P_{\beta_0}(q) \in C_1$.

Next, we define the subset $\widehat{S} \subset \mathrm{Sk}(\omega_1)$ using Lemma 4.21 such that the following fact is satisfied:

FACT 3. The set $\widehat{S} \subset \mathrm{Sk}(\omega_1)$ is separable, it contains the set S, and for any point $x \in \widehat{S}$ and any ordinal number β such that $x \in B(\mathrm{Sk}(\beta), 1/8)$, the unique ancestor $P_\beta(x)$ belongs to $\widehat{S}$.

To continue with the proof, since $\widehat{S}$ is separable, by Lemma 4.22 we can find a countable family of sequences $\{\gamma^n\}_{n\in\mathbb{N}}$ in Δ, and a countable set of pairs $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ in $\mathrm{Sk}(\omega_1)\times\mathrm{Sk}(\omega_1)$ such that, denoting by T^n the thread $T_{\gamma^n}(x_n, y_n) \subset \mathrm{Sk}(\omega_1)$ for each $n \in \mathbb{N}$, the countable family of threads $\mathcal{T}_0 = \{T^n\}_{n\in\mathbb{N}}$ satisfies that any point $x \in \widehat{S}$ belonging to a successor ordinal generation is contained in the interior of at least one thread T^{n_x} for some $n_x \in \mathbb{N}$. For every $n \in \mathbb{N}$, the thread $T^n = T_{\gamma^n}(x_n, y_n) \in \mathcal{T}_0$ has length 1, and so the open subsets given by $[x_n, x_n + 1/8)_{T^n}$ and $(y_n - 1/8, y_n]_{T^n}$ are separable subsets that do not intersect. Define for every $n \in \mathbb{N}$ two countable sets D_1^n and D_2^n such that D_1^n is dense in $[x_n, x_n + 1/8)_{T^n}$ and D_2^n is dense in $(y_n - 1/8, y_n]_{T^n}$.

Finally, we can define the countable family of threads given by

$$\mathcal{T} = \bigcup_{n\in\mathbb{N}} \left(\bigcup_{(x,y)\in D_1^n\times D_2^n} \{[x, y]_{T^n}\} \right).$$

Notice that each thread in $\mathcal{T}_0$ has Lebesgue measure of at least $1/2$. Therefore, the measure of the threads in $\mathcal{T}$ is bounded below by $1/4$. We can apply now Theorem 4.9 with $\varepsilon = 1/4$ and $2K \geq 1$ to find a sequence $\gamma^* \in \Delta$ with the following property:

FACT 4. There exists a sequence $\gamma^* = \{\gamma_k^*\}_{k\in\mathbb{N}} \in \Delta$ such that:

For any thread S of length l_S whose sequence of gaps $\{C_k^S\}_{k\in\mathbb{N}}$ in decreasing length order satisfies $\mathrm{length}(C_k^S) < \gamma_k^*$ for all $k \in \mathbb{N}$, it holds that: For every $K' \leq 2K$, if there exists a K'-Lipschitz function $F\colon S \to [x, y]_{T^n}$ such that $F(0) = x$ and $F(l_s) = y$, where $n \in \mathbb{N}$ and $(x, y) \in D_1^n \times D_2^n$; then the thread S has a gap of length greater than or equal to $d(x, y)/K'$.

Moreover, without loss of generality we may choose γ^* such that

$$\gamma_k^* < \min\left\{\frac{1}{16K}, \frac{d_0}{2K}\right\}$$

for all $k \in \mathbb{N}$.

Since the sequence γ^* belongs to Δ, the associated thread $T_{\gamma^*}(p, q)$ belongs to the threading space $\mathrm{Th}(p, q)$, and is therefore a subset of $\mathrm{Sk}(\omega_1)$. Put $T^* = T_{\gamma^*}(p, q)$. This is the problematic thread we will use to reach a contradiction. Recall that the length of the gaps of the thread T^* is given by the sequence $\gamma^* \in \Delta$. Hence, we have the following result by the choice of γ^* and Fact 2:

FACT 5. There does not exist any $2K$-Lipschitz map $F\colon T^* \to \mathrm{Sk}(\beta_0)$ such that $F(p) = P_{\beta_0}(p)$ and $F(q) = P_{\beta_0}(q)$.

In the next section we will find a function in contradiction with this last fact. The retraction R from $\mathrm{Sk}(\omega_1)$ onto S can be restricted to T^* to obtain a K-Lipschitz map $R_{|T^*}\colon T^* \to \widehat{S}$ such that $R(p) = p$ and $R(q) = q$. This restriction will be the starting point in the process to define the contradicting function.

2.- Transforming the map $R_{|T^*}$

We can only ensure that the image of the map $R_{|T^*}$ is contained in S, and so the order of $R_{|T^*}(T^*)$ is less than the order of S, but it can still be higher than β_0. We are going to transform inductively the map $R_{|T^*}$ to reduce the order of its image until we arrive at β_0, where we will reach a contradiction.

In order to do this, we need the following result:

CLAIM 1. *Let T^*, $\widehat{S}$, K and β_0 be defined as above. Let $F\colon T^* \to \widehat{S}$ be a Lipschitz map such that $\|F\|_{Lip} < 2K$, and $F(p) = P_\beta(p)$ and $F(q) = P_\beta(q)$ for some ordinal $\beta \geq \beta_0$. Then we have the three following results:*

- *(A) If $ord(F(T^*))$ is a limit ordinal then there exists an ordinal $\widehat{\beta} \geq \beta_0$, and a K-Lipschitz function $\widehat{F}\colon T^* \to \widehat{S}$ such that $\widehat{F}(p) = P_{\widehat{\beta}}(p)$, $\widehat{F}(q) = P_{\widehat{\beta}}(q)$, and $ord(\widehat{F}(T^*)) < ord(F(T^*))$.*
- *(B) If $ord(F(T^*))$ is a successor ordinal $\alpha + 1$ such that $\beta < \alpha + 1$ then for every $\varepsilon > 0$ such that $\|F\|_{Lip} + \varepsilon < 2K$, there exists a Lipschitz function $\widehat{F}\colon T^* \to \widehat{S}$ such that $\|\widehat{F}\|_{Lip} < \|F\|_{Lip} + \varepsilon$, $\widehat{F}(p) = P_\beta(p)$, $\widehat{F}(q) = P_\beta(q)$, and $ord(\widehat{F}(T^*)) \leq \alpha < ord(F(T^*))$.*
- *(C) If $ord(F(T^*))$ is a successor ordinal $\alpha+1$ such that $\beta = \alpha+1$, and $\beta > \beta_0$, then for every $\varepsilon > 0$ such that $\|F\|_{Lip} + \varepsilon < 2K$, there exists a Lipschitz function $\widehat{F}\colon T^* \to \widehat{S}$ such that $\|\widehat{F}\|_{Lip} < \|F\|_{Lip} + \varepsilon$, $\widehat{F}(p) = P_\alpha(p)$, $\widehat{F}(q) = P_\alpha(q)$ and $ord(\widehat{F}(T^*)) \leq \alpha < ord(F(T^*))$.*

Before proving this claim let us discuss its implications: The map $R_{|T^*}$ satisfies the general hypothesis of the claim with $\beta = \mathrm{ord}\{p, q\}$. Notice as well that if a function F satisfies either of the conditions (A), (B) or (C) then the resulting map $\widehat{F}$ for any valid $\varepsilon > 0$ satisfies again the general conditions of the claim. In all three cases, the order of the image of the map $\widehat{F}$ produced is an ordinal strictly lower than the order of the image of F.

This means that putting $F_0 = R_{|T^*}$, we can define inductively K-Lipschitz maps $\{F_n\}_{n\in\mathbb{N}}$ such that

$$F_{n+1} = \begin{cases} \widehat{F_n} & \text{if } F_n \text{ satisfies (A), (B) or (C),} \\ F_n & \text{otherwise .} \end{cases}$$

We may choose any valid $\varepsilon > 0$ at the steps that require it. There must exist $n_0 \in \mathbb{N}$ such that $F_n = F_{n_0}$ for all $n \geq n_0$. Indeed, otherwise the sequence $\{\mathrm{ord}(F_n)(T^*)\}_{n\in\mathbb{N}}$ is an infinite strictly decreasing sequence of ordinal numbers, resulting in a contradiction with the well ordering of the ordinals.

Therefore, the map $F_{n_0}\colon T^* \to \widehat{S}$ is a Lipschitz map with $\|F_{n_0}\|_{\mathrm{Lip}} < 2K$ such that $F(p) = P_\beta(p)$ and $F(q) = P_\beta(q)$ for some $\beta \geq \beta_0$ that does not satisfy one of (A), (B) or (C). Since it does not satisfy (A), we have that $\mathrm{ord}(F_{n_0})(T^*)$ is successor ordinal $\alpha + 1$. This means that in order to fail (B), the ordinal $\alpha + 1$ must equal β. In turn, since F_{n_0} does not meet the requirements of (C) either, we conclude that β (that is, the ordinal such that $F_{n_0}(p) = P_\beta(p)$, $F_{n_0}(q) = P_\beta(q)$, and the order of $F_{n_0}(T^*)$) equals β_0.

In conclusion, $F_{n_0}\colon T^* \to \widehat{S} \cap \mathrm{Sk}(\beta_0)$ is a Lipschitz map with $\|F_{n_0}\|_{\mathrm{Lip}} < 2K$ from the thread $T^* = T_{\gamma^*}(p, q)$ into $\mathrm{Sk}(\beta_0)$ such that $F_{n_0}(p) = P_{\beta_0}(p)$ and $F_{n_0}(q) = P_{\beta_0}(q)$. This contradicts Fact 5, which leads to the desired contradiction. It only remains to prove Claim 1.

Proof of Claim 1. We prove each statement separately:

Proof of (A). Put $\alpha = \mathrm{ord}(F(T^*))$. Then $\alpha \geq \beta$. Since T^* is compact, the image $F(T^*)$ is also compact in $\mathrm{Sk}(\omega_1)$. Therefore, there exists a point $x_0 \in T^*$ such that $F(x_0)$ belongs to the limit generation G_α in $\mathrm{Sk}(\omega_1)$. Put $r_0 = \min\{d(F(x_0), \mathrm{Sk}(\beta_0)), 1/8\}$. Since β_0 is a successor ordinal, the number r_0 is strictly positive.

Now, choose a finite set of points $\{x_i\}_{i=1}^n \subset T^*$ such that

$$F(T^*) \subset \bigcup_{i=1}^n B(F(x_i), r_0/2).$$

For each $i = 1, \ldots, n$, the ordinal number

$$\alpha_i = \min\{\gamma < \omega_1\colon d\big(F(x_i), \mathrm{Sk}(\gamma)\big) < r_0/2\}$$

is a successor ordinal strictly smaller than the order of $F(T^*)$. Hence, if $\widehat{\beta} = \max\{\alpha_i\colon i = 1, \ldots, n\}$, we have that $\widehat{\beta} < \mathrm{ord}(F(T^*))$ and $d\big(F(x), \mathrm{Sk}(\widehat{\beta})\big) < r_0$ for all $x \in T^*$. This implies that $\widehat{\beta}$ is greater than β_0. Applying Corollary 4.19, since $r_0 < 1/8$, we have that the map

$$\begin{aligned} \widehat{F}\colon T^* &\longrightarrow \widehat{S} \\ x &\longmapsto P_{\widehat{\beta}}\big(F(x)\big) \end{aligned}$$

is a K-Lipschitz map. It is well defined since the ancestor of order $\widehat{\beta}$ of each point in $F(T^*)$ is unique, and the image of any point $x \in T^*$ belongs to the set $\widehat{S}$ again since $d\big(F(x), \mathrm{Sk}(\widehat{\beta})\big) < r_0$ for all $x \in T^*$ (see Fact 3). We have then that

$\widehat{F}(p) = P_{\widehat{\beta}}(P_\beta(p)) = P_{\widehat{\beta}}(p)$, and similarly $\widehat{F}(q) = P_{\widehat{\beta}}(q)$. Finally, the order of $\widehat{F}(T^*)$ is at most $\widehat{\beta}$ as well, so it is satisfied that $\operatorname{ord}(\widehat{F}(T^*)) < \operatorname{ord}(F(T^*))$. $\square$

Proof of (B). Put $\overline{K} = \|F\|_{\mathrm{Lip}}$. Suppose that the order of $F(T^*)$ is $\alpha+1$, and that $\beta < \alpha+1$. Since the image of F is in $\widehat{S}$ and $\alpha+1$ is a successor ordinal, there exists a subsequence $\{n_k\}_{k\in\mathbb{N}}$ such that $F(T^*) \cap G_{\alpha+1} = F(T^*) \setminus \mathrm{Sk}(\alpha)$ is contained in the union of threads $\bigcup_{k\in\mathbb{N}} T^{n_k}$. Informally, the "problematic" part of $F(T^*)$ is contained in this countable set of threads (without the extreme points, since these always belong to a lower generation), which is a subfamily of the set $\mathcal{T}_0$ we have considered in the definition of T^*.

Hence, for any $t \in T^*$ such that $F(t) \in T^{n_{k(t)}}$ for some $n_{k(t)} \in \mathbb{N}$, we may find an extended interval I_t in T^* containing t such that I_t is maximal for F and $T^{n_{k(t)}}$. The extended interval I_t is actually of the form $[a_t, b_t]_{T^*}$ with $p \le a_t < b_t \le q$, since if I_t contains both extremes p and q of T^*, then necessarily $\{F(p), F(q)\} = \{P_\beta(p), P_\beta(q)\} \in \Gamma_\alpha$, so $\alpha = \beta$, and thus $I_t = [p,q]_{T^*} = T^*$.

With this idea, since $F(T^*)$ is separable, we can define a countable family of maximal intervals $\{[a_i, b_i]_{T^*}\}_{i\in\mathbb{N}}$ in T^* such that $F\big([a_i,b_i]_{T^*}\big)$ is contained in $T^{n_{k(i)}}$ for all $i \in \mathbb{N}$, and every point $t \in T^*$ such that its image $F(t)$ is in generation $G_{\alpha+1}$ is contained in $[a_i,b_i]_{T^*}$ for some $i \in \mathbb{N}$. To simplify the notation, we abuse it and write $n_{k(i)} = i$. Therefore, we will write that $F\big([a_i,b_i]_{T^*}\big)$ is contained in the thread T^i. Recall that the thread $T^i \in \mathcal{T}_0$ belongs to the threading space $\mathrm{Th}(x_i, y_i)$ for every $i \in \mathbb{N}$. Again informally, we have identified a countable family of maximal intervals in T^* that contain all the points whose image we need to change to prove (B).

In the following Fact, we "correct" the image of this countable family of maximal intervals.

Fact 6. For every $i \in \mathbb{N}$, there exists a Lipschitz function $F_i \colon T^* \to \widehat{S}$ with $\|F_i\|_{\mathrm{Lip}} \le \|F\| + \varepsilon$ such that $F_i(t) = F(t)$ for all $t \in T^* \setminus [a_i,b_i]_{T^*}$ and $F_i(t) \in \{x_i, y_i\}$ for all $t \in [a_i,b_i]_{T^*}$.

Proof. Fix $i \in \mathbb{N}$. Since the order of $F(T^*)$ is $\alpha+1$, we may work directly on the skein $\mathrm{Sk}(\alpha+1)$. Here, the point $F(a_i)$, which belongs to the thread T_i, is bound to either x_i or y_i in T^i. To see this, notice that, since there are no gaps in T^* of length greater than $(2K)^{-1}/8$, by maximality of $[a_i,b_i]_{T^*}$ in $T_{\gamma^i}(x_i,y_i)$, we have that the distance from $F(a_i)$ to $\mathrm{Sk}(\alpha+1) \setminus T^*$ is smaller than $1/8$. Hence, by construction of the successor ordinal skein $\mathrm{Sk}(\alpha+1)$, the distance from $F(a_i)$ to one of the two extremes of the thread T^i is also smaller than $1/8$, which implies that $F(a_i)$ is bound to one of these extremes. Similarly, $F(b_i)$ is bound to either x_i or y_i in T^i. Suppose without loss of generality that $F(a_i)$ is bound to x_i.

There are two possibilities: either $F(b_i)$ is bound to x_i as well, or $F(b_i)$ is bound to the other extreme point y_i. If $F(b_i)$ is bound to x_i, then we can apply Proposition 4.16 and obtain F_i with the desired properties.

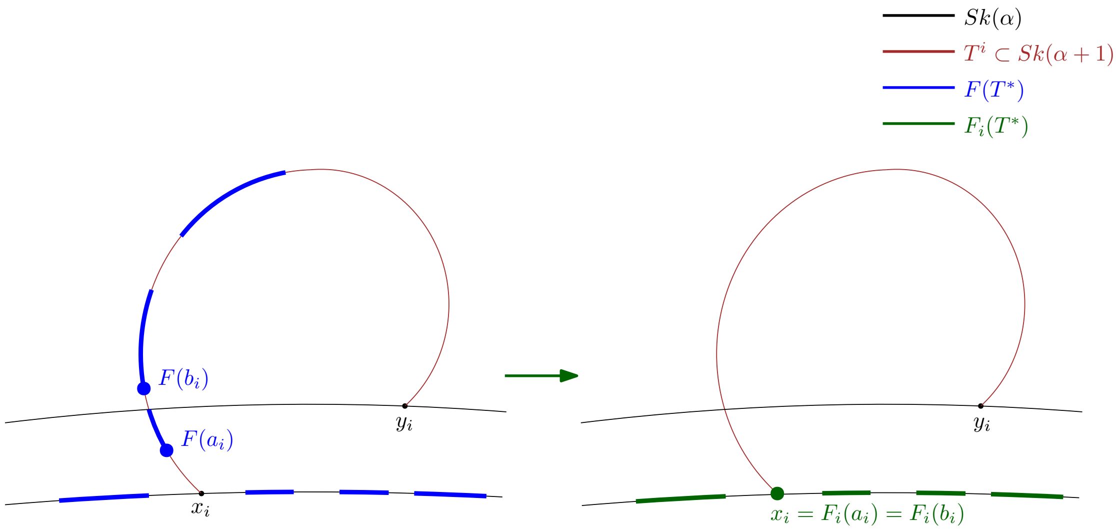

FIGURE 8. If both $F(a_i)$ and $F(b_i)$ are bound to the same anchor x_i, we may define F_i by sending all points in $[a_i, b_i]_{T^*}$ to x_i without increasing the Lipschitz constant.

In Figure 8 we observe this first possibility, and the resulting map F_i according to Proposition 4.16.

Suppose now that $F(b_i)$ is bound to y_i in T^i. We are going to show that there is a gap C_i in $[a_i, b_i]_{T^*}$ with length greater than $d(x_i, y_i)/(K+\varepsilon)$. Indeed, suppose by contradiction there is no such gap.

We have that $F(a_i)$ belongs to the interval $[x_i, x_i + 1/8)_{T^i}$, while $F(b_i)$ belongs to $(y_i - 1/8, y_i]_{T^i}$. Recall the definition (prior to Fact 4) of the dense and countable subsets $D_1^i \subset [x_i, x_i + 1/8)_{T^i}$ and $D_2^i \subset (y_i - 1/8, y_i]_{T^i}$ in T^i, which were used to define the sequence of gaps of the thread T^*. Since D_1^i is dense in $[x_i, x_i + 1/8)_{T^i}$, and D_2^i is dense in $(y_i - 1/8, y_i]_{T^i}$, considering the subset $[a_i, b_i]_{T^*}$ of T^* as a thread, and restricting F to this thread, we obtain by Proposition 4.13 that there exist two points $a'_i, b'_i \in [a_i, b_i]_{T^*}$, and two points $(x'_i, y'_i) \in D_1^i \times D_2^i$ with $x'_i < y'_i$, together with a $(K+\varepsilon)$-Lipschitz function $\overline{F}\colon [a'_i, b'_i]_{T^*} \to T_{\gamma^i}(x_i, y_i)$ such that $\overline{F}(a'_i) = x'_i$ and $\overline{F}(b'_i) = y'_i$. Notice that since the length of T^i is 1, and the points x'_i and y'_i belong to $[x_i, x_i + 1/8)_{T^i}$ and $(y_i - 1/8, y_i]_{T^i}$ respectively, the distance $d(x'_i, y'_i)$ is greater than $d(x_i, y_i)$.

Finally, since we are assuming that there is no gap in $[a_i, b_i]_{T^*}$ with length greater than $d(x_i, y_i)/(K+\varepsilon)$, we can apply Proposition 4.5 and assume that $\overline{F}$ has its image contained in the thread $[x'_i, y'_i]_{T^i}$, which belongs to the family $\mathcal{T}$ we have used to define γ^*. Since the thread $[a'_i, b'_i]_{T^*}$ is a subinterval of T^*, its decreasing sequence of gaps $\{C_n^i\}_{n\in\mathbb{N}}$ also satisfies that $\text{length}(C_n^i) < \gamma_n^*$ for all $n \in \mathbb{N}$. Hence, the existence of the function $\overline{F}$ whose Lipschitz constant does not

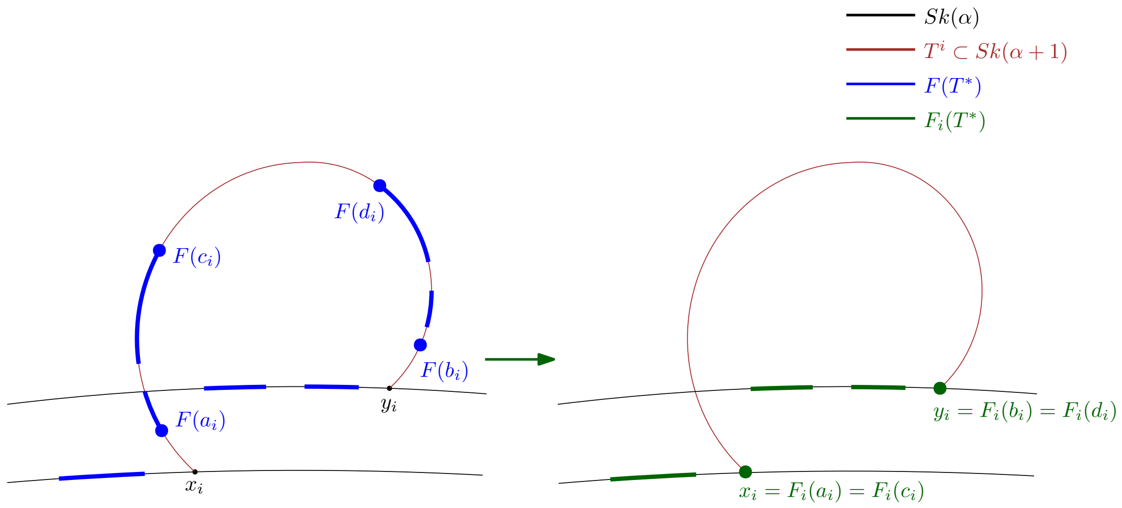

FIGURE 9. If $F(a_i)$ and $F(b_i)$ are bound to different anchors, then by choice of T^* there must exist a gap (c_i, d_i) in $[a_i, b_i]_{T^*}$ big enough to bridge the distance from x_i to y_i with a minimal increase of the Lipschitz constant.

exceed $K+\varepsilon < 2K$, implies by Fact 4 that there is a gap $C^i_{n_0}$ in $[a'_i, b'_i]_{T^*}$ such that $\mathrm{length}(C^i_{n_0}) \geq d(x'_i, y'_i)/(K+\varepsilon)$.

The fact that the gap $C^i_{n_0}$ is also a gap of $[a_i, b_i]_{T^*}$ and that $d(x'_i, y'_i) \geq d(x_i, y_i)$ results in the desired contradiction.

Hence, there exist two points $c_i, d_i \in [a_i, b_i]_{T^*}$ with $c_i < d_i$ such that $(c_i, d_i) \cap (a_i, b_i)_{T^*} = \emptyset$ and $d(c_i, d_i) > d(x_i, y_i)/(K+\varepsilon)$. Define now $F_i \colon T^* \to \widehat{S}$ by

$$F_i(t) = \begin{cases} F(t) & \text{if } t \in T^* \setminus [a_i, b_i]_{T^*}, \\ x_i & \text{if } t \in [a_i, c_i]_{T^*}, \\ y_i & \text{if } t \in [d_i, b_i]_{T^*}. \end{cases}$$

Using Proposition 4.2, maximality of $[a_i, b_i]_{T^*}$ for F and $T_{\gamma^i}(x_i, y_i)$, and the fact that $F(a_i)$ and $F(b_i)$ are bound to x_i and y_i respectively in $T_{\gamma^i}(x_i, y_i)$, it is straightforward to check that F_i satisfies $\|F_i\|_{\mathrm{Lip}} \leq K+\varepsilon$ (we use in fact the same argument as in the proof of Proposition 4.16).

Figure 9 intuitively summarizes the second possibility. Notice that in both Figures 8 and 9 the resulting map F_i avoids the thread T^i, thus reducing the order of the image of the maximal interval $[a_i, b_i]_{T^*}$. $\square$

To finish the proof of part (B) of the Claim, for each $t \in T^*$ such that $t \in [a_i, b_i]_{T^*}$ for some $i \in \mathbb{N}$, define $i(t) \in \mathbb{N}$ as the least of the natural numbers such that $t \in [a_{i(t)}, b_{i(t)}]_{T^*}$. Now, define $\widehat{F}\colon T^* \to \widehat{S}$ by

$$\widehat{F}(t) = \begin{cases} F(t), & \text{if } t \in T^* \setminus \left(\bigcup_{i\in\mathbb{N}} [a_i, b_i]_{T^*} \right), \\ F_{i(t)}(t), & \text{if } t \in [a_i, b_i]_{T^*} \text{ for some } i \in \mathbb{N}. \end{cases}$$

To check that $\|\widehat{F}\|_{\mathrm{Lip}} \le K + \varepsilon$, we only need to consider $t, s \in T^*$ with $t < s$ and such that $t \in [a_{i(t)}, b_{i(t)}]_{T^*}$ and $s \in [a_{i(s)}, b_{i(s)}]_{T^*}$ with $i(t) \ne i(s)$. We have then the following inequalities:

$$\begin{aligned} d\big(\widehat{F}(t), \widehat{F}(s)\big) &= d\big(F_{i(t)}(t), F_{i(s)}(s)\big) \\ &\le d\big(F_{i(t)}(t), y_{i(t)}\big) + d(y_{i(t)}, x_{i(s)}) + d\big(x_{i(s)}, F_{i(s)}(s)\big) \\ &\le d\big(F_{i(t)}(t), F_{i(t)}(b_{i(t)})\big) + d\big(F(b_{i(t)}), F(a_{i(s)})\big) + d\big(F_{i(s)}(a_{i(s)}), F_{i(s)}(s)\big) \\ &\le \big(K+\varepsilon\big)\big((b_{i(t)} - t) + (a_{i(s)} - b_{i(t)}) + (s - a_{i(s)})\big) \\ &= \big(K+\varepsilon\big)(s-t). \end{aligned}$$

Since $d\big(\widehat{F}(p), \widehat{F}(q)\big) = d\big(P_\beta(p), P_\beta(q)\big) \le d(p,q) = a_{T^*}$, we can apply Proposition 4.2 to obtain the desired Lipschitz constant for $\widehat{F}$ and finish the proof of (B).

□

Proof of (C). The proof of the third case (C) resembles the proof of (B). The difference is that in this case at least one of $F(p)$ and $F(q)$ is in generation $\mathrm{Sk}(\alpha+1)$, which is at the same time the order of $F(T^*)$. Intuitively, the idea of the proof of this last part is to first transform F to lower the order of the image of p and/or q. When we have done this, then we may simply apply the case (B) to the resulting map, thus obtaining a Lipschitz function whose image has a lower order than F.

Since $F(p) = P_{\alpha+1}(p)$, there exists $n \in \mathbb{N}$ such that $x_n = P_\alpha(p)$, $y_n = Q_\alpha(p)$, and $F(p)$ belongs to the thread $T^n = T_{\gamma^n}\big(P_\alpha(p), Q_\alpha(p)\big) \in \mathcal{T}_0$. We start by selecting $p' \in T^*$ such that $[p, p']_{T^*}$ is maximal for F and the thread T^n.

There are two possibilities: either $p' = q$, or the point p' is different from q. If $p' = q$, since $P_\alpha(p) \ne P_\alpha(q)$, we have that the range of F is contained in the single thread $T^n = T_{\gamma^n}\big(P_\alpha(p), P_\alpha(q)\big)$, and moreover $F(p) = P_{\alpha+1}(p)$ and $F(q) = P_{\alpha+1}(q)$ belong to this same thread. Since the distance from p and q to $\mathrm{Sk}(\alpha)$ is less than $1/8$, we have that $F(p)$ belongs to $\big[P_\alpha(p), P_\alpha(p) + 1/8\big)_{T^n}$ and $F(q)$ belongs to $\big[P_\alpha(q) - 1/8, P_\alpha(q)\big)_{T^n}$. Hence, we can apply Propositions 4.13 and 4.5 as we did in the proof of (B) to obtain two points $a', b' \in T^*$ with $a' < b'$ and two points $x', y' \in D_1^n \times D_2^n$ together with a $(K+\varepsilon)$-Lipschitz function $\overline{F}\colon [a', b']_{T^*} \to [x', y']_{T^n}$ with $\overline{F}(a') = x'$ and $\overline{F}(b') = y'$. Since the thread $[x', y']_{T^n}$ belongs to the family $\mathcal{T}$, by Theorem 4.9, there exists a gap $C = (c, d)$ in T^* such that $d(c,d) > d\big(P_\alpha(p), P_\alpha(q)\big)/(K+\varepsilon)$. Defining $\widehat{F}\colon T^* \to \widehat{S}$ as

$$\widehat{F}(t) = \begin{cases} P_\alpha(p), & \text{if } t \in [p,c]_{T^*}, \\ P_\alpha(q), & \text{if } t \in [d,q]_{T^*}, \end{cases}$$

finishes the proof of (C) if $p' = q$, without need for further discussion.

Hence, suppose now that p' is not q. We are going to define a $(K+\varepsilon/2)$-Lipschitz function $\overline{F}_1\colon T^* \to \widehat{S}$ such that $\overline{F}_1(p) = P_\alpha(p)$ and $\overline{F}_1(t) = F(t)$ for all $t \in (p', q]_{T^*}$.

In the space $\mathrm{Sk}(\alpha+1)$, the point $F(p) = P_{\alpha+1}(p)$ is bound to $P_\alpha(p)$ in T^n because the distance from p to $\mathrm{Sk}(\alpha)$ is less than $1/8$. In addition, the point $F(p')$ is also bound to one of the extremes $P_\alpha(p)$ or $Q_\alpha(q)$ in T^n. This is because there are no gaps in T^* bigger than $(2K)^{-1}/8$ and $[p,p']_{T^*}$ is maximal for F and T^n. We may consider again two possibilities: either $F(p')$ is bound to $P_\alpha(p)$ as well, or $F(p')$ is bound to $Q_\alpha(p)$.

If $F(p')$ is bound to $P_\alpha(p)$, we can use Proposition 4.16 to define a K-Lipschitz function $\overline{F}_1\colon T^* \to \widehat{S}$ with $\overline{F}_1(t) = P_\alpha(p)$ for all $t \in [p,p']_{T^*}$, and $\overline{F}_1(t) = F(t)$ for all $t \in (p', q]_{T^*}$; as desired.

Suppose then that $F(p')$ is bound to $Q_\alpha(p)$. Then, since $F(p)$ belongs to the interval $\big[P_\alpha(p), P_\alpha(p)+1/8\big)_{T^n}$ and $F(p') \in \big[Q_\alpha(p)-1/8, Q_\alpha(q)\big)_{T^n}$, we can repeat the process we did in the proof of (B) and in the case when $p' = q$ to find a gap $C = (c,d)$ in $[p,p']_{T^*}$ such that $d(c,d) > d\big(P_\alpha(p), Q_\alpha(p)\big)/(K+\varepsilon/2)$. Again, we use this gap to define $\overline{F}_1\colon T^* \to \widehat{S}$ by

$$\overline{F}_1(t) = \begin{cases} P_\alpha(p), & \text{if } t \in [p,c]_{T^*}, \\ Q_\alpha(p), & \text{if } t \in [d,p']_{T^*}, \\ F(p), & \text{if } t \in (p',q]_{T^*}. \end{cases}$$

The Lipschitz constant of $\overline{F}_1$ is less than or equal to $(K+\varepsilon/2)$ as desired.

We may repeat the same argument to find a point $q' \in T^*$ with $p' < q'$, together with a second $(K+\varepsilon/2)$-Lipschitz function $\overline{F}_2\colon T^* \to \widehat{S}$ such that $\overline{F}_2(q) = P_\alpha(q)$ and $\overline{F}_2(t) = F(t)$ for all $t \in [p, q')_{T^*}$. We combine $\overline{F}_1$ and $\overline{F}_2$ to form yet another Lipschitz function $\overline{F}\colon T^* \to \widehat{S}$ in the following way:

$$\overline{F}(t) = \begin{cases} \overline{F}_1(t), & \text{if } t \in [p,p']_{T^*}, \\ F(t), & \text{if } t \in (p',q')_{T^*}, \\ \overline{F}_2(t), & \text{if } t \in (q',q]_{T^*}. \end{cases}$$

It is again straightforward to prove that the Lipschitz constant of $\overline{F}$ is less than or equal to $K+\varepsilon/2$. It is possible that the order of $\overline{F}(T^*)$ is already the desired ordinal $\alpha < \alpha+1$, in which case the proof is finished. However, it might be that there are points in $\overline{F}(T^*)$ in the generation $G_{\alpha+1}$. If this is the case, notice that the function $\overline{F}$ satisfies the hypothesis of the claim and the conditions of (B). Hence, we may use the already proven case (B) with $\varepsilon/2 > 0$ to find a $(K+\varepsilon)$-Lipschitz function $\widehat{F}\colon T^* \to \widehat{S}$ such that $\widehat{F}(p) = P_\alpha(p)$, $\widehat{F}(q) = P_\alpha(q)$, and the order of $\widehat{F}(T^*)$ is α. The proof is now finished. $\square$

We have proven the three parts of the claim. $\square$

Having proven the claim, the theorem holds by the discussion after the statement of the claim.

$\square$

4.2. Failing the Lipschitz RP(Λ, Λ)

In the previous section we have constructed a metric space which fails the Lipschitz SRP in a strong sense. In this section we prove that for every infinite cardinal number Λ, we can find a metric space that fails the Lipschitz RP(Λ, Λ). Given a cardinal Λ, we want to construct a metric space M such that there exists a subset with density character Λ which is not contained in any subset of M with density character Λ that is a Lipschitz retraction of M.

Set $\Gamma = \{\gamma = (\gamma_\alpha)_{\alpha\in\Lambda} \colon 0 < \gamma_\alpha < 1/2,\ \forall \alpha \in \Lambda\}$. For every $\gamma \in \Gamma$ we are going to define a subset $M_\gamma \subset [0,1]^\Lambda \subset \ell_\infty(\Lambda)$ in the following way:

$$M_\gamma = \{(p_\alpha)_{\alpha\in\Lambda} \in [0,1]^\Lambda \colon p_\alpha \in [0, 1/2 - \gamma_\alpha] \cup [1/2 + \gamma_\alpha, 1],\ \forall \alpha \in \Lambda\},$$

endowed with the metric inherited from $\ell_\infty(\Lambda)$. Notice that if for a subset $A \subset \Lambda$ we write the point $e_A = ((e_A)_\alpha)_{\alpha\in\Lambda}$ (called a *vertex*) as the point such that $(e_A)_\alpha = 1$ if $\alpha \in A$, and $(e_A)_\alpha = 0$ if $\alpha \notin A$; then $e_A \in M_\gamma$, for any choice of $A \subset \Lambda$ and $\gamma \in \Gamma$. If $A = \{\alpha\}$ is a singleton, we write $e_{\{\alpha\}} = e_\alpha$. Notice also that $e_\emptyset = 0 \in M$.

As we did with in the construction of threading spaces, set for each $\gamma \in \Gamma$:

$$\widehat{M_\gamma} = \{(p, \gamma) \colon p \in M_\gamma,\ p \neq e_A,\ \text{for any} A \subset \Lambda\},$$

and consider

$$M = \left(\bigcup_{\gamma\in\Gamma} \widehat{M_\gamma} \right) \cup \{e_A\}_{A\subset\Lambda}.$$

Alternatively, the set M can be realized by considering the disjoint union of each M_γ and then identifying each vertex e_A with its corresponding copy in every M_γ. We will define a metric d on M "step-by-step". Let $p, q \in M$. If $p, q \in \widehat{M_\gamma} \cup \{e_A\}_{A\subset\Lambda}$ for a fixed $\gamma \in \Gamma$, then

$$d(p, q) = \|p - q\|_\infty,$$

where we make the identification $p = (p, \gamma) \in \widehat{M_\gamma}$ for any point in $\widehat{M_\gamma}$. If $p, q \in M$ belong to different $\widehat{M_{\gamma_1}}, \widehat{M_{\gamma_2}}$ respectively, then

$$d(p,q) = \inf_{A \subset \Lambda} \{d(p, e_A) + d(e_A, q)\}.$$

Notice that if a point p is not a vertex, then there exists a coordinate $\alpha \in \Lambda$ such that $0 < p_\alpha < 1$, and so $d(p, e_A) \geq \min\{p_\alpha, 1 - p_\alpha\} > 0$ for every $A \subset \Lambda$. This shows that $d(p,q) = 0$ if and only if $p = q$. The triangle inequality follows directly from the definition of the metric d.

Also note that for any $p \in M$ and any $\alpha \in \Gamma$, the coordinate p_α cannot be equal to $1/2$ by construction of M.

4.2.1. Arc-connected components of M. The metric space M as defined above is not arc-connected. Indeed, the points e_A and e_B are not connected by an arc if $A \neq B \subset \Lambda$. We will prove this in detail in this section. Let us first define precisely the concepts we will be using:

Recall that in a metric space M, an arc between two points $p, q \in M$ is a continuous map $F\colon [a,b] \to M$ with $a < b$, such that $F(a) = p$ and $F(a) = q$. We say that two points p, q are *arc-connected* if there exists an arc between p and q. This defines an equivalence relation in M. Moreover, if F is an arc that connects p and q, then it is straightforward to see that p is arc-connected with any point in $F([a,b])$. Therefore, given $C \subset M$ an equivalence class of this relation in M, we have that any two points in C are connected by an arc whose range is contained in C. We call the equivalence classes the *arc-connected components of* M, and they form a partition of M. M is said to be *arc-connected* if M is the only equivalence class.

Note that if $p, q \in M$ are connected by an arc $F\colon [a,b] \to M$, we can assume without loss of generality that $a = 0$ and $b = 1$.

LEMMA 4.23. *Let $p, q \in M$ be two arc-connected points in M such that $p \in M_{\gamma_1}$ and $q \in M_{\gamma_2}$ with $\gamma_1 \neq \gamma_2 \in \Gamma$. Then, for every arc $F\colon [0,1] \to M$ connecting p and q there exists $t_0 \in (0,1)$ and $A \subset \Lambda$ such that $F(t_0) = e_A$.*

PROOF. Consider

$$t_0 = \min\{t \in [0,1] \colon F(t) \notin \widehat{M}_{\gamma_1}\},$$

which exists by continuity of F and the fact that $F(1) = q \notin \widehat{M}_{\gamma_1}$. We claim that $F(t_0)$ is a vertex. Indeed, suppose there exists $\gamma_0 \in \Gamma$ such that $F(t_0) \in \widehat{M}_{\gamma_0}$. By definition of t_0, we have that $\gamma_0 \neq \gamma_1$. The set $\widehat{M}_{\gamma_0}$ is open in M, so by continuity of F, there exists $\varepsilon > 0$ such that $F((t_0 - \varepsilon, t_0 + \varepsilon)) \subset \widehat{M}_{\gamma_0}$. However, this contradicts the minimality of t_0. Therefore, $F(t_0) = e_A$ for some $A \subset \Lambda$. $\square$

PROPOSITION 4.24. *Let M be the metric space as defined above for the cardinal Λ. Then, for each $A \neq B \subset \Lambda$, the points e_A and e_B are inside different arc-connected components of M.*

PROOF. Suppose there is an arc $F\colon [0,1] \to M$ with $F(0) = e_A$ and $F(1) = e_B$. Consider the following points:

$$a_0 = \max\{t \in [0,1]\colon F(t) = e_A\}$$
$$b_0 = \min\{t \in [0,1]\colon F(t) = e_B\}$$

which exist by continuity of F. There are two possibilities: either there exists $\gamma_0 \in \Gamma$ such that $F(a_0, b_0) \subset \widehat{M}_{\gamma_0}$, or there exist $t_1 < t_2 \in (a_0, b_0)$ and $\gamma_1 \neq \gamma_2 \in \Gamma$ such that $F(t_1) \in \widehat{M}_{\gamma_1}$ and $F(t_2) \in \widehat{M}_{\gamma_2}$.

Notice that in the second case, the restriction of F to $[t_1, t_2]$ forms an arc between $F(t_1)$ and $F(t_2)$, so by the previous lemma there is $r \in (t_1, t_2)$ and $C \subset \Lambda$ such that $F(r) = e_C$. Taking the minimum over all such $r \in (a_0, b_0)$ yields a point $b_0^* \in (a_0, b_0)$ and $B^* \subset \Lambda$ such that $F(b_0^*) = e_{B^*}$ and $F(a_0, b_0^*)$ is contained in a $\widehat{M}_{\gamma_0^*}$ for some $\gamma_0^* \in \Gamma$. Moreover, $B^* \neq A$ by maximality of a_0.

In either case, without loss of generality we can assume that $F(0,1) \subset \widehat{M}_{\gamma_0}$ for some $\gamma_0 \in \Gamma$. Since $A \neq B$, we may assume without loss of generality that there exists an $\alpha \in B \setminus A$. Then $(e_A)_\alpha = 0$ and $(e_B)_\alpha = 1$. For each $\alpha \in \Lambda$, the projection $P_\alpha^{\gamma_0}\colon \widehat{M}_{\gamma_0} \cup \{e_A\colon A \subset \Lambda\} \to [0,1]$ given by $P_\alpha(p) = p_\alpha$, is a continuous map. Therefore, the composition map $\overline{F} = P_\alpha \circ F\colon [0,1] \to [0,1]$ is continuous too, and satisfies that $\overline{F}(0) = 0$ and $\overline{F}(1) = 1$. Therefore, there exists $t^* \in (0,1)$ such that $\overline{F}(t^*) = 1/2$. However, this means that the α-th coordinate of $F(t^*) \in \widehat{M}_{\gamma^*}$ is equal to $1/2$, which is a contraction. □

Thanks to this last result, we can properly define for each $A \subset \Lambda$ the arc-connected component C_A to be the arc-connected component of M that contains the vertex e_A. Moreover, given a point $p \in \widehat{M}_\gamma$ for some $\gamma \in \Gamma$, it is straightforward to see that there exists a (unique) vertex e_A such that $p \in C_A$. Indeed, we have that

$$C_A = \{(p,\gamma) \in M\colon |p_\alpha - (e_A)_\alpha| < 1/2,\ \forall \alpha \in \Lambda,\ \gamma \in \Gamma\} \cup \{e_A\}.$$

For convenience, we write $C_{\{\alpha\}} = C_\alpha$ for every $\alpha \in \Lambda$, and $C_\emptyset = C_0$.

4.2.2. Non-existence of Lipschitz retracts of cardinality Λ containing a set of vertices. We now prove that certain subsets of M with density character Λ are not contained in any subset of the same density character which is a Lipschitz retract of M.

THEOREM 4.25 ([25]). *Let Λ be an infinite cardinal. There exists a metric space M and a subspace $N \subset M$ with density character Λ such that every intermediate subset containing N with density character Λ is not a Lipschitz retract of M.*

Proof. Let M be the metric space we have defined in this section associated with the cardinal Λ. Put

$$N = \left(\bigcup_{\alpha \in \Lambda} e_\alpha \right) \cup \{0\},$$

which clearly satisfies $\mathrm{dens}(N) = \Lambda$. Let S be a subset of M such that $N \subset S$ and $\mathrm{dens}(S) = \Lambda$, and let $K \geq 1$. We are going to prove that S is not a K-Lipschitz retract of M.

Since $\mathrm{dens}(S) = \Lambda$, there exists a subset $\Gamma' \subset \Gamma$ with $\mathrm{card}(\Gamma') = \Lambda$ such that

$$S \subset \left(\bigcup_{\gamma \in \Gamma'} \widehat{M}_\gamma \right) \cup \{e_A \colon A \subset \Lambda\}.$$

We write $\Gamma' = (\gamma^\beta)_{\beta \in \Lambda}$ and $\gamma^\beta = (\gamma^\beta_\alpha)_{\alpha \in \Lambda}$ for $\beta \in \Lambda$.

Define $\gamma^* \in \Gamma$ as follows: $\gamma^*_\alpha = (2K)^{-1} \gamma^\alpha_\alpha$ for each $\alpha \in \Lambda$. We obtain directly that $\gamma^* \notin \Gamma'$. Now, suppose that $F \colon M \to S$ is a K-Lipschitz retraction onto S. The image of an arc-connected set under a continuous function is still arc-connected, so in particular, $F(C_\alpha) \subset C_\alpha$, since we know that $F(e_\alpha) = e_\alpha$ for every $\alpha \in \Lambda$. By the same argument, we have that $F(C_0) \subset C_0$.

Consider now the point $p^* = \Big(\big(1/2 - \gamma^*_\alpha\big)_{\alpha \in \Lambda}, \gamma^* \Big) \in \widehat{M}_{\gamma^*}$. Since each coordinate of p^* is less than $1/2$, we have that $p^* \in C_0$, which in turn implies that $F(p^*) \in C_0$. There are two possibilities: either $F(p^*) = 0$, or there exists a $\beta_0 \in \Lambda$ such that $F(p^*) \in C_0 \cap \widehat{M}_{\gamma^{\beta_0}}$.

Suppose first that $F(p^*) = 0$. Choose any $\beta_0 \in \Lambda$. The point $q^* = \big(q, \gamma^*\big) \in \widehat{M}_{\gamma^*}$ defined by $q_\alpha = 1/2 - \gamma^*_\alpha$ if $\alpha \neq \beta_0$ and $q_{\beta_0} = 1/2 + \gamma^*_{\beta_0}$ satisfies $q^* \in C_{\beta_0}$, so $F(q^*) \subset C_{\beta_0} \cap S$. However, now we have that $d(p^*, q^*) = 2\gamma^*_{\beta_0} < K^{-1} \gamma^{\beta_0}_{\beta_0} < K^{-1} 1/2$, and on the other hand

$$d(F(p^*), F(q^*)) > d(0, F(q^*)) > 1/2,$$

which contradicts the fact that F is K-Lipschitz.

Suppose now that there exists a $\beta_0 \in \Lambda$ such that $F(p^*) \in C_0 \cap \widehat{M}_{\gamma^{\beta_0}}$. Consider, as in the other case, the point $q^* = \big(q, \gamma^*\big) \in \widehat{M}_{\gamma^*}$ defined by $q_\alpha = 1/2 - \gamma^*_\alpha$ if $\alpha \neq \beta_0$ and $q_{\beta_0} = 1/2 + \gamma^*_{\beta_0}$. Similarly, we have that $q^* \in C_{\beta_0}$, and $d(p^*, q^*) < K^{-1} \gamma^{\beta_0}_{\beta_0}$. Since $F(q^*) \in C_{\beta_0} \cap S$ and $F(p^*) \in \widehat{M}_{\gamma^{\beta_0}}$, we have that the distance between $F(p^*)$ and $F(q^*)$ is bigger than the distance from $F(p^*)$ to $C_{\beta_0} \cap \widehat{M}_{\gamma^{\beta_0}}$. Looking at the coordinate β_0 of p^*, we have that $\big(F(p^*)\big)_{\beta_0} < 1/2 - \gamma^{\beta_0}_{\beta_0}$ and the coordinate β_0 of any point in $C_{\beta_0} \cap \widehat{M}_{\gamma^{\beta_0}}$ is bigger than $1/2 + \gamma^{\beta_0}_{\beta_0}$. Therefore, we obtain that

$$2\gamma^{\beta_0}_{\beta_0} < d\big(F(p^*), F(q^*)\big) < K d(p^*, q^*) < \gamma^{\beta_0}_{\beta_0},$$

a contradiction. $\square$

4.3. Open problems

For the open problems related to this chapter, we present two kinds of open problems related to the two constructions we have discussed.

On the one hand, we do not know the behaviour of the Lipschitz-free spaces associated to the two classes of metric spaces we have constructed. They could provide counterexamples to Problem 3.20.

PROBLEM 4.26. Does the Lipschitz-free space of the skein space $\mathrm{Sk}(\omega_1)$ have the SCP?

PROBLEM 4.27. Given a cardinal Λ, does the Lipschitz-free space of the metric space constructed in Theorem 4.25 have the $CP(\Lambda, \Lambda)$?

Notice that in general, we cannot obtain Lipschitz retractions from linear projections in Lipschitz-free spaces. For instance, it is not difficult to show that the Lipschitz-free space of any threading space we used to construct the skein space admits a commutative 1-projectional skeleton. However, threading spaces themselves do not have the Lipschitz SRP. This was shown in more detail in Remark 3.8 in [25].

In [4], Banakh, Vovk and Wójcik construct a class of connected complete metric spaces where every separable subset is disconnected. Such metric spaces clearly satisfy that every non-trivial separable subset is not a Lipschitz retract, and thus, they are also natural candidates to produce Lipschitz-free spaces without the SCP.

On the other hand, we may ask as well if the examples we have constructed can be generalized and improved.

PROBLEM 4.28. Given an uncountable cardinal Λ, does there exist a complete metric space such that any non-trivial subset of density character strictly smaller than Λ is not a Lipschitz retract?

The techniques we used in the construction of the skein space do not appear to be easily adaptable to uncountable cardinals. A complete metric space with this property would yield a stronger result than both Theorems 4.1 and 4.25. A weaker version of the previous question would also improve Theorem 4.25:

PROBLEM 4.29. Given an uncountable cardinal Λ, does there exist a complete metric space without the Lipschitz $\mathrm{RP}(\alpha, \beta)$ for all $\alpha \leq \beta$ strictly smaller than Λ?

We do not know if the metric space failing the Lipschitz $\mathrm{RP}(\Lambda, \Lambda)$ constructed in 4.25 has the Lipschitz $\mathrm{RP}(\alpha, \beta)$ for some $\alpha \leq \beta$ smaller than Λ.

CHAPTER 5

Local complementation in metric spaces

We turn our attention to the concept of local complementation, introduced by Kalton in [31] for Banach spaces. Divided into two sections, the first goal of this chapter is to extend this concept to the setting of metric spaces naturally, by describing the relationship between local complementation and Lipschitz maps. This first section is short and expository.

In the second section, we show that, as in the Banach space case, every metric space has a rich structure of locally complemented subsets for any given density character.

5.1. Local complementation and linear extension operators

Let us start with the standard definition of local complementation in Banach spaces:

DEFINITION 5.1. Let X be a Banach space and $\lambda \geq 1$. We say that a subspace $Y \subset X$ is *λ-locally complemented in X* if for every finite-dimensional subspace $F \subset X$ and every $\varepsilon > 0$ there exists a linear operator $T\colon F \to Y$ with $\|T\| \leq \lambda$ such that $\|Tf - f\| < \varepsilon\|f\|$ for all $f \in Y \cap F$.

We say that Y is locally complemented in X if Y is λ-locally complemented in X for some λ.

Note that the Principle of Local Reflexivity implies that every Banach space is 1-locally complemented in its bidual.

Several equivalent formulations of this concept are known in the literature. We include the most important and the most relevant to our discussion in the following result. The proof of these equivalent formulations can be found in [31, 13, 19].

THEOREM 5.2. *Let X be a Banach space, $Y \subset X$ a linear subspace, and $\lambda \geq 1$. The following statements are equivalent:*

(1) Y is λ-locally complemented in X.
(2) There exists a linear projection $P\colon X^* \to Y^\perp$ such that $\|Id_{X^*} - P\| \leq \lambda$.
(3) Y^{**} is λ-complemented in X^{**} in its natural embedding.
(4) Y has the Compact Extension Property in X, i.e.: for every Banach space Z and every linear compact operator $K\colon Y \to Z$, there exists a compact operator $\widehat{K}\colon X \to Z$ that extends K and such that $\|\widehat{K}\| \leq \lambda\|K\|$.
(5) There exists a linear extension operator $E\colon Y^* \to X^*$ with $\|E\| \leq \lambda$.

(6) There exists a linear extension operator $E\colon Lip_0(Y) \to Lip_0(X)$ with $\|E\| \le \lambda$.

The equivalence between (1) and (6) in Theorem 5.2 shows that, even though local complementation is a linear concept, it is characterized by a relationship between spaces of Lipschitz functions. Therefore, we can extend this concept naturally to the metric space setting:

Definition 5.3. Let M be a metric space and let $\lambda \ge 1$. We say that a subset N is λ-*locally complemented* in M if there exists a linear extension operator $E\colon \mathrm{Lip}_0(N) \to \mathrm{Lip}_0(M)$ with $\|E\| \le \lambda$. We say that N is locally complemented if it is λ-locally complemented for some $\lambda \ge 1$.

If N is a Lipschitz retract of M and $R\colon M \to N$ is a Lipschitz retraction, the operator $E\colon \mathrm{Lip}_0(N) \to \mathrm{Lip}_0(M)$ given by $Ef = f \circ R$ for every $f \in \mathrm{Lip}_0(N)$ is a linear extension operator with $\|E\| = \|R\|_{\mathrm{Lip}}$. Therefore, it holds that every Lipschitz retract is locally complemented.

In metric spaces, it is straightforward to show that the converse does not hold: Consider $M = [0,1] \subset \mathbb{R}$ with the usual metric. Then the set $N = \{0,1\}$ is 1-locally complemented, since the linear extension operator given by linear interpolation on $[0,1]$ does not increase the Lipschitz constant. However, N is not a Lipschitz retract of M since M is connected and N is not.

The answer is less obvious for linear subspaces of Banach spaces. We have the following result:

Proposition 5.4. *Let X be a Banach space and let $Y \subset X$ be a linear subspace. Suppose that Y is locally complemented in X. If Y is a Lipschitz retract of its bidual Y^{**}, then Y is a Lipschitz retract of X.*

Proof. By (3) in Theorem 5.2, there exists a linear projection $P\colon X^{**} \to Y^{**}$. If $R\colon Y^{**} \to Y$ is a Lipschitz retraction, the restriction of the composition map $(R \circ P)_{|X}\colon X \to Y$ is a Lipschitz retraction. $\square$

As discussed in Chapter 2, Kalton showed in [30] that there exists a (non-separable) Banach space which is not a Lipschitz retract of its bidual. Since every Banach space is locally complemented in its bidual, the nontrivial result of Kalton produces an example of a locally complemented space which is not a Lipschitz retract.

5.2. The local CP(Λ, Λ) in metric spaces

With a natural definition of local complementation in metric spaces, we can study local concepts analogous to the Lipschitz Retractional Properties and the Linear Complementation Properties:

Definition 5.5. Given α, β two cardinal numbers with $\alpha \le \beta$, we say that a metric space M has the *(α, β) Local Complementation Property (Local CP(α, β)*

for short), if for every closed subset $N \subset M$ with $\mathrm{dens}(N) = \alpha$ there exists another subset S that contains N, such that $\mathrm{dens}(S) \leq \beta$ and S is locally complemented in M. We say that M has the *Local Separable Complementation Property (Local SCP)* if it has the Local CP$(\aleph_0, \aleph_0)$.

Lindenstrauss and Tzafriri proved in [40] that if a Banach space X is not a Hilbert space, then it contains a closed linear subspace which does not satisfy the Compact Extension Property in X. Therefore, by (4) in Theorem 5.2, in every non-Hilbert Banach space we can find a linear subspace which is not locally complemented. However, in [26], Heinrich and Mankiewicz showed using Model Theory that every Banach space X has that Local CP(Λ, Λ) for every cardinal $\Lambda \leq \mathrm{dens}(X)$. Later, in [48], Sims and Yost proved the same result with a different approach.

The purpose of this section is to extend this result to the metric space setting. We will do so by adapting the techniques from [48] to this more general framework. We start by proving an auxiliary statement analogous to Lemma 1 in [39].

Lemma 5.6 ([25]). *Let M be a bounded complete metric space. Let $F \subset M$ be a finite subset of M, and let $k \in \mathbb{N}$ and $0 < \varepsilon \leq inf_{p \neq q \in F} d(p, q)$ be given. Then there exists a finite subset $Z \subset M$ with $F \subset Z$ such that for every ε-separated subset $E \subset M$ with $F \subset E$ and card$(E \setminus F) \leq k$ there is a Lipschitz map $L \colon E \to Z$ with $\|L\|_{Lip} \leq 1 + \varepsilon$ and $L(f) = f$ for all $f \in F$.*

Proof. Write $R = \mathrm{diam}(M)$ and $F = \{f_1, \ldots, f_n\}$. We may assume that $\varepsilon < 1$. Consider $E \subset M$ a ε-separated subset with $F \subset E$ and $\mathrm{card}(E \setminus F) \leq k$. We can write this set as $E = \{f_1, \ldots, f_n, p_1^E, \ldots, p_{l_E}^E\}$ with $l_E \leq k$. Consider now the real valued vector:

$$a_E = (d(f_1, p_1^E), \ldots, d(f_1, p_{l_E}^E), \ldots, d(p_{l_E}^E, p_1^E), \ldots, d(p_{l_E}^E, p_{l_E}^E)) \in \mathbb{R}^{(n+l_E)l_E}.$$

Since M has diameter $R < \infty$, the point a_E belongs to $RB_{\ell_\infty^{(n+l_E)l_E}}$. Hence, if we set

$$C = \bigsqcup_{l=1}^{k} RB_{\ell_\infty^{(n+l)l}},$$

that is, the disjoint union of $RB_{\ell_\infty^{(n+l)l}}$ for $l = 1, \ldots, k$, then for every set $E \subset M$ with $F \subset E$ and $\mathrm{card}(E \setminus F) \leq k$, the vector a_E belongs to C. Since we are working with a finite disjoint union, we can endow C with a metric d_∞ such that C is compact, the restriction of this metric to each $RB_{\ell_\infty^{(n+l)l}}$ coincides with the metric given by the supremum norm, and each $RB_{\ell_\infty^{(n+l)l}}$ is separated at least by ε from its complementary in C. Since C is compact, the subset

$$A_F = \{a_E \in C \colon F \subset E \text{ and } \mathrm{card}(E \setminus F) \leq k\} \subset C$$

is totally bounded in C. Hence, given $\varepsilon > 0$ there exist $\{E_1, \dots, E_s\}$ with $F \subset E_j$ and $\operatorname{card}(E_j \setminus F) \leq k$ such that $A_F = \bigcup_{j=1}^s B_\infty(a_{E_j}, \varepsilon^2)$. Set $Z = \bigcup_{j=1}^s E_j$. Let us prove that Z satisfies the thesis of the Lemma.

Clearly, Z is finite and contains F. Consider any ε-separated subset $E \subset M$ with $F \subset E$ and $\operatorname{card}(E \setminus F) \leq k$. There exists a $j_0 \in \{1, \dots, j\}$ such that $d_\infty(a_E, a_{E_j}) \leq \varepsilon^2$ and $E_j \subset F$. Moreover, since a_E and a_{E_j} are closer than ε, they must belong to the same ball $RB_{\ell_\infty^{(n+l_0)l_0}}$, so $d_\infty(a_E, a_{E_j}) = \|a_E - a_{E_j}\|_\infty \leq \varepsilon^2$, and $\operatorname{card}(E_j) = \operatorname{card}(E) = n + l_0$. Thus, we can write $E = \{f_1, \dots, f_n, p_1^E, \dots, p_{l_0}^E\}$ and $E_{j_0} = \{f_1, \dots, f_n, p_1^{E_{j_0}}, \dots, p_{l_0}^{E_{j_0}}\}$.

Define now $L\colon E \to Z$ by $L(f) = f$ if $f \in F$, and $L(p_i^E) = p_i^{E_{j_0}}$ for $i = 1, \dots, l_0$. The map L satisfies $L(f) = f$ for all $f \in F$ by definition, so it only remains to check that it is $(1+\varepsilon)$-Lipschitz. Since L is the identity on F, it is sufficient to check the Lipschitz constant for pairs of points $x, y \in E$ where $x \notin F$. Then $x = p_{i_1}^E$ for some $1 \leq i_1 \leq l_0$. If $y = p_{i_2}^E$ for some $1 \leq i_2 \leq l_0$, then

$$\begin{aligned} d(L(x), L(y)) &= d(p_{i_1}^{E_{j_0}}, p_{i_2}^{E_{j_0}}) - d(p_{i_1}^E, p_{i_2}^E) + d(p_{i_1}^E, p_{i_2}^E) \\ &\leq \|a_{E_{j_0}} - a_E\|_\infty + d(p_{i_1}^E, p_{i_2}^E) \leq \varepsilon\varepsilon + d(p_{i_1}^E, p_{i_2}^E) \leq (1+\varepsilon) d(x,y), \end{aligned}$$

using the fact that E is ε-separated. If $y \in F$, then the inequality is proved similarly. We conclude that $\|L\|_{\mathrm{Lip}} \leq 1 + \varepsilon$, and the proof is complete. □

Let us note two things about the previous lemma: first, we have restricted ourselves to bounded metric spaces, and secondly, we need the set E to be ε-separated for some $\varepsilon > 0$. The boundedness problem, although it has an effect on the construction of the linear extensions in the proof of Theorem 5.8 for unbounded metric spaces, is easy to work around as we will see. However, to solve the separation issue we need to alter the construction in a more meaningful way: instead of defining linear extensions to the whole metric space M, we will extend functions to some dense subset of M that has certain separation properties. The dense subset we will use is well defined thanks to the following lemma.

LEMMA 5.7 ([25]). *Let M be a complete metric space and $(F_n)_{n=1}^\infty$ be a sequence of finite subsets of M with $F_n \subset F_{n+1}$ for all $n \in \mathbb{N}$, and let $(\varepsilon_n)_{n=1}^\infty$ be a decreasing sequence of positive real numbers such that $\varepsilon_n < \inf_{p \neq q \in F_n} d(p,q)$. Then there exists a sequence of sets $(D_n)_{n=1}^\infty$ with the following properties:*

(i) $D_n \subset D_{n+1}$ for all $n \in \mathbb{N}$,
(ii) $F_n \subset D_n$ for all $n \in \mathbb{N}$,
(iii) $D_n \cup F_{n+k}$ is ε_{n+k}-separated for all $n \in \mathbb{N}$ and $k \geq 0$,
(iv) $D = \bigcup_{n \in \mathbb{N}} D_n$ is dense in M,

PROOF. Consider the family of sets

$$A_1 = \{D \subset M \colon F_1 \subset D,\ D \cup F_{1+k} \text{ is } \varepsilon_{1+k}\text{-separated },\forall k \geq 0\}.$$

Since $(F_n)_{n=1}^{\infty}$ is increasing and F_n is at least ε_n-separated, the set $F_1 \in A_1$, so A_1 is non-empty. Consider now a chain of subsets $(C_\alpha)_{\alpha\in I}$ in A_1. If we define $C = \bigcup_{\alpha\in I} C_\alpha$, then $C_\alpha \subset C$ and $C \in A_1$, so it is an upper bound for the chain. By Zorn's Lemma, we can choose $D_1 \in A_1$ to be maximal for the inclusion.

Suppose we have defined D_{n-1}, then we define

$$A_n = \{D \subset M \colon D_{n-1} \cup F_n \subset D,\ D \cup F_{n+k} \text{ is } \varepsilon_{n+k}\text{-separated },\forall k \geq 0\}.$$

Note that $D_{n-1} \cup F_n \in A_n$, so $A_n \neq \emptyset$. Arguing as before, we can choose a maximal set $D_n \in A_n$. Let $(D_n)_{n=1}^{\infty}$ be the sequence of sets obtained by this inductive process. Let us check that it satisfies properties (i)-(iv). The first three properties are clearly satisfied by definition of A_n. It only remains to check that $D = \bigcup_{n\in\mathbb{N}} D_n$ is dense in M. Suppose by contradiction that there exist an $x \in M$ and $\delta_0 > 0$ such that $D \cap B(x,\delta) = \emptyset$. Take an $n_0 \in \mathbb{N}$ such that $\varepsilon_n < \delta$ for all $n \geq n_0$. Then clearly $D \cap B(x,\varepsilon_n) = \emptyset$ for all $n \geq n_0$, which in particular means that $D_{n_0} \cap B(x,\varepsilon_{n_0}) = \emptyset$ and $F_n \cap B(x,\varepsilon_n) = \emptyset$ for all $n \geq n_0$. But then $D_{n_0} \cup \{x\} \in A_{n_0}$, contradicting the maximality of D_{n_0}. We conclude that D is dense, which finishes the proof. □

Now we can finally prove the main result of the section.

THEOREM 5.8 ([25]). *Let M be a complete metric space, and let $N \subset M$ be a subset of M with $0 \in N$. Then there exists a subspace $S \subset M$ with dens$(N) =$ dens(S) and a linear extension operator $T\colon Lip_0(S) \to Lip_0(M)$ such that $\|T\| = 1$. In particular, every metric space has the Local CP(Λ,Λ) for every infinite cardinal Λ smaller than the density character of M.*

PROOF. We first assume that N is separable.

We are going to find a linear extension operator $T\colon \mathrm{Lip}_0(D\cap S) \to \mathrm{Lip}_0(D)$ with $\|T\| = 1$, where D is a dense subset of M such that $D \cap S$ is dense in S. This will suffice to prove the result, since it is known that we can extend linearly Lipschitz functions from dense subsets preserving the Lipschitz constant, and this extension is unique by continuity (check Proposition 1.6 in [50], for instance).

Let $(p_n)_{n=1}^{\infty}$ be a dense sequence in N. For $n = 0$, put $S_0 = \{0\}$. Inductively, suppose we have defined S_{n-1}, which is finite. Put $F_n = S_{n-1} \cup \{p_n\}$ as a finite set, $\theta_n = \inf_{p\neq q\in F_n} d(p,q)$ as the separation of said set, and $r_n = \mathrm{rad}(F_n)$ its radius. Set $\varepsilon_n = \min\{1/n, \theta_n\}$ and $R_n = \max\{r_n, n\}$. We choose S_n to be the set Z given by Lemma 5.6 applied to $M \cap B(0,R_n)$, which is bounded, with $F = F_n$, $k = n$ and $\varepsilon = \varepsilon_n$. Set $S = \overline{\bigcup_{n\in\mathbb{N}} S_n}$. Then clearly S is separable and contains N.

Let $(D_n)_{n=1}^{\infty}$ be the increasing sequence of sets given by Lemma 5.7 applied to $(F_n)_{n=1}^{\infty}$ and $(\varepsilon_n)_{n=1}^{\infty}$. Notice that if $D = \bigcup_{n\in\mathbb{N}} D_n$, then D is dense in M by the Lemma, and $D \cap S$ is dense in S because $S_n \subset D_{n+1}$ for all $n \in \mathbb{N}$.

Fix $n \in \mathbb{N}$, and define the family of subsets:

$$I_n = \{E \subset D_n \cap B(0,R_n) \colon F_n \subset E,\ E \text{ is } \varepsilon_n\text{-separated, and } \mathrm{card}(E \setminus F_n) \leq n\}.$$

Note that if $E \subset D_n$, the condition that E is ε_n-separated is redundant, but we state it for clarity. Indeed, now it is clear that if $E \in I_n$, then there exists a Lipschitz map $L_E \colon E \to S_n$ with $(L_E)_{|F_n} = \mathrm{Id}_{F_n}$ and $\|L_E\|_{\mathrm{Lip}} \leq 1 + \varepsilon_n$.

Let

$$I = \bigcup_{n\in\mathbb{N}} I_n.$$

Notice that if $E_1 \in I_{n_1}$, $E_2 \in I_{n_2}$, then all three of E_1, E_2 and $F_{n_1+n_2}$ are contained in $D_{n_1+n_2}$. Hence, $E_0 = E_1 \cup E_2 \cup F_{n_1+n_2}$ is $\varepsilon_{n_1+n_2}$-separated, and moreover $E_0 \setminus F_{n_1+n_2} \subset (E_1 \setminus F_{n_1}) \cup (E_2 \setminus F_{n_2})$, which means that $\mathrm{card}(E_0 \setminus F_{n_1+n_2}) \leq n_1 + n_2$. Therefore, $E_0 \in I_{n_1+n_2} \subset I$. Thus, I, with the order given by inclusion, is a directed set.

For a set $E \in I$, consider $I_E := \{Z \in I \colon E \subset Z\}$, which is a subset of I. Since I is directed, the family $B = \{I_E\}_{E\in I}$ is the subbase of a filter on $\mathcal{P}(I)$. Let U be an ultrafilter that extends this filter. For any $p \in D$, there exists a minimum $n_p \in \mathbb{N}$ such that $p \in D_{n_p} \cap B(0, n_p)$, so the set $I_p := \{Z \in I \colon p \in Z\}$ belongs to $B \subset U$, since it can be written as $I_p = I_{E_p} = \{Z \in I \colon E_p \subset Z\}$, where $E_p = \{p\} \cup F_{n_p}$, which is a member of $I_{n_p} \subset I$.

Also note that for every $n \in \mathbb{N}$, the set I_n can be written as I_{F_n}, so $I_n \in U$ as well.

For each $E \in I$ define $n(E) := \max\{n \in \mathbb{N} \colon E \in I_n\}$ which exists since E is finite. Since $E \in I_{n(E)}$, there exists a Lipschitz map $L_E \colon E \to S_{n(E)}$ with $(L_E)_{|F_{n(E)}} = \mathrm{Id}_{F_{n(E)}}$ and $\|L_E\|_{\mathrm{Lip}} \leq 1 + \varepsilon_{n(E)}$. We can extend each L_E to a non-Lipschitz function defined on the dense subset $D = \bigcup_{n\in\mathbb{N}} D_n$ by simply defining $\widetilde{L}_E \colon M \to S_{n(E)}$ as

$$\widetilde{L}_E(p) = \begin{cases} L_E(p), & \text{if } p \in E, \\ 0, & \text{if } p \in D \setminus E. \end{cases}$$

Finally, the linear extension operator $T \colon \mathrm{Lip}_0(S) \to \mathrm{Lip}_0(D)$ is defined for each $f \in \mathrm{Lip}_0(D \cap S)$ by:

$$(Tf)(p) = \lim_U f(\widetilde{L}_E(p)), \qquad p \in D.$$

This limit exists since U is an ultrafilter and $\bigcup_{E\in I} f(\widetilde{L}_E(p))$ is relatively compact in $\mathbb{R}$ for all $p \in D$.

Let us check that T is a linear extension operator with norm 1. First, it follows that T is linear by the linearity of the limit with respect to an ultrafilter.

By the definition of limit with respect to an ultrafilter and by definition of S, to prove that T is an extension operator from $\mathrm{Lip}_0(D \cap S)$, it is enough to show that given $f \in \mathrm{Lip}_0(S)$, $n_0 \in \mathbb{N}$ and $p_0 \in S_{n_0}$, there is a set $I_0 \in U$ such that $f(\widetilde{L}_E)(p) = f(p)$ for all $E \in I_0$. Consider $I_0 = I_{F_{n_0+1}} \in U$, and note that $F_{n_0+1} \in I$ and satisfies $S_{n_0} \subset F_{n_0+1}$. For every $E \in I_0$, we have that $F_{n_0+1} \subset E$, so

$(L_E)_{|F_{n_0+1}} = \mathrm{Id}_{F_{n_0+1}}$. In particular, $L_E(p) = p$ for all $p \in S_{n_0}$, so $f(\widetilde{L}_E)(p) = f(p)$ as desired.

It only remains to prove that $\|T\| = 1$. Again, it suffices to show that for any pair of points $x, y \in D$ and any $\delta > 0$, there exists $I_1 \in U$ such that for all $E \in I_1$, the inequality $|f(\widetilde{L}_E(x)) - f(\widetilde{L}_E(y))| \leq (1+\delta)\|f\|_{\mathrm{Lip}} d(x,y)$ holds. Consider $n_0 \in \mathbb{N}$ such that $\varepsilon_{n_0} < \delta$, and set $I_1 = I_x \cap I_y \cap I_{n_0} \in U$. If $E \in I_1$, then $x, y \in E$, so $\widetilde{L}_E(x) = L_E(x)$ and $\widetilde{L}_E(y) = L_E(y)$. Therefore, since L_E is $(1+\varepsilon_{n_0})$-Lipschitz in E, we have that

$$\begin{aligned}|f(\widetilde{L}_E(x)) - f(\widetilde{L}_E(y))| &= \|f\|_{\mathrm{Lip}}|\widetilde{L}_E(x) - \widetilde{L}_E(y)| = \|f\|_{\mathrm{Lip}}|L_E(x) - L_E(y)| \\ &\leq (1+\varepsilon_{n_0})\|f\|_{\mathrm{Lip}} d(x,y) \leq (1+\delta)\|f\|_{\mathrm{Lip}} d(x,y),\end{aligned}$$

and the proof for the separable case is complete.

For the general case we use transfinite induction. Suppose that N is non-separable, let $\lambda = \mathrm{dens}(N)$, and suppose that we have proved the result for every cardinal α with $\omega_0 \leq \alpha < \lambda$. Choose $\{p_\alpha\}_{\alpha<\lambda}$. Since $\{p_\alpha\}_{\alpha<\omega_0}$ is countable, there exists a separable subset $S_{\omega_0} \subset M$ with $p_\alpha \in S_{\omega_0}$ for all $\alpha < \omega_0$ and a norm 1 linear extension operator $T_{\omega_0}\colon \mathrm{Lip}_0(S_{\omega_0}) \to \mathrm{Lip}_0(M)$. Similarly, for $\omega_0 < \alpha < \lambda$, we can find a subset $S_\alpha \subset M$ containing $\bigcup_{\omega_0<\beta<\alpha} S_\beta \cup \{p_\alpha\}$ with $\mathrm{dens}(S_\alpha) \leq \alpha$ and a norm 1 linear extension operator $T_\alpha\colon \mathrm{Lip}_0(S_\alpha) \to \mathrm{Lip}_0(M)$. Set

$$S = \overline{\bigcup_{\omega_0<\alpha<\lambda} S_\alpha}.$$

We have that $N \subset S$ and $\mathrm{dens}(S) = \lambda$. For any $\omega_0 < \alpha < \lambda$, consider the linear map $R_\alpha\colon \mathrm{Lip}_0(S) \to \mathrm{Lip}_0(S_\alpha)$ given by the restriction to S_α. We have that $E_\alpha = T_\alpha R_\alpha\colon \mathrm{Lip}_0(S) \to \mathrm{Lip}_0(M)$ is a bounded linear map with $\|E_\alpha\| \leq 1$. Therefore, for any $f \in \mathrm{Lip}_0(S)$, the set $\{E_\alpha f\colon \omega_0 \leq \alpha < \lambda\}$ is bounded in $\mathrm{Lip}_0(M)$, and thus relatively compact for the weak* topology.

Let U be a non-principal ultrafilter on $\{\alpha\colon \omega_0 \leq \alpha < \mu\}$. Then we define $E\colon \mathrm{Lip}_0(S) \to \mathrm{Lip}_0(M)$ by

$$Ef := w^*\text{-}\lim_U E_\alpha f,$$

which is well defined by the discussed compactness and the fact that U is an ultrafilter. It is straightforward to check that E is a linear extension operator with $\|E\| \leq 1$. Note that E can also be obtained as the weak* limit of a subnet of E_α using the compactness of $B_{\mathrm{Lip}_0(M)}$ in the weak* topology. $\square$

5.3. Open problems

Since every metric space has the Local $\mathrm{CP}(\Lambda, \Lambda)$ for every suitable Λ, we remark only the following important problem:

PROBLEM 5.9. Is every separable locally complemented Banach space a Lipschitz retract?

This question has been asked several times in the literature (see e.g.: [33] Problem 10 and Proposition 3.22). Since every Banach space is locally complemented in its bidual, by Proposition 5.4, it is equivalent to asking if every separable Banach space is a Lipschitz retract of its bidual. That is, Problem 5.9 is equivalent to the separable case of the Lindenstrauss conjecture. Recall that the non-separable case was solved in the negative by Kalton in [30]

On the other hand, since every Banach space has the Local SCP, if there exists a Banach space X without the Lipschitz SRP, then X contains a separable subspace which is locally complemented but is not a Lipschitz retract. In other words, a positive solution to the separable case of Problem 3.19 would yield a negative solution to Problem 5.9 and the Lindenstrauss conjecture.

CHAPTER 6

Conclusion

We have studied the Lipschitz retractional structure of non-separable metric and Banach spaces, generalizing classical concepts of the linear theory to the class of metric spaces. Specifically, we have defined the concepts of Lipschitz retractional skeletons and the Lipschitz Retractional Property (α, β) for a pair of cardinals $\alpha \leq \beta$. We have shown the relationship between these concepts and the classical ones in the context of Lipschitz-free spaces, and we have related them to some long-standing open questions in the theory of Nonlinear Functional Analysis.

In Chapter 3 we have characterized the Plichko property witnessed by Dirac measures in Lipschitz-free spaces, and have proven that this property is preserved for Lipschitz-free spaces associated to Banach spaces. We have observed that $C(K)$ Banach spaces have the Lipschitz SRP, and have pointed out the relevance of determining whether every Banach space possesses the Lipschitz SRP.

In Chapter 4 we have observed that, for metric spaces, the Lipschitz SRP can fail in a very strong sense: we have constructed a complete metric space whose only separable Lipschitz retracts are singletons. We have also produced, for every infinite cardinal Λ, a complete metric space failing the Lipschitz RP(Λ, Λ).

Finally, in Chapter 5 we have extended results of Heinrich and Mankiewicz to the nonlinear setting, by showing that in every complete metric space, each closed subset is contained in a bigger closed subset of the same density character which admits a linear extension operator for Lipschitz functions preserving the Lipschitz constant.

Open questions and further work has been discussed at the end of every chapter in the main body of the thesis.

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