Operational Research manuscript No.
(will be inserted by the editor)

Risk-sensitive control of cash management systems

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Received: date / Accepted: date

Abstract Firms manage cash for operational, precautionary and speculative purposes. Stat-of-the-art cash management models usually focus on cost minimization by means of a set of controlling bounds. In this paper, we propose a multiobjective model to control cash management systems with multiple accounts characterized by generalized cash flow processes. In addition, we replace the customary use of bounds with cash balance reference trajectories. The model considers two objectives such as cost minimization, measured by the sum of transaction and holding costs, and risk control, measured by the sum of deviations from a given cash balance reference. We also present theoretical results on the stability of the model for deterministic, predictable and purely random cash flow processes. By means of numerical examples, we analyze the robustness of different risk-sensitive models to mean-variance misspecifications. The results show that tuning a parameter of our model can be of help to find more robust cash management policies. Finally, we present a case study showing how our risk-sensitive model can be used to adjust policies according to risk preferences.

Keywords Finance · multiobjective decision-making · stability · robustness.

1 Introduction

Cash management focuses on planning and controlling a firm's cash resources. To this end, a number of cash management models have been proposed in the literature. Most of them present a common feature: they follow a inventory approach in which cash balances are controlled by means of bounds. Cash balances are allowed to wander around until one of these bounds, usually a

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higher bound and a lower bound, is reached. Then, a control action is made to restore the balance to a given target.

From the initial inventory approach to the cash management problem (CMP) by Baumol (1952), most models have attempted to derive optimal policies within the framework of some simple policy, typically employing time-invariant cash balance bounds. A slight departure of this framework was considered by Stone (1972); Gormley and Meade (2007) to introduce forecasts as key inputs to bound-based models. Cash management models are usually linked to a particular cash flow process ranging from the uniform and perfectly known cash flow in Baumol (1952) and Tobin (1956), to purely stochastic behavior in Miller and Orr (1966); Eppen and Fama (1969); Constantinides and Richard (1978); Baccarin (2009); Premachandra (2004); da Costa Moraes and Nagano (2014), which usually implies a normal, independent and stationary cash flow distribution. Surprisingly, little and/or contradictory empirical evidence on the assumption of normal, independent and stationary cash flows has been provided (Homonoff and Mullins, 1975; Emery, 1981; Pindado and Vico, 1996).

Recent cash management research proposed: (i) the use of the standard deviation of daily cost as a measure of risk within a multiobjective approach in which both cost and risk are goals to minimize (Salas-Molina et al, 2016); (ii) genetic algorithms and particle swarm optimization techniques to derive policies (Gormley and Meade, 2007; da Costa Moraes and Nagano, 2014); (iii) to set cash balance reference trajectories as a control strategy (Herrera-Cáceres and Ibeas, 2016); and (iv) cash flow forecasting as a way to reduce both uncertainty and costs in cash management (Salas-Molina et al, 2017).

However, there are some open research questions that are worth tackling in cash management from a multiobjective perspective. On the one hand, we can think of alternative measures of risk to standard deviation of daily cost used in Salas-Molina et al (2016) to better capture the notion of risk-sensitive control. In addition, Monte Carlo methods used in Salas-Molina et al (2016) and genetic algorithms proposed in Gormley and Meade (2007); da Costa Moraes and Nagano (2014) do not guarantee the optimality of solutions. On the other hand, unlike Herrera-Cáceres and Ibeas (2016), cash balance reference trajectories as a control strategy must consider the cost of control as suggested by Camacho and Bordons (2007) to be really useful in financial practice. Finally, cash flow characteristics are crucial in cash management since most models rely on these assumptions to provide solutions to the problem. Indeed, real-world cash flow process are neither completely predictable nor totally random (Stone, 1972). Thus, an exploration of the particular characteristics of the cash flow process under consideration is a key issue in cash management.

In this paper, we propose a framework for modelling and controlling cash management systems with multiple bank accounts and general cash flow processes. Unlike most of models presented in the literature, we consider cash management systems with both multiple accounts and multiple transactions between accounts. Our multidimensional cash management model describes the behavior of the system by discrete-time linear difference equations. In addition, we follow a multiobjective control approach in which both the cost and the risk of alternative policies are minimized by means of linear-quadratic objective functions. To this end, we consider linear holding and transaction costs, but also total squared deviations from a reference cash balance as a measure of risk. We focus on the common situation faced by cash managers characterized by much higher penalty costs for negative cash balances in comparison to holding costs for positive cash balances. We guarantee the optimality of solutions by encoding the multiobjective cash management problem as a Mixed Integer Quadratic Program (MIQP). Finally, we analyze the stability for different cash flow processes and the robustness of the model to misspecifications in cash flow dynamics.

This paper is organized as follows. In Section 2, we describe our linear-quadratic modelling framework. In Section 3, we provide a linear reformulation of our model. Next, in Section 4, we analyze the stability cash management systems. In Section 5, we numerically explore the robustness of our model to cash flow dynamics misspecifications. In Section 6, we present a case study showing how our risk-sensitive model can be used to adjust policies according to risk preferences. Finally, we conclude in Section 7 suggesting natural extensions of our work.

2 Multiobjective control of cash management systems

In this section, we formally introduce our multiobjective cash management model. Within the common two-assets setting in cash management (Miller and Orr, 1966), consider a company with two bank accounts as depicted in Figure 1. Account 1 receives payments from customers (inflows) and it is also used to send payments to suppliers (outflows). Both inflows and outflows are summarized through the net cash flow f_{1t}. Account 2 represents the amount of alternative investments available to be converted into cash through transaction u_{1t} when needed. In addition, idle cash balances from account 1 can be allocated in account 2 for a profit through transaction u_{2t}.

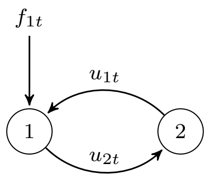

Fig. 1 The common two-assets setting in the cash management problem.

Let $\{\boldsymbol{b}_t : t = 1, 2, \ldots, T\}$ a sequence of m-dimensional vectors describing the state of a cash management system at time t in terms of available cash balance for m different accounts and a planning horizon of T time steps. Let $\{\boldsymbol{f}_t : t = 1, 2, \ldots, T\}$ a sequence of m-dimensional vectors with cash flows for each account within a cash management system. Then, cash managers can deploy policy $\{\boldsymbol{u}_t : t = 1, 2, \ldots, T\}$ as a sequence of n-dimensional control actions through n possible transactions between accounts. As a result, the law of motion of such a cash management system can be expressed as:

$$\boldsymbol{b}_t = A \cdot \boldsymbol{b}_{t-1} + B \cdot \boldsymbol{u}_t + C \cdot \boldsymbol{f}_t \tag{1}$$

where A is an $m \times m$ matrix; B is an $m \times n$ incidence matrix with element B_{ij} set to: 1 if transaction j adds cash to account i, -1 if transaction j removes cash from account i, and zero otherwise; and C is another $m \times m$ matrix. As an illustrative example, we can represent the system in Figure 1 by setting:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{2}$$

Then, the system behavior for each particular bank account is given by the following system of linear difference equations:

$$\begin{bmatrix} b_{1t} \\ b_{2t} \end{bmatrix} = \begin{bmatrix} b_{1t-1} \\ b_{2t-1} \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f_{1t} \\ f_{2t} \end{bmatrix} \tag{3}$$

The objective of cash managers is to minimize holding and transaction costs:

$$y_t = \boldsymbol{h}' \cdot \boldsymbol{b}_t + \boldsymbol{\gamma}_1' \cdot \boldsymbol{u}_t + \boldsymbol{\gamma}_0' \cdot \boldsymbol{z}_t \tag{4}$$

where $\boldsymbol{h}$ is an $m \times 1$ vector with holding costs per money unit for each account, $\boldsymbol{\gamma}_1$ is and $n \times 1$ vector with variable transaction costs, and $\boldsymbol{\gamma}_0$ is and $n \times 1$ vector with fixed transaction costs. Since fixed transaction costs are only charged when a control action occurs, we need an auxiliary n-dimensional binary vector $\boldsymbol{z}_t \in \{0, 1\}^n$ satisfying the following constraints (Bemporad and Morari, 1999):

$$k \cdot \boldsymbol{z}_t \leq \boldsymbol{u}_t \leq K \cdot \boldsymbol{z}_t \tag{5}$$

where K (k) is a very large (small) number.

However, cash managers may be interested not only in the cost but also in the risk of alternative policies. In the usual context of much higher penalty costs for negative cash balances in comparison to holding costs for positive cash balances, minimum cash balances must be kept as references for precautionary purposes. Reference signals are typically used to determine the necessary control actions in cash management systems (Herrera-Cáceres and Ibeas, 2016) and other control systems (Camacho and Bordons, 2007). Furthermore, the sum of deviations from these references at each time step can be viewed as a measure of risk from the cash manager perspective. Indeed, the higher the deviation from a given reference, the riskier the policy since the probability to be charged with unexpected penalty costs is higher. As a result, we here follow the approach of controlling risk by setting a positive target cash balance as a reference for each bank account. Then, we rewrite the cost function in equation (4) as the following cost-risk function:

$$y_t = w_1 \left(\boldsymbol{h}'\boldsymbol{b}_t + \boldsymbol{\gamma}_1'\boldsymbol{u}_t + \boldsymbol{\gamma}_0'\boldsymbol{z}_t \right) + w_2 \left((\boldsymbol{b}_t - \boldsymbol{b}_r)' Q (\boldsymbol{b}_t - \boldsymbol{b}_r) \right) \tag{6}$$

where $\boldsymbol{b}_r$ is an $m \times 1$ vector with cash balance references for each account; and w_1, w_2 are weights reflecting risk preferences of cash managers subject to $w_1 + w_2 = 1$. Under this framework, risk control takes the form of minimizing squared deviations from a reference balance. Note also that Q is used to select which accounts need to be controlled (or even to weight such a control). For instance, setting:

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{7}$$

implies that we are controlling balances for account 1, but not for account 2.

As a result, we are facing a multiobjective optimization problem in which cost and risk are desired goals to minimize. In order to guarantee the optimality of solutions, we encode the cash management problem as the following Mixed Integer Quadratic Program (MIQP):

$$\min \sum_{t=1}^{T} \left[\frac{w_1}{C_{max}} \left(\boldsymbol{h}'\boldsymbol{b}_t + \boldsymbol{\gamma}_1'\boldsymbol{u}_t + \boldsymbol{\gamma}_0'\boldsymbol{z}_t \right) + \frac{w_2}{R_{max}} \left((\boldsymbol{b}_t - \boldsymbol{b}_r)' Q (\boldsymbol{b}_t - \boldsymbol{b}_r) \right) \right] \tag{8}$$

subject to:

$$\boldsymbol{b}_t = A \cdot \boldsymbol{b}_{t-1} + B \cdot \boldsymbol{u}_t + C \cdot \boldsymbol{f}_t \tag{9}$$

$$\boldsymbol{b}_t, \boldsymbol{b}_r \in \mathbb{R}^m_{\geq 0}, \quad \boldsymbol{u}_t \in \mathbb{R}^n_{\geq 0}, \quad \boldsymbol{z}_t \in \{0, 1\}^n \tag{10}$$

$$t = 1, 2, \ldots, T \tag{11}$$

$$w_1 + w_2 = 1 \tag{12}$$

where C_{max}, R_{max} are normalization factors to avoid meaningless comparison between cost and risk goals. These factors may also be viewed as budget constraints for the whole planning horizon T in terms of both cost and risk leading to an unfeasible policy when exceeded. For comparative purposes, we can set these factors to the cost ($w_2 = 0$) and risk ($w_1 = 0$) of deploying a trivial policy such as taking no control action by means of equation (6) when both $\boldsymbol{u}_t$ and $\boldsymbol{z}_t$ are vectors with all entries set to zero for the whole planning horizon.

In practice, cash flow shocks randomly occur between the beginning and the end of each time-step, e.g., a working day. Similarly, cash managers make control decisions after these random shocks. However, new random shocks occur immediately after the last control action requiring a new policy. In order to model this common situation, we assume that both random shocks $\boldsymbol{f}_t$ and control actions $\boldsymbol{u}_t$ occur simultaneously. The duration of the time-step, being either days or minutes, will determine the frequency of control. Finally, it is also important to highlight that the MIQP encoded from equations (8) to (12) can be solved using state-of-the-art mathematical programming solvers such as Gurobi (Gurobi Optimization, Inc, 2016).

3 An equivalent linear reformulation

Since linearity is usually a desired feature in mathematical programming for computational reasons, we next provide an equivalent linear reformulation of the optimization described in Section 2 by relying on goal programming (Abdelaziz et al, 2007; Ballestero and Garcia-Bernabeu, 2012; Aouni et al, 2014). Briefly, goal programming aggregates goals to obtain a solution that minimizes the sum of deviations between achievement and the aspiration levels (or targets) of the goals. The underlying idea behind goal programming is that the decision-maker follows a satisfying logic expressed by means of targets. By establishing an achievement objective function, goal programming aims to conciliate the achievement of a set of goals instead of optimizing every goal.

Then, for each goal g_i, it is necessary to determine the aspiration level or target $G_i \in \mathbb{R}$, with $i = 1, 2, \ldots q$, being q the number of different goals considered. Next, positive δ_i^+ and negative δ_i^- deviation auxiliary variables are introduced to connect goal achievement and targets. Then, we express a general goal programming model as follows:

$$\min \sum_{i=1}^{q} (\delta_i^+ + \delta_i^-) \tag{13}$$

subject to:

$$g_i(\boldsymbol{x}) + \delta_i^- - \delta_i^+ = G_i \tag{14}$$

$$\delta_i^-, \delta_i^+ \geq 0 \tag{15}$$

where each $g_i(\boldsymbol{x})$ is a particular goal defined for any feasible solution $\boldsymbol{x}$. In order to linearize the objective function from equation (8), let us consider two goals, namely, cost and risk. On the one hand, the cost target is obviously zero and we are interested in minimizing only the sum of positive deviations from this target. On the other hand, the risk target is the cash balance reference $\boldsymbol{b}_r$ and we aim to minimize the sum of both positive and negative deviations from this target. As a result, we can reformulate the quadratic objective function (8) to encode the cash management problem as a Mixed Integer Linear Program (MILP) as follows:

$$\min \left[\frac{w_1}{C_{max}} \sum_{t=1}^{T} \delta_{1t}^+ + \frac{w_2}{R_{max}} \sum_{t=1}^{T} \left(\delta_{2t}^+ + \delta_{2t}^- \right) \right] \tag{16}$$

subject to:

$$\boldsymbol{b}_t = A \cdot \boldsymbol{b}_{t-1} + B \cdot \boldsymbol{u}_t + C \cdot \boldsymbol{f}_t \tag{17}$$

$$\boldsymbol{h}'\boldsymbol{b}_t + \boldsymbol{\gamma}_1'\boldsymbol{u}_t + \boldsymbol{\gamma}_0'\boldsymbol{z}_t - \delta_{1t}^+ \leq 0 \tag{18}$$

$$\mathrm{diag}(Q)'\boldsymbol{b}_t - \delta_{2t}^+ + \delta_{2t}^- = \mathrm{diag}(Q)'\boldsymbol{b}_r \tag{19}$$

$$\delta_{1t}^+, \delta_{2t}^+, \delta_{2t}^- \geq 0 \tag{20}$$

$$\boldsymbol{b}_t, \boldsymbol{b}_r \in \mathbb{R}_{\geq 0}^m, \quad \boldsymbol{u}_t \in \mathbb{R}_{\geq 0}^n, \quad \boldsymbol{z}_t \in \{0, 1\}^n \tag{21}$$

$$t = 1, 2, \ldots, T \tag{22}$$

$$w_1 + w_2 = 1 \tag{23}$$

where $\mathrm{diag}(Q)$ is an $m \times 1$ vector with elements set to the main diagonal of matrix Q.

4 Stability of cash management systems

A critical issue in cash management control is the cash flow process under study. This process clearly determines the stability of the cash management system. In this section, we adapt the standard definitions of stability (Keerthi and Gilbert, 1988; Bemporad and Morari, 1999) to the case of cash management systems. To this end, we restrict ourselves to the usual situation of much higher penalty costs for negative cash balances in comparison to holding cots for positive cash balances. Thus, we here follow the approach of analyzing stability from below, i.e., non-negative cash balances determine the stability of the system and negative cash balances provoke the instability of the system.

Definition 1 A vector $\boldsymbol{b}_t \in \mathbb{R}^m$, defining the state of a cash management system at time t, is said to be an equilibrium state when $\boldsymbol{b}_t$ is non-negative, i.e., when all its entries are greater or equal to zero, $b_{it} \geq 0$, with $i = 1, 2, \ldots m$.

By considering only non-negative cash balances in constraint (10), we are implicitly assuming an infinite penalty cost for negative cash balances. Under this framework, negative cash balances lead to unfeasible policies.

Definition 2 A sequence of vectors $\boldsymbol{b}_t \in \mathbb{R}^m$ derived from policy $\boldsymbol{u}_t \in \mathbb{R}^n_{\geq 0}$ and cash flow $\boldsymbol{f}_t \in \mathbb{R}^m$ is said to be stable when $\boldsymbol{b}_t$ is an equilibrium state for $t = 1, 2, \ldots, T$. The pair $(\boldsymbol{u}_t, \boldsymbol{b}_t)$ is said to be a stable policy-state pair for $\boldsymbol{f}_t$.

However, the uncertainty associated to the cash flow process under consideration may lead to an eventual negative cash balance even after deploying the optimal policy. Indeed, we relax the strong assumption in the common two-assets setting of an infinite cash buffer (e.g. in account 2 in Figure 1). Thus, we consider that the only available cash is given by the initial state of the system and the cumulative net cash flow for the whole planning horizon. This fact leads us to consider the possibility that an unexpected cash flow swing results in an undesirable cash balance situation. As a result, we next introduce the concepts of weak α-stability and strong α-stability, which we define in terms of the particular probability distribution of the cash flow process.

Definition 3 A stable policy-state pair $(\boldsymbol{u}_t, \boldsymbol{b}_t)$ for $\boldsymbol{f}_t$ sampled from a multivariate cash flow process $\boldsymbol{f}$ with probability density function $h(\boldsymbol{f})$ is said to be weak α-stable when $\mathrm{P}(\boldsymbol{f} \leq \mathrm{E}[\boldsymbol{b}_t]) \geq \alpha$, where P denotes probability and operator E computes the element-wise average over t.

Definition 4 A stable policy-state pair $(\boldsymbol{u}_t, \boldsymbol{b}_t)$ for $\boldsymbol{f}_t$ sampled from a multivariate cash flow process $\boldsymbol{f}$ with probability density function $h(\boldsymbol{f})$ is said to be strong α-stable when $\mathrm{P}(\boldsymbol{f} \leq \min[\boldsymbol{b}_t]) \geq \alpha$, where the operator min computes the element-wise minimum over t.

In the previous two definitions, we assume that the inequality operator for vectors holds when it does for each pair of entries in the vectors. As an illustrative example, consider a Gaussian process $f \sim \mathcal{N}(0,1)$ for account 1 in Figure 1, and a deterministic distribution for account 2, which always takes a value of zero. Assume also that $\mathrm{E}[\boldsymbol{b}_t] = [3.2 \;\; 5]'$ and $\min[\boldsymbol{b}_t] = [1.5 \;\; 3]'$. Clearly:

$$\mathrm{P}(\mathcal{N}(0,1) \leq 3.2) \geq 0.99 \tag{24}$$

$$\mathrm{P}(0 \leq 5) = 1 \geq 0.99. \tag{25}$$

Then, we can say that the policy-state pair $(\boldsymbol{u}_t, \boldsymbol{b}_t)$ is weak 0.99-stable, but we cannot say that it is strong 0.99-stable since $\mathrm{P}(\mathcal{N}(0,1) \leq 1.5) \geq 0.99$ does not hold. Since the stability of cash management systems is necessarily associated to the cash flow process under consideration, we derive useful insights for different cash flow processes by means of the following remarks:

Remark 1. A stable policy-state pair $(\boldsymbol{u}_t, \boldsymbol{b}_t)$ for a deterministic cash flow process centered at $\boldsymbol{k}_0$ is strong 1-stable if $\boldsymbol{k}_0 \leq \min[\boldsymbol{b}_t]$. Indeed, a degenerate or deterministic multivariate distribution always takes value $\boldsymbol{k}_0$. Since $\min[\boldsymbol{b}_t]$ is $\boldsymbol{k}_0$ at least, then $\mathrm{P}(\boldsymbol{k}_0 \leq \min[\boldsymbol{b}_t]) = \mathrm{P}(\boldsymbol{k}_0 \leq \boldsymbol{k}_0) \geq 1$, hence guaranteeing strong 1-stability.

Obviously, a strong α-stable is also a weak α-stable policy-state pair due to the classical definitions of mean and minimum values. Note also that the initial state is not relevant because the definition of a policy-state pair assumes the feasibility of the policy disregarding its initial condition. When dealing with stochastic but predictable cash flow processes, actual cash balances are necessarily affected by an m-dimensional error term, usually expressed in terms of a Gaussian distribution $\mathcal{N}(0, \boldsymbol{\sigma_e})$ with zero mean and standard deviation $\boldsymbol{\sigma_e} \in \mathbb{R}^m$.

Remark 2. A stable policy-state pair $(\boldsymbol{u}_t, \boldsymbol{b}_t)$ for a predictable cash flow process for a given forecasting error distribution $h_e(\boldsymbol{f})$ is strong α-stable if $\boldsymbol{s}_\alpha \leq \min[\boldsymbol{b}_t]$ where $\boldsymbol{s}_\alpha$ is the m-dimensional quantile function for $h_e(\boldsymbol{f})$ and probability α. Recall that the quantile function $\boldsymbol{s}_\alpha$ for distribution $h_e(\boldsymbol{f})$ specifies the value at which the probability is less than or equal to α. Then, the uncertainty for $\boldsymbol{f}$ is given by $h_e(\boldsymbol{f})$. As a result, if $\boldsymbol{s}_\alpha \leq \min[\boldsymbol{b}_t]$, then $\mathrm{P}(h_e(\boldsymbol{f}) \leq \min[\boldsymbol{b}_t]) \geq \alpha$, hence guaranteeing strong α-stability.

Remark 3. A stable policy-state pair $(\boldsymbol{u}_t, \boldsymbol{b}_t)$ for a predictable cash flow process with Gaussian forecasting error distribution $\mathcal{N}(0, \boldsymbol{\sigma_e})$ is strong α-stable if $z_\alpha \boldsymbol{\sigma_e} \leq \min[\boldsymbol{b}_t]$ where z_α is the quantile function for the standard Gaussian

$\mathcal{N}(0,1)$ with probability α.

Note that for perfect predictions, $\boldsymbol{\sigma_e} = 0$ holds and the forecasting error distribution degenerates to a deterministic distribution centered at zero as it is the case of Remark 1 for $k_0 = 0$. In the opposite limit, purely random cash flow processes can be characterized by a forecasting error distribution equal to the cash flow distribution itself.

Remark 4. A stable policy-state pair $(\boldsymbol{u}_t, \boldsymbol{b}_t)$ for a purely random cash flow process with distribution $h_e(\boldsymbol{f}) = \boldsymbol{f}$ is strong α-stable if $\boldsymbol{s}_\alpha \leq \min[\boldsymbol{b}_t]$ where $\boldsymbol{s}_\alpha$ is the m-dimensional quantile function for $\boldsymbol{f}$ and probability α.

As a result, within the framework of the program encoded from equations (8) to (12), cash managers can deal with stability issues through cash balance reference $\boldsymbol{b}_r$. This reference plays the role of a cash buffer for precautionary purposes so that the higher the reference the lower the risk of instability. Ultimately, risk is also associated to the assumptions made on the cash flow processes. However, cash flow process misspecifications may lead to undesirable situations as we next consider.

5 Robustness and risk-sensitive control

In this section, we graphically explore the robustness of different instances of our risk-sensitive model to a misspecification of the cash flow process. This analysis is closely related to the concept of stability. To this end, we define robust control as a risk-sensitive optimization procedure as proposed by Hansen and Sargent (2008). In this sense, our cash management model described in Section 2 can easily accommodate risk-sensitive analysis by tunning weight parameters w_1 and w_2.

Note that an additional source of uncertainty in the model encoded from equation (8) to (12) may be any misspecification in holding and transaction costs. Beyond the discussion about the relative importance of alternative sources of uncertainty, any cost misspecification usually results in a balance deviation with respect to the expected value. Then, an equivalent formulation is to summarize all sources of uncertainty in the model in a single random variable that affects balances, namely, cash flows. Thus, we here focus on the impact of cash flow misspecifications in an attempt to find robust models in cash management.

5.1 Model predictive control assumptions

In the following numerical experiments, we follow a Model Predictive Control (MPC) approach (Bemporad and Morari, 1999; Camacho and Bordons, 2007). The underlying idea behind MPC is to use a mathematical representation of the system (the model) to predict its future evolution. As described in

Section 2, the law of motion of a cash management system is given by equation (1). Based on this law, cash managers select a sequence of control actions (the policy) that minimizes some objective function according to their particular preferences. Even though short-term planning horizons, e.g., the next five working days, are usually considered in cash management, an on-line optimization procedure is recommended to achieve the desired control feedback. Thus, we here use MPC as an on-line optimization procedure in which only the first sample of the optimal policy for a given planning horizon is actually deployed at each time-step. At the next time-step, a new policy is obtained replacing the previous one, hence providing the desired feedback control.

As an illustrative example, we next evaluate the impact on cost-risk performance for alternative policies. Then, we apply a MPC strategy for a total deployment horizon of 500 days in steps of 50 days, equivalent to more than two working years. To this end, we experiment on the cash management system from Figure 1 and the cost structure summarized in Table 1. We select particular cost values from those recently used in da Costa Moraes and Nagano (2014) for experimental purposes. This analysis can be repeated as many times as desired for any misspecification parameter and different weights w_1 and w_2 in an attempt provide the desired risk-sensitive control. In our experiments, we consider two classes of possible misspecifications in cash flow processes: (i) mean distortions; and (ii) variance distortions.

Transaction	γ_0 (€)	γ_1 (%)	Account	h (%)
1	50	0.01	1	0.02
2	50	0.005	2	0

Table 1 Cost structure data for the examples.

5.2 Robustness to misspecification in cash flow means

Assume a Gaussian cash flow process $\mathcal{N}(\mu, \sigma)$ for account 1. Time-varying circumstances may lead to a misspecification of μ in one way or another. Within the framework of much higher penalty costs for negative cash balances than holding costs for positive cash balances, an overestimation of μ may have dramatic consequences. To face such a problem, cash managers can set a cash balance reference proportional to the variance. Following the recommendations in Ben-Tal and Nemirovski (1999) and Ben-Tal et al (2009), we set cash balance references proportional to the respective volatility of cash flow processes as follows:

$$\boldsymbol{b}_r = D \cdot \boldsymbol{\sigma} \tag{26}$$

where $\boldsymbol{\sigma}$ is an $m \times 1$ vector with the expected standard deviation for each cash flow process; and D is an $m \times m$ diagonal matrix with each element D_{ij}, with $i = j$, being a non-negative parameter reflecting the aversion to risk of a hypothetical cash manager for each account i. For instance, $D_{ij} = 3$ means that a particular cash balance can fall from b_r to zero in one time-step with probability 0.99.

Proportional cash balance references to cash flow volatility provide a first risk control tool for cash managers, but the key parameters to risk-sensitive control of our model are weights w_1 and w_2. Let us now consider a Gaussian cash flow process $\mathcal{N}(0.1, 1)$. We want to evaluate the impact of a negative swing of the mean from 0.1 to zero. To this end, we consider two different risk-sensitive models, model m_1 with $w_1 = 0.5$ and model m_2 with $w_1 = 0.25$. We set a reference balance equivalent to three standard deviations of the cumulative cash flow process over a planning horizon of ten days for optimization purposes ($T = 10$). Then, $\boldsymbol{b}_r = \left(3\sqrt{10},\ 0\right)' = (9.5,\ 0)'$. We also set an initial cash balance for both accounts of $\boldsymbol{b}_0 = (9.5,\ 9.5)'$. Once guaranteed feasibility of the policy, the selection of an initial cash balance does not interfere on the results of the experiment due to the quick adjustment provided by the optimal policy (da Costa Moraes and Nagano, 2014). Note also that the planning horizon T may be different to the deployment horizon for the MPC strategy. In this example, we consider a deployment horizon of 50 days. This setting implies that the optimization procedure for 10 days is repeated 50 times.

We can depict our estimate of the cost-risk robustness of both models as shown in Figure 2. We obtain normalized average cost-risk points by dividing both the cost (holding and transaction costs) and the risk (squared deviation from the references) by the respective maximum values obtained for each of the evaluations. In addition, we represent the uncertainty associated to each model by plotting an ellipse centered in the cost-risk point with horizontal semiaxis equal to the standard deviation of cost and vertical semiaxis equal to the standard deviation of risk. Finally, we measure robustness by the Euclidean distance between model performance due to a misspecification in the mean of the cash flow process.

From Figure 2, we can infer that model m_2 is more robust than m_1 for a negative swing in the mean because of the shorter distance d_2 between m_2 and m_2' than d_1 between m_1 and m_1'. Furthermore, the uncertainty around m_2 and m_2' is remarkably smaller than that of m_1 and m_1'. Indeed, from this simulation we can say that m_2 and m_2' are not significantly different in terms of cost and risk. However, robustness has a price since the average cost for m_1 is slightly lower than that of m_2.

5.3 Robustness to misspecification in cash flow variances

In addition to a change in the mean, a misspecification in the variance of a cash flow process may lead to dramatic consequences in terms of penalty costs for negative cash balances. Within the same setting of Section 5.2, our aim here is to compare the impact of a variance change in the cash flow process under consideration. Note that this variance can be viewed as a misspecification either in the cash flow process or in the forecasting error process. We mentioned

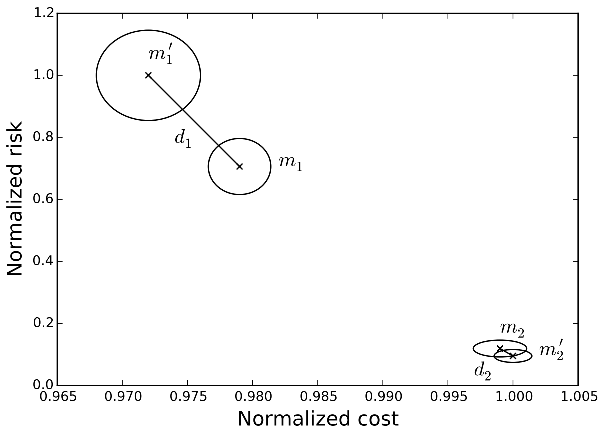

Fig. 2 Robustness to misspecification in cash flow means for two risk-sensitive models.

in the introduction that cash managers can leverage forecasts to achieve cost savings (Salas-Molina et al, 2017). However, forecasts are inevitably affected by some forecasting error, which is usually assumed to be Gausssian and expressed as $\mathcal{N}(0, \sigma_e)$ with zero mean and a given and standard deviation σ_e. Forecasts can be viewed as a reduction in the uncertainty associated to a cash flow process. The lower the uncertainty the lower the variance introduced by forecasting errors ranging from zero for a perfect prediction to the variance of the cash flow itself for a trivial prediction.

As an illustrative example, let us consider a Gaussian cash flow process $\mathcal{N}(0.1, 1)$ for account 1 in Figure 1. We want to evaluate the impact of a 20% increase in the cash flow standard deviation. As in Section 5.2, we consider two different risk-sensitive models, model m_1 with $w_1 = 0.5$ and model m_2 with $w_1 = 0.25$. We set both the same reference balances and the initial conditions to compute the normalized average cost-risk performance as shown in Figure 3.

From Figure 3, we infer that model m_2 is more robust than m_1 for an increase in cash flow variance because of the shorter distance d_2 between m_2 and m_2' than d_1 between m_1 and m_1'. Unlike in the case of mean misspecification shown in Figure 2, the uncertainty around model performance after the change remarkably increased for both models. This fact can be explained by the increase in variance that we evaluate in this second numerical example.

6 Case study

In this section, we show how our risk-sensitive control model can be used to satisfy the requirements of a wide range of cash managers according to their particular risk preferences. As a benchmark, we use the Miller and Orr (1966) model. The bound-based approach proposed by Miller and Orr is at the core

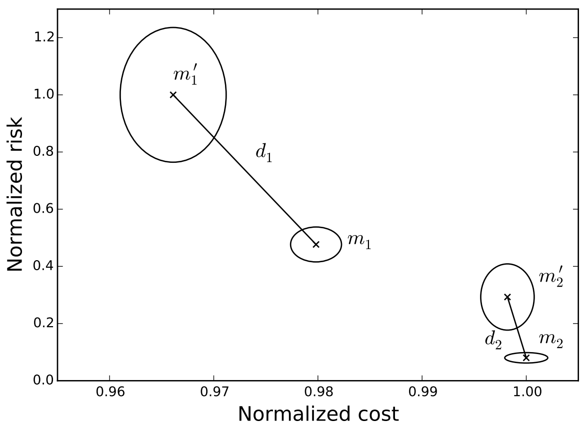

Fig. 3 Robustness to misspecification in cash flow variances for two risk-sensitive models.

of many recent cash management models dealing with stochastic cash flows such as Penttinen (1991); Premachandra (2004); Gormley and Meade (2007). A further advantage of the selection of the Miller and Orr model for illustrative purposes is the possibility to set control bounds based on the statistical properties of cash flows. In what follows, we briefly describe the Miller and Orr (1966) model that we later compare to our risk-sensitive approach.

6.1 The Miller and Orr model

Miller and Orr (1966) proposed a simple policy based on three bounds to control cash balances. This policy implies that when cash balance reaches the upper bound U, a control action is taken to restore the balance to a target level Z. Similarly, when cash balance reaches the lower bound L, a positive transfer is made to restore the balance to Z:

$$u_t = \begin{cases} Z - b_t - f_t & \text{if } b_t + f_t \geq U \\ 0 & \text{if } L < b_t + f_t < U \\ Z - b_t - f_t & \text{if } b_t + f_t \leq L \end{cases} \tag{27}$$

Note that control actions are taken at the end of each time step, when the current balance is known after cash flow f_t occurs. A similar policy could be deployed before knowing the actual cash flow by removing f_t from equation (27). Although Miller and Orr set the lower limit L to zero, cash managers should set a lower limit above zero for precautionary motives (Ross et al, 2002). To this end, we set a lower bound L proportional to the standard deviation (σ) of the particular cash flow process under consideration:

$$L = \xi \cdot \sigma \tag{28}$$

where ξ is a parameter reflecting the aversion to risk of cash managers. The higher the value of ξ, the more averse to risk they are. Typical values for ξ are 2 or 3 resulting in either a probability of 0.95 or 0.99, respectively, of bringing balance from the lower bound L to a negative value for Gaussian cash flows. Later, we can set Z and U according to the formulas proposed by Miller and Orr:

$$Z = L + \left(\frac{3\gamma_0 \sigma^2}{4h} \right)^{1/3} \tag{29}$$

and

$$U = 3Z - 2L \tag{30}$$

where γ_0 is the fixed transaction cost and h is the holding cost per money unit. We are now in a position to compare the Miller-Orr model to our risk-sensitive approach by means of the following numerical exercise.

6.2 Risk-sensitive control example

In the following numerical example, we show what is risk-sensitive control in practice. In this experiment, we use a real-world data set of 2717 net cash flows with standard deviation $\sigma = 0.096$ millions of Euros from an industrial company. Consider again a simple cash management system such as the one depicted in Figure 1, under the cost structure described in Table 1. Recall that we are interested in controlling balances for cash account 1 by means of transactions from/to an investment account 2.

A typical approach to control balances based on a set of bounds is described in Section 6.1 for the case of the Miller and Orr (1966) model. After selecting a typical value $\xi = 2$, we set $L = 0.193$, $Z = 0.313$ and $U = 0.555$ (figures in millions of Euros) according to equations (28), (29) and (30), respectively. Starting at an arbitrary initial cash balance Z, and randomly sampling 100 observations from our cash flow data set, we can simulate the resulting cash balance for account 1 by applying the Miller and Orr model described in equation (27). The evolution of the Miller and Orr controlled balance is depicted in Figure 4 using a dotted line.

Alternatively, we can follow a risk-sensitive approach by solving the MIQP encoded from equation (8) to (12) according to some particular cost-risk preferences summarized in weights w_1 and w_2. As in Section 5, we here follow an MPC strategy updating the optimal policy at each time-step for a given planning horizon. As mentioned in Section 2, we set C_{max} and R_{max} to the cost and risk of taking no control action for the whole planning horizon. For comparative purposes, we use target value Z as the cash balance reference to minimize deviations through objective function (8). The resulting cash balance derived from the application of our risk-sensitive model for a neutral cash manager represented by weights $w_1, w_2 = 0.5$ is shown in Figure 4 using a solid line. Note that the cash balance variability is lower than in the case of the Miller and Orr model.

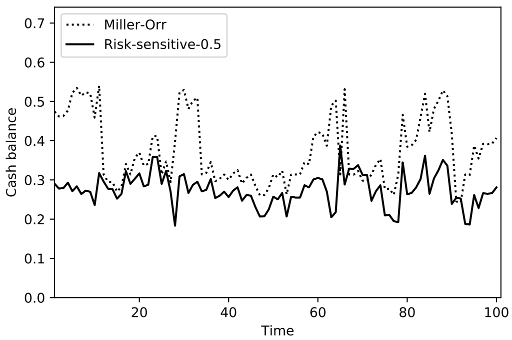

Fig. 4 Balance comparison with Miller and Orr for weights $w_1 = 0.5$ and $w_2 = 0.5$.

We can perform an interesting additional exercise by varying weights to $w_1 = 0.2$ and $w_2 = 0.8$ as in the case of a conservative cash manager that gives more importance to risk than to cost. The resulting cash balance derived from the application of our risk-sensitive model for a conservative cash manager represented by these weights is shown in Figure 5 by means of a solid line and it is compared again to the benchmark. In this case, our risk-sensitive model for conservative managers produces balances with remarkable less variability than both the model for neutral managers in Figure 4 and the Miller and Orr model.

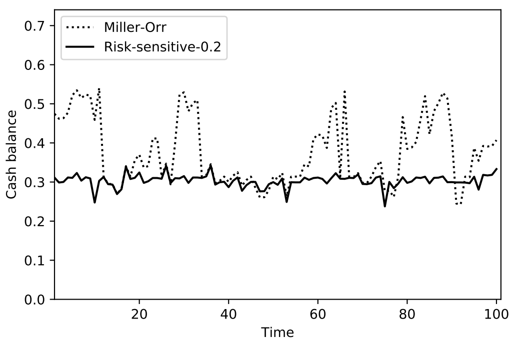

Fig. 5 Balance comparison with Miller and Orr for weights $w_1 = 0.2$ and $w_2 = 0.8$.

As a result, we can vary weights w_1 and w_2 in our risk-sensitive model to satisfy the requirements of a wide range of cash managers in terms of risk preferences. Indeed, there is a trade-off between cost and risk achievement as it is customary in multiple criteria decision-making. Following with our numerical example, Table 2 summarizes the cost and risk achievements derived from our risk-sensitive model expressed as a proportion of normalization factors C_{max} and R_{max} for different weights. Note that cost can be notably reduced by accepting higher levels of risk and that risk can be almost reduced to zero by accepting higher levels of cost. Interestingly, the best combined goal achievement is attained when risk is almost zero due to a constant balance evolution over time.

Table 2 Cost and risk proportions of C_{max} and R_{max} for different weights w_1 and w_2.

w_1	w_2	Cost	Risk	Objective
0.20	0.80	0.48	0.01	0.10
0.40	0.60	0.36	0.02	0.16
0.50	0.50	0.35	0.04	0.19
0.60	0.40	0.33	0.07	0.23
0.80	0.20	0.19	0.36	0.23

7 Concluding remarks

In this paper, we consider short-term decision-making in cash management from a cost-risk multiobjective perspective. More precisely, we focus on the common situation faced by cash managers characterized by much higher penalty costs for negative cash balances in comparison to holding costs for positive cash balances. Within this framework, we propose a multiobjective cash management model to deal with both multiple accounts and transaction between accounts. Since our model focuses on both the cost and the risk of alternative policies, we compute transaction and holding costs as a measure of cost and squared deviations from a given reference trajectory as a measure of risk. Unlike recent research on cash management, we guarantee the optimality of solutions by encoding the problem as a multiobjective linear-quadratic program. Since linearity is a desired feature in mathematical programming, we also provide an equivalent linear reformulation based on goal programming.

We present further insights on the stability of cash management systems from a theoretical point of view. In order to adapt the standard definition of stability to the cash management problem with high penalty costs for negative cash balances, we follow the approach of analyzing stability from below. We also study the stability of different cash flow processes from a probabilistic perspective through the concepts of weak and strong α-stability, which we introduce in this paper. Furthermore, we graphically explore the robustness of our model to misspecifications in both means and variances of multidimensional cash flow processes. We show how tuning a simple parameter of our model can be used to find more robust policies in cash management. Our graphical approach rely on representing uncertainty in the cost-risk space by means of ellipses. Further research can rely on this method to elicit a more general method to select more robust cash management models.

On the applicability of our model for real-world cash management, it is important to highlight that stability or absence of variability is usually a desired objective in financial contexts. From that fact, one can extract the conclusion that cash management models considering not only the cost but also the balance variability of alternative policies are helpful tools in practice. By means of a numerical case study, we show how our risk-sensitive control model can be tuned to accommodate the requirements of cash managers according to their particular risk preferences. On the other hand, we emphasize the importance of cash flow forecasting in cash management as a suitable way to reduce uncertainty about the near future. Thus, models accepting forecasts as a key input such as the one proposed in this paper are closer to real practice by relying on data-driven techniques such as forecasting rather than models assuming a theoretical probability distribution. A further advantage of our approach is its computational tractability. By relying on linear-quadratic programming, our model can be solved by state-of-the-art commercial solvers such as Gurobi and CPLEX. This fact implies that our model can be easily embedded in decision support systems for cash management.

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